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Subcritical, transcritical and supercritical flows over a step

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Free-surface flow over a bottom topography with an asymptotic depth change (a 'step') is considered for different ranges of Froude numbers varying from subcritical, transcritical, to supercritical. For the subcritical case, a linear model indicates that a train of transient waves propagates upstream and eventually alters the conditions there. This leading-order upstream influence is shown to have profound effects on higher-order perturbation models as well as on the Froude number which has been conventionally defined in terms of the steady-state upstream depth. For the transcritical case, a forced Korteweg–de Vries (fKdV) equation is derived, and the numerical solution of this equation reveals a surprisingly conspicuous distinction between positive and negative forcings. It is shown that for a negative forcing, there exists a physically realistic nonlinear steady state and our preliminary results indicate that this steady state is very likely to be stable. Clearly in contrast to previous findings associated with other types of forcings, such a steady state in the transcritical regime has never been reported before. For transcritical flows with Froude number less than one, the upstream influence discovered for the subcritical case reappears.

1. Introduction

Problems of free-surface flow over bottom topographies have been studied by many authors. A review on the linear theory can be found in Wehausen & Laitone (1960). Other approximate theories include the cnoidal wave theory proposed by Benjamin & Lighthill (1954) and the directed fluid sheet theory by Green & Naghdi (1976). Recently, a number of authors have also solved the fully nonlinear problem using numerical methods (e.g. Forbes & Schwartz 1982 for flow over a semi-circular obstacle; King & Bloor 1987 for flow over a step; Moni & King 1994 for interfacial flow over a step; King & Bloor 1990 and Zhang & Zhu 1996*a* for arbitrary topographies).

The topographies studied so far largely fall into the category in which there is no asymptotic change in depth between far upstream and downstream (which shall be referred to as an 'isolated bump' hereafter). For this type of topography, Stoker (1958) has shown, using a linear transient theory, that for a steady subcritical or supercritical streaming (or equivalently, a steadily moving topography), a steady state eventually emerges from transient states and the transient motion dies out far upstream and thus the conditions there are the same as the undisturbed ones. With a horizontal momentum flux argument, Benjamin (1970) later found the 'upstream influence' for the subcritical case, which is of the second order in a perturbation expansion for an isolated bump. In an investigation for flow past a semi-circular trench embedded in an otherwise flat bottom, Zhang & Zhu (1996*b*) confirmed, with a perturbation expansion up to the second order, Benjamin's findings on the upstream influence with the results comparing favourably with their numerical solutions.

It is well known that linear models break down when the Froude number F is close to one, where a resonance takes place. For a moving pressure distribution, Akylas (1984) showed, with a matched asymptotic analysis, that the large-time motion is governed by a forced Korteweg–de Vries (fKdV) equation, the numerical solution of which reveals the generation of a series of upstream-running solitons, which forms part of an undular bore in front of the moving pressure. This was later confirmed by a number of authors. Cole (1985) followed Akylas' approach to study a moving isolated bump on the bottom and found qualitatively the same results. Wu (1987) and Lee, Yates & Wu (1989) studied the bump case both experimentally and theoretically with a generalized long-wave equation derived by Wu (1981). Grimshaw & Smyth (1986) studied resonant stratified flows over an isolated bump. Shen (1991) examined critical flows in a channel with arbitrary cross-section but with localized forcing. Experimental results by Huang *et al.* (1982) and Sun (1985) qualitatively confirmed the previous theoretical findings. A common feature in this transcritical regime is the unsteadiness of the resulting motion generated by a steadily moving disturbance, regardless of the net volume displaced by the disturbance (i.e. either a positive or negative forcing). Although a number of time-independent solutions to the fKdV equation have been artificially constructed (Wu 1987; Camassa & Wu 1991; Djordjevic & Redekopp 1992), they were shown to be unstable.

In this paper, we study flow over a 'step' geometry. This problem has some applications in hydraulic and coastal engineering (e.g. it can be regarded as a simple model for flow such as tide over a continental slope followed by a continental shelf) and also is expected to provide a qualitative description of the flow caused by a long body moving close to the sea bed (King & Bloor 1987). For the subcritical case, we prove that the upstream influence in this case appears at the first order. As a result, the far upstream condition in a steady-state model (the so-called radiation condition) is different from the conventional one. Although one need not worry about this leading-order upstream effect in the first-order problem by adopting the upstream depth corresponding to the steady state as the length scale (which was implied in King & Bloor's 1987 paper), the upstream influence is certainly needed in a second-order model. More importantly, adopting the steady-state upstream depth as the length scale makes it impossible to distinguish classes of flows (i.e. 'subcritical', 'transcritical', or 'supercritical') according to the Froude number, since the Froude number will then vary with the topography. In contrast, we follow Benjamin (1970) and consistently adopt the undisturbed upstream depth as the length scale in this paper. Consequently, such a conventionally defined Froude number is conceptually clear and will not cause any confusion in differentiating classes of flows.

For the transcritical case, we shall show that the flow motion is governed by an fKdV equation with the forcing term being proportional to the Dirac delta function. Although it is mathematically similar to the 'isolated bump' case studied before, the resulting physics is quite different. In particular, a surprising contrast is revealed between the system's responses to a positive and negative forcing, which respectively correspond to a 'step up' and 'step down'. For the case of a positive forcing, the induced motion is qualitatively similar to the previous findings: a series of solitons marches upstream in procession. On the other hand, a nonlinear steady state eventually emerges from transient states for a negative-forcing case, and our preliminary results from a numerical stability analysis indicate that the resulting steady state is very likely to be stable. This new finding is significant since so far no physically realistic and stable steady state in the transcritical regime has been reported in the literature, and it may shed some light on the theoretical study of transcritical flows. By solving the time-independent fKdV equation, the steady-state solution, found analytically, indicates an inevitable upstream influence for Froude numbers less than one, which is confirmed by the numerical solutions. The upstream influence is realized via a train of transient nonlinear long waves which is a counterpart of the linear transient waves found in the subcritical range.

2. Linear theory for subcritical and supercritical flows

We consider a layer of fluid with a free surface flowing over a 'step' topography (figure 1). The fluid is assumed to be inviscid, incompressible, and the motion irrotational. It is assumed that there exists an asymptotic change ($c \neq 0$) between downstream and upstream depths. Before $t = 0$, the fluid and the 'step' are quiescent and at $t = 0^+$ a uniform current with constant speed U comes from far upstream towards the stationary step. For ease of approaching the problem mathematically, the problem can be viewed as the step starting impulsively to move with a speed U. A reference frame Oxy is then chosen to be moving with the step, with the x-axis pointing to the opposite direction to the step's motion, the y-axis pointing upwards, and the origin located on the undisturbed free surface. All variables are made dimensionless with reference to the length scale h, the undisturbed upstream depth, and velocity scale $(gh)^{1/2}$ (with g being the gravitational acceleration). It should be emphasized here that h is not necessarily the upstream depth corresponding to a steady state (if it exists), as will become clear in the following discussions. The non-dimensionalized initial-boundary-value problem now reads

$$\Phi_{xx} + \Phi_{yy} = 0 \text{ in the fluid region,} \tag{2.1}$$

$$\Phi_y = -d'(x)(F + \Phi_x), \;\; y = -d(x) = -1 + \epsilon D(x), \tag{2.2}$$

$$\left.\begin{aligned} \Phi_y &= H_t + (\Phi_x + F)H_x, \\ 0 &= H + \Phi_t + F\Phi_x + \tfrac{1}{2}(\Phi_x^2 + \Phi_y^2), \end{aligned}\right\} y = H(x,t), \tag{2.3}$$

$$\Phi_x, \Phi_y \to 0, \;\; |x| \to \infty, \tag{2.4}$$

$$\Phi|_{y=0} = \Phi_t|_{y=0} = H = 0, \;\; t = 0, \tag{2.5}$$

where Φ is the velocity potential, subscripts denote partial differentiation, $y = H(x,t)$ is the instantaneous position of the free surface, $y = -1 + \epsilon D(x)$ represents the bottom profile, ϵ is a dimensionless quantity characterizing the small depth change $c/h = \epsilon C$, with $C = D(+\infty)$, $D(-\infty) = 0$, $D(x) = O(1)$, and $F = U/(gh)^{1/2}$ is the Froude number, which characterizes the nature of the induced motion. It should be emphasized here that the Froude number defined in this way does not change with the topography, and consequently it is fundamentally different from the one defined in King & Bloor (1987). $C > 0 (< 0)$ corresponds to a step up (down) or positive (negative) forcing.

It is well known that for $F - 1 = O(1)$ as $\epsilon \to 0$, the problem can be solved with a perturbation expansion in ϵ:

$$\Phi = \epsilon\phi + O(\epsilon^2), \tag{2.6}$$

$$H = \epsilon\eta + O(\epsilon^2), \tag{2.7}$$

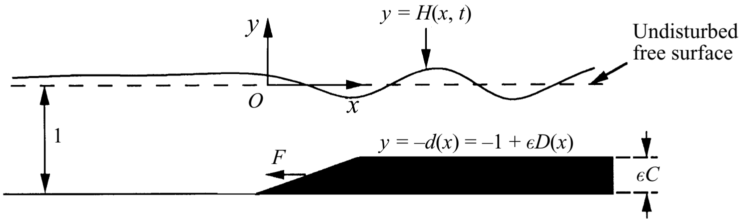


FIGURE 1. Definition sketch for flow over a step.

with which the first-order problem reads

$$\phi_{xx} + \phi_{yy} = 0 \text{ in the fluid region,} \tag{2.8}$$

$$\phi_y = FD'(x), \;\; y = -1, \tag{2.9}$$

$$\left.\begin{array}{l} 0 = \phi_y + \phi_{tt} + 2F\phi_{xt} + F^2\phi_{xx}, \\ \eta = -F\phi_x - \phi_t, \end{array}\right\} y = 0, \tag{2.10}$$

$$\phi_x, \phi_y \to 0, \;\; |x| \to \infty, \tag{2.11}$$

$$\phi|_{y=0} = \phi_t|_{y=0} = \eta = 0, \;\; t = 0. \tag{2.12}$$

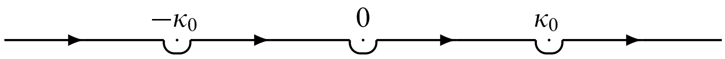
The solution of (2.8)–(2.12) can be obtained via a Fourier transform as

$$\eta(x,t) = -\frac{\mathrm{i}F^2}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \frac{\hat{D}'(\kappa)\mathrm{sech}\,\kappa}{F^2\kappa - \tanh\kappa} \mathrm{e}^{\mathrm{i}\kappa x} \left(1 - \frac{\omega_-}{2\kappa F}\mathrm{e}^{-\mathrm{i}\omega_+ t} - \frac{\omega_+}{2\kappa F}\mathrm{e}^{-\mathrm{i}\omega_- t}\right) \mathrm{d}\kappa, \tag{2.13}$$

where $\omega_\pm = F\kappa \pm (\kappa\tanh\kappa)^{1/2}$ and $\hat{D}'(\kappa)$ is the Fourier transform of the derivative $D'(x)$. It can be verified that $\eta \to 0$ as $|x| \to \infty$ for any finite t. The large time solution is crucially dependent on the poles of the first factor of the integrand. For the subcritical case $F < 1$, there are three poles at $\kappa = 0, \pm\kappa_0$, where κ_0 is the wavenumber of the progressive wave with a phase velocity U:

$$F^2\kappa_0 = \tanh\kappa_0. \tag{2.14}$$

The path of integration is now indented with semi-circles of vanishingly small radius placed below the three poles $\kappa = 0, \pm\kappa_0$ so that the total contribution from these three semi-circles is zero. The indented path Γ^- looks like



With the path of integration being properly defined, we then split the solution into two parts:

$$\eta = \eta^s + \eta^t, \tag{2.15}$$

$$\eta^s = -\frac{\mathrm{i}F^2}{(2\pi)^{1/2}} \int_{\Gamma^-} \frac{\hat{D}'(\kappa)\mathrm{sech}\,\kappa}{F^2\kappa - \tanh\kappa} \mathrm{e}^{\mathrm{i}\kappa x} \mathrm{d}\kappa, \tag{2.16}$$

$$\eta^t = \frac{\mathrm{i}F^2}{(2\pi)^{1/2}} \int_{\Gamma^-} \frac{\hat{D}'(\kappa)\mathrm{sech}\,\kappa}{F^2\kappa - \tanh\kappa} \mathrm{e}^{\mathrm{i}\kappa x} \frac{\omega_-\mathrm{e}^{-\mathrm{i}\omega_+ t} + \omega_+\mathrm{e}^{-\mathrm{i}\omega_- t}}{2\kappa F} \mathrm{d}\kappa, \tag{2.17}$$

where η^s represents a time-independent state, η^t represents the transient motion, which diminishes as $t \to \infty$ in the classical case for an isolated bump (Stoker 1958). In the present case, we can similarly show, using the method of stationary phase, that as $t \to \infty$, the contributions to η^t from the straight-line part and the two semi-circles centred at $\kappa = \pm\kappa_0$ of Γ^- vanish like $t^{-1/2}$. To examine the asymptotic form of the integral in (2.17) along the semi-circle centred at $\kappa = 0$, we expand $\omega_\pm$ for small κ as

$$\omega_\pm = (F \pm 1)\kappa \mp \tfrac{1}{6}\kappa^3 + O(\kappa^5). \tag{2.18}$$

It is seen that for $F < 1$, $\mathrm{Im}\{\omega_+\} < 0$, $\mathrm{Im}\{\omega_-\} > 0$ on the semi-circle. Therefore the first term in η^t (the one proportional to $\exp(-\mathrm{i}\omega_+ t)$) vanishes as $t \to \infty$ by a similar argument as before, but not the second term since its exponent has a positive real part on the semi-circle. The large-time asymptotic form of this term, which represents the leading contribution from η^t, is found by Cauchy's residue theorem as

$$\begin{aligned}\eta^t &\sim \frac{\mathrm{i}F^2}{(2\pi)^{1/2}} \int_{\Gamma^-} \frac{\hat{D}'(\kappa)\operatorname{sech}\kappa}{F^2\kappa - \tanh\kappa} \frac{\omega_+}{2\kappa F} \mathrm{e}^{\mathrm{i}(\kappa x - \omega_- t)} \mathrm{d}\kappa \\ &= (\pi/2)^{1/2} \frac{\hat{D}'(0)F}{1-F} + \frac{\mathrm{i}F^2}{(2\pi)^{1/2}} \int_{\Gamma^{+0}} \frac{\hat{D}'(\kappa)\operatorname{sech}\kappa}{F^2\kappa - \tanh\kappa} \frac{\omega_+}{2\kappa F} \mathrm{e}^{\mathrm{i}(\kappa x - \omega_- t)} \mathrm{d}\kappa,\end{aligned} \tag{2.19}$$

where the path Γ^{+0} curves around $\kappa = 0$ from above. The second term can be shown to vanish as $t \to \infty$ and therefore we obtain the steady-state solution

$$\eta_s = \eta^s + (\pi/2)^{1/2} \frac{\hat{D}'(0)F}{1-F}. \tag{2.20}$$

The second term represents a change in the mean free-surface level and is zero for an isolated bump since $\hat{D}'(\kappa) = \mathrm{i}\kappa\hat{D}(\kappa)$ and $\hat{D}(0)$ is finite. But if there is a change in depth, $\hat{D}'(0) = C/(2\pi)^{1/2} \neq 0$. Since $\eta^s \to 0$ as $x \to -\infty$, the second term also represents an upstream influence which is positive or negative depending on the sign of C. Unlike the isolated bump case, the upstream influence in this extreme case appears at the first order and thus should be taken into consideration in a *linear* model. A closely related problem is the choice of radiation condition in a steady-state problem. Strictly speaking, only through a transient model can a radiation condition be justified. The correct radiation condition for a topography with an asymptotic depth change should thus be

$$\eta \to \frac{CF}{2(1-F)} \quad \text{as} \quad x \to -\infty, \tag{2.21}$$

for $F < 1$.

Since the upstream depth in the steady state remains a constant for this particular case, one can take the final steady-state upstream depth instead of the undisturbed depth as the length scale, as King & Bloor (1987) did. In that case, the upstream influence is automatically accounted for in the first-order problem. However, a great disadvantage of defining the Froude number in terms of an unknown length scale, which is supposed to be part of the solution of the problem, is that the Froude number must now be based on the upstream depth in the steady state and thus must be a function of the topography ϵ, which makes it impossible to distinguish 'subcritical', 'transcritical', and 'supercritical' flows in terms of this Froude number. This definition of Froude number certainly does not meet the criterion of Benjamin (1970) in distinguishing classes of flows. Furthermore, the upstream influence will still be needed in the second-order problem through the Froude number. Hence we believe the current approach is conceptually clearer and easier to use in a perturbation model than that of King & Bloor (1987).

The upstream influence is realized by a train of transient waves with a leading long-wave envelope in the front (called "forward surge" by Benjamin 1970). To see this, let $x = mt$ with $m < 0$. The exponent in (2.19) then becomes $E_-(\kappa, t) = \mathrm{i}t(m\kappa - \omega_-) \equiv \mathrm{i}tf_-(\kappa)$. The stationary points of $f_-(\kappa)$ are given by $\omega'_-(\kappa) = m$. Therefore $f_-(\kappa)$ has a stationary point if $F - 1 \leqslant m < 0$ and in this case the second term in (2.19) decays like $t^{-1/2}$ for large t. From (2.19) we can see that in addition to the oscillatory wave motion, there is also a change of mean free-surface level after the passage of the transient waves which are led by a long-wave envelope (travelling with a speed $F - 1$). In front of this long-wave envelope the motion decays like t^{-1}. It is then easy to understand that for the supercritical case $F > 1$ there is no such upstream influence since no (linear) waves can propagate upstream at a speed greater than the long-wave speed and thus the energy can no longer be radiated upstream.

In summary, the steady-state solution in the far field is

$$\eta \to \begin{cases} \dfrac{CF}{2(1-F)}, & x \to -\infty, \\ \dfrac{CF}{2(1+F)} + \dfrac{2(2\pi)^{1/2}F^2 \mathrm{sech}\kappa_0}{F^2 - \mathrm{sech}^2\kappa_0} \mathrm{Re}\{\hat{D}'(\kappa_0)\mathrm{e}^{\mathrm{i}\kappa_0 x}\}, & x \to +\infty, \end{cases} \tag{2.22}$$

for $F < 1$, and

$$\eta \to \begin{cases} 0, & x \to -\infty, \\ \dfrac{CF^2}{F^2 - 1}, & x \to +\infty, \end{cases} \tag{2.23}$$

for $F > 1$. The results are basically consistent with King & Bloor's (1987). It is interesting to note that while the upstream influence becomes unbounded as $F \to 1^-$, the mean height of the downstream free surface approaches $\frac{1}{4}C$.

3. Transcritical case

3.1. *The forced Korteweg–de Vries equation*

In the vicinity of $F = 1$, the response predicted by the linear model becomes unbounded at large time. Using a matched asymptotic analysis (cf. Akylas 1984), one can show that the small-time linear solution can be matched to that from a weakly nonlinear theory (with the nonlinearity and dispersion effects being assumed to be weak and of the same order) at a time scale proportional to $\epsilon^{-3/4}$. Upon introducing the following rescaled variables:

$$T = \epsilon^{3/4}t, \; X = \epsilon^{1/4}x, \; \Phi = \epsilon^{1/4}\bar{\phi}, \; H = \epsilon^{1/2}Y, \tag{3.1}$$

and rewriting the governing equations (2.1)–(2.5) in terms of these 'slow' variables and slightly detuning the Froude number off the exact linear resonance as $F = 1 + \epsilon^{1/2}\Omega$ (with $\Omega = O(1)$), one can derive the forced KdV equation:

$$Y_T - \tfrac{3}{2}YY_X + \Omega Y_X - \tfrac{1}{6}Y_{XXX} = \tfrac{1}{2}C\delta(X), \tag{3.2}$$

subject to the boundary and initial conditions

$$Y, Y_X, Y_{XX} \to 0 \text{ as } |x| \to \infty, \tag{3.3}$$

$$Y(X, 0) = 0. \tag{3.4}$$

Equation (3.2) is identical to Djordjevic & Redekopp's (1992) result derived in a different way. The only difference between this fKdV equation and that for an isolated bump is the forcing term, which in the present case is proportional to the depth change and the delta function. $C > 0$ (< 0) corresponds to a positive (negative)

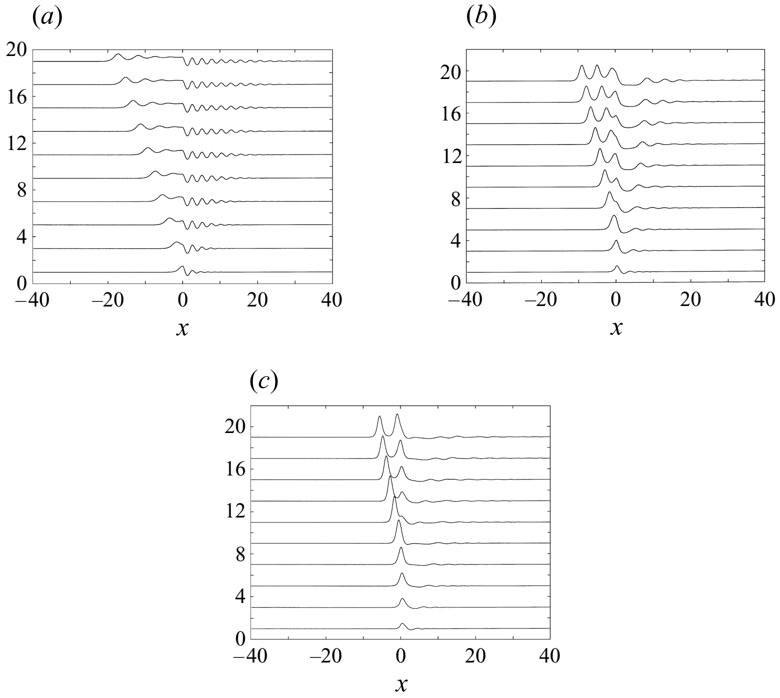


FIGURE 2. Evolution of the free surface for a positive forcing $C = 1$. The time interval between neighbouring curves is 1.5. (*a*) $\Omega = -1$; (*b*) $\Omega = 0$; (*c*) $\Omega = 0.5$.

forcing. Two conservation equations for mass and energy can be easily derived from (3.2) as

$$\int_{-\infty}^{+\infty} Y\,\mathrm{d}X = \frac{CT}{2}, \tag{3.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}T}\int_{-\infty}^{+\infty} Y^2\mathrm{d}X = CY(0,T). \tag{3.6}$$

Therefore unlike the case of an isolated bump where the total mass does not change with time, the total mass is either increasing ($C > 0$) or decreasing ($C < 0$) with time for the problem under consideration. If we postulate that at the downstream side of the disturbance the mass is always decreasing due to the ever lengthening region of depression of the free surface and a train of weakly dispersive waves oscillating around the undisturbed level (cf. figures 2 and 3), we can deduce that for a positive forcing $C > 0$, the mass in front of the step must be increasing in order to render an increasing total mass; a series of solitary waves is therefore expected to appear upstream. On the other hand, for a negative forcing $C < 0$ (corresponding to a step down), the upstream mass need not be increasing since the total mass decreases with time. In fact, we shall show that the upstream mass is always non-increasing and no solitary wave is emitted upstream.

3.2. *Numerical method and results*

We adopt a simple numerical method to solve the fKdV equation (3.2). This method is similar to that used by Johnson (1972). We use a central difference for the spatial derivatives and fourth-order Runge–Kutta method to march with time. Similar to

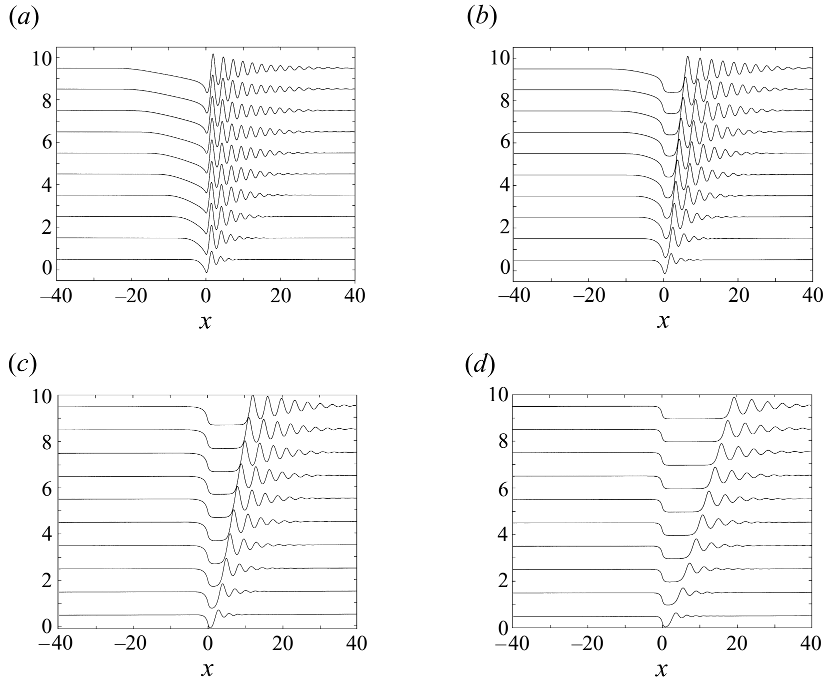


FIGURE 3. Evolution of the free surface for a negative forcing $C=-1$. The time interval between neighbouring curves is 2. (*a*) $\Omega=-1$; (*b*) $\Omega=-0.5$; (*c*) $\Omega=0$; (*d*) $\Omega=0.5$.

Akylas (1984), the jump condition across $X=0$:

$$Y_{XX}|_{X=0^+} - Y_{XX}|_{X=0^-} = -3C, \tag{3.7}$$

is replaced by a finite difference. As is common for explicit finite-difference schemes, smoothing or numerical damping must be used to suppress the unphysical high-frequency oscillations. With the X-axis being discretized into a number of grid points and Y^j denoting the Y-value on grid point j, we apply a five-point smoothing formula used by Longuet-Higgins & Cokelet (1976):

$$Y^j = \tfrac{1}{16}(-Y^{j-2} + 4Y^{j-1} + 10Y^j + 4Y^{j+1} - Y^{j+2}), \tag{3.8}$$

at all grid points except for the last two points far upstream and downstream. Our numerical experiments showed that although the errors for the computation of the conservation equations (3.5) and (3.6) could reach 5% and 10%, respectively, the maximum errors only occurred after the simulation time became extremely long. The main cause of error is believed to be the instability of the KdV equation for short waves. However, as an ultimate verification, we were satisfied with the overall agreement between the numerical and analytical solutions (e.g. figure 4), on top of what has been reported by Longuet-Higgins & Cokelet on the smoothing formula.

Figures 2 and 3 show the evolution of the free surface for both positive and negative forcings and for different values of the detuning factor Ω. It should be noted that in these figures, the vertical axis has been shifted so that the time evolution can be readily viewed. For a positive forcing, the solution is qualitatively similar to that for an isolated bump: a series of solitary waves is found to advance upstream with an ever lengthening region of depression of the free surface and a train of weakly nonlinear and weakly dispersive waves being shed towards the downstream region. On the

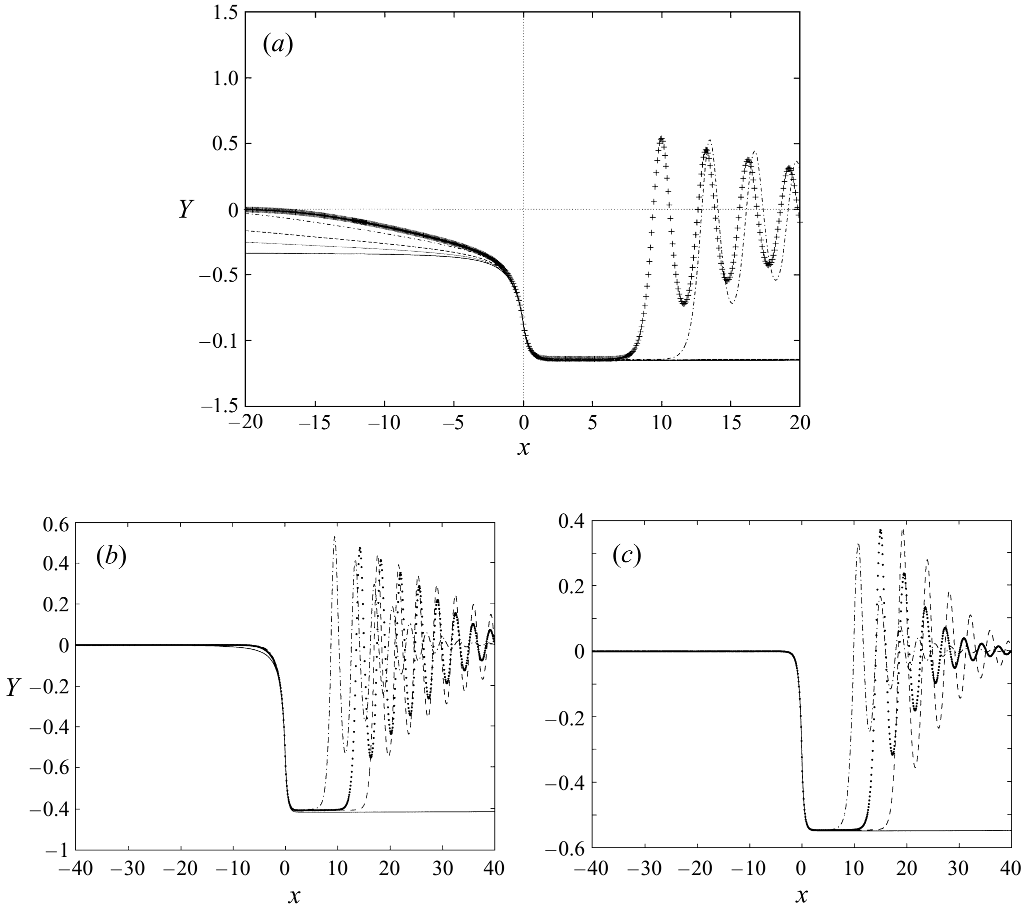


FIGURE 4. Comparison of the analytical steady-state solutions (solid line) and numerical large-time solutions for $C = -1$. (*a*) $\Omega = -0.5$, $T = 30$ (plus signs), $T = 40$ (dash-dotted line), $T = 75$ (dash line), $T = 150$ (dotted line); (*b*) $\Omega = 0$, $T = 15$ (dash-dotted line), $T = 24$ (dotted line), $T = 30$ (dash line); (*c*) $\Omega = 0.5$, $T = 10$ (dash-dotted line), $T = 15$ (dotted line), $T = 20$ (dash line).

other hand, a very interesting result is that steady states are always reached when the flow is negatively forced. It is found that there is no upstream influence for the case $\Omega \geqslant 0$ and there is an upstream advancing long wave for the case $\Omega < 0$, although the propagating speed is very small. This can be seen in figure 3 for different Ω values. Note that for the case $\Omega = -1$, the transcritical solution seems to have merged with the subcritical solution, i.e. with a train of lee waves developed downstream of the disturbance, and a train of transient long waves propagating upstream.

The existence of a steady-state solution for the case of negative forcing can also be explained qualitatively with the modulation theory (cf. Smyth 1987, and private communication). It can be shown that the depth fall is equivalent to a decrease in the detuning factor Ω (and therefore the Froude number) and according to Smyth, a ‘steady state’ (with a constant elevation trailing an expansion fan of cnoidal waves) can be reached if the Froude number is smaller than a critical value in the subcritical range. But this cannot explain the fact that in the present case there is no such expansion fan of cnoidal waves as in the case of an isolated bump. Furthermore, according to his theory, the shift in the Froude number is proportional to C and as $|C| \downarrow 0$, the soliton-type solution is expected to reappear. But our numerical evidence indicates otherwise: as $|C| \downarrow 0$, the solution is qualitatively the same as that shown in figure 3 with a diminishing gap between upstream and downstream free-surface levels. This is consistent with the conservation equation (3.5). In other words, steady-state solutions can always be reached for arbitrary negative forcings $C < 0$. The steady-state solution to the nonlinear transient fKdV equation found here is of great significance as it is the only one found so far which is physically realistic in the transcritical regime.

3.3. *Steady-state solution for negative forcings* $C < 0$

In this subsection, the steady-state solution for $C < 0$ is to be found analytically and compared with the numerical solution. The steady state is governed by the time-independent fKdV equation:

$$\tfrac{3}{2}Y_sY_s' - \Omega Y_s' + \tfrac{1}{6}Y_s''' = -\tfrac{1}{2}C\delta(X), \tag{3.9}$$

subject to appropriate boundary conditions which depend on the value of Ω. It is noteworthy that the parameter C can be explicitly eliminated from (3.9) via the following scalings:

$$X = |C|^{-1/4}X^*, \ \ Y_s = |C|^{1/2}Y^*, \tag{3.10}$$

with which (3.9) becomes

$$\tfrac{3}{2}Y^*\frac{\mathrm{d}Y^*}{\mathrm{d}X^*} - \Omega_1\frac{\mathrm{d}Y^*}{\mathrm{d}X^*} + \frac{1}{6}\frac{\mathrm{d}^3Y^*}{\mathrm{d}X^{*3}} = \tfrac{1}{2}\delta(X^*), \tag{3.11}$$

with $\Omega_1 = \Omega|C|^{-1/2}$.

(i) $\Omega > 0$

We impose the far-field boundary conditions

$$Y_s, Y_s', Y_s'' \to 0 \text{ as } X \to -\infty, \tag{3.12}$$

$$Y_s \to -|C|^{1/2}\alpha, Y_s', Y_s'' \to 0 \text{ as } X \to +\infty, \tag{3.13}$$

where α is a constant and can be found by integrating (3.9) once from $X = -\infty$ to $X = +\infty$:

$$\alpha = \frac{(4\Omega_1^2 + 6)^{1/2} - 2\Omega_1}{3}. \tag{3.14}$$

Upon solving (3.9) respectively in $X < 0$ and $X > 0$ and invoking two continuity conditions across $X = 0$, we obtain the solution

$$Y_s = \begin{cases} -2\Omega \operatorname{cosech}^2\left(\tfrac{1}{2}(6\Omega)^{1/2}X + k\right), & X \leqslant 0, \\ |C|^{1/2}[-\alpha + \beta \operatorname{sech}^2(a|C|^{1/4}X + b)], & X > 0, \end{cases} \tag{3.15}$$

where

$$\beta = (4\Omega_1^2 + 6)^{1/2}, \ a = \frac{1}{2}(3\beta)^{1/2}, \ \xi_0 = -\frac{2}{3}\alpha\frac{\beta + \Omega_1}{\beta + 2\Omega_1}(> -\alpha),$$

$$b = \operatorname{sech}^{-1}\left(\frac{\xi_0 + \alpha}{\beta}\right)^{1/2}, \ k = -\operatorname{cosech}^{-1}\left(\frac{-\xi_0}{2\Omega_1}\right)^{1/2}. \tag{3.16}$$

Note that Y'' is discontinuous at $X = 0$ and the solution approaches constant values *exponentially* both upstream and downstream. This asymptotic behaviour is consistent with that of the supercritical case, which can be worked out with a careful analysis as shown in the Appendix.

(ii) $\Omega = 0$

This case has been solved by Djordjevic & Redekopp (1992) with some obvious errors in their results (the first derivative is not continuous at the origin in their solution). With the same boundary conditions as those in the previous case, we obtain

$$Y_s = \begin{cases} -|C|^{1/2}\left(k - \frac{1}{2}\sqrt{3}|C|^{1/4}X\right)^{-2}, & X \leqslant 0, \\ -|C|^{1/2}[\alpha + 3\alpha \operatorname{sech}^2(a|C|^{1/4}X + b)], & X > 0, \end{cases} \tag{3.17}$$

where

$$\alpha = \tfrac{1}{3}\sqrt{6},\ a = \tfrac{3}{2}\alpha^{1/2},\ b = \operatorname{sech}^{-1}(\tfrac{1}{3}),\ k = \alpha^{-3/2}. \tag{3.18}$$

Note that the solution decays algebraically upstream but approaches a non-zero constant exponentially downstream.

(iii) $\Omega < 0$

In light of the results for the subcritical case in §2, we can no longer demand that there is no upstream influence ($Y_s \to 0$ as $X \to -\infty$). Instead, the appropriate boundary conditions must be

$$Y_s \to -|C|^{1/2}\gamma,\ \ Y_s', Y_s'' \to 0 \text{ as } X \to -\infty, \tag{3.19}$$

$$Y_s \to -|C|^{1/2}\alpha,\ \ Y_s', Y_s'' \to 0 \text{ as } X \to +\infty, \tag{3.20}$$

where γ and α are two constants which are related by

$$\alpha = \frac{[(2\Omega_1 + 3\gamma)^2 + 6]^{1/2} - 2\Omega_1}{3}. \tag{3.21}$$

Obviously an extra condition is needed to determine γ. We solve the equation for $X < 0$ and $X > 0$ respectively.

(*a*) $X < 0$

Integrating (3.9) twice gives

$$u' = u[3(2\Omega + 3\gamma|C|^{1/2} - u)]^{1/2}, \tag{3.22}$$

where $u = Y_s + |C|^{1/2}\gamma \uparrow 0$, as $X \to -\infty$. If $2\Omega + 3\gamma|C|^{1/2} \neq 0$, Y_s approaches $-|C|^{1/2}\gamma$ exponentially upstream. With an asymptotic analysis one can prove that the steady-state free-surface elevation approaches a constant *algebraically* (like const. $+\,O(x^{-1})$) as $x \to -\infty$ for both the subcritical case and the critical case $\Omega = 0$. Consequently, a sensible hypothesis is that the solution for the present transcritical case with $\Omega < 0$ approaches a constant algebraically upstream as well. Hence,

$$\gamma = -\tfrac{2}{3}\Omega|C|^{-1/2} = -\tfrac{2}{3}\Omega_1, \tag{3.23}$$

which confirms that there is no upstream influence for the critical case $\Omega = 0$. It is seen that the upstream influence is proportional to the detuning factor Ω, and one can actually define two threshold Froude numbers F_1 and F_2 such that $0 < F < F_1$, $F_1 < F < F_2$, and $F > F_2$ correspond respectively to the subcritical, transcritical, and supercritical flows, in the sense that the upstream or downstream height (cf. (2.22)–(2.23)) is continuous across these two Froude numbers. Consequently, the steady-state solution could vary 'smoothly' from one region to another, as if the resonance never took place.

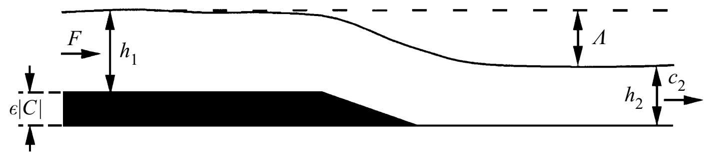


FIGURE 5. Steady transcritical flow for a negative forcing.

The upstream solution now reads

$$Y_s = |C|^{1/2}[-\gamma - (k - \tfrac{1}{2}\sqrt{3}|C|^{1/4}X)^{-2}], \tag{3.24}$$

which indeed varies algebraically as $X \to -\infty$.

(b) $X > 0$

The solution can be found in a similar fashion as in the cases $\Omega \geqslant 0$ as

$$Y_s = |C|^{1/2}[-\alpha + \sqrt{6}\mathrm{sech}^2(a|C|^{1/4}X + b)]. \tag{3.25}$$

Matching the two solutions at $X = 0$ yields the final solution:

$$Y_s = \begin{cases} |C|^{1/2}[-\gamma - (k - \frac{1}{2}\sqrt{3}|C|^{1/4}X)^{-2}], & X \leqslant 0, \\ |C|^{1/2}[-\alpha + \sqrt{6}\mathrm{sech}^2(a|C|^{1/4}X + b)], & X > 0, \end{cases} \tag{3.26}$$

where

$$\gamma = -\tfrac{2}{3}\Omega_1, \;\; \alpha = \tfrac{1}{3}(\sqrt{6} - 2\Omega_1), \;\; a = \tfrac{1}{2}\sqrt{3\sqrt{6}},$$

$$\xi_0 = -\tfrac{1}{2}[\gamma^3 + 2\alpha^2(\alpha + \Omega_1)], \;\; b = \mathrm{sech}^{-1}\left((\xi_0 + \alpha)/\sqrt{6}\right)^{1/2}, k = (-\gamma - \xi_0)^{-1/2}. \tag{3.27}$$

It can be verified that as $\Omega \to 0^-$, the critical solution is recovered.

The comparison of the steady-state solutions obtained above and from the numerical method presented in §3.2 (strictly speaking, the numerical solutions are only 'quasi-steady') is shown in figure 4. The asymptotic agreement is very good. In particular, we can see an upstream advancing transient long wave for the case $\Omega < 0$, although its propagating speed is very small indeed due to the closeness to the critical condition; about 10 minutes CPU time was spent on a VP2200 supercomputer to obtain the transient solution closest to the steady-state solution since we had to use a very small time step ($= 5 \times 10^{-4}$) required by the stability of the numerical scheme. This transient wave is a nonlinear counterpart of the linear wave discovered in the subcritical case. Figure 4(a) also supports the hypothesis made in the process of finding γ in (a).

The change of the free-surface level between upstream and downstream can also be calculated with an exact theory (cf. King & Bloor 1987). With reference to figure 5, we can obtain a cubic equation for the downstream depth h_2 via the continuity and Bernoulli equations as

$$h_2^3 - h_2^2\left(\tfrac{1}{2}F^2 + h_1 + \epsilon|C|\right) + \tfrac{1}{2}F^2h_1^2 = 0, \tag{3.28}$$

where h_1 can be found from (3.15), (3.17) and (3.26). Note that we could have set $h_1 = 1$ as King & Bloor (1987) did; we did not do this in order to facilitate the following comparison. After h_2 is found, the change of the free-surface level is given by

$$\Lambda = h_2 - h_1 - \epsilon|C|. \tag{3.29}$$

Shown in figure 6 is the comparison of the changes of the free-surface level calculated

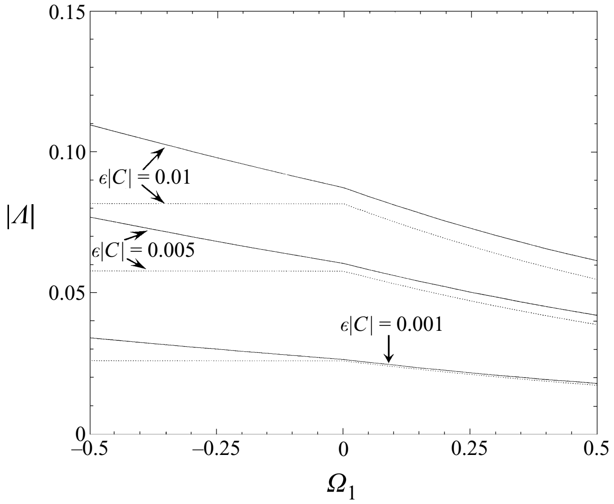


FIGURE 6. Comparison of the change in the free-surface level computed from the fKdV theory (dotted lines) and the exact theory (solid lines).

from the fKdV theory and the exact theory for different values of Ω_1 and the step height $\epsilon|C|$. As expected, the agreement is better for a smaller step height. In the range $\Omega_1 < 0$, the fKdV theory predicts a constant value for Λ while in the exact theory Λ still varies with the detuning factor Ω_1.

Time-independent solutions to the fKdV equation have also been found before for the isolated bump case but they were shown to be generally unstable (Wu 1987; Camassa & Wu 1991; Gong & Shen 1994). To find whether the present steady-state solutions are stable or not, we followed Wu (1987) and carried out several stability tests. Our preliminary test results indicate that if the flow is negatively forced, the steady states found in the transcritical regime are very likely to be stable. However, a comprehensive stability analysis, which is by no means trivial in this case, is needed before a definite conclusion can be drawn.

4. Conclusions

In this paper, a linear theory and a weakly nonlinear theory are presented for subcritical, transcritical, and supercritical flows over a step topography. For the subcritical case, it is found that the so-called upstream influence appears at the first order. Unlike King & Bloor (1987), we define the Froude number in terms of the initially undisturbed upstream depth, which is conceptually clearer and more systematic. The upstream influence is also present for the transcritical case with the Froude number being less than one. In both cases, the upstream influence is realized by a train of transient long waves propagating upstream.

In the transcritical regime, it is shown that steady states are always reached if the flow is negatively forced. That is to say, in the negative-forcing case, one will always observe steady responses *for all Froude numbers*, if a sufficient amount of time has elapsed. Since so far we have not found any disturbance under which these steady states become unstable, it seems reasonable to postulate the stability of these steady states.

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Appendix. Upstream asymptotic free-surface profile for supercritical case

For $F > 1$, the steady-state free-surface elevation is

$$\eta = \frac{CF^2}{2(F^2-1)} + \frac{F^2}{\pi} I(x), \tag{A 1}$$

where

$$I(x) = (2\pi)^{1/2} \int_0^{+\infty} \frac{\operatorname{sech}\kappa}{F^2\kappa - \tanh\kappa} \operatorname{Im}\{\hat{D}'(\kappa)\mathrm{e}^{\mathrm{i}\kappa x}\}\mathrm{d}\kappa. \tag{A 2}$$

It should be noted that the integrand possesses no singularity. Since we are mainly interested in the asymptotic form as $x \to -\infty$, we write

$$\begin{aligned} I(x) &= \frac{(2\pi)^{1/2}}{x} \int_0^{+\infty} \frac{\operatorname{sech}(u/x)}{F^2 u/x - \tanh(u/x)} \operatorname{Im}\{\hat{D}'(-u/x)\mathrm{e}^{-\mathrm{i}u}\}\mathrm{d}u \\ &\sim \frac{C}{F^2-1} \int_0^{+\infty} \frac{\operatorname{sech}\kappa}{\kappa} \left[1 - \frac{1}{3(F^2-1)}\kappa^2 + O(\kappa^4)\right] \sin\kappa x \mathrm{d}\kappa. \end{aligned} \tag{A 3}$$

From the known integrals (Gradshteyn & Ryzhik 1965)

$$\int_0^{+\infty} \frac{\operatorname{sech}\kappa}{\kappa} \sin\kappa x \mathrm{d}\kappa = -2\arctan\left(\mathrm{e}^{-\pi x/2}\right) + \tfrac{1}{2}\pi \ (x < 0) \tag{A 4}$$

$$\int_0^{+\infty} \kappa^{2m+1} \operatorname{sech}\kappa \sin\kappa x \mathrm{d}\kappa = (-1)^{m+1} \frac{\pi}{2} \frac{\mathrm{d}^{2m+1}}{\mathrm{d}x^{2m+1}} \operatorname{sech}\tfrac{1}{2}\pi x, \tag{A 5}$$

it is clear that starting from the second term in (A 3), we get exponential decays. The first term can be written as

$$I_1(x) = \frac{C}{F^2-1}\left[\tfrac{1}{2}\pi - 2\arctan\mathrm{e}^{-\pi x/2}\right]. \tag{A 6}$$

But since

$$\arctan\theta - \tfrac{1}{2}\pi = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\theta^{-2n-1}}{2n+1} \tag{A 7}$$

for $\theta \gg 1$, we have

$$I_1(x) = \frac{C}{F^2-1}\left[-\tfrac{1}{2}\pi + \frac{2}{\theta} - \frac{2}{3\theta^3} + O(\theta^{-5})\right], \tag{A 8}$$

where $\theta = \mathrm{e}^{-\pi x/2} \gg 1$. The first term cancels the semi-residue in (A 1) and the remaining terms decay exponentially as $x \to -\infty$.

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