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# A NOTE ON JUDICIOUS BISECTIONS OF GRAPHS

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**Abstract**

Let $G$ be a graph with $m$ edges, minimum degree $\delta$ and containing no cycle of length 4. Answering a question of Bollobás and Scott, Fan *et al.* ['Bisections of graphs without short cycles', *Combinatorics, Probability and Computing* **27**(1) (2018), 44–59] showed that if (i) $G$ is 2-connected, or (ii) $\delta \geq 3$, or (iii) $\delta \geq 2$ and the girth of $G$ is at least 5, then $G$ admits a bisection such that $\max\{e(V_1), e(V_2)\} \leq (1/4 + o(1))m$, where $e(V_i)$ denotes the number of edges of $G$ with both ends in $V_i$. Let $s \geq 2$ be an integer. In this note, we prove that if $\delta \geq 2s - 1$ and $G$ contains no $K_{2,s}$ as a subgraph, then $G$ admits a bisection such that $\max\{e(V_1), e(V_2)\} \leq (1/4 + o(1))m$.

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## 1. Introduction

Many classical partitioning problems in combinatorics seek a partition of a combinatorial object (for example, a graph, directed graph, hypergraph and so on) which optimises a single quantity. For example, the well-known *max-cut problem* asks for a *bipartition* $(V_1, V_2)$ of $G$ which maximises the *size of the cut* $e(V_1, V_2)$, the number of edges with one end in $V_1$ and the other in $V_2$. It is easy to see that every graph with $m$ edges has a cut of size at least $m/2$. Edwards [5, 6] proved the best possible result that the max-cut of graphs with $m$ edges is at least

$$\frac{m}{2} + \frac{1}{4}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right). \tag{1.1}$$

*Judicious partitioning problems* ask for partitions of graphs that maximise or minimise several quantities simultaneously. Bollobás and Scott initiated a systematic study of such problems. It was proved in [2] that every graph with $m$ edges has a bipartition satisfying (1.1) in which each vertex class contains at most

$$\frac{m}{4} + \sqrt{\frac{m}{32} + \frac{1}{256}} - \frac{1}{16}$$

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edges. The extremal graphs are the complete graphs of odd order. For more such results and problems, we refer the reader to [3, 11, 12].

In this paper, we focus on bisections of graphs. Let $G$ be a graph. A *bisection* of $G$ is a bipartition $(V_1, V_2)$ of its vertex set $V(G)$ with $||V_1| - |V_2|| \leq 1$, and *judicious bisection problems* usually ask for bisections in which both parts induce few edges. Considering $K_{1,n-1}$ shows that we cannot in general demand a bisection with fewer than $\lfloor m/2 \rfloor$ edges in each part. To circumvent this issue, a natural idea is to add a minimum degree condition for the graphs under consideration. Specifically, Bollobás and Scott conjectured in [3] that every graph with $m$ edges and minimum degree at least 2 admits a bisection such that the number of edges in each part is at most $m/3$. This problem was studied by several authors [4, 9, 13, 14], and the conjecture was finally confirmed by Xu and Yu [15].

In [9], Lee *et al.* studied how the bound changes as the minimum degree condition imposed on the graph grows. They proved that if $\delta$ is even, then every graph $G$ with $m$ edges and minimum degree $\delta$ admits a bisection such that each part induces at most $((\delta + 2)/4(\delta + 1) + o(1))m$ edges. One of their main contributions for analysing bisections is the introduction of the notion of tight component in a graph. Let $T$ be a connected graph. We say that $T$ is *tight* if it has the following properties:

(i) for every vertex $v \in V(T)$, $T - v$ contains a perfect matching; and
(ii) for every vertex $v \in V(T)$ and every perfect matching $M$ of $T - v$, no edge in $M$ has exactly one end adjacent to $v$.

If $G$ is disconnected, the components which are tight are called *tight components* of $G$. Answering a question of Lee *et al.* [9], Lu *et al.* [10] gave the following characterisation of tight graphs.

LEMMA 1.1 (Lu *et al.* [10]). *A connected graph $G$ is tight if and only if every block of $G$ is an odd clique.*

REMARK 1.2. Each tight graph has an odd number of vertices and the degree of each vertex is even. Obviously, $K_1$ is tight and we call it *trivial*.

Note that by taking a random bisection $(V_1, V_2)$, one expects $m/4$ edges in each part. However, $e(V_1)$ and $e(V_1)$ are dependent and the extremal graphs for the result of Lee *et al.* [9] indicate that, in general, both $V_1$ and $V_2$ cannot simultaneously induce at most $(1/4 + o(1))m$ edges. This leads to the following problem, which was posed by Bollobás and Scott [3].

PROBLEM 1.3. Under what conditions can we guarantee a bisection $(V_1, V_2)$ of a graph $G$ of $m$ edges such that $\max\{e(V_1), e(V_2)\} \leq (1/4 + o(1))m$?

This problem was studied by Fan *et al.* [7]. They proved the following result. Let $G$ be a graph with $m$ edges, minimum degree $\delta$ and containing no cycle of length 4. If (i) $G$ is 2-connected, or (ii) $\delta \geq 3$, or (iii) $\delta \geq 2$ and the girth of $G$ is at least 5, then $G$

admits a bisection $(V_1, V_2)$ such that $\max\{e(V_1), e(V_2)\} \leq (1/4 + o(1))m$. For a set $\mathcal{H}$ of graphs, we say $G$ is *$\mathcal{H}$-free* if $G$ contains no member of $\mathcal{H}$ as a subgraph. If $\mathcal{H} = \{H\}$, we simply write *$H$-free* instead of *$\mathcal{H}$-free*. In [8], Hou and Wu improved property (iii) by considering $\{K_3, K_{\delta,l}\}$-free graphs with minimum degree $\delta$. In this note, we improve property (ii).

THEOREM 1.4. *For any fixed integer $s \geq 2$, if $G$ is $K_{2,s}$-free and $\delta(G) \geq 2s - 1$, then $G$ admits a bisection $(V_1, V_2)$ such that $\max\{e(V_1), e(V_2)\} \leq (1/4 + o(1))m$.*

We end this section with some notation and definitions. All graphs considered here are finite, undirected, and have no loops and no parallel edges. Let $G$ be a graph with edge set $E(G)$ and vertex set $V(G)$. The set of neighbours of a vertex $v \in V(G)$ is denoted by $N_G(v)$ and $d(v) = |N_G(v)|$ is the *degree* of $v$ in $G$. Let $\Delta(G)$ and $\delta(G)$ be the *maximum* and *minimum degree* of $G$, respectively. For disjoint subsets $X, Y$ of $V(G)$, we denote by $E(X)$ the set of edges of $G$ with both ends in $X$, and by $E(X, Y)$ the set of edges of $G$ with one end in $X$ and the other end in $Y$. The cardinalities of $E(X)$ and $E(X, Y)$ are $e(X)$ and $e(X, Y)$, respectively. When $X = \{v\}$, we write $e(v, Y)$ instead of $e(\{v\}, Y)$ for simplicity. Let $N_Y(v)$ denote the set of neighbours of $v$ in $Y$ and $d_Y(v) = |N_Y(v)|$ the $Y$-degree of $v$ in $G$. Clearly, $d_Y(v) = e(v, Y)$.

## 2. Proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4. Let $G$ be a $K_{2,s}$-free graph with $n$ vertices, $m$ edges and $\delta(G) \geq 2s - 1$. It suffices to show that for any small real $\varepsilon > 0$, there exists an integer $n_0 > 0$ such that if $n \geq n_0$, then $G$ has a bisection $(V_1, V_2)$ such that $e(V_i) \leq (1/4 + \varepsilon)m$ for $i = 1, 2$. Throughout the proof, we tacitly assume that the number of vertices $n$ is large enough. Since $\delta(G) \geq 2s - 1$, we have $m \geq (2s - 1)n/2$, which indicates that $m$ is also large enough.

As a starting point for Problem 1.3, Bollobás and Scott [3] (see also [11]) suggested that one of $\Delta(G) = o(n)$ or $\delta(G) \to \infty$ might suffice. This was confirmed independently by several authors [9, 14, 16]. We use the following result of Lee *et al.* [9].

LEMMA 2.1 (Lee *et al.* [9]). *Let $\varepsilon$ be a fixed positive constant and let $G$ be a graph with $n$ vertices and $m$ edges such that (i) $m \geq n/\varepsilon^2$ or (ii) $\Delta(G) \leq \varepsilon^2 n/2$. If $n$ is sufficiently large, then $G$ admits a bisection $(V_1, V_2)$ such that $\max\{e(V_1), e(V_2)\} \leq (1/4 + \varepsilon)m$.*

It follows from Lemma 2.1 that to prove Theorem 1.4, we need only consider sparse graphs with large maximum degree. More formally, we may assume that

$$m < \frac{n}{\varepsilon^2} \quad \text{and} \quad \Delta(G) > \frac{\varepsilon^2 n}{2}.$$

In fact, for sparse graphs with small maximum degree, Lee *et al.* [9] gave the following strengthening of Lemma 2.1. The key benefit is its parametrisation in terms of the number of tight components.

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LEMMA 2.2 (Lee *et al.* [9]). *Given any real constants $C, \varepsilon > 0$, there exist $\gamma, n_0 > 0$ for which the following holds. Every graph $G$ with $n \geq n_0$ vertices, $m \leq Cn$ edges, maximum degree at most $\gamma n$ and $\tau$ tight components admits a bisection $(V_1, V_2)$ such that* $\max\{e(V_1), e(V_2)\} \leq m/4 - (n-\tau)/8 + \varepsilon n$.

Combining Lemmas 2.1 and 2.2, we see that the main obstacle for Problem 1.3 is the maximum degree condition. To work around this, we use the natural idea of Lee *et al.* [9], which was first used by Bollobás and Scott [1] and then by several others. First, partition $V(G)$ into $A$ and $\overline{A}$, where $A$ consists of certain high degree vertices; then partition $A$ into $A_1$ and $A_2$ with certain properties and partition $\overline{A}$ by Lemma 2.2; finally appropriately combine the vertex subsets of the two partitions. This leads to the following result.

LEMMA 2.3 (Lee *et al.* [9]). *Given any real constants $C, \varepsilon > 0$, there exist $\gamma, n_0 > 0$ for which the following holds. Let $G$ be a given graph with $n \geq n_0$ vertices and at most $Cn$ edges, and let $A \subseteq V(G)$ be a set of $\leq \gamma n$ vertices which has already been partitioned into $A_1$ and $A_2$. Let $\overline{A} = V(G) \setminus A$, and suppose that every vertex in $\overline{A}$ has degree at most $\gamma n$ (with respect to the full $G$). Let $\tau$ be the number of tight components in $G[\overline{A}]$. Then there is a bisection $(V_1, V_2)$ with $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$, such that for $i = 1, 2$,*

$$e(V_i) \leq e(A_i) + \frac{e(A_i, \overline{A})}{2} + \frac{e(\overline{A})}{4} - \frac{n-\tau}{8} + \varepsilon n.$$

Now we use Lemma 2.3 and some additional ideas to prove Theorem 1.4. Let

$$A = \{v \in V(G) : d_G(v) \geq n^{3/4}\} \quad \text{and} \quad \overline{A} = V(G) \setminus A.$$

Suppose $A \neq \emptyset$, otherwise we are already done by Lemma 2.2. Note that

$$2m = \sum_{v \in V(G)} d(v) \geq \sum_{v \in A} d(v) \geq |A| n^{3/4},$$

which, together with $m < n/\varepsilon^2$, yields

$$|A| < \frac{2n^{1/4}}{\varepsilon^2}, \tag{2.1}$$

and hence

$$e(A) \leq \binom{|A|}{2} = O(n^{1/2}).$$

Partition $A$ into $(A_1, A_2)$ such that $e(A_1, \overline{A}) \geq e(A_2, \overline{A})$ and, subject to this,

$$\theta := e(A_1, \overline{A}) - e(A_2, \overline{A}) \tag{2.2}$$

is minimised. Since $e(A_1, \overline{A}) + e(A_2, \overline{A}) = e(A, \overline{A})$, from (2.2), we see that

$$e(A_2, \overline{A}) \leq e(A_1, \overline{A}) = \frac{e(A, \overline{A}) + \theta}{2}.$$

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By (2.1), $|A| = O(n^{1/4})$. For each $v \in \overline{A}$, we have $d_G(v) < n^{3/4}$. Since $n$ is sufficiently large (by choosing $n_0$ large), it follows from Lemma 2.3 (with $C = 1/\varepsilon^2$) that $G$ has a bisection $(V_1, V_2)$ with $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$, such that for $i = 1, 2$,

$$e(V_i) \le e(A_i) + \frac{e(A, \overline{A}) + \theta}{4} + \frac{e(\overline{A})}{4} - \frac{n-\tau}{8} + \varepsilon n \le \frac{1}{4}\Big(\theta + \frac{\tau}{2} - \frac{n}{2}\Big) + \frac{m}{4} + \frac{3\varepsilon}{4}m,$$

where $\tau$ is the number of tight components in $G[\overline{A}]$. The last inequality holds as $e(A_i) = O(n^{1/2})$ and $m \ge (2s-1)n/2$. Then, to prove $e(V_i) \le (1/4 + \varepsilon)m$, it suffices to show

$$\theta + \frac{\tau}{2} \le \frac{n}{2} + \varepsilon m. \tag{2.3}$$

Now we prove (2.3) through carefully bounding $\theta$ and $\tau$. Consider the partition $(T, K)$ of $\overline{A}$, where $T$ consists of all vertices of the tight components in $G[\overline{A}]$ and $K := \overline{A} \setminus T$. Let $T_0$ be the set of isolated vertices in $G[\overline{A}]$ and denote $T_1 = T \setminus T_0$. By Lemma 1.1, each component of $G[T_1]$ has at least three vertices. Therefore,

$$\tau \le |T_0| + \frac{|T_1|}{3}.$$

Let

$$S = \{v \in \overline{A} : d_A(v) \ge 2\}.$$

Then $T_0 \subset S$ since $\delta(G) \ge 2s - 1 \ge 3$. To give a reasonable bound for $\tau$, we bound $|S|$ by using the condition that $G$ is $K_{2,s}$-free.

*Claim 1.* $|S| = O(n^{1/2})$ and thus $\tau \le \frac{1}{3}|T_1| + O(n^{1/2})$.

Since $G$ is $K_{2,s}$-free, any pair of vertices in $A$ has at most $s-1$ common neighbours in $G$ (and thus in $S$). Through (double) counting the number of $K_{1,2}$ with the 2-degree vertex in $S$ and the two pendent vertices in $A$, we have

$$(s-1)\binom{|A|}{2} \ge \sum_{v \in S} \binom{d_A(v)}{2} \ge |S|.$$

Since $|A| = O(n^{1/4})$ by (2.1), we see that $|S| = O(n^{1/2})$. This proves Claim 1.

For $s \ge 3$, we give a better bound for $\tau$.

*Claim 2.* For $s \ge 3$, $\tau \le |S| = O(n^{1/2})$.

Clearly, each vertex in $S$ falls in at most one tight component in $G[\overline{A}]$. Now we show that each tight component in $G[\overline{A}]$ has a vertex in $S$, which implies that $\tau \le |S| = O(n^{1/2})$. Suppose in contrast that $T'$ is a tight component in $G[\overline{A}]$ which does not contain a vertex of $S$. This means each vertex of $T'$ has at most one neighbour in $A$. Considering one of the endblocks of $T'$, by Lemma 1.1, it is an odd clique with minimum degree at least $2s-2$, and hence contains a $K_{2s-1}$. Since $s \ge 3$, it also contains a $K_{2,s}$, which gives a contradiction. This proves Claim 2.

Now we bound $\theta$. In the partition $(A_1, A_2)$ of $A$, since $A \ne \emptyset$, we have $A_1 \ne \emptyset$.

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*Claim 3.* For any $v \in A_1$, we have $d_{\overline{A}}(v) \geq \theta$.

For otherwise, through moving $v$ from $A_1$ to $A_2$,

$$\begin{aligned}\theta' &= e(A_1 \setminus \{v\}, \overline{A}) - e(A_2 \cup \{v\}, \overline{A}) \\ &= e(A_1, \overline{A}) - d_{\overline{A}}(v) - e(A_2, \overline{A}) - d_{\overline{A}}(v) \\ &= \theta - 2d_{\overline{A}}(v) \\ &> -\theta.\end{aligned}$$

However, $\theta' = \theta - 2d_{\overline{A}}(v) < \theta$. This implies that $|\theta'| < \theta$, which is a contradiction to the optimality of the partition $(A_1, A_2)$. This proves Claim 3.

For some fixed $v_0 \in A_1$, by Claim 3, $\theta \leq d_{\overline{A}}(v_0)$. We give a bound for $d_{\overline{A}}(v_0)$.

*Claim 4.* $d_{\overline{A}}(v_0) \leq \frac{1}{2}|\overline{A}| + |S|$. Moreover, if $s = 2$, then $d_{\overline{A}}(v_0) \leq |T_0| + \frac{1}{3}|T_1| + \frac{1}{2}|K| + |S|$.

Denote $X = N_{\overline{A}}(v_0) \setminus S$ and $Y = \overline{A} \setminus X$. We show that

$$|X| \leq |Y|.$$

Then the claim follows immediately.

For any connected component $B$ of $G[\overline{A}]$, no matter whether it belongs to $G[T]$ or $G[K]$, let $B \cap X = C$ and $B \cap Y = D$. To prove $|X| \leq |Y|$, it suffices to show $|C| \leq |D|$.

Summing up the degrees of all vertices in $C$,

$$\sum_{v \in C} d(v) = 2e(C) + e(C, A) + e(C, D).$$

Since $G$ contains no $K_{2,s}$, the maximum degree of $G[C]$ is no more than $s - 1$, which implies

$$e(C) \leq \frac{(s-1)|C|}{2}.$$

Note that $(C \cap S) \subset (X \cap S) = \emptyset$, so that $e(C, A) \leq |C|$. Therefore, on the one hand,

$$\begin{aligned}e(C, D) &= \sum_{v \in C} d(v) - 2e(C) - e(C, A) \\ &\geq (2s-1)|C| - (s-1)|C| - |C| \\ &\geq (s-1)|C|.\end{aligned}$$

On the other hand, for any vertex $y$ of $D$, if $d_C(y) \geq s$, then a $K_{2,s}$ can be found easily in $G[v_0 \cup y \cup N_C(y)]$. Hence, $y$ has at most $s - 1$ neighbours in $C$, which implies

$$e(C, D) \leq (s-1)|D|.$$

We conclude that

$$|C| \leq |D|.$$

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For the second inequality, by Lemma 1.1, if $G$ is $K_{2,2}$-free, each block of a tight component in $G[\overline{A}]$ has three vertices. Therefore, $v_0$ has at most one neighbour in each such block. It is easy to see $d_T(v_0) \leq |T_0| + \frac{1}{3}|T_1|$. Through considering nontight components $B$ (restrict $B$ in $G[K]$), our proof above implies that $d_K(v_0) \leq \frac{1}{2}|K| + |S|$. Thus, $d_{\overline{A}}(v_0) = d_T(v_0) + d_K(v_0) \leq |T_0| + \frac{1}{3}|T_1| + \frac{1}{2}|K| + |S|$ when $s = 2$. This proves Claim 4.

When $s \geq 3$, combining Claims 2–4,

$$\theta + \frac{\tau}{2} \leq \frac{|\overline{A}|}{2} + |S| + \frac{|S|}{2} \leq \frac{n}{2} + \varepsilon m.$$

When $s = 2$, by Claims 1 and 3 and the second inequality of Claim 4,

$$\theta + \frac{\tau}{2} \leq |T_0| + \frac{|T_1|}{3} + \frac{|K|}{2} + |S| + \frac{|T_1|/3 + O(n^{1/2})}{2} \leq \frac{n}{2} + \varepsilon m,$$

where the final inequality follows as $T_0 \subset S$. This completes the proof of (2.3).

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