ON THE ASYMPTOTIC EXPANSION OF AIRY'S INTEGRAL

by E. T. COPSON

(Received 19 November, 1962)

1. Introduction. The integral function

$$Ai(z) = \frac{1}{3^{\frac{2}{3}}\pi} \sum_{0}^{\infty} \frac{\Gamma(n+\frac{1}{3})}{n!} \sin \tfrac{2}{3}(n+1)\pi \,.\, (3^{\frac{1}{3}}z)^n \tag{1.1}$$

is known as Airy's Integral since, when z is real, it is equal to the integral

$$\frac{1}{\pi}\int_0^\infty \cos(\tfrac{1}{3}t^3 + zt)\,dt \tag{1.2}$$

which first arose in Airy's researches on optics. It is readily seen that $w = Ai(z)$ satisfies the differential equation $d^2w/dz^2 = zw$, an equation which also has solutions $Ai(\omega z)$, $Ai(\omega^2 z)$, where ω is the complex cube root of unity, $\exp \frac{2}{3}\pi i$. The three solutions are connected by the relation

$$Ai(z) + \omega Ai(\omega z) + \omega^2 Ai(\omega^2 z) = 0. \tag{1.3}$$

Instead of using $Ai(\omega z)$ or $Ai(\omega^2 z)$ as second solution, the function

$$Bi(z) = i\omega^2 Ai(\omega^2 z) - i\omega Ai(\omega z), \tag{1.4}$$

which is real when z is real, is commonly employed.

If we write $z = v^2$, it follows from (1.2) that, when $v > 0$,

$$Ai(v^2) = \frac{1}{2\pi i}\int_I e^{v^2 s - \frac{1}{3}s^3}\,ds, \tag{1.5}$$

where I is the imaginary axis from $-\infty i$ to ∞i. By Cauchy's Theorem, the path can be replaced by any path L, such as a pair of radii, from $\infty \exp \frac{4}{3}\pi i$ to $\infty \exp \frac{2}{3}\pi i$, and the resulting integral represents $Ai(v^2)$, not merely for $v > 0$, but for all values of ph v.

When v is positive, it is convenient to make the change of variable $s = vw$, which gives

$$Ai(v^2) = \frac{v}{2\pi i}\int_C e^{v^3(w - \frac{1}{3}w^3)}\,dw,$$

where C is the path I or a path L. The integrand has saddle points $w = \pm 1$; and the path of steepest descents through $w = -1$ is $3u^2 - v^2 = 3$ (if we write $w = u + iv$), and this is a path L. The asymptotic expansion of $Ai(v^2)$ for large v when $v > 0$, or, more generally, when $|\text{ph}\, v| < \frac{1}{6}\pi$ is obtainable by integrating along this hyperbolic path. The discussion is straight-

https://doi.org/10.1017/S2040618500034821 Published online by Cambridge University Press

forward but rather tedious.† It gives the asymptotic expansion of $Ai(z)$ when $|\operatorname{ph} z| < \frac{1}{3}\pi$.

By a similar argument, the asymptotic expansion can be found when $|\operatorname{ph}(-z)| < \frac{1}{3}\pi$. To fill in the gap between these two angles is difficult.

It is the purpose of this note to show that the asymptotic expansion of $Ai(z)$ when $|\operatorname{ph} z| < \pi$ can be obtained by a very much simpler argument, and that the asymptotic expansions of $Ai(z)$ and $Bi(z)$ for all values of ph z can be readily deduced by using formulae (1.3) and (1.4).

2. Another integral representation of $Ai(v^2)$. We suppose first that $v > 0$ and start from the formula (1.5). The point in the s plane corresponding to the saddle point $w = -1$ is $s = -v$. We show that the integral is unaltered in value if the path I is deformed into a parallel line through $s = -v$. By Cauchy's Theorem, all we have to show is that the integral along the straight line from $s = -v+it$ to $s = it$ tends to zero as $t \to \pm\infty$.

Writing $s = \sigma + it$, we have then to show that

$$\int_{-v}^{0} e^{v^2(\sigma+it)-\frac{1}{3}(\sigma+it)^3}\, d\sigma$$

tends to zero. The absolute value of this integral does not exceed

$$\int_{-v}^{0} e^{v^2\sigma-\frac{1}{3}\sigma^3+\sigma t^2}\, d\sigma \leqq e^{\frac{2}{3}v^3}\int_{-v}^{0} e^{\sigma t^2}\, d\sigma < \frac{e^{\frac{2}{3}v^3}}{t^2},$$

which tends to zero as $t \to \pm\infty$.

We may therefore put $s = -v+it$ in (1.5), where t varies from $-\infty$ to $+\infty$. This gives

$$Ai(v^2) = \frac{1}{2\pi} e^{-\frac{2}{3}v^3} \int_{-\infty}^{\infty} e^{-vt^2+\frac{1}{3}it^3}\, dt$$

or

$$Ai(v^2) = \frac{1}{\pi} e^{-\frac{2}{3}v^3} \int_{0}^{\infty} e^{-vt^2} \cos\left(\tfrac{1}{3}t^3\right) dt \tag{2.1}$$

when $v > 0$.

Before we apply Watson's Lemma, we observe that, when v is complex, the integral (2.1) converges uniformly in $0 < v_0 \leqq |v| \leqq v_1$, $|\operatorname{ph} v| \leqq \frac{1}{2}\pi - \delta < \frac{1}{2}\pi$ by Weierstrass's M-test; and as $Ai(v^2)$ is an integral function of v, it follows that (2.1) holds in the half-plane $|\operatorname{ph} v| < \frac{1}{2}\pi$.

If we now write $t^2 = u$, we have

$$Ai(v^2) = \frac{1}{2\pi} e^{-\frac{2}{3}v^3} \int_{0}^{\infty} e^{-vu} \cos\left(\tfrac{1}{3}u^{\frac{3}{2}}\right) \frac{du}{\sqrt{u}}.$$

As the conditions of Watson's Lemma are evidently satisfied, it follows at once that

† This steepest descents proof is due to Brillouin, *Ann. Sci. de l'École Norm. Sup.* 33 (1916), 17–69.

$$Ai(v^2) \sim \frac{1}{2\pi} e^{-\frac{2}{3}v^3} \sum_0^\infty \frac{\Gamma(3n+\frac{1}{2})}{3^{2n}(2n)!} \frac{(-1)^n}{v^{3n+\frac{1}{2}}},$$

and hence that

$$Ai(z) \sim \frac{1}{2\pi z^{\frac{1}{4}}} e^{-\frac{2}{3}z^{\frac{3}{2}}} \sum_0^\infty \frac{\Gamma(3n+\frac{1}{2})}{3^{2n}(2n)!} \frac{(-1)^n}{z^{\frac{3}{2}n}}$$

when $|\,\mathrm{ph}\, z\,| < \pi$.

It will be observed that this proof is not only simpler than that of Brillouin; it also gives a much better result.

3. Extension of the range of values of ph z.

The result just proved holds in any angle not containing the negative real axis. To extend the range of values of ph z, we use equation (1.3), viz.

$$Ai(z) = -\omega Ai(\omega z) = \omega^2 Ai(\omega^2 z)$$

with $\omega = e^{\frac{2}{3}\pi i}$, $\omega^2 = e^{\frac{4}{3}\pi i}$. It follows that, if $-\frac{5}{3}\pi < \mathrm{ph}\, z < -\frac{1}{3}\pi$,

$$Ai(z) \sim F(z) - iG(z),$$

where

$$F(z) \sim \frac{1}{2\pi z^{\frac{1}{4}}} e^{-\frac{2}{3}z^{\frac{3}{2}}} \sum_0^\infty \frac{\Gamma(3n+\frac{1}{2})}{3^{2n}(2n)!} \frac{(-1)^n}{z^{\frac{3}{2}n}},$$

$$G(z) \sim \frac{1}{2\pi z^{\frac{1}{4}}} e^{\frac{2}{3}z^{\frac{3}{2}}} \sum_0^\infty \frac{\Gamma(3n+\frac{1}{2})}{3^{2n}(2n)!} \frac{1}{z^{\frac{3}{2}n}}.$$

But if we take $\omega = e^{-\frac{2}{3}\pi i}$, $\omega^2 = e^{-\frac{4}{3}\pi i}$, we find that

$$Ai(z) \sim F(z) + iG(z)$$

when $\frac{1}{3}\pi < \mathrm{ph}\, z < \frac{5}{3}\pi$.

We have thus obtained three different asymptotic expansions, namely $F(z)$ and $F(z) \pm iG(z)$, for the integral function $Ai(z)$ valid in three overlapping angles. The expansions are consistent, the change of form as ph z varies being well known—it is the Stokes Phenomenon.

In a similar way, we can obtain asymptotic expansions for $Bi(z)$ by using (1.4); but as the results are well known, we shall not discuss them further.

St Salvator's College
University of St Andrews

https://doi.org/10.1017/S2040618500034821 Published online by Cambridge University Press