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On supercritical problems involving the Laplace operator

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We discuss the existence, nonexistence and multiplicity of solutions for a class of elliptic equations in the unit ball with zero Dirichlet boundary conditions involving nonlinearities with supercritical growth. By using Pohozaev type identity we prove a nonexistence result for a class of supercritical problems with variable exponent which allow us to complement the analysis developed in (Calc. Var. (2016) 55:83). Moreover, we establish existence results of positive solutions for semilinear elliptic equations involving nonlinearities which are subcritical at infinity just in a part of the domain, and can be supercritical in a suitable sense.

Keywords: Critical Sobolev exponent; Supercritical elliptic problems; Radial solutions

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1. Introduction

Let Ω be a smooth domain in $\mathbb{R}^N, N \geqslant 3$. In the theory of semilinear elliptic equations, the phenomena of subcritical and critical growth in the nonlinearity are well known. For instance, the simplest model correspond to

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where existence results were obtained for the subcritical case $1 < p < 2^* - 1 = (N+2)/(N-2)$ (see for instance [1]) when Ω is bounded, while for the supercritical

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case $p \geqslant 2^* - 1$, in the classical work due to Pohozaev [7], it was proven that no positive solutions exist for (1.1) when $\Omega \neq \mathbb{R}^N$ and is strictly star-shaped. After that, existence issues about critical problems have been the focus of an active research area. For instance, Brezis and Nirenberg [3] showed that the nonexistence of solution may be reverted by adding a linear perturbation to the critical nonlinearity, while Ambrosetti et al. [2] showed existence of solutions for a problem involving a critical power perturbed by a concave nonlinearity. For more results related with this class of problems, the reader can see [4, 5].

1.1. Supercritical elliptic problems

Our approach in this paper allow us to study elliptic equations with variable exponent, for instance,

$$\begin{cases} -\Delta u = u^{p(r)} & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.2}$$

where B_1 is the unit ball centred at the origin, $r = |x|$ and $p(r)$ is a differentiable function in $[0, 1]$. Observe that, as a consequence of the previous comments involving the Pohozaev identity, it is natural to conjecture the following:

Problem (1.2) does not have any solution as $p(r) \geqslant 2^* - 1$.

Surprisingly, this conjecture is false due to theorem 1.5 in [6], where it was proved existence by considering the following increasing function

$$p(r) = 2^* - 1 + r^{\alpha}, \quad \text{with } 0 < \alpha < \min\{N/2, N-2\}.$$

However, we have proved here that the conjecture holds for the class of nonincreasing supercritical variable powers $p(r)$. This will be a consequence of the following Pohozaev type identity.

THEOREM 1.1. *Let $u \in H^1_{0,\mathrm{rad}}(B_1)$ be a solution of problem (1.2). Then u satisfies*

$$\frac{1}{2}\int_{\partial B_1} |\nabla u|^2 \sigma\nu \, \mathrm{d}\sigma = \int_{B_1} \left(\frac{N}{p(r)+1} - \frac{(N-2)}{2}\right) u^{p(r)+1} + \frac{r p'(r) u^{p(r)+1}}{(p(r)+1)^2}\left(\ln u^{p(r)+1} - 1\right) \mathrm{d}x,$$

where ν denotes the unit outward normal to ∂B_1.

As a consequence, we have the following nonexistence result.

THEOREM 1.2. *Suppose that $p(r)$ is a nonincreasing function satisfying $p(r) \geqslant 2^* - 1$. Then problem (1.2) has no weak solution in $H^1_{0,\mathrm{rad}}(B_1)$.*

Because of theorem 1.2, a natural question arises: does problem (1.2) have a nonradial weak solution in the case $p(r) \geqslant 2^* - 1$?

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Our second goal in this paper is to investigate the existence of positive radial solution for the following class of problems

$$\begin{cases} -\Delta u = f(r,u) & \text{in } B_1,\\ u>0 & \text{in } B_1,\\ u=0 & \text{on } \partial B_1, \end{cases} \tag{$\mathcal{P}$}$$

where the nonlinear term $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying the following conditions:

(f_1) There exists $1<p<2^*-1$ such that

$$\limsup_{\substack{s\to+\infty\\ r\to 0}} \frac{f(r,s)}{s^p} < +\infty,$$

where $2^* = 2N/(N-2)$ is the critical Sobolev exponent.

(f_2) $\displaystyle\lim_{s\to 0}\frac{f(r,s)}{s} = 0$ uniformly in $r\in[0,1]$.

(f_3) There exists $\theta>2$ and $s_0>0$ such that

$$0<\theta F(r,s) := \theta\int_0^s f(r,t)\,\mathrm{d}t \leqslant s f(r,s), \quad \forall r\in[0,1] \quad \text{and} \quad s>s_0.$$

Note that (f_1) is a new notion of subcritical growth at infinity, since the nonlinear term must be subcritical only in a neighbourhood of the origin, and note also that $f(r,u)$ may have supercritical growth at infinity in another subset of $(0,1)$. Nevertheless, we can directly associate a well-defined energy functional, without a truncation argument, which is the usual technique in order to apply variational methods for problems involving nonlinearities with the lack of subcritical growth. Assumptions (f_2)–(f_3) are the usual superlinearity condition at zero and the classical Ambrosetti–Rabinowitz, respectively. Now let us precisely state our existence result which is more in line with the classical work due to Ambrosetti and Rabinowitz [**1**], where subcritical behaviour was assumed in the whole domain.

THEOREM 1.3. *Suppose that (f_1)–(f_3) holds. Then problem ($\mathcal{P}$) has a nontrivial radial weak solution.*

As an application of theorem 1.3, we can obtain a nontrivial positive solution of problem (1.2) when $p(r)>1$.

1.2. Supercritical concave and convex problems

Our third target in this work is to analyse multiplicity results for supercritical problems in the presence of a concave perturbation. Indeed, we will consider the

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following two classes of concave and convex nonlinearities:

$$f(r,s) + \lambda a(r)u^q \quad \text{with} \quad 0 < q < 1. \tag{1.3}$$

$$K(r)u^{q(r)} \quad \text{where} \quad q(0) < 1 < q(1). \tag{1.4}$$

We will show that when we introduce a concave perturbation in the nonlinearity $f(r,u)$ of type (1.3), it produces some effect in the multiplicity of solutions. Note that, in this case, the nonlinearity is concave–convex in the same subset of the domain.

THEOREM 1.4. *Suppose that* (f_1)–(f_3) *holds. Then there exists* $\Lambda_0 > 0$ *such that for all* $\lambda \in (0, \Lambda_0)$ *problem*

$$\begin{cases} -\Delta u = f(r,u) + \lambda a(r)u^q & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{$\mathcal{C}$}$$

where $0 < q < 1$ *and* $a(r)$ *is a continuous satisfying* $a(r_0) > 0$, *for some* $r_0 \in [0,1]$ *possesses at least two nontrivial radial weak solutions.*

The above result is more in line with the famous Ambrosetti–Brezis–Cerami problem studied in [**2**] where a concave perturbation of the critical power was considered. Now, we state the following nonexistence result corresponding to problem ($\mathcal{C}$) by considering concave–convex nonlinearities of the first type.

THEOREM 1.5. *Assume that* (f_1)–(f_3). *Then there exists* λ_0 *such that problem* ($\mathcal{C}$) *has no radial weak solutions for any* $\lambda > \lambda_0$.

Our second type of concave–convex nonlinearity of type (1.4) is concave in a subset of the domain and convex in another. In order to generalize this kind of nonlinearity, we consider the problem

$$\begin{cases} -\Delta u = \lambda f(r,u) & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{$\mathcal{D}$}$$

where $f(r,s)$ satisfies

(f_4) There exist $0 < r_1 < 1$, $s_0 > 0$, $\theta > 2$ and $0 < q < 1$ such that for all $s \geqslant s_0$ we have

$$\theta F(r,s) - sf(r,s) \leqslant c_1 s^q, \quad \text{for all } r \in [0, r_1]$$

and

$$\theta F(r,s) - sf(r,s) \leqslant 0, \quad \text{for all } r \in [r_1, 1].$$

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(f_5) There exist $0 < \overline{q} < 1$ such that

$$\limsup_{r\to 0^+} \frac{f(r,s)}{s^{\overline{q}}} < +\infty, \quad \text{uniformly in s}$$

and

$$0 < \liminf_{\substack{s\to 0\\ r\to 0}} \frac{F(r,s)}{s^{\overline{q}+1}}.$$

(f_6) There exist $p > 1$ such that

$$\limsup_{\substack{s\to 0\\ r\to 1}} \frac{f(r,s)}{s^{p}} < +\infty.$$

(f_7) For all $[a,b] \subset (0,1)$ there exists $l \geqslant 1$ such that

$$\limsup_{s\to 0} \frac{f(r,s)}{s^{l}} < +\infty, \quad \text{uniformly in } r \in [a,b].$$

A basic model verifying all the above hypotheses (f_4)–(f_7) is given by

$$\begin{cases} -\Delta u = \lambda K(r) u^{q(r)} & \text{in } B_1,\\ u > 0 & \text{in } B_1,\\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.5}$$

where q is a increasing function with $q(0) < 1 < q(1)$, $q(r_1) = 1$ and K is a continuous function satisfying $K(0), K(1) > 0$ and $K(r_1) < 0$. It is worth mentioning that a main feature of problem (1.5) is the presence of the interphase $q(r_1) = 1$ for some $r_1 \in (0,1)$. This fact raises technical problems in several key points of our variational approach, for instance, the validity of Palais–Smale condition. Note that $f(r,u) = K(r)u^{q(r)}$ is a concave function for $r \in (0,\varepsilon)$ and it is a convex function for $r \in (1-\varepsilon, 1)$ for $\varepsilon > 0$ small enough. Now let us state our last result.

THEOREM 1.6. *Suppose that (f_4)–(f_7) holds. Then, there exists $\Lambda_1 > 0$ such that for all $\lambda \in (0,\Lambda_1)$, problem $(\mathcal{D})$ possesses at least two nontrivial radial weak solutions.*

The paper is structured as follows: In § 2, we prove a Pohozaev-type identity and our nonexistence result. In § 3, we show the variational structure needed to apply the min–max methods for our supercritical problems and the proof of theorem 1.3. Section 4 is dedicated to prove theorems 1.4, 1.5 and 1.6 related with the concave–convex problems.

2. Pohozaev-type identity and nonexistence result

Here we present a Pohozaev identity for a class of nonlinear elliptic problems involving variable exponent. As a consequence of this identity, we prove the nonexistence of positive solutions to semilinear problems with supercritical nonlinearities in a ball.

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Proof of theorem 1.1. Let $u \in H^1_{0,\mathrm{rad}}(B_1)$ be a solution of problem (1.2). Observe that

$$\operatorname{div}\left(\frac{xu^{p(r)+1}}{p(r)+1}\right) = \frac{Nu^{p(r)+1}}{p(r)+1} + \frac{rp'(r)u^{p(r)+1}\ln u}{p(r)+1} + \langle x, \nabla u\rangle u^{p(r)} - \frac{rp'(r)u^{p(r)+1}}{(p(r)+1)^2}.$$

and

$$\langle x, \nabla u\rangle \Delta u = \operatorname{div}\left(\langle x, \nabla u\rangle \nabla u - \frac{x|\nabla u|^2}{2}\right) + \frac{N-2}{2}|\nabla u|^2.$$

Multiplying the equation (1.2) by $\langle x, \nabla u\rangle$ and integrating by parts we obtain

$$\begin{aligned}\int_{\partial B_1} &\left[\langle \sigma, \nabla u\rangle \nabla u - \frac{\sigma|\nabla u|^2}{2} + \frac{\sigma u^{p(r)+1}}{p(r)+1}\right]\nu \,\mathrm{d}\sigma \\ &= \int_{B_1} \frac{Nu^{p(r)+1}}{p(r)+1} - \frac{rp'(r)u^{p(r)+1}}{(p(r)+1)^2}\,\mathrm{d}x \\ &\quad + \int_{B_1} \frac{rp'(r)u^{p(r)+1}\ln u}{p(r)+1} - \frac{(N-2)|\nabla u|^2}{2}\,\mathrm{d}x.\end{aligned}$$

Since $u = 0$ on ∂B_1 and $\nabla u = \langle \nabla u, \nu\rangle \nu$ we have

$$\begin{aligned}\frac{1}{2}\int_{\partial B_1} |\nabla u|^2 \sigma\nu \,\mathrm{d}\sigma = \int_{B_1} &\left(\frac{N}{p(r)+1} - \frac{(N-2)}{2}\right) u^{p(r)+1} \\ &+ \frac{rp'(r)u^{p(r)+1}}{(p(r)+1)^2}\left(\ln u^{p(r)+1} - 1\right)\mathrm{d}x,\end{aligned}$$

and we finish the proof. □

Proof of theorem 1.2. We can use the identity in theorem 1.1 to obtain

$$0 < \int_0^1 \left[\frac{N}{p(r)+1} - \frac{N-2}{2}\right] r^{N-1}u^{p(r)+1} + \frac{p'(r)r^N}{(p(r)+1)^2}u^{p(r)+1}\left(\ln u^{p(r)+1} - 1\right)\mathrm{d}r.$$

Observe that the function $s \to s(\ln s - 1)$ is bounded from below. Thus, since $p(r) \geqslant 2^* - 1$ and $p(r)$ is nonincreasing we have

$$\begin{aligned}0 < -\int_0^1 \frac{p'(r)r^N}{(p(r)+1)^2}\,\mathrm{d}r &= \int_0^1 \left(\frac{r^N}{p(r)+1}\right)' - \frac{Nr^{N-1}}{p(r)+1}\,\mathrm{d}r \\ &= \frac{1}{p(1)+1} - N\int_0^1 \frac{r^{N-1}}{p(r)+1} \leqslant 0,\end{aligned}$$

which is a contradiction and we finish the proof. □

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3. Problem ($\mathcal{P}$)

3.1. Variational background for problem ($\mathcal{P}$)

Assumption (f_1) implies that there exist constants $\rho \in (0,1)$ and $C_1, C_2 > 0$ such that

$$|f(r,s)| \leqslant C_1|s|^p + C_2 \quad \forall\, (r,s) \in (0,\rho) \times \mathbb{R}. \tag{3.1}$$

The inequality (3.1) is a subcriticality condition just around a neighbourhood of the origin. It is easy to see that using assumption (f_2) we have

$$\lim_{s\to 0} \frac{F(r,s)}{s^2} = 0. \tag{3.2}$$

Hereafter, for each $u \in H^1_{0,\mathrm{rad}}(B_1)$ we denote $\|u\|_{H^1_{0,\mathrm{rad}}(B_1)} := \|u\|$. We are going to use the following version of the radial lemma (see [8]).

Lemma 3.1 Radial Lemma. *If $u \in H^1_{0,\mathrm{rad}}(B_1)$, then*

$$|u(r)| \leqslant \|u\| \frac{(1-r)^{1/2}}{r^{(N-2)/2}}.$$

Proof. Assume that $u \in C_c^\infty(B_1)$ is radial, since $u(r) = \int_1^r u'(s)\,\mathrm{d}s$, by using Cauchy–Schwartz inequality we have

$$\begin{aligned}
|u(r)| &\leqslant \left|\int_r^1 u'(s)\,\mathrm{d}s\right| \leqslant \int_r^1 \left|u'(s)s^{(N-1)/2}\frac{1}{s^{(N-1)/2}}\right| \mathrm{d}s \\
&\leqslant \|u\| \cdot \|1/s^{(N-1)/2}\|_{L^2([r,1])} = \|u\| \left(\frac{r^{2-N}-1}{N-2}\right)^{1/2} \\
&= \|u\| \frac{1}{\sqrt{N-2}} \frac{1}{r^{N-2/2}} \left((1+r)(1+r+\cdots+r^{N-3})\right)^{1/2} \\
&\leqslant \|u\| \frac{(1-r)^{\frac{1}{2}}}{r^{\frac{N-2}{2}}},
\end{aligned}$$

which finishes the proof. □

We employ variational methods in order to prove the existence of solutions. Our next lemma shows that the energy functional for problem ($\mathcal{P}$) $J : H^1_{0,\mathrm{rad}}(B_1) \to \mathbb{R}$ given by

$$J(u) := \frac{1}{2}\int_{B_1} |\nabla u|^2\,\mathrm{d}x - \int_{B_1} F(r,u)\,\mathrm{d}x$$

is well-defined and of class C^1, which brought some complication since the nonlinear term is not necessarily subcritical.

Lemma 3.2. *Under our assumptions, the associated energy functional J is well-defined in $H^1_{0,\mathrm{rad}}(B_1)$ and belongs to class C^1.*

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Proof. First we will prove that J is well-defined in $H^1_{0,\mathrm{rad}}(B_1)$. Indeed,

$$\begin{aligned}\int_{B_1} F(r,u)\,\mathrm{d}x &= C\int_0^1 F(r,u)r^{N-1}\,\mathrm{d}r\\ &= C\int_0^\rho F(r,u)r^{N-1}\,\mathrm{d}r + C\int_\rho^1 F(r,u)r^{N-1}\,\mathrm{d}r.\end{aligned}$$

Note that the assumption (f_1) implies that there exist constants $C_3, C_4 > 0$ such that

$$|F(r,u)| \leqslant C_3|u|^{p+1} + C_4, \quad 0 \leqslant r \leqslant \rho.$$

Then, by Sobolev embedding,

$$\begin{aligned}\left|\int_0^\rho F(r,u)r^{N-1}\,\mathrm{d}r\right| &\leqslant C_2\int_0^\rho |u|^{p+1}r^{N-1}\,\mathrm{d}r\\ &\leqslant C_3\|u\|^{p+1}.\end{aligned}$$

On the other hand, using Radial lemma 3.1,

$$|u(r)| \leqslant C_\rho\|u\|, \quad \forall\,\rho \leqslant r \leqslant 1.$$

Since $F(r,s)$ is continuous in $[\rho,1]\times[-C_\rho\|u\|, C_\rho\|u\|]$, we obtain that $F(\cdot,u(\cdot)) \in L^\infty[\rho,1]$. Thus $\int_{B_1} F(r,u)\,\mathrm{d}x$ is well-defined. To conclude we need to prove that

$$I(u) := \int_{B_1} F(r,u)\,\mathrm{d}x$$

belongs to class C^1. The Gateaux derivative of functional I is given by

$$\langle Lu, v\rangle := \int_{B_1} f(r,u)v\,\mathrm{d}x, \quad \forall\,u,v \in H^1_{0,\mathrm{rad}}(B_1) \tag{3.3}$$

In fact, we need to prove that

$$\begin{aligned}\lim_{t\to 0}\frac{I(u+tv)-I(u)}{t} &= \lim_{t\to 0}\int_{B_1}\frac{F(r,u+tv)-F(r,u)}{t}\,\mathrm{d}x\\ &= \int_{B_1} f(r,u)v\,\mathrm{d}x.\end{aligned}$$

Using the Mean Value Theorem, we have

$$\frac{F(r,u+tv)-F(r,u)}{t}\,\mathrm{d}x = f(r,u+t_x v)v$$

Note that if $r \in (0,\rho)$ we can estimate

$$|f(r,u+t_x v)v| \leqslant \left(C|u(x)+t_x v(x)|^p + C\right)v(x) \leqslant \left(C\left(|u(x)|+|v(x)|\right)^p + C\right)|v(x)|$$

and if $r \in [\rho,1]$, using the Radial Lemma we obtain

$$|f(r,u(x)+t_x v(x))v(x)| \leqslant M_\rho, \quad \forall\,r \in [\rho,1].$$

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Hence

$$|f(r,u+t_x v)v| \leqslant C\left(|u(x)|+|v(x)|\right)^p |v(x)| + M_\rho, \quad \forall\, r\in(0,1).$$

Note that the constant M_ρ also depends on $\|u\|$ and $\|v\|$. To complete it is sufficient to prove that L defined in (3.3) is continuous in $H^1_{0,\mathrm{rad}}(B_1)$. Assume that $u_n \to u$ in $H^1_{0,\mathrm{rad}}(B_1)$ and consider

$$|\langle L(u_n-u),v\rangle| = \left|\int_{B_1}\left(f(r,u_n)-f(r,u)\right)v\right|, \quad v\in H^1_{0,\mathrm{rad}}(B_1).$$

Note that $\|u_n\|\leqslant C$ and by using equation (3.1) and Radial lemma 3.1 we have

$$|f(r,u_n(x))| \leqslant \begin{cases} C|u_n(x)|^p + C & \text{if } r\in(0,\rho)\\ M_\rho & \text{if } r\in[\rho,1].\end{cases}$$

Thus,

$$|f(r,u_n(x))| \leqslant C|u_n(x)|^p + M_\rho, \quad \forall\, r\in[0,1]$$

and

$$|\langle L(u_n-u),v\rangle| \leqslant \|f(r,u_n)-f(r,u)\|_{L^{q/p}}\|v\|_{L^{(q/p)'}}$$

where $q=(p+1)/p$, which proves that L is continuous in $H^1_{0,\mathrm{rad}}(B_1)$. □

3.2. Existence results for problem ($\mathcal{P}$)

In the next two lemmas we will be concerned to prove that the energy functional J satisfies the mountain pass geometry.

Lemma 3.3. *There exist $\alpha>0$ and $\eta>0$ such that*

$$J(u)\geqslant \alpha>0, \quad \textit{if } \|u\|=\eta.$$

Proof. We first recall that using conditions (f_1) and (f_2), we have that given $\varepsilon_1>0$, there exists $M>0$ such that

$$F(r,u)\leqslant \varepsilon_1 u^2 + Mu^{p+1}, \quad \forall r\in(0,\rho).$$

Thus, using Sobolev embedding we obtain the following estimate

$$\int_0^\rho F(r,u)r^{N-1}\,\mathrm{d}r \leqslant C_1M\|u\|^{p+1} + \varepsilon_1 C_2\|u\|^2$$

On the other hand, from Radial lemma 3.1 we have

$$|u(r)|\leqslant \rho^{-(N+2)/2}\|u\| \quad \text{if} \quad r\in(\rho,1). \tag{3.4}$$

Thus, by using (3.4) and (f_2), given $\varepsilon_2>0$,

$$F(r,u)\leqslant \varepsilon_2 u^2,$$

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for all $r \in (\rho, 1)$ and $\|u\| = \eta$ with η sufficiently small. This implies that

$$\int_\rho^1 F(r,u) r^{N-1}\,\mathrm{d}r \leqslant \varepsilon_2 C \|u\|^2.$$

Hence

$$\begin{aligned} J(u) &\geqslant \frac{1}{2}\|u\|^2 - C_1\|u\|^{p+1} - \varepsilon_2 C\|u\|^2 - \varepsilon_1 C\|u\|^2 \\ &= \left(\frac{1}{2} - \varepsilon C\right)\|u\|^2 - C_1\|u\|^{p+1}. \end{aligned}$$

Now, choosing $\varepsilon = \varepsilon_1 + \varepsilon_2 < 1/(2C)$, there exists $\eta > 0$ sufficiently small such that if $\|u\| = \eta$ then $J(u) \geqslant \alpha > 0$ and the lemma is proved. □

Lemma 3.4. *Assume* (f_3). *Then there exist* $w \in H^1_{0,\mathrm{rad}}(B_1)$ *such that* $\|w\| > \eta$ *and* $J(w) < 0$.

Proof. It is well known that if condition (f_3) holds, then by integration we can prove that there exist constants $c, d \in \mathbb{R}$ such that $F(r,u) \geqslant c|u|^\theta - d$. Thus, given $\phi \in C_0^\infty(B_1, [0, +\infty))$ a nontrivial function it is easy to see that

$$\lim_{t \to +\infty} J(t\phi) = -\infty,$$

which proves the lemma. □

Lemma 3.5 Palais–Smale condition. *The functional* $J : H^1_{0,\mathrm{rad}}(B_1) \to \mathbb{R}$ *satisfies the Palais–Smale condition.*

Proof. We need to prove that any Palais–Smale sequence admits a convergent subsequence. Let $(u_n) \subset H^1_{0,\mathrm{rad}}(B_1)$ be a Palais–Smale sequence of the functional J at level c, that is,

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0.$$

Thus, we have $|J'(u_n)u_n| \leqslant \varepsilon_n \|u_n\|$ where $\varepsilon_n \to 0$ as $n \to \infty$. Hence,

$$\begin{aligned} \theta J(u_n) - J'(u_n)u_n &= \left(\frac{\theta}{2} - 1\right)\|u_n\|^2 + C_1 \int_0^1 \left(u_n f(r,u_n) - \theta F(r,u_n)\right) r^{N-1}\,\mathrm{d}r \\ &\leqslant \varepsilon_n \|u_n\| + C_2 \end{aligned}$$

Thus, by hypothesis (f_3), the sequence (u_n) is bounded in $H^1_{0,\mathrm{rad}}(B_1)$. Hence, up to a subsequence, we can assume that $u_n \rightharpoonup u$ weakly in $H^1_{0,\mathrm{rad}}(B_1)$. From $J'(u_n) \to 0$

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we obtain

$$\left|\int_{B_1} \nabla u_n \cdot \nabla \phi - \int_{B_1} f(r,u_n)\phi\right| \leqslant \varepsilon_n \|\phi\|, \quad \forall\, \phi \in H^1_{0,\mathrm{rad}}(B_1).$$

Taking $\phi = u_n - u$ we can get

$$\left|\int_{B_1} \nabla u_n \cdot \nabla (u_n - u) - \int_{B_1} f(r,u_n)(u_n - u)\right| \leqslant \varepsilon_n \|(u_n - u)\|.$$

To conclude our proof it is enough to verify that

$$\int_{B_1} f(r,u_n)(u_n - u) \to 0.$$

In fact,

$$\begin{aligned}\int_{B_1} f(r,u_n)(u_n - u) &= C\int_0^\rho f(r,u_n)(u_n-u)r^{N-1}\,\mathrm{d}r \\ &\quad + C\int_\rho^1 f(r,u_n)(u_n-u)r^{N-1}\,\mathrm{d}r\end{aligned}$$

But

$$\begin{aligned}\left|\int_0^\rho f(r,u_n)(u_n-u)r^{N-1}\,\mathrm{d}r\right| &\leqslant C\int_0^\rho |u_n|^p|u_n-u|r^{N-1}\,\mathrm{d}r \\ &\leqslant C\|u_n\|^p_{L^{p+1}}\|u_n-u\|_{L^{p+1}} \\ &\leqslant C\|u_n-u\|_{L^{p+1}} \to 0\end{aligned}$$

since $p+1 < 2^*$. On the other hand, $f(r,u_n)$ is bounded in $[\rho,1]$ by the Radial lemma 3.1 and that the sequence is bounded in $H^1_{0,\mathrm{rad}}(B_1)$, hence

$$\int_\rho^1 f(r,u_n)(u_n-u)r^{N-1}\,\mathrm{d}r \to 0$$

which completes the proof. □

Proof of theorem 1.3. In view of lemmas 3.3, 3.4 and 3.5, we can apply the Mountain Pass Theorem to obtain a positive level c and a nontrivial solution $u \in H^1_{0,\mathrm{rad}}(B_1)$ of problem ($\mathcal{P}$). □

4. Semilinear elliptic equations with concave–convex nonlinearities

4.1. Problem ($\mathcal{C}$)

Here we are going to prove our multiplicity and nonexistence results for supercritical problems in the presence of a concave perturbation.

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Proof of theorem 1.4. The energy functional associated to problem ($\mathcal{C}$) is given by

$$J_\lambda(u) := \frac{1}{2}\|u\|^2 - \int_{B_1} F(r,u)\,\mathrm{d}x - \frac{\lambda}{q+1}\int_{B_1} |u|^{q+1}\,\mathrm{d}x, \quad u \in H^1_{0,\mathrm{rad}}(B_1)$$

Note that it follows from similar calculations as lemma 3.2 that the functional J_λ is well-defined and belongs to class C^1. Since $q < 1$, we can use similar arguments to the lemma 3.3 to prove that

$$J_\lambda(u) \geqslant \left(\frac{1}{2} - \varepsilon\right)\|u\|^2 - \|u\|^{p+1} - \lambda C\|u\|^{q+1}.$$

Hence if we take $\|u\| = \lambda^\alpha$, we have

$$J_\lambda(u) \geqslant \left(\frac{1}{2} - \varepsilon\right)\lambda^{2\alpha} - \lambda^{\alpha(p+1)} - C\lambda^{\alpha(q+1)+1}.$$

Thus if $0 < \alpha < 1/(1-q)$, then there exists $\Lambda_0 > 0$ such that for all $\lambda \in (0, \Lambda_0)$,

$$J_\lambda(u) > \alpha_\lambda > 0, \quad \|u\| = \eta_\lambda$$

for some α_λ, $\eta_\lambda > 0$. We can also prove versions of lemmas 3.4 and 3.5 for our functional J_λ. Hence by using the Mountain Pass Theorem we obtain a nontrivial solution of positive energy for problem ($\mathcal{C}$). The second solution we will obtain through minimization on a small ball. In fact, since $a(r_0) > 0$, there exist $c_0, \delta > 0$ such that $a(r_0) > c_0$ and $a(r) > 0$ for all $r \in (r_0 - \delta, r_0 + \delta)$. Now, taking $\phi \in C^1_0(0,1)$ such that $\operatorname{supp}\phi \subset [r_0 - \delta, r_0 + \delta]$ then for $t > 0$,

$$J(t\phi) = \frac{t^2}{2}\|\phi\|^2 - C\int_0^1 F(r,t\phi)r^{N-1}\,\mathrm{d}r - \frac{\lambda C t^{q+1}}{q+1}\int_0^1 |\phi|^{q+1}r^{N-1}\,\mathrm{d}r.$$

Thus, by using (3.2), it is not difficult to prove that

$$\int_0^1 \frac{F(r,t\phi)r^{N-1}}{t^{q+1}}\,\mathrm{d}r \to 0 \quad \text{as} \quad t \to 0.$$

Thus, there exist points of negative energy $w \in H^1_{0,\mathrm{rad}}(B_1)$ with $\|w\| < \eta_\lambda$. Since the functional J_λ is lower semicontinuous we can obtain a second solution of negative energy. □

Proof of theorem 1.5. Since $a(r_0) > 0$ we can choose $\delta > 0$ sufficiently small such that $a(r) > 0$ for all $r \in (r_0 - \delta, r_0 + \delta)$. On the other hand, observe that (f_1) and (f_3) imply that $[f(|x|,s) + \lambda a(r)s^q]/s$ blows-up near the origin and at infinity for $r \in (r_0 - \delta, r_0 + \delta)$. By compactness, it must attain a minimum and for

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$r \in (r_0 - \delta, r_0 + \delta)$,

$$\frac{f(|x|, s) + \lambda a(r) s^q}{s} \geqslant c_1 s^{p-1} + \lambda c_0 s^{q-1} \geqslant C(p, q, c_0, c_1) \lambda^{(p-1)/(p-q)},$$

where

$$C(p, q, c_0, c_1) = \left(\frac{c_0^{p-1}}{c_1^{q-1}}\right)^{1/(p-q)} \left(\frac{1-q}{p-1}\right)^{(p-1)/(p-q)} \left[1 + \frac{p-1}{1-q}\right].$$

Now, denote by $A_\delta = B_{r_0+\delta} \setminus \overline{B_{r_0-\delta}}$ and let $\psi_1 > 0$ be the first eigenfunction for $(-\Delta, H_0^1(A_\delta))$. Thus,

$$\lambda_1 \int_{A_\delta} u\psi \, \mathrm{d}x = \int_{B_1} \nabla u \cdot \nabla \psi \, \mathrm{d}x = \int_{B_1} [f(r, u) + \lambda a(r) u^q] \, \psi \, \mathrm{d}x$$
$$\geqslant C(p, q, c_0, c_1) \lambda^{(p-1)/(p-q)} \int_{A_\delta} u\psi \, \mathrm{d}x.$$

Since $\int_{B_\delta} u\psi \, \mathrm{d}x$ is positive, one deduces that $C(p, q, c_0, c_1)\lambda^{(p-1)/(p-q)} \leqslant \lambda_1$, which is a contradiction if λ is large enough and the proof is complete. □

4.2. Problem $(\mathcal{D})$

Now we will prove our existence result related with problem $(\mathcal{D})$. For our purpose, we introduce the functional $I : H^1_{0,\mathrm{rad}}(B_1) \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2} \int_{B_1} |\nabla u|^2 \, \mathrm{d}x - \lambda \int_{B_1} F(r, u) \, \mathrm{d}x,$$

which is well-defined and of class C^1.

We emphasize that the classical superquadraticity condition of Ambrosetti–Rabinowitz [**1**] will not be satisfied for problem $(\mathcal{D})$. In order to overcome this difficulty, we use an alternative condition introduced by de Figueiredo et al. [**4**].

Lemma 4.1. *The Palais–Smale sequence for the functional I is bounded.*

Proof. Let $(u_n) \subset H^1_{0,\mathrm{rad}}(B_1)$ be a Palais–Smale sequence of the functional I at level c, that is,

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0.$$

Thus, we have $|I'(u_n)u_n| \leqslant \varepsilon_n \|u_n\|$ where $\varepsilon_n \to 0$ as $n \to \infty$ which implies

$$\left(\frac{\theta}{2} - 1\right) \|u_n\|^2 + \int_0^1 (sf(r, s) - \theta F(r, s)) \, r^{N-1} \, \mathrm{d}r = \theta I(u_n) - I'(u_n)u_n$$
$$\leqslant \varepsilon_n \|u_n\| + C. \tag{4.1}$$

On the other hand, by using hypothesis (f_4) we have the following inequalities

$$\int_0^{r_1} (sf(r, s) - \theta F(r, s)) \, r^{N-1} \, \mathrm{d}r \geqslant -C_1 \int_0^1 u_n^{q+1}, \tag{4.2}$$

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and

$$\int_{r_1}^{1} (sf(r,s) - \theta F(r,s))\, r^{N-1}\, \mathrm{d}r \geqslant 0,$$

for some $0 < r_1 < 1$. Thus, by using Sobolev embedding in (4.2) follows

$$\int_{0}^{r_1} (sf(r,s) - \theta F(r,s))\, r^{N-1}\, \mathrm{d}r \geqslant -C_2\|u_n\|^{q+1}.$$

Returning to (4.1) we obtain

$$\left(\frac{\theta}{2} - 1\right)\|u_n\|^2 \leqslant \varepsilon_n\|u_n\| + C_2\|u_n\|^{q+1} + C,$$

which implies that (u_n) is bounded and we finished the proof. □

Proof of theorem 1.6. Let $0 < \overline{q} < 1$ given by (f_5). Thus, there exists $\delta > 0$ such that

$$\begin{aligned}\int_0^{\delta} F(r,u) r^{N-1}\,\mathrm{d}x &\leqslant C_1\int_0^{\delta} u^{\overline{q}+1} r^{N-1}\,\mathrm{d}x\\ &\leqslant C_2\|u\|^{\overline{q}+1}.\end{aligned}$$

On the other hand, the Radial lemma 3.1 and (f_6) imply that

$$\int_{\overline{r}}^{1} F(r,u)\,\mathrm{d}x \leqslant C_2\int_{\overline{r}}^{1} u^{p+1} r^{N-1}\,\mathrm{d}x \leqslant C_3\|u\|^{p+1}.$$

Now, (f_7) implies that

$$\int_{\delta}^{\overline{r}} F(r,u)\,\mathrm{d}x \leqslant C_2\int_{\delta}^{\overline{r}} u^{l+1} r^{N-1}\,\mathrm{d}x \leqslant C_3\|u\|^{l+1}.$$

Gathering above inequalities we obtain that

$$I(u) \geqslant \frac{1}{2}\|u\|^2 - C\lambda\left(\|u\|^{p+1} + \|u\|^{l+1} + \|u\|^{\overline{q}+1}\right)$$

Taking $\|u\| = \lambda^{\alpha}$ and $0 < \alpha < 1/(1-\overline{q})$, there exists $\Lambda_1 > 0$ such that for all $\lambda \in (0, \Lambda_1)$,

$$I(u) > \alpha_\lambda > 0, \quad \|u\| = \eta_\lambda$$

for some α_λ, $\eta_\lambda > 0$. Hence by using lemma 4.1 and the Mountain Pass Theorem we obtain a nontrivial solution of positive energy of problem $(\mathcal{D})$. The second solution we will obtain through minimization on a small ball. By hypothesis (f_5), there exist

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$\overline{s}, \overline{\delta} > 0$ such that

$$F(r, s) \geqslant \tilde{c} s^{\overline{q}+1},$$

for all $0 < s \leqslant \overline{s}$ and $0 < r \leqslant \overline{\delta}$. Let $\phi \in C_0^1(0, 1)$ a nonnegative function and $\varepsilon > 0$ such that $\operatorname{supp} \phi \subset\subset (0, \overline{\delta})$ and $\varepsilon\phi \leqslant \overline{s}$. Then for $\varepsilon > 0$,

$$\begin{aligned} I(\varepsilon\phi) &= \frac{\|\varepsilon\phi\|^2}{2} - \lambda C \int_0^1 F(r, \varepsilon\phi) r^{N-1} \, \mathrm{d}x \\ &\leqslant \frac{\varepsilon^2}{2} \|\phi\|^2 - \lambda C \varepsilon^{\overline{q}+1} \int_0^1 \phi^{\overline{q}+1} r^{N-1} \, \mathrm{d}r. \end{aligned}$$

Thus, for $\varepsilon > 0$ sufficiently small there exist points of negative energy $v \in H^1_{0,\mathrm{rad}}(B_1)$ with $\|v\| < \eta_\lambda$. Since the functional I is lower semicontinuous we can obtain a second solution of negative energy and we finish the proof. □

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