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Strong approximation of sets in BV(Ω)

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Taking the argument used by De Giorgi to obtain the rectifiability of the reduced boundary of a set of finite perimeter in $\mathbb{R}^N$, we prove that a set E of finite perimeter in an open set Ω of $\mathbb{R}^N$ may be approached, in the sense of the BV(Ω) norm, by sets whose boundary is included in a finite union of $\mathcal{C}^1$ hypersurfaces; more precisely, arbitrarily large parts (for the $\mathcal{H}_{N-1}$ measure in Ω) of the essential boundaries of E and of the approximating set coincide and are included in a single $\mathcal{C}^1$ hypersurface.

1. Introduction

It is well known that the continuous functions are not dense in the normed space BV(Ω), and that $\mathcal{C}^\infty(\Omega)$ is dense in BV(Ω), only for the topology of the convergence in $L^1(\Omega)$ associated with the convergence of the total variations of the derivatives measures (see [2]). Next, for this topology (strict convergence in [1]), the sets of finite perimeter in Ω are known to be limits of sets with $\mathcal{C}^\infty$ boundary in Ω (see [10]).

Taking again the argument used by De Giorgi to obtain the rectifiability of the reduced boundary of a set of finite perimeter, we prove that a set of finite perimeter in an open set Ω of $\mathbb{R}^N$ may be approached, in the sense of the BV(Ω) norm, by sets whose boundaries are included in a finite union of $\mathcal{C}^1$ hypersurfaces.

From this fact, we can deduce (as indicated in [14]), the density of the semi-continuous functions in the normed space BV(Ω); this property was our earlier motivation in view of a Γ-convergence problem, studied by Picard [13].

The approximation theorem of integral currents by Federer (see [9, § 4.2.20]) is an analogous result. However, the following proof seems to be more direct and quite different from Federer's. Cerf has found a new use of this result in statistical mechanics, namely in the approximation of Cacciopoli partitions (see [4]). This has increased interest in the detailed proof presented here.

2. Notation

Let $E \subset \mathbb{R}^N$, $N \geqslant 2$. We denote by χ_E its characteristic function. If E is measurable, we denote by $m_N(E)$ or $m(E)$ its Lebesgue measure. For Ω an open subset of $\mathbb{R}^N$ and $u \in L^1_{\mathrm{loc}}(\Omega)$ we define

$$\int_\Omega |Du| = \sup\biggl\{ \int u(x)\,\mathrm{div}\,\phi(x)\,\mathrm{d}x \biggm| (\phi_1,\dots,\phi_N) \in \mathcal{C}^1_0(\Omega)^N,\ \sum_{1\leqslant i\leqslant N} \phi_i^2 \leqslant 1 \biggr\}.$$

In the case when $\int_\Omega |Du| < \infty$, we denote by Du the vector-bounded Radon measure defined by

$$\int \phi \cdot Du = -\int u(x)\,\mathrm{div}\,\phi(x)\,\mathrm{d}x \quad \forall \phi \in \mathcal{C}^1_0(\Omega)^N.$$

We denote by $|Du|$ the total variation measure associated to Du. We also set

$$\mathrm{BV}(\Omega) = \left\{ u \in L^1(\Omega) \;\middle|\; \int_\Omega |Du| < \infty \right\},$$

$$\mathrm{BV}_{\mathrm{loc}}(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) \;\middle|\; \int_{\Omega'} |Du| < \infty \text{ for all open sets } \Omega' \subset\subset \Omega \right\}.$$

We then define $\mathcal{P}_\Omega(E) = \int_\Omega |D\chi_E|$; E is said to have a finite perimeter (respectively, locally finite perimeter) in Ω if E is measurable and $\mathcal{P}_\Omega(E) < \infty$ (respectively, $\chi_E \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$). Let E be a set having locally finite perimeter in Ω. A point $x \in \Omega$ belongs to the reduced boundary of E in Ω, denoted by $\Omega \cap \mathcal{F}^*E$, if

(i) $|D\chi_E|(B(x,\rho)) > 0$ for all $\rho > 0$,

(ii) $n_E(x) = \lim_{\rho \to 0^+} \dfrac{D\chi_E(B(x,\rho))}{|D\chi_E|(B(x,\rho))}$ exists,

(iii) $|n_E(x)| = 1$.

$B(x,\rho)$ (respectively, $Q(x,\rho)$) denotes the open ball (respectively, the open cube) of radius ρ centred at x. We extend the map $n_E(x)$ by setting $n_E(x) = 0$ for $x \in \Omega \setminus \mathcal{F}^*E$. Besicovitch's theorem implies that $D\chi_E = n_E|D\chi_E|$. By localizing De Giorgi's results [5,6], we have the following.

(a) For any Borelian set $B \subset \Omega$,

$$|D\chi_E|(B) = \mathcal{H}_{N-1}(B \cap \mathcal{F}^*E),$$

where $\mathcal{H}_{N-1}$ is the $(N-1)$-dimensional Hausdorff measure.

(b) For any $x \in \Omega \cap \mathcal{F}^*E$,

$$\lim_{\rho \to 0^+} \rho^{-N} m(B(x,\rho) \cap E \cap \Pi^-_E(x)) = 0,$$

$$\lim_{\rho \to 0^+} \rho^{-N} m(B(x,\rho) \cap (\Omega \setminus E) \cap \Pi^+_E(x)) = 0,$$

$$\lim_{\rho \to 0^+} \rho^{1-N} |D\chi_E|(B(x,\rho)) = \omega_{N-1},$$

where

$$\omega_{N-1} = m_{N-1}(\{x \in \mathbb{R}^{N-1} \mid |x| \leqslant 1\})$$

and $\Pi^+_E(x)$, $\Pi^-_E(x)$ are the half-spaces defined for $x \in \Omega \cap \mathcal{F}^*E$ by

$$\Pi^+_E(x) = \{y \in \mathbb{R}^N \mid (y-x)\cdot n_E(x) > 0\},$$

$$\Pi^-_E(x) = \{y \in \mathbb{R}^N \mid (y-x)\cdot n_E(x) < 0\}.$$

In fact, when Ω is an arbitrary open set and B is an open ball whose closure is contained in Ω, $B \cap E$ has finite perimeter in $\mathbb{R}^N$ and $B \cap \mathcal{F}^*(B \cap E) = B \cap \mathcal{F}^* E$. We also set $\omega_N = m_N(\{x \in \mathbb{R}^N \mid |x| \leqslant 1\})$. For E a measurable subset of $\mathbb{R}^N$, we define the essential interior E_* of E by

$$x \in E_* \quad \Longleftrightarrow \quad \lim_{\rho \to 0^+} \frac{1}{\omega_N \rho^N} m(B(x,\rho) \cap E) = 1$$

and we call the essential boundary of E the set $\partial^* E$ which, together with E_* and $(\mathbb{R}^N \setminus E)_*$, forms a partition of $\mathbb{R}^N$.

Whenever $\mathcal{P}_\Omega(E) < \infty$, we have $\mathcal{H}_{N-1}(\Omega \cap (\partial^* E \setminus \mathcal{F}^* E)) = 0$ (see [9, 15]). We denote by ∂E the topological boundary of E. We have $\Omega \cap \mathcal{F}^* E \subset \Omega \cap \partial^* E \subset \Omega \cap \partial E$. In the special case where $E = \{x \in \Omega \mid g(x) \geqslant 0\}$ for some function g in $\mathcal{C}^1(\Omega)$ having 0 as a regular value (i.e. $Dg(x) \neq 0$ when $g(x) = 0$), $\Omega \cap \mathcal{F}^* E = \Omega \cap \partial E = \{x \in \Omega \mid g(x) = 0\}$, and if $x \in \Omega \cap \partial E$, then $n_E(x) = Dg(x)/|Dg(x)|$, which is the usual interior normal vector to E at x. Note that, for some subset E of Ω, the regularity of $\Omega \cap \partial E$ is not quite sufficient to imply the coincidence of $\Omega \cap \partial E$ with $\Omega \cap \mathcal{F}^* E$ (see [7, § 1.3]).

3. Strong approximation of sets of finite perimeter

THEOREM 3.1. *Let Ω be an open subset of $\mathbb{R}^N$. Let $E \subset \Omega$ be a set having finite perimeter in Ω (i.e. $\chi_E \in L^1_{\mathrm{loc}}(\Omega)$ and $\int_\Omega |D\chi_E| < \infty$). Let $\varepsilon > 0$. There exists a set $L \subset \Omega$, having finite perimeter in Ω, and a compact set $C \subset \Omega$ such that*

$$\Omega \cap \partial L \text{ is contained in a finite union of } \mathcal{C}^1 \text{ hypersurfaces,} \tag{3.1}$$

$$\int_\Omega |\chi_E - \chi_L| < \varepsilon, \tag{3.2}$$

$$\int_\Omega |D(\chi_E - \chi_L)| < \varepsilon, \tag{3.3}$$

$$\mathcal{H}_{N-1}(\Omega \cap \partial L \setminus \mathcal{F}^* L) < \varepsilon, \tag{3.4}$$

$$L \subset E + B(0,\varepsilon), \qquad \Omega \setminus L \subset (\Omega \setminus E) + B(0,\varepsilon), \tag{3.5}$$

$$C \subset \Omega \cap \mathcal{F}^* E \cap \mathcal{F}^* L, \tag{3.6}$$

$$n_E(x) = n_L(x) \quad \forall x \in C, \tag{3.7}$$

$$|D\chi_E|(\Omega \setminus C) < \varepsilon. \tag{3.8}$$

REMARK 3.2.

(i) We might assume that L is open or closed in Ω, for condition (3.1) implies that $m(\Omega \cap \partial L) = 0$; thus, when the boundary in Ω is added to or subtracted from L, conditions (3.1)–(3.8) hold (if necessary, ε is increased a little to ensure that (3.5) holds).

(ii) In fact, the proof of theorem 3.1 yields the following more detailed result, used in [4] by Cerf, who also suggested to the author condition (3.4), above.

Let Ω be an open subset of $\mathbb{R}^N$, let E be a set of finite perimeter in Ω and let ε be positive. There exists a set L of finite perimeter, a C^1 function

$f : \Omega \mapsto \mathbb{R}$, a compact set C, an open set V and an open bounded set B such that, setting $F = \{x \in \Omega : f(x) \geqslant 0\}$, the set $V \cap \partial F$ is the hypersurface $\{x \in V : f(x) = 0\}$ and

$$C \subset B \subset V \subset \{x \in \Omega : Df(x) \neq 0\}, \quad C \subset \Omega \cap \partial^* E \cap \partial F,$$
$$L \cap B = F \cap B, \quad V \cap \partial^* F = V \cap \partial F,$$
$$\nu_E(x) = \nu_F(x) = -|Df(x)|^{-1} Df(x) \quad \forall x \in C,$$
$$m(V) < \varepsilon, \quad m(E \Delta L) < \varepsilon,$$
$$\mathcal{H}_{N-1}(\partial F \cap (V \setminus C)) < \varepsilon,$$
$$\mathcal{H}_{N-1}(\Omega \cap \partial^* E \setminus C) < \varepsilon,$$
$$\mathcal{H}_{N-1}(\Omega \cap (\partial^* E \Delta \partial L)) < \varepsilon,$$
$$L \subset E + B(0, \varepsilon), \quad \Omega \setminus L \subset (\Omega \setminus E) + B(0, \varepsilon).$$

4. Preliminary lemmas

4.1. Results on the blow-up of the reduced boundary

LEMMA 4.1. *Let U be an open subset of $\mathbb{R}^N$, let $F = \{x \in U \mid f(x) \geqslant 0\}$ for some $f \in \mathcal{C}^1(U)$ for which 0 is a regular value and let C be a compact subset of $U \cap \partial F$. Then the limits*

$$\lim_{\rho \to 0^+} \rho^{-N} m(B(x, \rho) \cap F \cap \Pi_F^-(x)) = 0,$$
$$\lim_{\rho \to 0^+} \rho^{-N} m(B(x, \rho) \cap F^{\mathrm{c}} \cap \Pi_F^+(x)) = 0$$

are uniform over $x \in C$.

Proof. The compact set C is covered by a finite union of balls included in U, inside which ∂F is, up to an isometry, the graph of a function of class $\mathcal{C}^1$, whose differential map is uniformly continuous. □

LEMMA 4.2. *Let E be a set having finite perimeter in Ω. Let C be a closed subset of $\Omega \cap \mathcal{F}^* E$ such that the limits*

$$\lim_{\rho \to 0^+} \rho^{-N} m(B(x, \rho) \cap E \cap \Pi_E^-(x)) = 0,$$
$$\lim_{\rho \to 0^+} \rho^{-N} m(B(x, \rho) \cap E^{\mathrm{c}} \cap \Pi_E^+(x)) = 0$$

are uniform over $x \in C$, and such that the restriction of n_E to C is continuous. Then there exists $f \in \mathcal{C}^1(\Omega)$ such that

$$C \subset \{x \in \Omega \mid f(x) = 0,\ Df(x) = n_E(x)\}$$

Proof. See [6] or [11]. Let us recall that De Giorgi's proof consists in proving that

$$\lim_{|x-y| \to 0} n_E(x) \cdot \frac{x - y}{|x - y|} = 0$$

uniformly over $(x, y) \in C^2$, and then in applying Whitney's extension theorem to the germ $(0, n_E)$ on C. □

4.2. Weak approximation in BV(Ω)

The following results rely on classical techniques originating from [2] (see also, for example, [11]). We state and prove the precise versions that we need.

PROPOSITION 4.3. *Let Ω be an open set in $\mathbb{R}^N$ and let $u \in L^1_{\text{loc}}(\Omega)$ such that $\int_\Omega |Du| < \infty$. Let $\delta > 0$. There exists a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{C}^\infty(\Omega)$ such that*

$$u_n - u \in L^1(\Omega) \quad \forall n \in \mathbb{N} \quad \textit{and} \quad \lim_{n\to\infty} \int_\Omega |u_n - u| = 0, \tag{4.1}$$

$$\lim_{n\to\infty} \int_\Omega |Du_n| = \int_\Omega |Du|. \tag{4.2}$$

Moreover, if $a \leqslant u(y) \leqslant b$ for all $y \in \Omega \cap B(x, 1/n)$, then $a\delta \leqslant u_n(x) \leqslant b(2-\delta)$. In particular, if $u = \chi_E$, then

$$0 < u_n(x) < \delta \implies \operatorname{dist}(x, E) < \frac{1}{n} \textit{ and } \operatorname{dist}(x, \Omega \setminus E) < \frac{1}{n}. \tag{4.3}$$

Proof. Let $\varepsilon > 0$. There exists λ such that, if we set

$$\Omega_0 = \left\{ x \in \Omega \;\middle|\; \operatorname{dist}(x, \partial\Omega) > \frac{1}{\lambda},\ |x| < \lambda \right\},$$

then we have $\Omega_0 \neq \emptyset$. Now we set $\Omega_{-1} = \emptyset$ and

$$\Omega_k = \left\{ x \in \Omega \;\middle|\; \operatorname{dist}(x, \partial\Omega) > \frac{1}{(\lambda + k)},\ |x| < \lambda + k \right\}.$$

We have $\bar{\Omega}_{k-1} \subseteq \Omega_k$, $k = 0, 1, 2, \dots$, and $\bigcup_{k\in\mathbb{N}} \Omega_k = \Omega$. We define the relatively compact open sets $A_i = \Omega_{i+1} \setminus \bar{\Omega}_{i-1}$, $i = 0, 1, 2, \dots$, which cover Ω. We can find a sequence $(\phi_i)_{i\geqslant 0}$ in $\mathcal{C}^\infty_0(\Omega)$ such that $\sum_{i\geqslant 0} \phi_i = 1$ and

$$\forall i \geqslant 0, \quad (\operatorname{support} \phi_i) \subset A_i, \quad 0 \leqslant \phi_i \leqslant 1.$$

Let η_r be a regularizing kernel such that support $\eta_r \subseteq B(0, r)$. We choose a sequence $(r_i)_{i\geqslant 0}$ such that

$$0 < r_i < \varepsilon, \tag{4.4}$$

$$(\operatorname{support} \phi_i) + B(0, r_i) \subset A_i, \tag{4.5}$$

$$\|\eta_{r_i} * \phi_i - \phi_i\|_\infty < 2^{-i}(1 - \delta), \tag{4.6}$$

$$\int |\eta_{r_i} * (u\phi_i) - u\phi_i| < 2^{-i}\varepsilon, \tag{4.7}$$

$$\int |\eta_{r_i} * (uD\phi_i) - uD\phi_i| < 2^{-i}\varepsilon. \tag{4.8}$$

We define $g_\varepsilon = \sum_{i\geqslant 0} \eta_{r_i} * (u\phi_i)$. For any i, $\eta_{r_i} * (u\phi_i) \in \mathcal{C}^\infty(\Omega)$. In view of (4.5), locally at most three terms of this series do not vanish, because if $j \in \mathbb{N}$, we have $A_i \cap A_j = \emptyset$ for all $i \notin \{j-1, j, j+1\}$; thus $g_\varepsilon \in \mathcal{C}^\infty(\Omega)$. Next,

$$\int_{\Omega_k} |g_\varepsilon - u| = \int_{\Omega_k} \left| \sum_{0\leqslant i\leqslant k} \eta_{r_i} * (u\phi_i) - \sum_{0\leqslant i\leqslant k} u\phi_i \right| \leqslant \varepsilon \sum_{1\leqslant i\leqslant k} 2^{-i} = \varepsilon(1 - 2^{-k}).$$

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Hence, $g_\varepsilon - u \in L^1(\Omega)$ and $\int_\Omega |g_\varepsilon - u| \leqslant \varepsilon$. Moreover,

$$Dg_\varepsilon = \sum_{i\geqslant 0} \eta_{r_i} * (\phi_i Du) + \sum_{i\geqslant 0} \eta_{r_i} * (uD\phi_i) = \sum_{i\geqslant 0} \eta_{r_i} * (\phi_i Du) + \sum_{i\geqslant 0} (\eta_{r_i} * (uD\phi_i) - uD\phi_i),$$

since $\sum_{i\geqslant 0} D\phi_i = 0$. Thus,

$$|Dg_\varepsilon| \leqslant \sum_{i\geqslant 0} \eta_{r_i} * \phi_i |Du| + \sum_{i\geqslant 0} |\eta_{r_i} * (uD\phi_i) - uD\phi_i|.$$

Note that, for $r > 0$ and $\phi \in \mathcal{C}_0(\Omega)$ such that $(\operatorname{support}\phi) + B(0,r) \subset \Omega$, we have

$$\int_\Omega \eta_r * (\phi|Du|) = \int_\Omega \phi|Du|;$$

hence, recalling that $\sum_{i\geqslant 0} \phi_i = 1$ and in view of (4.8), we obtain

$$\int_\Omega |Dg_\varepsilon| \leqslant \int_\Omega |Du| + \varepsilon.$$

Thus,

$$\limsup_{\varepsilon\to 0^+} \int_\Omega |Dg_\varepsilon| \leqslant \int_\Omega |Du|.$$

Since g_ε converges towards u in $L^1_{\mathrm{loc}}(\Omega)$, then

$$\int_\Omega |Du| \leqslant \liminf_{\varepsilon\to 0^+} \int_\Omega |Dg_\varepsilon|$$

and thus

$$\lim_{\varepsilon\to 0^+} \int_\Omega |Dg_\varepsilon| = \int_\Omega |Du|.$$

We finally consider the case where $u = \chi_E$, E being a measurable subset of Ω. First, $g_\varepsilon(x) > 0$ implies that $\operatorname{dist}(x, E) < \varepsilon$; indeed, if $B(x,\varepsilon) \cap E = \emptyset$, then, by (4.4), for any $i \geqslant 0$, $B(x, r_i) \cap E = \emptyset$ and $\eta_{r_i} * (\phi_i \chi_E)(x) = 0$, whence $g_\varepsilon(x) = 0$. Let now $x \in \Omega$ be such that $B(x,\varepsilon) \cap (\Omega \setminus E) = \emptyset$. By (4.4), for any i, we have $\eta_{r_i} * (\phi_i \chi_E)(x) = \eta_{r_i} * \phi_i(x)$, and clearly $(\phi_i \chi_E)(x) = \phi_i(x)$; from (4.6) we may infer that

$$\left\| \sum_{i\geqslant 0} \eta_{r_i} * \phi_i - \sum_{i\geqslant 0} \phi_i \right\|_\infty < 1 - \delta$$

and hence $g_\varepsilon(x) > \delta$. Thus, $g_\varepsilon(x) < \delta$ that implies $\operatorname{dist}(x, \Omega \setminus E) < \varepsilon$. In the case (not used below) where $a \leqslant u \leqslant b$ on $\Omega \cap B(x,\varepsilon)$, we also clearly obtain $a\delta \leqslant g_\varepsilon(x) \leqslant b(2-\delta)$; taking $a = 1$ and $b = 0$, we should also again obtain the result (4.3) for a set E.

We conclude by setting $u_n = g_{1/n}$. □

LEMMA 4.4. *Let Ω be an open subset of $\mathbb{R}^N$ and let E be a set having finite perimeter in Ω. There exists a sequence $(G_n)_{n\in\mathbb{N}}$ of relatively closed subsets of Ω, having*

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finite perimeter in Ω, such that $\Omega \cap \partial G_n$ is a hypersurface of class C^∞ coinciding with $\Omega \cap \partial^* G_n$, and

$$\lim_{n\to\infty} m(E\Delta G_n) = 0, \qquad \lim_{n\to\infty} \mathcal{P}_\Omega(G_n) = \mathcal{P}_\Omega(E),$$

$$G_n \subset E + B\biggl(0, \frac{1}{n}\biggr), \quad \Omega \setminus G_n \subset (\Omega \setminus E) + B\biggl(0, \frac{1}{n}\biggr) \quad \forall n \in \mathbb{N}.$$

Remark 4.5. If the sets E and G_n, $n \in \mathbb{N}$, satisfy the hypothesis and the conclusion of lemma 4.4 and if A is an open subset of Ω such that $|D\chi_E|(\Omega \cap \partial A) = 0$, then

$$\lim_{n\to\infty} \int_A |D\chi_{G_n}| = \int_A |D\chi_E|.$$

This is a direct consequence of the lower semicontinuity of the perimeters relative to A and $\Omega \setminus \bar{A}$ (see also [11, § 1.13]).

Proof. We fix some δ in $]0,1[$ and apply proposition 4.3 to the function $u = \chi_E$ to get a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{C}^\infty(\Omega)$ satisfying conditions (4.1)–(4.3). We next reproduce the argument of De Giorgi (see [11, § 1.24]). We set $G_{n,t} = \{x \in \Omega \mid u_n(x) \geqslant t\}$. Since, for any $t \in]0,1[$,

$$\int_\Omega |u_n - \chi_E| \geqslant \min(t, 1-t) m(E\Delta G_{n,t}),$$

condition (4.1) readily implies that

$$\lim_{n\to\infty} m(E\Delta G_{n,t}) = 0 \quad \forall t \in]0,1[.$$

By lower semicontinuity, we then have

$$\liminf_{n\to\infty} \int_\Omega |D\chi_{G_{t,n}}| \geqslant \int_\Omega |D\chi_E| \quad \forall t \in]0,1[.$$

On the other hand, for any $n \in \mathbb{N}$, from the co-area formula we have

$$\int_\Omega |Du_n| \geqslant \int_0^1 \mathrm{d}t \int_\Omega |D\chi_{G_{t,n}}|,$$

whence, with (4.2) and by Fatou's lemma,

$$\int_\Omega |D\chi_E| = \lim_{n\to\infty} \int_\Omega |Du_n| \geqslant \int_0^1 \biggl(\liminf_{n\to\infty} \int_\Omega |D\chi_{G_{t,n}}|\biggr) \mathrm{d}t \geqslant \int_\Omega |D\chi_E|.$$

It follows that, for almost all $t \in]0,1[$,

$$\liminf_{n\to\infty} \int_\Omega |D\chi_{G_{t,n}}| = \int_\Omega |D\chi_E|.$$

However, for each n, by Sard's theorem, almost every real number t is a regular value of u_n. Therefore, we can choose t in $]0,\delta[$ such that, setting $G_n = G_{n,t}$, $\Omega \cap \partial G_n$ is a $\mathcal{C}^\infty$ hypersurface which coincides with $\Omega \cap \partial^* G_n$ for all n and, up to the extraction of a subsequence, $\lim_{n\to\infty} \mathcal{P}_\Omega(G_n) = \mathcal{P}_\Omega(E)$. Finally, from the condition (4.3), we deduce $G_n \subset E + B(0, 1/n)$ (because $t > 0$) and $\Omega \setminus G_n \subset (\Omega \setminus E) + B(0, 1/n)$ (because $t < \delta$). □

4.3. An elementary covering lemma

The use of the following covering lemma instead of Besicovitch's lemma (see [9, § 2.8.14]) was suggested to the author by Patrice Assouad.

LEMMA 4.6. *Let $\rho_0 > 0$, let C be a compact subset of $\mathbb{R}^N$ and let $x \mapsto \rho(x)$ be a map defined on C with values in $]\rho_0/2, \rho_0[$. We can cover C with a finite set of cubes $Q_i = Q(x_i, \rho(x_i))$, $i \in I$, such that, for any $a \in \mathbb{R}^N$, we have $|\{i \in I \mid a \in Q_i\}| \leqslant K(N)$ (where $K(N) \leqslant 8^N$).*

Proof. We choose a maximal set $\{x_i, i \in I\}$ of points of C such that $\|x_i - x_j\|_\infty \geqslant \rho_0/2$ for any i, j distinct and in such a way that the cubes $Q_i = Q(x_i, \rho(x_i))$, $i \in I$, cover C. The cubes Q_i contain the disjoint cubes $Q(x_i, \rho_0/4)$; therefore $|I| \leqslant (\rho_0/2)^{-N} m(C + Q(0, \rho_0/2)) < \infty$, and if $a \in Q_{i_1} \cap \cdots \cap Q_{i_k}$, then $k \leqslant 8^N$ since

$$k(\rho_0/2)^N \leqslant m(Q_{i_1} \cup \cdots \cup Q_{i_k}) \leqslant m(Q(a, 2\rho_0)) = (4\rho_0)^N.$$

□

4.4. Computation of $|D\chi_{E\Delta F}|$ and $|D(\chi_E - \chi_F)|$

LEMMA 4.7. *Let E, F be two sets having finite perimeter in an open set Ω. We have the following equalities between measures on Ω:*

$$|D\chi_{E\Delta F}| = (1 - \chi_{\partial^* F})|D\chi_E| + (1 - \chi_{\partial^* E})|D\chi_F|,$$
$$|D(\chi_E - \chi_F)| = |D\chi_{E\Delta F}| + |n_E - n_F|\chi_{\partial^* E \cap \partial^* F}\mathcal{H}_{N-1}.$$

COROLLARY 4.8. $|D(\chi_E - \chi_F)| = |D\chi_{E\Delta F}| + 2\chi_{(\partial^* E \cap \partial^* F)\setminus \partial^*(E \cap F)}\mathcal{H}_{N-1}$.

Proof. According to [15], whenever A and B are two sets having finite perimeter in Ω, the same holds for $A \cap B$ and, moreover, $D\chi_{A\cap B} = \bar{\chi}_A D\chi_B + \bar{\chi}_B D\chi_A$, where $\bar{\chi}_A = \chi_{A_*} + \frac{1}{2}\chi_{\partial^* A}$ and $D\chi_A = \chi_{\partial^* A} n_A \mathcal{H}_{N-1}$, with similar expressions for $\bar{\chi}_B$, $D\chi_B$. Therefore,

$$D\chi_{A\cap B} = \chi_{A_*} D\chi_B + \chi_{B_*} D\chi_A + \tfrac{1}{2}\chi_{\partial^* A \cap \partial^* B}(n_A + n_B)\mathcal{H}_{N-1}, \tag{4.9}$$

where it should be noted that (see [7, Ch. IV]), for $\mathcal{H}_{N-1}$ almost all $x \in \partial^* A \cap \partial^* B$,

$$n_A(x) = \pm n_B(x) \quad \text{and} \quad n_A(x) = n_B(x) \quad \Longleftrightarrow \quad x \in \partial^*(A \cap B). \tag{4.10}$$

We remark that $\chi_E - \chi_F = \chi_{E\setminus F} - \chi_{F\setminus E}$ and $\chi_{E\Delta F} = \chi_{E\setminus F} + \chi_{F\setminus E}$. We apply formula (4.9) to compute $D\chi_{E\cap F^c}$ and $D\chi_{F\cap E^c}$. We have $D\chi_{E^c} = -D\chi_E$, $n_{E^c} = -n_E$, $\partial^*(E^c) = \partial^* E$, $\chi_{E_*} + \chi_{(E^c)_*} = 1 - \chi_{\partial^* E}$, and similar equalities for F, so we obtain

$$D(\chi_E - \chi_F) = (1 - \chi_{\partial^* F})D\chi_E - (1 - \chi_{\partial^* E})D\chi_F + (n_E - n_F)\chi_{\partial^* E \cap \partial^* F}\mathcal{H}_{N-1},$$
$$D\chi_{E\Delta F} = (1 - \chi_{\partial^* F})D\chi_E + (1 - \chi_{\partial^* E})D\chi_F.$$

The total variations associated to mutually singular measures sum up and we obtain the conclusion of lemma 4.7. Together with property (4.10), we obtain corollary 4.8.

□

4.5. A trace inequality at the boundary of a cube

The following result is elementary and simpler than a genuine trace lemma (as in [17, § 5.10.7] or [8, § 5.3]).

LEMMA 4.9. *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $x_0 \in \Omega$. We set $Q_\rho = Q(x_0, \rho)$. For almost all $\rho > 0$, such that $Q(x_0, \rho) \subset \Omega$, we have*

$$\int_{\partial Q_\rho} |u| \,\mathrm{d}\mathcal{H}_{N-1} \leqslant \frac{N}{\rho} \int_{Q_\rho} |u| \,\mathrm{d}x + \sqrt{N} \int_{Q_\rho} |Du|.$$

Proof. We need only to prove the inequality for almost all $\rho \in\,]0, R[$ such that $\bar{Q}_R \subset \Omega$ and $\int_{Q_R} |Du| < \infty$. Let θ_j be a regularizing sequence. For sufficiently large j, $\phi_j = u * \theta_j \in \mathcal{C}^\infty(Q_R)$ and

$$\lim_{j\to\infty} \int_{Q_\rho} |u - \phi_j| = 0 \qquad \text{for all } \rho \in\,]0, R[, \tag{4.11}$$

$$\lim_{j\to\infty} \int_{Q_\rho} |D\phi_j| = \int_{Q_\rho} |Du| \quad \text{for almost all } \rho \in\,]0, R[. \tag{4.12}$$

In fact, (4.12) holds for all $\rho \in\,]0, R[$ such that $|Du|(\partial Q_\rho) = 0$. By Fatou's lemma, we have

$$\begin{aligned} 0 &= \lim_{j\to\infty} \int_{Q_R} |u - \phi_j| \\ &= \lim_{j\to\infty} \int_0^R \left(\int_{\partial Q_\rho} |u - \phi_j| \,\mathrm{d}\mathcal{H}_{N-1} \right) \mathrm{d}\rho \\ &\geqslant \int_0^R \left(\liminf_{j\to\infty} \int_{\partial Q_\rho} |u - \phi_j| \,\mathrm{d}\mathcal{H}_{N-1} \right) \mathrm{d}\rho; \end{aligned}$$

thus,

$$\liminf_{j\to\infty} \int_{\partial Q_\rho} |u - \phi_j| \,\mathrm{d}\mathcal{H}_{N-1} = 0$$

for almost all $\rho \in\,]0, R[$. In other words, for almost all $\rho \in\,]0, R[$, we may extract from the sequence ϕ_j a subsequence, still denoted by ϕ_j, which satisfies (4.11) and (4.12) as well as

$$\lim_{j\to\infty} \int_{\partial Q_\rho} |u - \phi_j| \,\mathrm{d}\mathcal{H}_{N-1} = 0. \tag{4.13}$$

To end the proof, it is sufficient to prove the inequality of the lemma for an arbitrary $\phi \in \mathcal{C}^1(Q_R)$ and for any $\rho \in\,]0, R[$. Therefore, let $\phi \in \mathcal{C}^1(Q_R)$, let $\rho \in\,]0, R[$ and suppose that $x_0 = 0$. For $i \in \{1, \dots, N\}$ and $x \in \mathbb{R}^N$, we set $x = (x_i, x'_i)$. Then

$$\phi(\rho, x'_i) = \phi(x_i, x'_i) + \int_{x_i}^{\rho} D_i\phi(t, x'_i) \,\mathrm{d}t \quad \text{if } 0 < x_i < \rho,$$

whence

$$|\phi(\rho, x'_i)| \leqslant |\phi(x_i, x'_i)| + \int_{x_i}^{\rho} |D_i\phi(t, x'_i)| \,\mathrm{d}t \quad \text{if } 0 < x_i < \rho.$$

By integrating with respect to x_i, we get

$$|\phi(\rho, x_i')| \leqslant \frac{1}{\rho}\int_0^\rho |\phi(x_i, x_i')|\,\mathrm{d}x_i + \int_0^\rho |D_i\phi(t, x_i')|\,\mathrm{d}t$$

and, similarly,

$$|\phi(-\rho, x_i')| \leqslant \frac{1}{\rho}\int_{-\rho}^0 |\phi(x_i, x_i')|\,\mathrm{d}x_i + \int_{-\rho}^0 |D_i\phi(t, x_i')|\,\mathrm{d}t,$$

whence

$$|\phi(\rho, x_i')| + |\phi(-\rho, x_i')| \leqslant \frac{1}{\rho}\int_{-\rho}^\rho |\phi(x_i, x_i')|\,\mathrm{d}x_i + \int_{-\rho}^\rho |D_i\phi(t, x_i')|\,\mathrm{d}t.$$

We integrate with respect to x_i', for $\|x_i'\|_\infty < \rho$, and we get

$$\int_{\|x_i'\|_\infty<\rho} (|\phi(\rho, x_i')| + |\phi(-\rho, x_i')|)\,\mathrm{d}x_i' \leqslant \frac{1}{\rho}\int_{Q_\rho} |\phi(x)|\,\mathrm{d}x + \int_{Q_\rho} |D_i\phi(x)|\,\mathrm{d}x,$$

whence, by summing over i in $\{1, \dots, N\}$, and remarking that $\sum_{1\leqslant i\leqslant N} |D_i\phi| \leqslant \sqrt{N}|D\phi|$,

$$\int_{\partial Q_\rho} |\phi|\,\mathrm{d}\mathcal{H}_{N-1} \leqslant \frac{N}{\rho}\int_{Q_\rho} |\phi|\,\mathrm{d}x + \sqrt{N}\int_{Q_\rho} |D\phi|.$$

□

4.6. A particular use of Egoroff's theorem

We want to apply Egoroff's theorem, not as usual to a sequence (f_n), but to a real indexed family (f_ρ) of measurable functions; some additional assumption is needed (see [16]). Such a situation occurs when differentiating measures. The following remark originates from a common work with Assouad (see [3]) but is not included there; it is implicitly supposed by De Giorgi in the case of continuity of the functions f_ρ that are considered below.

LEMMA 4.10. *Let μ be a positive bounded Radon measure in Ω, $k > 0$, $f_\rho(x) = \rho^{-k}\mu(\Omega \cap B(x, \rho))$ for each $x \in \Omega$ and $\rho \in\,]0, +\infty[$, and f a μ-measurable function such that $\lim_{\rho\to 0+} f_\rho(x) = f(x)$ for μ-almost every x in Ω. Then, for each $\varepsilon > 0$, there exists a compact set C in Ω, verifying $\mu(\Omega\setminus C) < \varepsilon$, such that the convergence of f_ρ to f is uniform on C.*

Proof. Let us set

$$u_r(x) = \sup_{0<\rho<r} f_\rho(x), \quad v_r(x) = \inf_{0<\rho<r} f_\rho(x), \quad h_r(x) = \sup_{0<\rho<r} |f_\rho(x) - f(x)|.$$

Clearly, $h_r = \max(u_r - f, f - v_r)$. One readily verifies that u_r (respectively, v_r) is lower semicontinuous (respectively, upper semicontinuous) on Ω. Thus, h_r are μ-measurable functions decreasing to 0 μ-almost everywhere on Ω. Then, following [16], one may apply the usual Egoroff theorem to the sequence $h_{1/n}$. □

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5. Proof of theorem 3.1

Let Ω be an open subset of $\mathbb{R}^N$ and let E be a set having finite perimeter in Ω. Let $\varepsilon > 0$. Following the ideas of De Giorgi (see [6,11]), applying Lusin and Egoroff theorems (see lemma 4.10), there exists a compact set $C \subset \Omega \cap \mathcal{F}^* E$ such that the restriction of n_E to C is continuous,

$$|D\chi_E|(\Omega \setminus C) < \varepsilon, \tag{5.1}$$

and the following limits are uniform over $x \in C$:

$$\lim_{\rho \to 0^+} \rho^{-N} m(B(x,\rho) \cap E \cap \Pi_E^-(x)) = 0, \tag{5.2}$$

$$\lim_{\rho \to 0^+} \rho^{-N} m(B(x,\rho) \cap E^{\mathrm{c}} \cap \Pi_E^+(x)) = 0, \tag{5.3}$$

$$\lim_{\rho \to 0^+} \rho^{1-N} |D\chi_E|(B(x,\rho)) = \omega_{N-1}. \tag{5.4}$$

By lemma 4.2, there exists $f \in \mathcal{C}^1(\Omega)$ such that

$$f(x) = 0 \quad \text{and} \quad Df(x) = n_E(x) \quad \forall x \in C.$$

Since $n_E(x) \neq 0$ for $x \in C$, we have $C \subset \partial F$, where $F = \{x \in \Omega \mid f(x) \geqslant 0\}$. There exists an open set U such that $C \subset U \subset\subset \Omega$ and $Df(x) \neq 0$ for $x \in U$. Thus, $U \cap \partial F$ is a hypersurface of class $\mathcal{C}^1$, coinciding with $U \cap \partial^* F$, $U \cap F$ is a set having finite perimeter in U and $m(C) = 0$. Therefore, there exists an open set V such that $C \subset V \subset U$ and

$$|D\chi_F|(V \setminus C) < \varepsilon, \tag{5.5}$$

$$m(V) < \varepsilon. \tag{5.6}$$

In addition, the limit $\lim_{\rho \to 0^+} \rho^{-N} m((E \Delta F) \cap B(x,\rho)) = 0$ is uniform over $x \in C$, according to lemma 4.1 and the limits (5.2), (5.3). From this remark, the limit (5.4) and the fact that the map $x \mapsto \mathrm{dist}(x, V^{\mathrm{c}})$ admits a minimum over C, we deduce the existence of a real number ρ_0, $0 < \rho_0 < \varepsilon/\sqrt{N}$, such that, for any $\rho \in \,]0, \rho_0[$ and any $x \in C$,

$$\bar{Q}(x,\rho) \subset V, \tag{5.7}$$

$$\omega_{N-1} \rho^{N-1} \leqslant 2|D\chi_E|(Q(x,\rho)), \tag{5.8}$$

$$\rho^{-N} m((E \Delta F) \cap Q(x,\rho)) < \varepsilon. \tag{5.9}$$

By lemma 4.4, there exists a sequence $(G_n)_{n \in \mathbb{N}}$ of closed subsets of Ω such that, for all n, $\Omega \cap \partial G_n$ is a hypersurface of class $\mathcal{C}^1$ coinciding with $\Omega \cap \partial^* G_n$. Furthermore, $G_n \subset E + B(0, 1/n)$, $\Omega \setminus G_n \subset (\Omega \setminus E) + B(0, 1/n)$ and

$$\lim_{n \to \infty} m(E \Delta G_n) = 0, \tag{5.10}$$

$$\lim_{n \to \infty} \mathcal{P}_\Omega(G_n) = \mathcal{P}_\Omega(E). \tag{5.11}$$

We fix $x \in C$. For all n, we have

$$m(E \Delta G_n) \geqslant \int_{Q(x,\rho_0)} \chi_{E \Delta G_n} \geqslant \int_0^{\rho_0} \left(\int_{\partial Q(x,\rho)} \chi_{E \Delta G_n} \, \mathrm{d}\mathcal{H}_{N-1} \right) \mathrm{d}\rho.$$

From this inequality and equation (5.10), with the help of Fatou's lemma, we deduce that

$$\liminf_{n\to\infty} \int_{\partial Q(x,\rho)} \chi_{E\Delta G_n}\,\mathrm{d}\mathcal{H}_{N-1} = 0 \quad \text{for almost all } \rho \in \,]0,\rho_0[.$$

If μ is a positive measure on $Q(x,\rho_0)$ such that $\mu(Q(x,\rho_0)) < \infty$, then the set

$$\{\rho \in \,]0,\rho_0[\; | \; \mu(\partial Q(x,\rho)) \neq 0\}$$

is at most countable. Therefore, we can choose a negligible subset $\sigma(x)$ of $]0,\rho_0[$ such that, for all $\rho \in \,]0,\rho_0[$, the trace inequality of lemma 4.9 holds for the cube $Q(x,\rho)$ and the function $u = \chi_{E\Delta F}$ and

$$\liminf_{n\to\infty} \int_{\partial Q(x,\rho)} \chi_{E\Delta G_n}\,\mathrm{d}\mathcal{H}_{N-1} = 0, \tag{5.12}$$

$$|D\chi_{G_n}|(\partial Q(x,\rho)) = 0 \quad \forall n \in \mathbb{N}, \tag{5.13}$$

$$|D\chi_E|(\partial Q(x,\rho)) = 0, \tag{5.14}$$

$$|D\chi_F|(\partial Q(x,\rho)) = 0. \tag{5.15}$$

Let $x \mapsto \rho(x)$ be a map from C to $]\rho_0/2,\rho_0[$ such that, for all $x \in C$, $\rho(x) \notin \sigma(x)$. According to lemma 4.6, there exists a finite collection of cubes $Q_i = Q(x_i,\rho(x_i))$, $i \in I$, which covers C and such that, for any $a \in \mathbb{R}^N$, we have $|\{i \in I \mid a \in Q_i\}| \leqslant K(N)$. We set $B = \bigcup_{i\in I} Q_i$ and $A = \Omega \setminus \bar{B}$, $\Gamma = \partial B$. We obtain a partition of Ω as $\Omega = A \cup B \cup \Gamma$, where A, B are open. Up to a finite number of subsequences extractions, we might assume that the sequence $(G_n)_{n\in\mathbb{N}}$ satisfies, for any index i,

$$\lim_{n\to\infty} \int_{\partial Q_i} \chi_{E\Delta G_n}\,\mathrm{d}\mathcal{H}_{N-1} = 0. \tag{5.16}$$

Moreover, since $\partial A = \Gamma \subset \bigcup_i \partial Q_i$, we deduce from (5.14) that $|D\chi_E|(\partial A) = 0$. Applying remark 4.5, we get

$$\lim_{n\to\infty} |D\chi_{G_n}|(A) = |D\chi_E|(A) \tag{5.17}$$

and, since $A \subset \Omega \setminus C$, equation (5.1) yields

$$|D\chi_E|(A) < \varepsilon. \tag{5.18}$$

Considering (5.16)–(5.18) and (5.10), we can choose n and define $G = G_n$ in such a way that $G \subset E + B(0,\varepsilon)$, $\Omega \setminus G \subset (\Omega \setminus E) + B(0,\varepsilon)$ and

$$\sum_{i\in I} \int_{\partial Q_i} \chi_{E\Delta G}\,\mathrm{d}\mathcal{H}_{N-1} < \varepsilon, \tag{5.19}$$

$$|D\chi_G|(A) < \varepsilon, \tag{5.20}$$

$$m(E\Delta G) < \varepsilon. \tag{5.21}$$

We set $L = (F \cap \bar{B}) \cup (G \cap \bar{A})$. First we have

$$L \subset G \cup \bar{B}, \qquad \Omega \setminus L \subset (\Omega \setminus G) \cup B, \qquad B \subset (\Omega \cap \partial E) + B(0,\varepsilon);$$

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thus,

$$L \subset E + B(0,\varepsilon), \qquad \Omega \setminus L \subset (\Omega \setminus E) + B(0,\varepsilon).$$

We will next prove that

$$m(L\Delta E) < 2\varepsilon, \tag{5.22}$$

$$\int_\Omega |D(\chi_L - \chi_E)| \leqslant K(N,\Omega,E)\varepsilon, \tag{5.23}$$

as well as the conclusions (3.1), (3.6)–(3.8) of theorem 3.1.

Since

$$L = (F \cap B) \cup (G \cap A) \cup \big((F \cup G) \cap \Gamma\big), \tag{5.24}$$

we then have $\Omega \cap \partial L \subset (B \cap \partial F) \cup (A \cap \partial G) \cup \Gamma$, whence

$$\Omega \cap \partial L \subset (U \cap \partial F) \cup (\Omega \cap \partial G) \cup \Gamma,$$

which gives (3.1). From [7, § 1.3, theorems 3.2 and 3.3], we have

$$B \cap \mathcal{F}^* L = B \cap \mathcal{F}^* F = B \cap \partial F.$$

Therefore, the inclusions $C \subset B \cap \partial F$ and $C \subset \Omega \cap \mathcal{F}^* E$ imply (3.6). Moreover, the same theorems of [7] yield

$$n_L(x) = n_F(x) = \frac{Df(x)}{|Df(x)|} \quad \forall x \in B \cap \partial F.$$

In particular, if $x \in C$, then $Df(x) = n_E(x)$, which proves (3.7), while (3.8) comes from (5.1). From (5.24) we obtain

$$m(E\Delta L) = m((E\Delta F) \cap B) + m((E\Delta G) \cap A),$$

whence $m(E\Delta L) \leqslant m(B) + m(E\Delta G)$ and, using (5.6) and (5.21), $m(E\Delta L) \leqslant 2\varepsilon$. It remains to prove (5.23), where $K(N,\Omega,\varepsilon)$ will be a constant depending only on N and $\mathcal{P}_\Omega(E)$. According to lemma 4.7,

$$|D(\chi_E - \chi_L)| = |D\chi_{E\Delta L}| + |n_E - n_L|\chi_{\partial^* E \cap \partial^* L}\mathcal{H}_{N-1}. \tag{5.25}$$

From (3.7) and (3.8), we deduce the inequalities

$$\int_\Omega |n_E - n_L|\chi_{\partial^* E \cap \partial^* L}\mathcal{H}_{N-1} \leqslant 2\mathcal{H}_{N-1}(\Omega \cap \partial^* E \setminus C) < 2\varepsilon. \tag{5.26}$$

It remains to evaluate $\int_\Omega |D\chi_{E\Delta L}|$ or, equivalently, the values of the measure $|D\chi_{E\Delta L}|$ on the sets A, B and Γ. From lemma 4.7, we get $|D\chi_{E\Delta L}| \leqslant |D\chi_E| + |D\chi_L|$. However, $|D\chi_L|(A) = |D\chi_G|(A)$, since $L \cap A = G \cap A$. Together with (5.18) and (5.20), we obtain

$$|D\chi_{E\Delta L}|(A) < 2\varepsilon. \tag{5.27}$$

From lemma 4.7, we also obtain

$$|D\chi_{E\Delta L}|(B) = \int_B (1 - \chi_{\partial^* E})|D\chi_L| + \int_B (1 - \chi_{\partial^* L})|D\chi_E|.$$

However, $L\cap B = F\cap B$; hence, $B\cap\partial^* L = B\cap\partial^* F = B\cap\partial F$ and the restrictions to B of the measures $|D\chi_L|$ and $|D\chi_F|$ are equal. Since $C \subseteq B\cap\partial^* E$ and $C \subseteq B\cap\partial^* L$, we have

$$|D\chi_{E\Delta L}|(B) \leqslant \int_{B\setminus C} |D\chi_L| + \int_{B\setminus C} |D\chi_E| = \int_{B\setminus C} |D\chi_F| + \int_{B\setminus C} |D\chi_E|,$$

whence, taking into account (5.1) and (5.5),

$$|D\chi_{E\Delta L}|(B) \leqslant 2\varepsilon. \tag{5.28}$$

It remains to evaluate $|D\chi_{E\Delta L}|(\varGamma)$. We have

$$|D\chi_E|(\varGamma) \leqslant \sum_{i\in I} |D\chi_E|(\partial Q_i)$$

which, together with (5.14), yields

$$|D\chi_E|(\varGamma) = 0. \tag{5.29}$$

We also have

$$\int_{\varGamma} \chi_{\partial^* E}|D\chi_L| \leqslant \mathcal{H}_{N-1}(\varGamma\cap\partial^* E) = |D\chi_E|(\varGamma),$$

whence, together with lemma 4.7, we obtain

$$|D\chi_{E\Delta L}|(\varGamma) = |D\chi_L|(\varGamma) = \int_{\varGamma} \chi_{\partial^* L}\,\mathrm{d}\mathcal{H}_{N-1}.$$

Since L is closed in $\varOmega$, we have $\varOmega\cap\partial^* L \subset L$ and

$$\begin{aligned} |D\chi_{E\Delta L}|(\varGamma) &= \int_{\varGamma\cap L} \chi_{\partial^* L}\,\mathrm{d}\mathcal{H}_{N-1} \\ &= \int_{\varGamma\cap(F\cup G)} \chi_{\partial^* L}\,\mathrm{d}\mathcal{H}_{N-1} \\ &= \int_{\varGamma\cap(F\cap G)} \chi_{\partial^* L}\,\mathrm{d}\mathcal{H}_{N-1} + \int_{\varGamma\cap(F\Delta G)} \chi_{\partial^* L}\,\mathrm{d}\mathcal{H}_{N-1}. \end{aligned}$$

However,

$$F\cap G \subseteq \partial F \cup \partial G \cup (\mathring{F}\cap\mathring{G}).$$

We deduce from (5.13) and (5.15), as in (5.29), that

$$|D\chi_F|(\varGamma) = |D\chi_G|(\varGamma) = 0,$$

whence

$$\mathcal{H}_{N-1}(\varGamma\cap\partial^* F) = \mathcal{H}_{N-1}(\varGamma\cap\partial^* G) = 0. \tag{5.30}$$

However, $\varGamma \subseteq U$ and $U\cap\partial F = U\cap\partial^* F$; hence,

$$\mathcal{H}_{N-1}(\varGamma\cap\partial F) = 0. \tag{5.31}$$

Similarly, $\varOmega\cap\partial G = \varOmega\cap\partial^* G$; hence,

$$\mathcal{H}_{N-1}(\varGamma\cap\partial G) = 0. \tag{5.32}$$

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We next prove that

$$\mathring{F} \cap \mathring{G} \cap \Gamma \cap \partial^* L = \emptyset. \tag{5.33}$$

Let $x \in \mathring{F} \cap \mathring{G} \cap \Gamma$. If $B(x,\rho) \subseteq \mathring{F} \cap \mathring{G}$, then

$$m(B(x,\rho)\cap L) = m(B(x,\rho)\cap B) + m(B(x,\rho)\cap A) = m(B(x,\rho))$$

(recall that $m(\Gamma)=0$). Therefore, $x \in L_*$ and, in particular, $x \notin \partial^* L$.

From (5.31)–(5.33), we obtain

$$\int_{\Gamma\cap F\cap G} \chi_{\partial^* L}\, \mathrm{d}\mathcal{H}_{N-1} = 0$$

and

$$|D\chi_{E\Delta L}|(\Gamma) \leqslant \mathcal{H}_{N-1}(\Gamma \cap (F\Delta G)) \leqslant \sum_{i\in I} \int_{\partial Q_i} \chi_{F\Delta G}\, \mathrm{d}\mathcal{H}_{N-1}. \tag{5.34}$$

Using the inequality $\chi_{F\Delta G} \leqslant \chi_{F\Delta E} + \chi_{E\Delta G}$, for any i in I we get

$$\int_{\partial Q_i} \chi_{F\Delta G}\, \mathrm{d}\mathcal{H}_{N-1} \leqslant \int_{\partial Q_i} \chi_{F\Delta E}\, \mathrm{d}\mathcal{H}_{N-1} + \int_{\partial Q_i} \chi_{E\Delta G}\, \mathrm{d}\mathcal{H}_{N-1}. \tag{5.35}$$

Applying lemma 4.9, we get

$$\int_{\partial Q_i} \chi_{F\Delta E}\, \mathrm{d}\mathcal{H}_{N-1} \leqslant \frac{N}{\rho_i}\int_{Q_i} \chi_{F\Delta E} + \sqrt{N}\int_{Q_i} |D\chi_{F\Delta E}|. \tag{5.36}$$

We examine next the term

$$\frac{N}{\rho_i}\int_{Q_i} \chi_{F\Delta E}.$$

According to (5.9),

$$\frac{1}{\rho_i}\int_{Q_i} \chi_{F\Delta E} = \frac{1}{\rho_i} m(Q_i \cap (F\Delta E)) \leqslant \varepsilon \rho_i^{N-1} \tag{5.37}$$

and, by (5.8),

$$\omega_{N-1}\rho_i^{N-1} \leqslant 2|D\chi_E|(Q_i).$$

The multiplicity of the covering Q_i, $i \in I$, is bounded by $K(N)$. Thus,

$$\begin{aligned} \sum_{i\in I} \tfrac{1}{2}\omega_{N-1}\rho_i^{N-1} &\leqslant \sum_{i\in I} |D\chi_E|(Q_i) \leqslant K(N)|D\chi_E|\biggl(\bigcup_{i\in I} Q_i\biggr) \\ &\leqslant K(N)\mathcal{P}_\Omega(E). \end{aligned} \tag{5.38}$$

Combining (5.37) and (5.38), we get

$$\sum_{i\in I} \frac{N}{\rho_i}\int_{Q_i} \chi_{F\Delta E} \leqslant 2\varepsilon \frac{NK(N)}{\omega_{N-1}} \mathcal{P}_\Omega(E). \tag{5.39}$$

Similarly,

$$\sum_{i\in I} \sqrt{N}\int_{Q_i} |D\chi_{F\Delta E}| \leqslant \sqrt{N}K(N)|D\chi_{F\Delta E}|(B) \tag{5.40}$$

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and, using lemma 4.7, as in (5.28), with (5.1) and (5.5), we have

$$\int_B |D\chi_{F\Delta E}| \leqslant |D\chi_F|(B\setminus C) + |D\chi_E|(B\setminus C) \leqslant 2\varepsilon. \tag{5.41}$$

From equations (5.19), (5.34)–(5.36) and (5.39)–(5.41), we deduce that

$$\begin{aligned}|D\chi_{E\Delta L}|(\Gamma) &\leqslant \mathcal{H}_{N-1}(\Gamma\cap(F\Delta G))\\ &\leqslant \left(1 + 2\frac{NK(N)}{\omega_{N-1}}\mathcal{P}_\Omega(E) + 2\sqrt{N}K(N)\right)\varepsilon.\end{aligned} \tag{5.42}$$

From (5.25)–(5.28) and (5.42), we conclude that

$$\int_\Omega |D(\chi_E - \chi_L)| \leqslant \left(7 + 2\frac{NK(N)}{\omega_{N-1}}\mathcal{P}_\Omega(E) + 2\sqrt{N}K(N)\right)\varepsilon,$$

which proves (5.23).

We finally prove (3.4). Firstly, since $L\cap B = F\cap B$, where B is open, and $B\cap\partial F = B\cap\mathcal{F}^*F$, we have $B\cap\mathcal{F}^*L = B\cap\partial L$. Secondly, $L\cap A = G\cap A$ and $A\cap\partial L = A\cap\partial G$, implying that $A\cap\mathcal{F}^*L = A\cap\partial L$. It remains to study $\mathcal{H}_{N-1}(\Gamma\cap(\partial L)\setminus\mathcal{F}^*L)$. Next, from (5.24), and since L is closed in Ω,

$$\begin{aligned}\mathcal{H}_{N-1}(\Gamma\cap\partial L) &= \mathcal{H}_{N-1}(\Gamma\cap(F\cup G)\cap\partial L)\\ &\leqslant \mathcal{H}_{N-1}(\Gamma\cap F\cap G\cap\partial L) + \mathcal{H}_{N-1}(\Gamma\cap(F\Delta G)\cap\partial L).\end{aligned}$$

Yet $F\cap G\subset\partial F\cup\partial G\cup(\mathring{F}\cap\mathring{G})$. Moreover, by (5.31), (5.32), $\mathcal{H}_{N-1}(\Gamma\cap\partial G) = 0$ and $\mathcal{H}_{N-1}(\Gamma\cap\partial F) = 0$. We claim that $\mathring{F}\cap\mathring{G}\cap\Gamma\cap\partial L = \emptyset$. Let $x\in\mathring{F}\cap\mathring{G}\cap\Gamma$. There exists $\rho > 0$ such that $B(x,\rho)\subseteq F\cap G$. Then $B(x,\rho)\subseteq L$ (we check separately that $B(x,\rho)\cap B\subseteq L$, $B(x,\rho)\cap A\subseteq L$, $B(x,\rho)\cap\Gamma\subseteq L$), and hence $x\notin\partial L$. Thus, $\mathcal{H}_{N-1}(\Gamma\cap F\cap G\cap\partial L) = 0$. Using (5.42), we conclude that

$$\mathcal{H}_{N-1}(\Gamma\cap\partial L) \leqslant \mathcal{H}_{N-1}(\Gamma\cap(F\Delta G)) \leqslant \left(1 + 2\frac{NK(N)}{\omega_{N-1}}\mathcal{P}_\Omega(E) + 2\sqrt{N}K(N)\right)\varepsilon,$$

and this ends the proof of theorem 3.1.

6. Strong approximation of functions in $\mathbf{BV}(\Omega)$

We want to approximate a function u of bounded variation in Ω by mean of its hypograph H (which lies in $\Omega\times\mathbb{R}$ and thus in $\mathbb{R}^{N+1}$). Only the first N coordinates of $D\chi_H$ depend on Du, so we first need to modify theorem 3.1 slightly.

Fix $\alpha = (1,\dots,k)$ with $1\leqslant k\leqslant N$. For $x = (x_1,\dots,x_N)$, we set

$$D_\alpha = (\partial/\partial x_1,\dots,\partial/\partial x_k), \qquad p_\alpha(x) = (x_1,\dots,x_k)$$

and $\Lambda_\alpha(x,\rho) = \{y\in\partial Q(x,\rho) \mid \forall h > k,\ |y_h - x_h| < \rho\}$. For Ω an open subset of $\mathbb{R}^N$ and $u\in L^1_{\mathrm{loc}}(\Omega)$ we define

$$\int_\Omega |D_\alpha u| = \sup\left\{\int \sum_{1\leqslant i\leqslant k} u(x)D_i\phi_i(x)\,\mathrm{d}x \;\middle|\; (\phi_1,\dots,\phi_k)\in\mathcal{C}^1_0(\Omega)^k,\ \sum_{1\leqslant i\leqslant k}\phi_i^2\leqslant 1\right\}$$

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and say that a set $E \subset \Omega$ has finite α-perimeter in Ω if E is measurable and $\int_\Omega |D_\alpha \chi_E| < \infty$.

For $\chi_E \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$, we have $D_\alpha \chi_E = p_\alpha(n_E)|D\chi_E|$. We set

$$\Omega \cap \mathcal{F}^*_\alpha E = \{x \in \Omega \cap \mathcal{F}^* E \mid p_\alpha(n_E(x)) \neq 0\}.$$

For all $x \in \Omega \cap \mathcal{F}^*_\alpha E$, in the same way as in [6], one obtains

$$\lim_{\rho \to 0^+} \rho^{1-N} |D_\alpha \chi_E|(B(x,\rho)) = \omega_{N-1} |p_\alpha(n_E(x))|.$$

6.1. Strong approximation of sets of finite α-perimeter

THEOREM 6.1. *Let Ω be an open subset of $\mathbb{R}^N$. Let $E \subset \Omega$, having locally finite perimeter in Ω (i.e. $\chi_E \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$) such that*

$$\int_\Omega |D_\alpha \chi_E| < \infty.$$

Let $\varepsilon > 0$. There exists a set $L \subset \Omega$, having locally finite perimeter in Ω, and a compact set $C \subset \Omega$ such that

$$\Omega \cap \partial L \text{ is contained in a finite union of } \mathcal{C}^1 \text{ hypersurfaces,} \tag{6.1}$$

$$\int_\Omega |\chi_E - \chi_L| < \varepsilon, \tag{6.2}$$

$$\int_\Omega |D_\alpha(\chi_E - \chi_L)| < \varepsilon, \tag{6.3}$$

$$L \subset E + B(0,\varepsilon), \qquad \Omega \setminus L \subset (\Omega \setminus E) + B(0,\varepsilon), \tag{6.4}$$

$$C \subset \Omega \cap \mathcal{F}^*_\alpha E \cap \mathcal{F}^*_\alpha L, \tag{6.5}$$

$$n_E(x) = n_L(x) \quad \forall x \in C, \tag{6.6}$$

$$|D_\alpha \chi_E|(\Omega \setminus C) < \varepsilon. \tag{6.7}$$

6.2. Modified preliminary lemma

When substituting D_α for D, proposition 4.3 remains valid, as does its proof. In the same way, lemma 4.4 becomes the following.

LEMMA 6.2. *Let Ω be an open subset of $\mathbb{R}^N$ and let E be a set having finite α-perimeter in Ω. There exists a sequence $(G_n)_{n\in\mathbb{N}}$ of relatively closed subsets of Ω, having finite α-perimeter in Ω, such that $\Omega \cap \partial G_n$ is a hypersurface of class C^∞ coinciding with $\Omega \cap \partial^* G_n$, and*

$$\lim_{n\to\infty} m(E \Delta G_n) = 0, \qquad \lim_{n\to\infty} \int_\Omega |D_\alpha \chi_{G_n}| = \int_\Omega |D_\alpha \chi_E|,$$

$$G_n \subset E + B\left(0, \frac{1}{n}\right) \quad \textit{and} \quad \Omega \setminus G_n \subset (\Omega \setminus E) + B\left(0, \frac{1}{n}\right) \quad \forall n \in \mathbb{N}.$$

We compute vector-valued measures involved in the proof of lemma 4.7, above, and take their projections under p_α, so that lemma 4.7 becomes the following.

Lemma 6.3. *Let E and F be two sets having locally finite perimeter in an open set Ω. We have the following equalities between measures on Ω:*

$$|D_\alpha \chi_{E\Delta F}| = (1 - \chi_{\partial^* F})|D_\alpha \chi_E| + (1 - \chi_{\partial^* E})|D_\alpha \chi_F|,$$
$$|D_\alpha(\chi_E - \chi_F)| = |D_\alpha \chi_{E\Delta F}| + |p_\alpha(n_E - n_F)|\chi_{\partial^* E\cap\partial^* F}\mathcal{H}_{N-1}.$$

Lemma 6.4. *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $x_0 \in \Omega$. We set $Q_\rho = Q(x_0, \rho)$ and $\Lambda_\rho = \Lambda_\alpha(x_0, \rho)$. For almost all $\rho > 0$, such that $Q(x_0, \rho) \subset \Omega$, we have*

$$\int_{\Lambda_\rho} |u| \,\mathrm{d}\mathcal{H}_{N-1} \leqslant \frac{k}{\rho}\int_{Q_\rho} |u| \,\mathrm{d}x + \sqrt{k}\int_{Q_\rho} |D_\alpha u|.$$

Proof. We take up the proof of lemma 4.9 again. For almost all $\rho \in]0, R[$, we find a sequence $\phi_j \in \mathcal{C}^\infty(Q_R)$ such that following conditions hold: (4.11), (4.12) with D_α in place of D; (4.13), which implies that

$$\lim_{j\to\infty}\int_{\Lambda_\rho} |u - \phi_j| \,\mathrm{d}\mathcal{H}_{N-1} = 0.$$

The proof of the asserted inequality for $\phi \in \mathcal{C}^\infty(Q_R)$ is quite similar to the corresponding one for lemma 4.9. □

6.3. Proof of theorem 6.1

We shall follow the proof of theorem 3.1, given in § 5 and indicate modifications if necessary. We choose a compact set $C \subset \Omega \cap \mathcal{F}^*_\alpha E$ such that $|D_\alpha \chi_E|(\Omega \setminus C) < \varepsilon$, the restriction of n_E to C is continuous and the limits (5.2), (5.3) as well as

$$\lim_{\rho\to 0^+} \rho^{1-N}|D_\alpha \chi_E|(B(x, \rho)) = \omega_{N-1}|p_\alpha(n_E(x))|$$

hold uniformly over $x \in C$. We have $\mu = \min_{x\in C} |p_\alpha(n_E(x))| > 0$ and choose $\rho_0 > 0$ in the same manner but with conditions (5.8), (5.9) replaced by

$$\mu\omega_{N-1}\rho^{N-1} \leqslant 2|D_\alpha \chi_E|(Q(x, \rho)) \quad \text{and} \quad \rho^{-N}m((E\Delta F) \cap Q(x, \rho)) < \mu\varepsilon.$$

We construct a sequence $(G_n)_{n\in\mathbb{N}}$ as in § 5, but with (5.11) replaced by

$$\lim_{n\to\infty}\int_\Omega |D_\alpha \chi_{G_n}| = \int_\Omega |D_\alpha \chi_E|$$

and associate to each $x \in C$ a negligible subset $\sigma(x)$ of $]0, \rho_0[$ such that, for all $\rho \in]0, \rho_0[$, the trace inequality of lemma 6.4 holds for the cube $Q(x, \rho)$ and the function $u = \chi_{E\Delta F}$, together with conditions (5.12)–(5.15). Clearly, $|D\chi_E|(\partial A) = 0$ implies $|D_\alpha \chi_E|(\partial A) = 0$ and thus

$$\lim_{n\to\infty} |D_\alpha \chi_{G_n}|(A) = |D_\alpha \chi_E|(A) < \varepsilon.$$

We choose the set G with the same properties as before, but with D_α replacing D in (5.20), and define L in the same manner. If $x \in C$, then $n_E(x) = n_F(x) = n_L(x)$ and hence $C \subset \Omega \cap \mathcal{F}^*_\alpha E \cap \mathcal{F}^*_\alpha L$. We have essentially to evaluate $\int_\Omega |D(\chi_L - \chi_E)|$. According to lemma 6.3,

$$\int_\Omega |D_\alpha(\chi_E - \chi_L)| = \int_\Omega |D_\alpha \chi_{E\Delta L}| + \int_\Omega |p_\alpha(n_E - n_L)|\chi_{\partial^* E\cap\partial^* L} \,\mathrm{d}\mathcal{H}_{N-1}.$$

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The second term is not greater than

$$\int_{\Omega\setminus C}(|p_\alpha(n_E)|+|p_\alpha(n_L)|)\chi_{\partial^*E\cap\partial^*L}\,\mathrm{d}\mathcal{H}_{N-1},$$

where

$$\int_{\Omega\setminus C}|p_\alpha(n_E)|\chi_{\partial^*E}\,\mathrm{d}\mathcal{H}_{N-1}=|D_\alpha\chi_E|(\Omega\setminus C)<\varepsilon$$

and, as $\mathcal{H}_{N-1}(\Gamma\cap\partial^*E)=0$, $L\cap B=F\cap B$, $L\cap A=G\cap A$, we have

$$\int_{\Omega\setminus C}|p_\alpha(n_L)|\chi_{\partial^*E\cap\partial^*L}\,\mathrm{d}\mathcal{H}_{N-1}\leqslant|D_\alpha\chi_F|(B\setminus C)+|D_\alpha\chi_G|(A)<2\varepsilon.$$

Then we consider $\int_\Omega|D_\alpha\chi_{E\Delta L}|$. At first the values of $|D_\alpha\chi_{E\Delta L}|$ on A and B, taking account of lemma 6.3, are less than 2ε. Secondly, we compute

$$|D_\alpha\chi_{E\Delta L}|(\Gamma)=|D_\alpha\chi_L|(\Gamma)=|D_\alpha\chi_L|(\Gamma\cap(F\Delta G)).$$

As $L\cap(\mathring{F}\setminus G)=\bar{B}\cap(\mathring{F}\setminus G)$ (respectively, $L\cap(\mathring{G}\setminus F)=\bar{A}\cap(\mathring{G}\setminus F)$), if they are restricted to $\mathring{F}\setminus G$ (respectively, $\mathring{G}\setminus F$), the measures $D\chi_L$ and $D\chi_B$ (respectively, $D\chi_A$) coincide; thus, if $x\in(\mathring{F}\setminus G)\cap\mathcal{F}^*L$ (respectively, $x\in(\mathring{G}\setminus F)\cap\mathcal{F}^*L$), then $x\in\mathcal{F}^*B$ and $n_L(x)=n_B(x)$ (respectively, $n_L(x)=n_A(x)=-n_B(x)$), so in each case $|p_\alpha(n_L(x))|=|p_\alpha(n_B(x))|$. Hence, as $\mathcal{H}_{N-1}(\Gamma\cap\partial F)=\mathcal{H}_{N-1}(\Gamma\cap\partial G)=0$, we obtain

$$|D_\alpha\chi_{E\Delta L}|(\Gamma)\leqslant\int_{\Gamma\cap\mathcal{F}^*B}\chi_{F\Delta G}(x)|p_\alpha(n_B(x))|\,\mathrm{d}\mathcal{H}_{N-1}(x).$$

We set $\Lambda_i=\Lambda_\alpha(x_i,\rho(x_i))$ for $i\in I$, recall that $\Gamma\subset\bigcup_{i\in I}\partial Q_i$ and observe that if x is in $(\partial Q_i)\cap\mathcal{F}^*B$, then $|p_\alpha(n_B(x))|$ equals 1 if $x\in\Lambda_i$, and 0 otherwise. Thus,

$$|D_\alpha\chi_{E\Delta L}|(\Gamma)\leqslant\sum_{i\in I}\int_{\Lambda_i}\chi_{F\Delta G}\,\mathrm{d}\mathcal{H}_{N-1}\leqslant\varepsilon+\sum_{i\in I}\int_{\Lambda_i}\chi_{F\Delta E}\,\mathrm{d}\mathcal{H}_{N-1}.$$

Applying lemma 6.4 and recalling that we have, for each index i,

$$\frac{1}{\rho_i}m((E\Delta F)\cap Q_i)\leqslant\mu\varepsilon\rho_i^{N-1}\leqslant\frac{2\varepsilon}{\omega_{N-1}}|D_\alpha\chi_E|(Q_i),$$

we conclude in the same way that

$$\int_\Omega|D(\chi_E-\chi_L)|\leqslant\biggl(7+2kK(N)\omega_{N-1}^{-1}\int_\Omega|D_\alpha\chi_E|+2\sqrt{k}K(N)\biggr)\varepsilon.$$

6.4. Strong approximation of functions of bounded variation

For each real-valued function u defined on the open set Ω of $\mathbb{R}^N$, we set

$$H(u)=\{(x,y)\in\Omega\times\mathbb{R}\mid u(x)>y\}$$

and take $\alpha=(1,\dots,N)$. We know (see [12]) that if $u\in\mathrm{BV}(\Omega)$, then $H(u)$ has locally finite perimeter in $\Omega\times\mathbb{R}$,

$$\int_{\Omega\times\mathbb{R}}|D_\alpha\chi_{H(u)}|<\infty$$

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and, for each Borelian subset S of Ω,

$$Du(S) = D_\alpha \chi_{H(u)}(S \times \mathbb{R}).$$

Lemma 6.5. *Let $M > 0$. Let L be a closed subset of $\Omega \times \mathbb{R}$ having a locally finite perimeter in $\Omega \times \mathbb{R}$, such that*

$$\int_{\Omega\times\mathbb{R}} |D_\alpha \chi_L| < \infty \quad \text{and} \quad \Omega \times]-\infty, -M[\subset L \subset \Omega \times]-\infty, M[.$$

For $x \in \Omega$, we set

$$f(x) = \int_{-M}^{M} \chi_L(x, y)\,\mathrm{d}y - M.$$

Then

(i) *for each $x \in \Omega$, $\int_{\mathbb{R}} (\chi_{H(f)}(x, y) - \chi_L(x, y))\,\mathrm{d}y = 0$,*

(ii) *$f \in L^1_{\mathrm{loc}}(\Omega)$ and $\int_\Omega |Df| < \infty$,*

(iii) *for each Borelian subset S of Ω, $Df(S) = D_\alpha \chi_L(S \times \mathbb{R})$,*

(iv) *f is upper semicontinuous in Ω.*

Proof. Although the following techniques essentially derive from [12], for the sake of completeness we provide details. For $x \in \Omega$, we have $-M \leqslant f(x) \leqslant M$ and infer that $\chi_{H(f)}(x, y) - \chi_L(x, y) = 0$ if $|y| > M$ and that

$$\int_{-M}^{M} \chi_{H(f)}(x, y)\,\mathrm{d}y = f(x) + M = \int_{-M}^{M} \chi_L(x, y)\,\mathrm{d}y.$$

Hence, (i) is obvious. Suppose that $\phi \in \mathcal{C}^1_0(\Omega)$ and set $\eta(y) = 1$ if $|y| \leqslant M$ and $\eta(y) = 0$ if $|y| > M + 1$; otherwise, η is linear. For $1 \leqslant i \leqslant N$, the measure $D_i\chi_L$ has its support in $\Omega \times [-M, M]$. Thus, using, in addition, some regularization, we obtain

$$\begin{aligned}
\int \phi(x) D_i \chi_L(\mathrm{d}x\,\mathrm{d}y) &= \int \phi(x)\eta(y) D_i \chi_L(\mathrm{d}x\,\mathrm{d}y) \\
&= -\int D_i\phi(x) \biggl(\int \eta(y)\chi_L(x, y)\,\mathrm{d}y \biggr) \mathrm{d}x \\
&= -\int D_i\phi(x) \biggl(\int \eta(y)\chi_L(x, y)\,\mathrm{d}y - M - \frac{1}{2} \biggr) \mathrm{d}x \\
&= -\int D_i\phi(x) f(x)\,\mathrm{d}x.
\end{aligned}$$

However,

$$\int_{\Omega\times\mathbb{R}} |D_\alpha \chi_L| < \infty,$$

and thus we deduce $\int_\Omega |Df| < \infty$ together with conclusion (iii). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence which converges to some $x \in \Omega$. The function $1 - \chi_L$ is lower semicontinuous. Thus, by Fatou's lemma,

$$\int_{-M}^{M}(1-\chi_L(x,y))\,\mathrm{d}y \leqslant \liminf_{n\to\infty}\int_{-M}^{M}(1-\chi_L(x_n,y))\,\mathrm{d}y$$

i.e. $f(x) \geqslant \limsup_{n\to\infty} f(x_n)$. Then f is upper semicontinuous. □

REMARK 6.6. If we suppose L to be an open subset of $\Omega\times\mathbb{R}$, in the same way we find that f is lower semicontinuous.

THEOREM 6.7. *Let Ω be an open subset of $\mathbb{R}^N$ and let $u \in \mathrm{BV}(\Omega)$. There exists a sequence $(u_n)_{n\in\mathbb{N}}$ in* BV(Ω) *of upper semicontinuous functions (respectively, lower semicontinuous functions) converging towards u with respect to the norm of the space* BV(Ω), *i.e.*

$$\lim_{n\to\infty}\int_{\Omega}|u_n-u| = \lim_{n\to\infty}\int_{\Omega}|D(u_n-u)| = 0.$$

Proof. As a consequence of the co-area formula, we know that $\mathrm{BV}(\Omega)\cap L^\infty(\Omega)$ is dense in the space BV(Ω) with respect to its norm (see [10]). Therefore, we may suppose that $u \in \mathrm{BV}(\Omega)\cap L^\infty(\Omega)$. We apply theorem 6.1 in the open set $\Omega\times\mathbb{R}$ of $\mathbb{R}^{N+1}$, with $\alpha = (1,\dots,N)$. Let $\varepsilon \in\,]0,1[$. We associate to $E = H(u)$ a set L, having a locally finite perimeter in $\Omega\times\mathbb{R}$ and verifying properties (6.1)–(6.4). If we set $M = \|u\|_\infty + 1$, then L verifies the hypothesis of lemma 6.5 and we consider the corresponding function f, satisfying conditions (i)–(iv). We know that for each Borelian subset S of Ω

$$\int_S D(u-f) = \int_{S\times\mathbb{R}} D_\alpha(\chi_{H(u)}-\chi_L).$$

Thus,

$$\int_\Omega |D(u-f)| \leqslant \int_{\Omega\times\mathbb{R}} |D_\alpha(\chi_{H(u)}-\chi_L)|,$$

which, with condition (6.3), gives

$$\int_\Omega |D(u-f)| < \varepsilon.$$

On the other hand, for each $x\in\Omega$,

$$u(x)-f(x) = \int_{\mathbb{R}}(\chi_{H(u)}(x,y)-\chi_{H(f)}(x,y))\,\mathrm{d}y = \int_{\mathbb{R}}(\chi_{H(u)}(x,y)-\chi_L(x,y))\,\mathrm{d}y,$$

which, with condition (6.2), gives

$$\int_\Omega |u(x)-f(x)|\,\mathrm{d}x \leqslant \int_{\Omega\times\mathbb{R}}|\chi_{H(u)}(x,y)-\chi_L(x,y)|\,\mathrm{d}x\,\mathrm{d}y < \varepsilon$$

and completes the proof of theorem 6.7. □

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