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On the number of topological orbits of complex germs in $\mathcal{K}$ classes $(xy, x^a + y^b)$

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We show that there exist an infinite number of topological orbits in $\mathcal{K}$ classes of complex map germs from the plane to the plane that have a representative of type $(xy, x^a + y^b)$, with $(a, b) \neq (2, 3)$ or $(2, 5)$. Our key tool to prove this existence is the existence (or not) of *stems* in the $\mathcal{K}$ class; these germs are not $\mathcal{A}$-finitely determined and allow the determination of a non-finite number of topological orbits. We also show that the $\mathcal{K}$ class $(xy, x^2 + y^5)$ has two topological orbits.

Keywords: topological classification; invariants;
Newton non-degeneracy conditions

2010 *Mathematics subject classification:* Primary 32S15; 32S05; 32S30
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1. Introduction

The description of the topological orbits of map germs is a central problem in singularity theory; even finding when a $\mathcal{K}$ class has a finite (or not) number of topological orbits is, in general, an open problem. Concerning complex map germs from the plane to the plane, Gaffney and Mond described in [4] the topological orbits of *semi-quasi-homogeneous* map germs that have a representative that is finitely determined.

In the co-rank 2 case, of interest here, there are germs that belong to a given $\mathcal{K}$ class, but are not semi-quasi-homogenous, particularly if the germ belongs to a

$\mathcal{K}$ class with a representative $(xy, x^a + y^b)$ with $\gcd(a, b) = 1$. The simplest case is the $\mathcal{K}$ class $(xy, x^2 + y^3)$, where the germs $(xy, x^2 + \alpha xy + y^3)$ are not semi-quasi-homogeneous for any $\alpha \neq 0$, but are $\mathcal{A}$-equivalent to $(xy, x^2 + y^3)$ and there exists only one topological orbit (see [4, example 5.11]).

Another simple case is the $\mathcal{K}$ class $(xy, x^3 + y^4)$ [4, example 5.12]: if α or $\beta \neq 0$, the germs $g_{\alpha,\beta}(xy, x^3 + y^4 + \alpha xy^2 + \beta x^2 y)$ are not semi-quasi-homogeneous and we cannot apply the results of [4, example 5.11] to describe the topological orbits of these germs. The expectation here was to obtain only a finite number of topological orbits. However, we show in § 3 that the number of different topological orbits in this $\mathcal{K}$ class is not finite. Our key tool is the existence of stems, i.e. germs that are not $\mathcal{A}$-finitely determined; using these we can construct a non-finite family of $\mathcal{A}$-finitely determined germs that are in different topological orbits.

We show that the number of topological orbits in any $\mathcal{K}$ class of type $(xy, x^a + y^b)$ is not finite, with two exceptions: the classes $(xy, x^2 + y^3)$ and $(xy, x^2 + y^5)$. We describe how to obtain stems in all other classes. For the class $(xy, x^2 + y^5)$ we show that there are two topological orbits.

The method of obtaining these orbits is to study the vanishing cusps and transversal double-fold points that appear in the discriminant curve of any generic deformation of the germ.

Such numbers may be thought of as reflecting the complexity of the original map germs. Whitney showed in [15] that any real stable map germ in these dimensions has only a finite number of cusps and double folds as singular points of the discriminant curve. In [3] Gaffney and Mond showed sufficient conditions for finite determinacy in terms of the finiteness of the number of these singularities. Moreover, the constancy of these singularities is a necessary and sufficient condition for topological triviality in a family [3, corollary 1.10].

First, we show formulae to compute the number of cusps. For this we describe the Milnor number of the critical curve using geometric conditions given by its Newton polygon and apply the relationship between this Milnor number and the number of cusps.

Alternatively, the transversal double-fold points are related to the Milnor number and the Fitting ideals of the discriminant curve. With the aid of computational methods we can describe the Fitting ideals and thus the defining equation of the discriminant curve. Then we use geometric conditions given by the Newton polygon of the discriminant curve, which allow us to determine whether or not the Milnor number of the discriminant is constant in a family.

2. Cusps and double folds

Given a map germ $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, denote its critical set by $\Sigma(f)$. The Jacobian ideal, denoted by $J(f)$, is the ideal generated by the determinant of the matrix of the partial derivatives of f, and $\Delta(f) = f(\Sigma(f))$ is the discriminant of f.

For any finitely determined germ f with $\Sigma(f) \neq \emptyset$, the curves $\Sigma(f)$ and $\Delta(f)$ have isolated singularities. When f is perturbed so that it becomes stable, a finite number of cusps, denoted by $c(f)$, and double folds, denoted by $d(f)$, appear on the discriminant curve. In [3, corollary 1.10] it is shown that the constancy of the numbers $d(g_t)$ and $c(g_t)$ is a necessary and sufficient condition for the topological triviality in a family of finitely determined map germs g_t. Moreover, in [2, theorem 9.9] it is shown that the family is topologically trivial if and only if the Milnor number of the discriminant curve is constant in the family.

2.1. Cusps and the Milnor number $\mu(\Sigma(g))$

Denote by $\mu(\Sigma(f))$ the Milnor number of the critical curve, and by $\mu(\Delta(f))$ the Milnor number of the discriminant curve. As usual $m(f)$ denotes the degree of f. Then we see in [3] that

$$c(f) = \mu(\Sigma(f)) + m(f) - 2. \tag{2.1}$$

A formula for the number of cusps of semi-quasi-homogeneous maps is given in [4].

THEOREM 2.1 (Gaffney and Mond [4, theorem 2.1.i]). *Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a semi-quasi-homogeneous map germ of type $(w_1, w_2; d_1, d_2)$. Then*

$$c(f) = \frac{\{(d_1 + d_2 - (w_1 + w_2))(d_1 + d_2 - 2(w_1 + w_2)) + d_1 d_2 - w_1 w_2\}}{w_1 w_2}.$$

In this section we apply (2.1) to show formulae to compute the number of cusps of some map germs in a $\mathcal{K}$ class $(xy, x^a + y^b)$ with $\gcd(a, b) = 1$. Here we consider only germs of type $(xy, x^a + y^b + \sum \alpha_{r,s} x^r y^s)$ with $as + br < ab$. As we see in [12], these germs, called pre-quasi-homogeneous germs, are finitely determined.

First, we compute the Milnor number of the critical curve when the Jacobian ideal is Newton non-degenerate. Remember that $m(f) = a + b$ for any germ f in the $\mathcal{K}$ class $(xy, x^a + y^b)$. We now recall some ideas that help us to compute the Milnor number.

Let $\mathcal{O}_n$ be the local ring of germs from $(\mathbb{C}^n, 0)$ to $\mathbb{C}$ for any $g = \sum_k a_k x^k$ in $\mathcal{O}_n$. The *support* of g, denoted by $\operatorname{supp} g$, is the set of points k in $\mathbb{Z}^n$ such that $a_k \neq 0$. If I is an ideal in $\mathcal{O}_n$, the *support* of I is defined as $\operatorname{supp} I := \bigcup_{g \in I} \operatorname{supp} g$.

DEFINITION 2.2. The *Newton polyhedron* of an ideal I in $\mathcal{O}_n$, denoted by $\Gamma_+(I)$, is the convex hull in $\mathbb{R}^n_+$ of $\{k + v\colon v \in \mathbb{R}^n_+,\ k \in \operatorname{supp}(I)\}$. $\Gamma(I)$ denotes the union of the compact faces of $\Gamma_+(I)$, and $\Gamma_-(I)$ is the closure of $\mathbb{R}^n \setminus \Gamma_+(I)$.

If D is a fixed subset of $\Gamma_+(I)$ and $g = \sum_k a_k x^k$ in $\mathcal{O}_n$, we define

$$g_D = \sum_{k \in D} a_k x^k.$$

Given a face $\Delta \subseteq \Gamma(I)$, denote the union of half-rays emanating from the origin and passing through Δ by $C(\Delta)$. $\mathcal{O}_\Delta$ denotes the set of all germs g_Δ.

DEFINITION 2.3 (Saia [11]). An ideal I of finite codimension in $\mathcal{O}_n$ is *Newton non-degenerate* if there exists a system of generators $g_1, \dots, g_s$ of I such that, for each compact face $\Delta \subseteq \Gamma(I)$, the ideal I_Δ generated by $g_{1_\Delta}, \dots, g_{s_\Delta}$ has finite codimension in $\mathcal{O}_\Delta$. A germ $f \in \mathcal{O}_2$ is *Newton non-degenerate* if the ideal $\langle x\partial f/\partial x, y\partial f/\partial y \rangle$ is Newton non-degenerate (see [7]).

THEOREM 2.4 (Kouchnirenko [7, theorem I(ii)]). *A germ f is Newton non-degenerate in $\mathcal{O}_2$ if and only if $\mu(f) = 2S - i - j + 1$. Here S denotes the volume of the Newton polygon $\Gamma_-(f)$ and the numbers i and j are those for which $\Gamma_-(g)$ meets the coordinate axis in the points $(i, 0)$ and $(0, j)$.*

LEMMA 2.5. *The germs $J(g)$ of the germs below are Newton non-degenerate.*

(1) $g(x, y) = (xy, x^a + y^b + kx^r y^s)$.

(2) $g(x, y) = (xy, x^a + y^b + k_1 x^{r_1} y^{s_1} + k_2 x^{r_2} y^{s_2})$, *with* $k_1, k_2 \neq 0$.

(3) $g(x, y) = (xy, x^a + y^b + \sum_{i=1}^{3} k_i x^{r_i} y^{s_i})$ *with* $r_1 < r_2 < r_3$ *and* $s_3 < s_2 < s_1$.

(4) $g(x, y) = (xy, x^a + y^b + \sum_{i=1}^{t} k_i x^{r_i} y^{s_i})$ *with* $k_i \neq 0$ *for all* $i = 1, 2, \ldots, t$, $t \geqslant 4$, $r_1 < r_2 < \cdots < r_t$, $s_t < s_{t-1} < \cdots < s_2 < s_1$ *and* $\Gamma_+(J(g))$ *has* $t + 1$ *facets.*

Proof. The Newton non-degeneracy of (1) follows from [12, proposition 2.6], since all monomials of $J(g)$ correspond to vertices of its Newton polygon.

To prove (2) and (3) we consider the number of compact facets of the Newton polygon of $J(g)$.

For the germ given in (2), if $\Gamma_+(J(g))$ has three compact facets, again from [12, proposition 2.6] we obtain the result, since all monomials of $J(g)$ correspond to vertices of its Newton polygon. If $\Gamma_+(J(g))$ has two compact facets, a straightforward calculation also shows that the germ $J(g)$ is Newton non-degenerate.

To show the Newton non-degeneracy of the germ in (3), if $\Gamma_+(J(g))$ has three or four compact facets, the proof is analogous to that of (2).

If $\Gamma_+(J(g))$ has two compact facets, we consider the following cases:

- (r_2, s_2) is a vertex;
- (r_1, s_1) or (r_3, s_3) is a vertex with $x^a + k_1 x^{r_1} y^{s_1} + k_2 x^{r_2} y^{s_2} + k_3 x^{r_3} y^{s_3} \neq x^r (x^m - y^q)^3$; or
- $y^b + k_1 x^{r_1} y^{s_1} + k_2 x^{r_2} y^{s_2} + k_3 x^{r_3} y^{s_3} \neq y^s (x^n - y^\ell)^3$.

The proof of the two first cases is straightforward. We show that the third case does not hold. Suppose that it holds and the Newton polygon has two compact facets. Then we can have $y^b + k_1 x^{r_1} y^{s_1} + k_2 x^{r_2} y^{s_2} = y^s (x^n - y^\ell)^2$ or $x^a + k_2 x^{r_2} y^{s_2} + k_3 x^{r_3} y^{s_3} = x^r (x^m - y^q)^2$, but these equalities hold if $r = 2n$ and $s = 2q$, and this implies that $a = 2(n + m)$ and $b = 2(q + \ell)$. Therefore, $\gcd(a, b) \neq 1$, and the germ g is not finitely determined.

The Newton non-degeneracy of the germ in (4) is immediate from [12, proposition 2.6]. □

In table 1 we show the formulae for computing $c(g)$ and $\mu(\Sigma(g))$. In these cases, $\#\Gamma$ denotes the number of one-dimensional facets of the Newton polygon.

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Table 1. Source invariants.

Germ g	$\#\Gamma$	$\mu(\Sigma(g))$	$c(g)$
$(xy, x^a + y^b + kx^ry^s)$	2	$(b-1)(r-1) + (a-1)(s-1) + (r+s) - 1$	$br + as - 1$
$(xy, x^a + y^b + \sum_{i=1}^2 k_ix^{r_i}y^{s_i})$	3	$(a-r_1)s_2 + br_1 + r_2s_1 - (a+b) + 1$	$(a-r_1)s_2 + br_1 + r_2s_1 - 1$
$(xy, x^a + y^b + \sum_{i=1}^2 k_ix^{r_i}y^{s_i})$	2	(r_ℓ, s_ℓ), vertex for $\ell = 1, 2$. $(b-1)(r_1-1) + (a-1)(s_1-1) + (r_\ell + s_\ell) - 1$	$br_\ell + as_\ell - 1$
$(xy, x^a + y^b + y^k(x^m - y^t)^2)$	2	$(b-1)(2m-1) + (a-1)(k-1) + (k+2m) - 1$	$2bm + ak - 1$
$(xy, x^a + y^b + x^k(x^m - y^t)^2)$	2	$(b-1)(k-1) + (a-1)(2t-1) + (k+2t) - 1$	$bk + 2at - 1$
$(xy, x^a + y^b + \sum_{i=1}^3 k_ix^{r_i}y^{s_i})$	4	$(a-r_2)s_3 + br_1 + s_2(r_3 - r_1)$ $+ s_1r_2 - (a+b) + 1$	$(a-r_2)s_3 + s_2(r_3 - r_1)$ $+ s_1r_2 + br_1 - 1$
$(xy, x^a + y^b + \sum_{i=1}^3 k_ix^{r_i}y^{s_i})$	3	(r_1, s_1) and (r_2, s_2), vertices. $(a-r_1)s_2 + br_1 + r_3s_1 - (a+b) - 1$	$(a-r_1)s_2 + br_1 + r_2s_1 - 1$
$(xy, x^a + y^b + \sum_{i=1}^3 k_ix^{r_i}y^{s_i})$	3	(r_1, s_1) and (r_3, s_3), vertices. $(a-r_1)s_3 + br_1 + r_3s_1 - (a+b) - 1$	$(a-r_1)s_3 + br_1 + r_3s_1 - 1$
$(xy, x^a + y^b + \sum_{i=1}^3 k_ix^{r_i}y^{s_i})$	3	(r_2, s_2) and (r_3, s_3), vertices. $(a-r_2)s_3 + br_2 + r_3s_2 - (a+b) - 1$	$(a-r_2)s_3 + br_2 + r_3s_2 - 1$
$(xy, x^a + y^b + x^ry^s(x^m - y^q)^2)$	3	$(a-r)s + br + (r+2m)(s+2q) - (a+b) + 1$	$(r+2m)(s+2q)$ $+ (a-r)s + br - 1$
$(xy, x^a + y^b + k_1x^{r_1}y^{s_1} + x^r(x^m - y^q)^2)$	3	$(a-r_1)2q + br_1 + rs_1 - (a+b) + 1$	$(a-r_1)2q + br_1 + rs_1 - 1$
$(xy, x^a + y^b + k_3x^{r_3}y^{s_3} + y^s(x^m - y^q)^2)$	3	$(a-2m)s_3 + 2bm + r_3s - (a+b) + 1$	$(a-2m)s_3 + 2bm + r_3s - 1$
$(xy, x^a + y^b + \sum_{i=1}^3 k_ix^{r_i}y^{s_i})$	2	(r_ℓ, s_ℓ), vertex for $\ell = 1, 3$. $(b-1)(r_\ell - 1) + (a-1)(s_\ell - 1) + (r_\ell + s_\ell) - 1$	$br_\ell + as_\ell - 1$
$(xy, x^a + y^b + \sum_{i=1}^t k_ix^{r_i}y^{s_i})$	$t+1$	$(a+r_t)s_t + (b+s_1)r_1 - (a+b) + 1$ $+ \sum_{j=2}^t (s_j + s_{j-1})(r_j - r_{j-1})$	$(a+r_t)s_t + (b+s_1)r_1 - 1$ $+ \sum_{j=2}^t (s_j + s_{j-1})(r_j - r_{j-1})$
$(xy, x^a + y^b + \sum_{i=1}^t k_ix^{r_i}y^{s_i})$	2	(r_ℓ, s_ℓ), vertex for $\ell = 1, t$. $(b-1)(r_\ell - 1) + (a-1)(s_\ell - 1) + (r_\ell + s_\ell) - 1$	$br_\ell + as_\ell - 1$

2.2. Double-fold points and Milnor number $\mu(\Delta(g))$

For any finitely determined map germ $g\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$, a formula to compute the number $d(g)$ is shown in [3, 1.2′]:

$$d(g) + c(g) = \tfrac{1}{2}\{\mu(\Delta(g)) - \mu(\Sigma(g))\}. \tag{2.2}$$

For semi-quasi-homogeneous map germs from the plane to the plane, in [4, theorem 2.1(ii)] we see the following.

Theorem 2.6. *Let $g\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ be a semi-quasi-homogeneous germ of type $(w_1,w_2;d_1,d_2)$. Then*

$$\begin{aligned} d(g) = \frac{1}{2(w_1w_2)^2}&((d_1d_2 - 4w_1w_2)(d_1+d_2-(w_1+w_2))^2 \\ &+ 2w_1w_2((w_1+w_2)(d_1+d_2-(w_1+w_2)) - (d_1d_2 - w_1w_2))). \end{aligned}$$

This formula is possible because the defining equation of the discriminant curve is also semi-quasi-homogeneous; hence, its Milnor number is written in terms of the weights w_1, w_2 and the degrees d_1, d_2. However, if the germ is not semi-quasi-homogeneous, it is harder to describe the defining equation of the discriminant. We remark that the defining equation of the discriminant curve is the defining equation of the zeroth Fitting ideal of the $\mathcal{O}_2$-module $g_*\mathcal{O}_{\Sigma(g)}$, which we denote by $\mathcal{F}_0(g) = \mathcal{F}_0(g_*\mathcal{O}_{\Sigma(g)})$.

Another way to obtain the number $d(g)$ is shown in [3, remark 1.5], using the first Fitting ideal of the $\mathcal{O}_2$-module $g_*\mathcal{O}_{\Sigma(g)}$, defined as $\mathcal{F}_1(g) = \mathcal{F}_1(g_*\mathcal{O}_{\Sigma(g)})$:

$$c(g) + d(g) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\mathcal{F}_1(g)}. \tag{2.3}$$

Equation (2.3) is a particular case of a more general result of Mond and Pellikaan relating the sum of the 0-stable invariants of map germs and the Fitting ideals of the discriminant [9, §§ 1 and 2]. The key tool is the determination of the matrix of a presentation of $g_*\mathcal{O}_{\Sigma(g)}$ over $\mathcal{O}_2$. We recall these concepts here.

For any multi-germ of a Cohen–Macaulay variety $(X,x) \in \mathbb{C}^m$ of dimension $(m-1)$, call $\mathcal{O}_{(X,x)}$ the set of germs h in $\mathcal{O}_m$ such that $h(X) = 0$ for a fixed finite analytic map $f\colon (X,x) \to (\mathbb{C}^m,0)$. From the Weierstrass preparation theorem, $\mathcal{O}_{(X,x)}$ is a finite $\mathcal{O}_m$-module via the function f^*.

A presentation of $\mathcal{O}_{(X,x)}$ over $\mathcal{O}_m$ is an exact sequence of $\mathcal{O}_m$-modules:

$$\mathcal{O}_m^h \xrightarrow{\lambda} \mathcal{O}_m^h \xrightarrow{\alpha} \mathcal{O}_{(X,x)} \to 0. \tag{2.4}$$

The *presentation matrix* λ is given by the relations between the set of generators $g_1,\dots,g_h$ of $\mathcal{O}_{(X,x)}$ as an $\mathcal{O}_m$-module.

Definition 2.7. The kth fitting ideal of $\mathcal{O}_{(X,x)}$, denoted by $\mathcal{F}_k(\lambda)$, is the ideal in $\mathcal{O}_m$ generated by all $(q-k)\times(q-k)$ minors of the matrix λ. For $q > k \geqslant q-p$, $\mathcal{F}_k(\lambda) = \mathcal{O}_m$, and for $k \geqslant q$ or $k < q-p$ we have $\mathcal{F}_k(\lambda) = 0$.

To compute the presentation matrix we use Maple [14] and Singular [5] software and apply the algorithm developed by Hernandes *et al.* in [6].

Table 2. *First orbits in $\mathcal{K}(xy, x^3 + y^4)$.*

Germ g	$\mu(\Sigma(g))$	$\mu(\Delta(g))$	$c(g)$	$d(g)$
Orbit (0): $(xy, x^3 + y^4)$	6	66	11	19
Orbit (1): $(xy, x^3 + y^4 + xy^2)$	4	52	9	15
Orbit (2): $(xy, x^3 + y^4 + x^2y)$	5	59	10	17
Orbit (3): $(xy, y^4 + x(x + y)^2)$	4	54	9	16

3. Topological orbits in the $\mathcal{K}$ class $(xy, x^3 + y^4)$

In this section we obtain a non-finite number of topological orbits in the $\mathcal{K}$ class $(xy, x^3 + y^4)$. First we describe the topological orbits for the germs $f_{u,v}(x, y) = (xy, x^3 + y^4 + uxy^2 + vx^2y)$.

For the particular values of $(u, v) = (0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 2)$ the normal forms in this family are shown in table 2.

The next step is to consider the family $f_{u,v}(x, y) = (xy, x^3 + y^4 + uxy^2 + vx^2y)$.

3.1. The family $f_{u,v}(x, y) = (xy, x^3 + y^4 + uxy^2 + vx^2y)$

For each pair (u, v), the defining equation of the discriminant curve denoted by $G_{u,v}(X, Y)$ is

$$\begin{aligned} &G_{u,v}(X, Y)\\ &\quad= (\tfrac{1}{2}v^2u^6 - \tfrac{1}{16}v^4u^5 - u^7)X^{10} + Y^7 + (\tfrac{4}{27}v^3 - \tfrac{91}{12}uv)X^3Y^5 + \tfrac{27}{256}u^4X^4Y^4\\ &\qquad - \tfrac{343}{12}vX^5Y^4 + (\tfrac{2401}{48}vu^3 - \tfrac{94\,325}{1728}v^3u^2 + \tfrac{30\,625}{1728}v^5 - \tfrac{3125}{1728}v^7)X^{11}\\ &\qquad + (\tfrac{1}{64}u^4v^3 - \tfrac{9}{16}u^5v)X^7Y^2 + (\tfrac{57\,127}{3456}v^2u^2 - \tfrac{25}{27}uv^4 - \tfrac{18\,571}{1728}u^3)X^6Y^3\\ &\qquad\quad \times (\tfrac{420\,175}{3456}v^2 - \tfrac{117\,649}{576}u)X^{10}Y + (-\tfrac{1015}{96}u^3v^3 + \tfrac{125}{96}u^2v^5 + \tfrac{539}{24}u^4v)X^9Y\\ &\qquad - \tfrac{823\,543}{6912}X^{12} + (-\tfrac{112\,847}{1152}u^2 - \tfrac{116\,375}{6912}v^4 + \tfrac{70\,315}{1728}uv^2)X^8Y^2. \end{aligned}$$

When $G_{u,v}$ is Newton non-degenerate we apply theorem 2.4 to compute the Milnor number $\mu(G_{u,v}) = \mu(\Delta(g_{u,v}))$. The possible monomials that can contribute to vertices in the Newton polygon $\Gamma_+(G_{u,v})$ are

$$\{Y^7, \tfrac{27}{256}u^4X^4Y^4, -\tfrac{1}{64}u^4v(-v^2 + 36u)X^7Y^2, -\tfrac{1}{16}u^5(4u - v^2)^2X^{10}\}.$$

Then one has the following.

(i) The points on the line $(u, 0)$ are in orbit (1) (see figure 1(a)). The points on the line $(0, v)$ are in orbit (2) (see figure 1(b)). In both cases the discriminant curve is Newton non-degenerate.

(ii) When $36u - v^2 = 0$, the monomial $-\frac{1}{64}X^7Y^2u^4v(-v^2 + 36u)$ does not appear in the defining equation for $G_{v^2/36,v}$. Hence, its corresponding vertex is eliminated from the Newton polygon and one has orbit (1). Here the discriminant curve is Newton non-degenerate. Note that the Newton polygon of $G_{v^2/36,v}$ is equal to that in the case $(u, 0)$.

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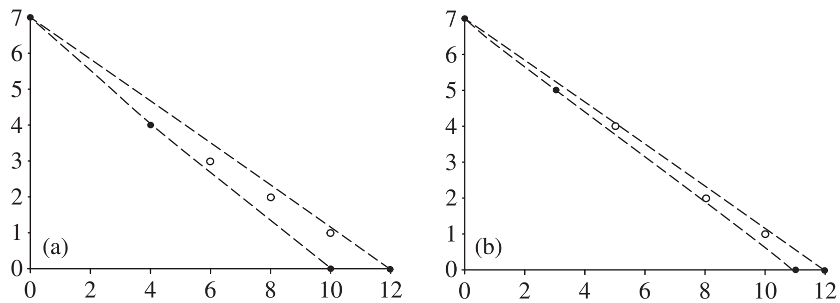

Figure 1. Newton polygon: (a) case $(u, 0)$; (b) case $(0, v)$.

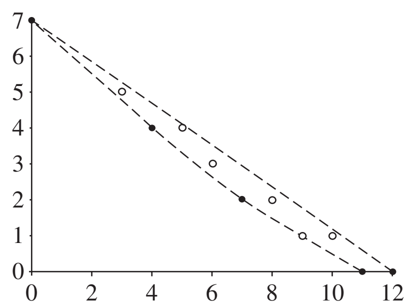

Figure 2. Newton polygon: case $(\frac{1}{4}v^2, v)$.

(iii) When $u = \frac{1}{4}v^2$, the monomial X^{10} does not appear in the germ $G_{v^2/4,v}(X, Y)$. In this case the discriminant curve is Newton degenerate. Hence, it is not possible to apply the theorem 2.4.

We now show in figure 2 that for $u = \frac{1}{4}v^2$ the germs that appear are in orbit (3). In fact, we show that the Milnor number of the discriminant curve $G_{v^2/4,v}$, denoted by G_v, is constant for these values.

To do this we can apply the results given in [1] by Damon and Gaffney, who show how to compute families that have constant Milnor number using *jump conditions*. We can also apply the results of Yoshinaga [16] and Kouchnirenko [7], or we can show that the integral closure of the ideals $I_v := \langle X\partial G_v/\partial X, Y\partial G_v/\partial Y\rangle$ does not depend on $v \neq 0$ (see [10]). According to Teissier [13], one has the following.

Remark 3.1 (valuative criterion). An element h in $\mathcal{O}_2$ is in the integral closure of the ideal I if, for each analytic curve $\varphi\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, x_0)$, $h \circ \varphi$ is in $\varphi^*(I)\mathcal{O}_1$.

The curves φ that we need to study appear when the germ is not Newton non-degenerate and are obviously associated with the degenerate faces of the Newton polygon of the germ G_v, which is given by

$$\begin{aligned} G_v(X, Y) = {} & -\tfrac{1}{2048}v^{11}X^7Y^2 + \tfrac{1}{256}v^9X^9Y - \tfrac{1}{128}v^7X^{11} + \tfrac{27}{65\,536}v^8X^4Y^4 \\ & - \tfrac{343}{12}vX^5Y^4 + Y^7 + \tfrac{7787}{12\,288}v^6X^6Y^3 - \tfrac{707\,021}{55\,296}v^4X^8Y^2 \\ & + \tfrac{487\,403}{6912}v^2X^{10}Y - \tfrac{823\,543}{6912}X^{12} - \tfrac{755}{432}v^3X^3Y^5. \end{aligned}$$

Here there exists one degenerate face of the Newton polygon of G_v with vertices $\{(11,0),(7,2)\}$; we denote this face by F.

To show that this face is degenerate we consider the restriction of the germ G_v to the monomials that correspond to elements in the face F:

$$G_v|_F := -\tfrac{1}{2048}v^7X^7(4X^2 - v^2Y)^2.$$

Since this restriction has the square factor $(4X^2 - v^2Y)^2$, the face is degenerate.

Now, for any parametrization φ of the curve $4X^2 - v^2Y = 0$, $G_v|_F \circ \varphi = 0$. Thus, we consider the restriction of the germ G_v to the parallel lines above the face F, denoted by F^i, and find the first parallel line such that the restriction of G_v to it does not have the polynomial $(4X^2 - Yv^2)$ as a square factor. In this case, we see that the first parallel line F^1 satisfies this condition, since

$$\begin{aligned} G_v|_{F^1} = -\frac{X^4}{1769\,472}(&210\,827\,008X^8 - 124\,775\,168v^2X^6Y + 22\,624\,672v^4X^4Y^2 \\ &- 1121\,328v^6X^2Y^3 - 729v^8Y^4) \end{aligned}$$

and there is no v such that $(4X^2 - Yv^2)$ is a factor of $G_v|_{F^1}$.

Therefore, for $u = \frac{1}{4}v^2$ and $v \neq 0$, we conclude that the multiplicity of the integral closure of the ideals, $I_v := \langle X\partial G_v/\partial X, Y\partial G_v/\partial Y\rangle$, does not depend on v. Hence, $\mu(G_v) = \mu(\Delta(f_{v^2/4,v})) = 54$, and these germs are in topological orbit (3).

To conclude, this two-dimensional space of variables (u, v) splits into four topological orbits:

(0) the origin, with normal form $(xy, x^3 + y^4)$,

(3) the curve $4u = v^2$, with normal form $(xy, y^4 + x(x + y)^2)$,

(2) the line $u = 0$, with normal form $(xy, x^3 + y^4 + x^2y)$, and

(1) the two-dimensional space formed by the complement of these curves, with normal form $(xy, x^3 + y^4 + xy^2)$.

Other orbits appear when we add more monomials to the second coordinate of these normal forms and proceed analogously.

For the orbits (0), (1) and (2) the discriminant curve is Newton non-degenerate. Therefore, if we add monomials with higher degree to the second coordinate germ, the Milnor numbers of the discriminant are constant.

Next we add monomials to the germ $(xy, y^4 + x(x + y)^2)$. First, we consider monomials of degree 4.

3.2. The family $f_{u,v,w} = (xy, y^4 + x(x + y)^2 + uxy^3 + vx^3y + wx^4)$

For generic values of (u, v, w), the defining equation of the discriminant of the germ $f_{u,v,w} = (xy, y^4 + x(x+y)^2 + uxy^3 + vx^3y + wx^4)$ has 39 monomials with huge coefficients depending on the parameters (u, v, w). We do not include this equation here, but we show the monomials that contribute to vertices of the Newton polygon of the germ $G_{u,v,w}$:

$$\{Y^7, \tfrac{27}{256}X^4Y^4, -X^7Y^2, -2(v - w + u - 1)X^9Y, -(v - w + u - 1)^2X^{11}\}.$$

We study these Newton polygons in the space of variables (u, v, w).

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Table 3. *More orbits in* $\mathcal{K}(xy, x^3 + y^4)$.

Germ g	$\mu(\Sigma(g))$	$\mu(\Delta(g))$	$c(g)$	$d(g)$
Orbit (0): $(xy, x^3 + y^4)$	6	66	11	19
Orbit (1): $(xy, x^3 + y^4 + xy^2)$	4	52	9	15
Orbit (2): $(xy, x^3 + y^4 + x^2y)$	5	59	10	17
Orbit (3): $(xy, y^4 + x(x+y)^2)$	4	54	9	16
Orbit (4): $(xy, y^4 + x(x+y)^2 + x^5)$	4	56	9	17
Orbit (5): $(xy, y^4 + x(x+y)^2 + 2xy^3 + x^6)$	4	58	9	18
Orbit (6): $(xy, y^4 + x(x+y)^2 + 2xy^3 + x^{11})$	4	60	9	19

Case 1 $(v - w + u - 1 = 0)$. The Newton polygon has vertices associated with the following monomials:

$$\{Y^7, \tfrac{27}{256}X^4Y^4, -X^7Y^2, -\tfrac{1}{16}(2v - 3w + 1)^4 X^{12}\}.$$

If $2v - 3w + 1 \neq 0$, the germ is Newton non-degenerate and $\mu(G_{u,v,w}) = 54$; hence, these germs are in orbit (3).

If $2v - 3w + 1 = 0$, then $\mu(G_{u,v,w}) = \infty$ and the germs $f_{u,v,w}$ are not finitely determined.

Case 2 $(v - w + u - 1 \neq 0)$. In this case there exists one face, denoted by F, that is degenerate. The monomials of the germ that correspond to points in the face F are

$$\{-Y^2X^7, -2(v - w + u - 1)YX^9, -(v - w + u - 1)^2X^{11}\}$$

and the restriction of the germ $G_{u,v,w}$ to the monomials of F gives

$$G_{u,v,w}|_F := -X^7(X^2(v - w + u - 1) + Y)^2.$$

Hence, this face is degenerate.

Analogously, we find the special set

$$u = v - 2w + 2 \quad \text{with } G_{v-2w+2,v,w} := ((2v - 3w + 1)X^2 + Y)^2 K_{v,w}(X, Y)$$

and $\mu(G_{v-2w+2,v,w}) = \mu(\Delta(f_{v-2w+2,v,w})) = \infty$. Hence, we obtain a new set of germs in this $\mathcal{K}$ class that are not $\mathcal{A}$-finitely determined.

On the other hand, if we consider $u \neq v - 2w + 2$, the restriction of the germ $G_{u,v,w}$ to the first parallel line to F, denoted by F^1, shows that $G_{u,v,w}|_{F^1}$ does not have the polynomial $(X^2(v - w + u - 1) + Y)$ as a common factor. Therefore, we conclude that the Milnor numbers $\mu(G_{u,v,w})$ are constant with $\mu(G_{u,v,w}) = 54$, and these germs are in orbit (3).

Following this method, we describe in table 3 the orbits with $15 \leqslant d(g) \leqslant 19$. Note that 15 is the smallest number of double folds that appear in any germ in this $\mathcal{K}$ class, and 19 is the corresponding number of double folds of the quasi-homogeneous germ $(xy, x^3 + y^4)$.

We also show some germs that belong to these orbits.

- Orbit (3): $(xy, x^3 + y^4 + uxy^2 + vx^2y)$, with $u = \frac{1}{4}v^2$.

- Orbit (4): $(xy, y^4 + x(x+y)^2 + uxy^3 + vx^3y + wx^4 + \sum_{r+s=5} \alpha_{r,s}x^ry^s)$, with $\frac{1}{4}(u - v + 2w - 2)^2 + \alpha_{0,5} - \alpha_{1,4} + \alpha_{2,3} - \alpha_{3,2} + \alpha_{4,1} - \alpha_{5,0} = 0$ and $\alpha_{r,s} \neq 0$ for some r, s.

- Orbit (5): $(xy, y^4 + x(x+y)^2 + uxy^3 + vx^3y + wx^4 + \sum_{r+s=6} \alpha_{r,s}x^ry^s)$, with $v - 2w - u + 2 = 0$ and $\alpha_{r,s} \neq 0$ for some r, s.

- Orbit (6): $(xy, y^4 + x(x+y)^2 + uxy^3 + vx^3y + wx^4 + \sum_{r+s=11} \alpha_{r,s}x^ry^s)$, with $v - 2w - u + 2 = 0$ and $\alpha_{r,s} \neq 0$ for some r, s.

3.3. Stems in $\mathcal{K}(xy, x^3 + y^4)$

The germs $(xy, y^4 + x(x+y)^2 + uxy^3 + vx^3y + wx^4)$ with $v - w + u - 1 \neq 0$ and $u - v + 2w - 2 = 0$ are the key tool for answering the question of Gaffney and Mond about the number of topological orbits in this $\mathcal{K}$ class. These germs are *stems* in this family; they are not $\mathcal{A}$-finitely determined and from them we construct a non-finite family of $\mathcal{A}$-finitely determined germs in this $\mathcal{K}$ class such that the Milnor number of the discriminant is increasing.

The stems are well known in the class of germs of maps from surfaces to 3-space. For example, in [8] we see the stems S_∞, B_∞ and H_∞. Concerning map germs from the plane to the plane, (x^2, y^2) is a stem; other stems are $(xy, x^a + y^b)$ with $\gcd(a, b) \neq 1$.

For simplicity we consider $u = 2$ and $v = w = 0$ to show the following.

THEOREM 3.2. *The $\mathcal{K}$ class of the germ $f(x, y) = (xy, x^3 + y^4)$ has infinitely many topological types of $\mathcal{A}$-finitely determined germs.*

Proof. Consider the family

$$f_s(x, y) = (xy, y^4 + x(x+y)^2 + 2xy^3 + x^sy^{s+1}) \quad \text{with } s > 3.$$

The defining equation of the discriminant, denoted by $G_s(X, Y)$, is the determinant of the presentation matrix λ with respect to f, which is the 7×7 matrix:

$$\begin{bmatrix} Y & -\frac{4}{3}X & 0 & -\frac{4}{3}X - \frac{4}{3}X^s & -\frac{10}{3}X & 0 & -\frac{7}{3} \\ \varepsilon & Y & -\frac{4}{3}X & -\frac{10}{3}X^2 & 0 & -\frac{7}{3}X & 0 \\ \delta & \gamma & Y & \frac{6}{5}X^{s+1} + \frac{6}{5}X^2 & 0 & 0 & \frac{11}{10}X \\ -\frac{5}{2}X^2 & 0 & -\frac{7}{4}X & Y & \theta & -X & 0 \\ 0 & -\frac{7}{4}X^2 & 0 & -\frac{5}{2}X^2 & Y & \theta & -X \\ -\frac{37}{12}X^3 & 0 & -\frac{4}{3}X^2 & \frac{1}{3}XY & -\frac{5}{2}X^2 & Y + \frac{2}{3}X^2 & \theta \\ \nu & \beta & X^2(\frac{21}{16} - X^{s-1}) & \alpha & X(\frac{9}{16}X^s + Y) & 0 & Y - \frac{63}{160}X \end{bmatrix}.$$

Here

$$\alpha = -\tfrac{1}{2}XY - \tfrac{9}{4}X^3 - \tfrac{9}{40}X^2 - \tfrac{1}{40}X^s(9X - 10Y),$$
$$\beta = -\tfrac{5}{2}X^3 - \tfrac{9}{40}X^2 - \tfrac{3}{4}XY,$$
$$\gamma = -\tfrac{4}{3}X^{s+1} + \tfrac{6}{5}X^2,$$

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$$\theta = -\tfrac{3}{4}X^s - \tfrac{3}{4}X,$$
$$\varepsilon = -\tfrac{4}{3}X^{s+1} - \tfrac{4}{3}X^2,$$
$$\delta = -\tfrac{10}{3}X^3 - \tfrac{37}{30}XY$$

and

$$\nu = \tfrac{15}{8}X^3 + \tfrac{27}{160}XY.$$

We show that the Milnor number $\mu(G_s(X,Y)) > 2s+7$ for all $s > 3$.

First, we recall that

$$\mu(G_s(X,Y)) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{(\mathbb{C}^2,0)}}{\langle \partial G_s/\partial X, \partial G_s/\partial Y\rangle},$$

and this dimension is the multiplicity of the integral closure of this ideal.

Then one has $G_s(X,Y) := (X^2+Y)^2 P(X,Y) - X^{2s+8} + R_s(X,Y)$, where

$$\begin{aligned} P(X,Y) := {}& -X^7 + \tfrac{1}{256}X^4(6587X^4 + 7382X^2Y + 27Y^2) + Y^5 - \tfrac{41}{8}X^{s+6} \\ & - \tfrac{1}{216}X^3(50\,000X^6 + 82\,500X^4Y + 35\,925X^2Y^2 + 2777Y^3) \\ & + \tfrac{1}{192}X^{s+3}(26\,050X^4 + 9307X^2Y + 81Y^2) \\ & - \tfrac{5}{12}X^{s+2}Y^2(100X^2 + 31Y) \end{aligned}$$

and

$$\begin{aligned} R_s(X,Y) := {}& \tfrac{1}{4}X^{2s+5}(-42Y^2 + 31X^2Y + 61X^4) \\ & - \tfrac{1}{3456}X^{2s+2}(-2187Y^4 + 11\,474Y^3X^2 + 903\,309Y^2X^4 \\ & \qquad + 2066\,280X^6Y + 1187\,000X^8) \\ & + \tfrac{1}{8}X^{2s+1}Y^4(9Y + X^2) - 5X^{3s+7} \\ & + \tfrac{1}{24}X^{3s+4}(-258Y^2 + 554X^2Y + 2921X^4) \\ & - \tfrac{1}{1728}X^{3s+1}Y(-729Y^3 + 40\,765X^2Y^2 + 116\,558X^4Y + 25\,600X^6) \\ & - 10X^{4s+6} - \tfrac{1}{12}X^{4s+3}(66Y^2 + 97X^2Y + 1564X^4) + \tfrac{27}{256}X^{4s}Y^4 \\ & - 10X^{5s+5} - 5X^{6s+4} - X^{7s+3} \\ & - \tfrac{1}{216}X^{5s+2}(243Y^2 + 3276X^2Y + 256X^4). \end{aligned}$$

We apply the valuative criterion to show that, for all $\ell < 2s+7$, any monomial X^ℓ is not in the integral closure of the ideal $\langle \partial G_s/\partial X, \partial G_s/\partial Y\rangle$.

Then

$$\begin{aligned} \frac{\partial G_s}{\partial X}(X,Y) := {}& (X^2+Y)[4XP(X,Y) + (X^2+Y)P_X(X,Y)] \\ & - (2s+8)X^{2s+7} + \frac{\partial R_s}{\partial X}(X,Y) \end{aligned}$$

and

$$\frac{\partial G_s}{\partial Y}(X,Y) := (X^2+Y)[2P(X,Y) + (X^2+Y)P_Y(X,Y)] + \frac{\partial R_s}{\partial Y}(X,Y).$$

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Consider the parametrization $\varphi(t) = (t, -t^2)$ of the curve $X^2 + Y = 0$. Therefore,

$$\varphi^* \frac{\partial G_s}{\partial X} = \frac{\partial G_s}{\partial X} \circ \varphi = \frac{\partial G_s}{\partial X}(t, -t^2) = -(2s+8)t^{2s+7} + \varphi^* \frac{\partial R_s}{\partial X}$$

and

$$\varphi^* \frac{\partial G_s}{\partial Y} = \varphi^* \frac{\partial R_s}{\partial Y}.$$

Since the monomials (in the variable t) of $\varphi^* \partial R_s/\partial X$ and $\varphi^* \partial R_s/\partial Y$ have order greater than $2s+7$, X^ℓ is not in the integral closure of the ideal $\langle \partial G_s/\partial X, \partial G_s/\partial Y \rangle$ for all $\ell < 2s + 7$. Hence, $\mu(G_s) > 2s + 7$. □

To obtain the exact Milnor number for each s, we compute the Milnor number for some fixed values of s. Then we obtain $\mu(\Delta(f_s)) = 2s + 50$. Computing all invariants of this family, one has $\mu(\Sigma(f_s)) = 4$, $c(f_s) = 9$ and $d(f_s) = s + 14$.

4. Number of topological orbits in $\mathcal{K}(xy, x^a + y^b)$

In this section we show how to obtain stems in a $\mathcal{K}$-class $(xy, x^a + y^b)$. Moreover, we show that there exists a non-finite number of topological types of finitely determined map germs, answering the question posed by Gaffney and Mond.

The only exceptions are the classes $\mathcal{K}(xy, x^2 + y^3)$ and $\mathcal{K}(xy, x^2 + y^5)$, where it is not possible to obtain non-$\mathcal{A}$-finitely determined map germs. Hence, there are no stems. In the $\mathcal{K}$ class $(xy, x^2 + y^3)$ there is only one topological class, and in § 4.3 we show that the $\mathcal{K}$ class $(xy, x^2 + y^5)$ has two topological types. We note that in any $\mathcal{K}$ class that has a representative of type $(xy, x^a + y^b)$ with $\gcd(a, b) > 1$, the germ $(xy, x^a + y^b)$ is a stem.

First, we consider $a > 2$ and, to describe the remaining $\mathcal{K}$ classes $(xy, x^2 + y^b)$ with $b \geqslant 7$, we show a result that holds for any a even.

4.1. The $\mathcal{K}$ classes $(xy, x^a + y^b)$ with $a > 2$

In theorem 4.1 we show the stems for each $\mathcal{K}$ class $(xy, x^a + y^b)$ with $a > 2$, and in theorem 4.2 we show the corresponding family of $\mathcal{A}$-finitely determined map germs f_s that have Milnor number $\mu(\Sigma(f_s))$ depending on the parameter s.

Theorem 4.1.

(i) *For a or b not equal to 2 or 4, the germ*

$$f^{\infty}_{a,b}(x, y) = \left(xy, -\frac{x^a}{a} + \frac{2}{2-a} y x^{a-1} + \frac{y^2 x^{a-2}}{4-a} + \frac{y^b}{b} + \frac{2xy^{b-1}}{b-2} + \frac{x^2 y^{b-2}}{b-4}\right)$$

is a stem in the class $\mathcal{K}(xy, x^a + y^b)$.

(ii) *The germ*

$$f^{\infty}_{3,4}(x, y) = (xy, 3x^4 + 6x^3 y - 3y^4 - 6xy^3 + 4x^3 + 24x^2 y - 12xy^2)$$

is a stem in the class $\mathcal{K}(xy, x^3 + y^4)$.

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(iii) *The germ*

$$f^{\infty}_{4,b}(x,y) = \left(xy, -\frac{x^4}{4} - 2x^3y^2 + \frac{x^2y^4}{2} + \frac{y^b}{b} + \frac{2xy^{b-2}}{b-3} + \frac{x^2y^{b-4}}{b-6}\right)$$

is a stem in the class $\mathcal{K}(xy, x^4 + y^b)$.

If $b = 6$, the germ $(xy, x^4 + y^6)$ is not $\mathcal{A}$-finite and is a stem in its $\mathcal{K}$ class. The case $b = 3$ is obviously not considered.

Proof.
(i) Here $J(f^{\infty}_{a,b}(x,y)) := (x^{a-2} + y^{b-2})(x+y)^2$ and $\mu(\Sigma(f^{\infty}_{a,b}(x,y))) = \infty$. Therefore, $c(f^{\infty}_{a,b}(x,y)) = \infty$ and the germ is not $\mathcal{A}$-finitely determined. To show that it is a stem, consider the family

$$f_{a,b,s}(x,y) = f^{\infty}_{a,b}(x,y) + (0, x^s) \quad \text{with } s > a.$$

Hence,

$$J(f_s) := (x^{a-2} + y^{b-2})(x+y)^2 - sx^s,$$

as $\Sigma(f_{s,a,b})$ is reduced and $f_{a,b,s}\colon \Sigma(f_{s,a,b}) \to \Delta(f_{s,a,b})$ is one to one, according to [3, proposition 1.1]. The germ f_s is $\mathcal{A}$-finitely determined for all $s > a$.

(ii) In this case $J(f^{\infty}_{3,4}(x,y)) := -12(x^2 - xy + x + y^2)(y+x)^2$ is not reduced. Hence, $f^{\infty}_{3,4}$ is not $\mathcal{A}$-finitely determined. The corresponding family of $\mathcal{A}$-finitely determined map germs is

$$f_{3,4,s}(x,y) = f^{\infty}_{3,4}(x,y) + (0, x^s) \quad \text{with } s > 3.$$

(iii) The germ $J(f^{\infty}_{4,b}(x,y)) := (x^2 + y^{b-4})(x + y^2)^2$ is not reduced. Hence, $f^{\infty}_{3,4}$ is not $\mathcal{A}$-finitely determined. The corresponding family is

$$f_{4,b,s}(x,y) = f^{\infty}_{4,b}(x,y) + (0, x^s) \quad \text{with } s > 4.$$

□

The answer to the question by Gaffney and Mond in these cases is given below.

THEOREM 4.2. *The* $\mathcal{K}$ *class of any* $\mathcal{A}$*-finitely determined map germ* $f(x,y) = (xy, x^a + y^b)$ *with* $a > 2$ *has infinitely many topological types of* $\mathcal{A}$*-finitely determined germs.*

Proof. We show that, for each pair (a,b), the Milnor number $\mu(\Sigma(f_{a,b,s}))$ depends on s, and the number of cusps also depends on s.

We apply the *valuative criterion* to show that x^{ℓ} is not in the integral closure of the Jacobian ideals $\langle \partial J(f_{a,b,s})/\partial x, \partial J(f_{a,b,s})/\partial y \rangle$ for all $\ell < s-1$ to conclude that $\mu(\Sigma(f_{a,b,s})) > s-1$.

If $a \neq 2, 4$ and $b \neq 2, 4$,

$$\frac{\partial J(f_{a,b,s})}{\partial x} := 2(x+y)(x^{a-2} + y^{b-2}) + (a-2)x^{a-3}(x+y)^2 - s^2x^{s-1}$$

and

$$\frac{\partial J(f_{a,b,s})}{\partial y} := 2(x+y)(x^{a-2} + y^{b-2}) + (b-2)y^{a-3}(x+y)^2,$$

for $\varphi(t) = (t, -t)$, one has

$$\varphi^*(x^\ell) = t^\ell, \qquad \varphi^*\left(\frac{\partial J(f_{a,b,s})}{\partial x}\right) = -s^2 t^{s-1}, \qquad \varphi^*\left(\frac{\partial J(f_{a,b,s})}{\partial y}\right) = 0,$$

and the result follows.

For the $\mathcal{K}$ class (xy, x^3+y^4), since $J(f_{3,4,s}) := -12(x^2-xy+x+y^2)(x+y)^2 - sx^s$ we consider the curve $\varphi(t) = (t, -t)$.

For $\mathcal{K}(xy, x^4 + y^b)$, since $J(f_{4,b,s}) := (x^2 + y^{b-4})(x + y^2)^2 - sx^s$ we consider $\varphi(t) = (-t^2, t)$.

To conclude we recall that $m_{f_{a,b,s}} = a + b$. Then

$$c(f_{a,b,s}) = \mu(\Sigma(f_{a,b,s})) + m_{f_{a,b,s}} - 2 > (s-1) + (a+b) - 2 = s + a + b - 3.$$

Therefore, for any $a > 2$, the $\mathcal{K}$ class $(xy, x^a + y^b)$ has a non-finite number of topological types. □

4.2. The $\mathcal{K}$ classes $(xy, x^a + y^b)$ with a even

Here we describe stems such that the Milnor number of the discriminant curve is not finite. The only exceptions are $(xy, x^2 + y^3)$ and $(xy, x^2 + y^5)$.

THEOREM 4.3. *The $\mathcal{K}$ class $f(x, y) = (xy, x^{2n} + y^{m+6})$ with $n, m > 0$ has a non-finite number of $\mathcal{A}$-finitely determined topological types.*

Proof. Consider the following family:

$$f_{2n,m+6,s}(x, y) = \left(xy, x^{2n} + y^m(x - y^3)^2 + \frac{n}{3n-2} xy^{6n-3} + \left(x - \frac{y^3}{5}\right) x^s y^{s+2}\right)$$

with a corresponding defining equation for the critical curve:

$$\begin{aligned} J(f_{2n,m+6,s}) = -2nx^{2n} + 2nxy^{6n-3} + (m+6)y^{m+6} + (-4-2m)xy^{m+3} \\ + (-2+m)y^m x^2 + x^s y^{s+2}(x - y^3). \end{aligned}$$

We write $J(f_{2n,m+6,s})(x, y) = (x - y^3)\eta_s(x, y)$ and show that the image of the restriction to the curve $x - y^3$ has a Milnor number depending on s.

If n is even, then the presentation matrix $\lambda_{2n,m+6,s}|_{V(x-y^3)}$ of the restriction of $f_{2n,m+6,s}$ to the component $V(x - y^3)$ is

$$\begin{bmatrix} Y - \dfrac{4n-2}{3n-2} X^{3n/2} & -\dfrac{4X^{s+1}}{5} & 0 & 0 \\ 0 & Y - \dfrac{4n-2}{3n-2} X^{3n/2} & -\dfrac{4X^{s+1}}{5} & 0 \\ 0 & 0 & Y - \dfrac{4n-2}{3n-2} X^{3n/2} & -\dfrac{4X^{s+1}}{5} \\ -\dfrac{4X^{s+2}}{5} & 0 & 0 & Y - \dfrac{4n-2}{3n-2} X^{3n/2} \end{bmatrix}.$$

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Therefore, we can write the defining equation of the discriminant as

$$G_{2n,m+6,s}(X,Y) = G_1(X,Y).P(X,Y)$$

with

$$G_1(X,Y) = \frac{16}{(3n-2)^4}\left(\left(1-\frac{3n}{2}\right)Y + (2n-1)X^{3n/2}\right)^4 - \frac{256}{625}X^{4s+5}.$$

If n is odd, then $\lambda_{2n,m+6,s}|_{V(x-y^3)}$ is

$$\begin{bmatrix} Y & -\dfrac{4X^{s+1}}{5} & -\dfrac{4n-2}{3n-2}X^{(3n-1)/2} & 0 \\ 0 & Y & -\dfrac{4X^{s+1}}{5} & -\dfrac{4n-2}{3n-2}X^{(3n-1)/2} \\ -\dfrac{4n-2}{3n-2}X^{(3n+1)/2} & 0 & Y & -\dfrac{4X^{s+1}}{5} \\ -\dfrac{4X^{s+2}}{5} & -\dfrac{4n-2}{3n-2}X^{(3n+1)/2} & 0 & Y \end{bmatrix}.$$

Therefore, we write $G_{2n,m+6,s}(X,Y) = G_2(X,Y)\cdot Q(X,Y)$ with

$$G_2(X,Y) = \left(-Y^2+\left(\frac{4n-2}{3n-2}\right)^2 X^{3n}\right)^2 - \frac{128}{25}\left(\frac{2n-1}{3n-2}\right)YX^{(3n+5)/2+2s} - \frac{256}{625}X^{4s+5}.$$

Now we apply the valuative criterion to show which monomials X^ℓ are not in the integral closure of the ideals $\langle \partial G_i/\partial X, \partial G_i/\partial Y\rangle$, $i=1,2$, with ℓ depending on s.

Consider the parametrization

$$\varphi_1(t) = \left(t, \frac{4n-2}{3n-2}t^{3n/2}\right)$$

for even n and the parametrization

$$\varphi_2(t) = \left(t^2, \frac{4n-2}{3n-2}t^{3n}\right)$$

for odd n.

Then

$$\varphi_1^*\left(\frac{\partial G_1}{\partial X}\right) = \frac{\partial G_1}{\partial X}\circ\varphi_1 = t^{4s+4}, \qquad \varphi_1^*\left(\frac{\partial G_1}{\partial Y}\right) = \frac{\partial G_1}{\partial Y}\circ\varphi_1 = 0$$

and

$$\varphi_2^*\left(\frac{\partial G_2}{\partial X}\right) = \frac{\partial G_2}{\partial X}\circ\varphi = t^{8s}, \qquad \varphi_2^*\left(\frac{\partial G_2}{\partial Y}\right) = \frac{\partial G_2}{\partial Y}\circ\varphi_1 = t^{4s+3n+5}.$$

Hence, if $\ell < 4s+4$, X^ℓ is not in the integral closure of the ideal $\langle \partial G_1/\partial X, \partial G_1/\partial Y\rangle$, and if $\ell < \min\{8s, 4s+3n+5\}$, X^ℓ is not in the integral closure of the ideal $\langle \partial G_2/\partial X, \partial G_2/\partial Y\rangle$.

Therefore, as $\mu(G_{2n,m+6,s}(X,Y)) \geqslant \mu(G_i(X,Y)) \geqslant 4s+4$, these $\mathcal{K}$ classes have a non-finite number of different topological types. □

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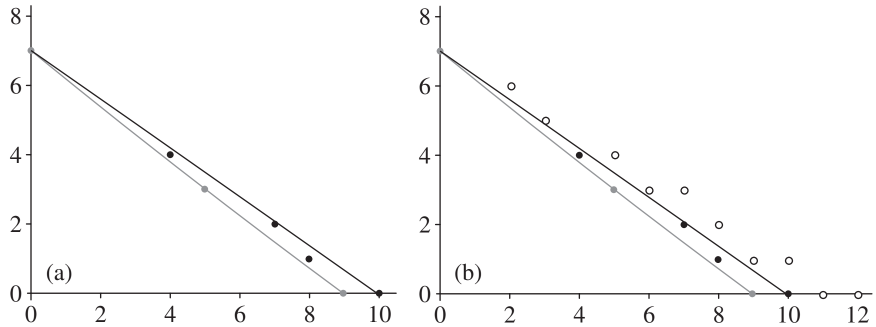

Figure 3. Newton polygons of the discriminant curves of (a) the germ $(xy, x^2 + y^5 + xy^2)$ and (b) the germ $(xy, x^2 + y^5 + xy^2 + xy^3 + x^2y)$.

4.3. The $\mathcal{K}$ class $(xy, x^2 + y^5)$

We show that this $\mathcal{K}$ class has only two topological orbits. Note that it has at least two orbits, since the germ $(xy, x^2 + y^5 + xy^2)$ is not in the topological orbit of $(xy, x^2 + y^5)$. The invariants of these germs are as follows.

(i) Germ $g(x, y) = (xy, x^2 + y^5)$, $\mu(\Sigma(g)) = 4$, $\mu(\Delta(g)) = 54$, $c(g) = 9$ and $d(g) = 16$.

(ii) Germ $g(x, y) = (xy, x^2 + y^5 + xy^2)$, $\mu(\Sigma(g)) = 3$, $\mu(\Delta(g)) = 47$, $c(g) = 8$ and $d(g) = 14$.

To show that any other germ is in one of these orbits, first we consider the family $g_t(x, y) = (xy, x^2 + y^5 + txy^2)$. For $t \neq 0$ the defining equation $G_t(X, Y)$ of the discriminant curve of $g_t(x, y) = (xy, x^2 + y^5 + txy^2)$ is given by

$$G_t(X, Y) = -\tfrac{1728}{3125}t^7X^9 + \tfrac{256}{3125}t^5Y^3X^5 + \tfrac{49\,392}{3125}t^4X^8Y - \tfrac{931}{100}t^2X^4Y^4 - \tfrac{16\,807}{250}tX^7Y^2 - \tfrac{823\,543}{12\,500}X^{10} + Y^7.$$

For all $t \neq 0$, the germ G_t is Newton non-degenerate. Then all germs g_t are in the topological orbit of $g_1(x, y) = (xy, x^2 + y^5 + xy^2)$ and the result follows.

Now we consider a deformation $(xy, x^2 + y^5 + xy^2 + \sum_{\alpha,\beta} a_{\alpha,\beta}x^\alpha y^\beta)$ of the germ $(xy, x^2+y^5+xy^2)$ with (α, β) above the Newton polygon of $x^2+y^5+xy^2$. In this case the corresponding monomials that appear in the defining equation $G_{\alpha,\beta}(X, Y)$ of the discriminant curve are also above the Newton polygon of the defining equation of the germ $(xy, x^2 + y^5 + xy^2)$.

Since for any (α, β) satisfying this condition the germ $G_{\alpha,\beta}(X, Y)$ is Newton non-degenerate and the Newton polygon of $G_{\alpha,\beta}$ does not change, the result follows.

To illustrate, we show in figure 3 the Newton polygons of the discriminant curves of the germs $(xy, x^2 + y^5 + xy^2)$ and $(xy, x^2 + y^5 + xy^2 + xy^3 + x^2y)$, respectively.

5. How to obtain stems in a $\mathcal{K}$ class $(xy, x^a + y^b)$

We developed an algorithm to compute stems in a $\mathcal{K}$ class $(xy, x^a + y^b)$ with $a > 2$ and $\gcd(a, b) = 1$. This algorithm is obtained using the Newton polygon of the germ $x^a + y^b$.

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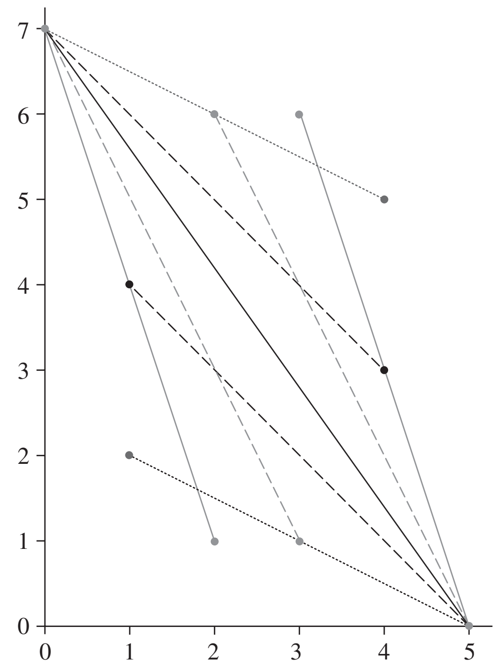

Figure 4. Possible exponents of the germs for stems in the $\mathcal{K}$ class $(xy, x^5 + y^7)$.

The main idea is to find stems f in the $\mathcal{K}$ class with a non-reduced defining equation of the critical curve, and hence with a non-finite Milnor number of the critical curve.

We begin by fixing the germ $J(f)$ of the critical curve and find the appropriate germs f that have this critical curve. Then we obtain the families of germs $f_s = f + (0, x^s)$, which are finitely determined, with Milnor numbers of the critical and of the discriminant curves depending on s.

As an application we describe some stems in the $\mathcal{K}$ class $(xy, x^5 + y^7)$.

(i) Critical curve $J(f) := (x^3 + y^5)(x + y)^2$, germ

$$f(x, y) = (xy, -\tfrac{1}{5}x^5 + \tfrac{1}{7}y^7 - \tfrac{2}{3}x^4y - x^3y^2 + \tfrac{1}{3}x^2y^5 + \tfrac{2}{5}xy^6),$$

with Milnor numbers $\mu(\Sigma(f_s)) = s + 15$ and $\mu(\Delta(f_s)) = 3s + 311$.

(ii) Critical curve $J(f) := -(x - y^2)(x + y^2)(x + y)^3$, germ

$$f(x, y) = (xy, \tfrac{1}{5}x^5 + \tfrac{1}{7}y^7 + yx^4 + 3y^2x^3 - x^2y^3 + y^4x^3 + x^2y^5 + \tfrac{3}{5}xy^6),$$

with Milnor numbers $\mu(\Sigma(f_s)) = 2s + 8$ and $\mu(\Delta(f_s)) = 8s + 262$.

(iii) Critical curve: $J(f) := (x + y^3)(x + y)^4$, germ

$$f(x, y) = (xy, -\tfrac{1}{5}x^5 + \tfrac{1}{7}y^7 - \tfrac{4}{3}x^4y - 6x^3y^2 + 4x^2y^3 + \tfrac{1}{3}xy^4 - x^4y^3 + 4x^3y^4 + 2x^2y^5 + \tfrac{4}{5}xy^6),$$

with Milnor numbers $\mu(\Sigma(f_s)) = 3s + 1$ and $\mu(\Delta(f_s)) = 15s + 187$.

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(iv) Critical curve $J(f) := -(x-y)(x^2+xy+y^2)(x+y^2)^2$, germ

$$f(x,y) = (xy, \tfrac{1}{5}x^5 + \tfrac{1}{7}y^7 + x^4y^2 - x^3y^4 + \tfrac{1}{2}xy^5 + x^2y^3),$$

with Milnor numbers $\mu(\Sigma(f_s)) = 2s + 11$ and $\mu(\Delta(f_s)) = 6s + 281$.

(v) Critical curve $J(f) := -(x-y^5)(x^2+y)^2$, germ

$$f(x,y) = (xy, \tfrac{1}{5}x^5 + \tfrac{1}{7}y^7 + x^3y - xy^2 + x^4y^5 + \tfrac{1}{2}x^2y^6),$$

with Milnor numbers $\mu(\Sigma(f_s)) = s + 1$ and $\mu(\Delta(f_s)) = 3s + 143$.

(vi) Critical curve: $J(f) := -(x^2-y)(x+y^2)^3$, germ

$$f(x,y) = (xy, \tfrac{1}{5}x^5 + \tfrac{1}{7}y^7 + \tfrac{3}{4}xy^5 + 3x^2y^3 - \tfrac{1}{2}x^3y - 3y^4x^3 + \tfrac{3}{2}y^2x^4 - \tfrac{1}{4}x^2y^6),$$

with Milnor numbers $\mu(\Sigma(f_s)) = 4s + 1$ and $\mu(\Delta(f_s)) = 16s + 201$.

(vii) Critical curve $J(f) := -(x^3-y)(x+y^3)^2$, germ

$$f(x,y) = (xy, \tfrac{1}{5}x^5 + \tfrac{1}{7}y^7 + 2x^4y^3 - \tfrac{1}{3}x^3y^6 + \tfrac{2}{3}xy^4 - x^2y),$$

with Milnor numbers $\mu(\Sigma(f_s)) = 3s + 1$ and $\mu(\Delta(f_s)) = 9s + 161$.

(viii) Critical curve $J(f) := (x+y^3)(x^2+y^2)^2$, germ

$$f(x,y) = (xy, -\tfrac{1}{5}x^5 + \tfrac{1}{7}y^7 - 2x^3y^2 + \tfrac{1}{3}xy^4 - x^4y^3 + \tfrac{2}{3}x^2y^5),$$

with Milnor numbers: $\mu(\Sigma(f_s)) = 2s + 6$ and $\mu(\Delta(f_s)) = 6s + 232$.

In figure 4 we show the possible exponents of the germs above:

- the solid black line corresponds to the Newton polygon of the germ $x^5 + y^7$,
- the dashed black line corresponds to factors $(x+y)$ or (x^2+y^2) in the critical curve,
- the solid grey line corresponds to the factor $(x+y^3)$ in the critical curve,
- the dashed grey line corresponds to the factor $(x+y^2)$ in critical curve and
- the dotted grey line corresponds to the factor (x^2+y) in the critical curve.

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