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A NOTE ON UNIFORM ORDERED SPACES

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Abstract

We characterize the generalized ordered topological spaces X for which the uniformity $\mathscr{C}(X)$ is convex. Moreover, we show that a uniform ordered space for which every compatible convex uniformity is totally bounded, need not be pseudocompact.

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We answer two questions of P. Fletcher and W. F. Lindgren. In [2, Problem F, page 94] they ask for necessary and sufficient conditions such that the uniformity $\mathscr{C}(X)$ of a GO space $(X, \mathscr{T}, \leqslant)$ is convex with respect to $\leqslant$. In this note we show that the uniformity $\mathscr{C}(X)$ of a GO spaces X is convex if and only if each closed discrete subset of X is countable. In [2, Problem I, p. 95] they ask whether a uniform ordered space $(X, \mathscr{T}, \leqslant)$ for which every convex uniformity compatible with $\mathscr{T}$ is totally bounded, is necessarily pseudocompact. They observe that the answer is positive, if $\leqslant$ is a linear order on X. In this note we show that the answer is negative in general. We will use the notation and the terminology of [2].

1.

In the first part of this note we will need the following lemma.

LEMMA. *Let $(X, \leqslant)$ be an uncountable linearly ordered set. Then there exists an uncountable subset A of X such that, if $a, b \in A$ and $a \leqslant b$, then there is a $c \in X \setminus A$ with $a \leqslant c \leqslant b$.*

PROOF. Denote by $[X]^3$ the set of the subsets of X with three elements. If $B \in [X]^3$, we denote the minimal element of B by L_B and the maximal element of B by R_B. Finally, M_B denotes the element of B such that $L_B < M_B < R_B$.

Case 1. There exists a countable subset D of X so that for each $B \in [X]^3$ there is a $d \in D$ such that $L_B \leqslant d \leqslant R_B$. Set $C = X \setminus D$ and $A = \{x \in X \mid x$ is the smallest element of a convexity-component of C in $X\}$. Since each convexity-component of C in X has at most two elements, A is an uncountable set that satisfies the condition of the lemma.

Case 2. For each countable subset D of X there is a $B \in [X]^3$ such that $D \cap \{x \in X \mid L_B \leqslant x \leqslant R_B\} = \varnothing$. Define by transfinite induction for each $\beta < \omega_1$ a set $B(\beta) \in [X]^3$: Suppose that $B(\alpha)$ has been defined for each $\alpha < \beta$. There is a set $B \in [X]^3$ such that $\{x \in X \mid L_B \leqslant x \leqslant R_B\}$ contains no element of the countable set $\bigcup\{B(\alpha) \mid \alpha < \beta\}$. Set $B(\beta) = B$. Then $A = \{M_{B(\beta)} \mid \beta < \omega_1\}$ is uncountable and satisfies the condition of the lemma.

It is known that each *GO* space is normal. In the next proof we will use the result that for a normal T_2-space X the uniformity $\mathscr{C}(X)$ is the finest compatible uniformity on X if and only if each locally finite open cover of X has a countable open refinement of finite order [3, Remark following the proof of the theorem; compare 1 and 2, p. 190, §5.28]. Recall that a topological space X is called ω_1-*compact*, if each closed discrete subset of X is countable.

PROPOSITION. *Let* $(X, \mathscr{T}, \leqslant)$ *be a GO space. Then the following conditions are equivalent*:

(a) X *is* ω_1-*compact*.

(b) $\mathscr{C}(X)$ *is the finest uniformity for* $(X, \mathscr{T})$.

(c) $\mathscr{C}(X)$ *is convex*.

PROOF. (a) → (b). Since X is ω_1-compact and $\dim X \leqslant 1$, every locally finite open cover of X is refined by a countable open refinement of finite order. We conclude that $\mathscr{C}(X)$ is the finest uniformity for $(X, \mathscr{T})$.

(b) → (c). Since the finest uniformity for a *GO* space is convex [2, Theorem 4.33], $\mathscr{C}(X)$ is convex.

(c) → (a). Let $\mathscr{C}(X)$ be convex. Assume that X is not ω_1-compact. Then X has an uncountable closed discrete subset A. By the lemma there exists an uncountable subset A' of A such that every subset of A' that is convex in A contains at most one point. Define a function f: $A \to R$ by $f(x) = 0$ if $x \in A'$, and $f(x) = 1$ if $x \in A \setminus A'$. Let g: $X \to R$ be continuous such that $g \mid A = f$. Let

$V \in \mathscr{C}(X)$ such that $V \subset \{(x, y) \in X \times X: |g(x) - g(y)| < \frac{1}{2}\}$ and $V(x)$ is convex in X for each $x \in X$. Since $V \in \mathscr{C}(X)$, there is a countable subset D of X such that $X = \bigcup\{V(d) \mid d \in D\}$. Clearly, each $V(d)$ contains at most one element of A'—a contradiction. We conclude that X is ω_1-compact.

EXAMPLE 1. Let R denote the set of the reals and let $\leqslant$ denote the usual order on R. Consider the GO space $(R, \mathscr{T}, \leqslant)$ where $\mathscr{T}$ denotes the discrete topology on R. Clearly $(R, \mathscr{T})$ is not ω_1-compact. Hence $\mathscr{C}(R)$ is not convex.

2.

We construct a uniform ordered space $(X, \mathscr{T}, \leqslant)$ such that every convex uniformity compatible with $(X, \mathscr{T}, \leqslant)$ is totally bounded, but X is not pseudocompact.

EXAMPLE 2. We use a modification of the well-known pseudocompact space ψ [see 4, 5I]. Let N be the set of the positive integers and let Γ be an infinite maximal almost disjoint family of infinite subsets of N. As usual set $\psi = N \cup \Gamma$. Let $(a_{2n-1})_{n \in N}$ be a sequence of pairwise different elements of Γ. Set $a_{2n} = -n$ for each $n \in N$, $A = \{a_{2n} \mid n \in N\}$, and $D = \{a_n \mid n \in N\}$. Let $X = N \cup \Gamma \cup A$. Consider the following collection of subsets of X:

$\mathscr{B} = \{\{n\} \mid n \in N\} \cup \{(E \setminus F) \cup \{E\} \mid E \in \Gamma \setminus D,\ F$ is a finite subset of $N\} \cup \{\bigcup_{n=1}^{k} G(a_n) \mid k \in N;\ G(a_{2m}) = \{a_{2m}\}$ (for each $m \in N$ such that $2m \leqslant k$); $G(a_{2m-1}) = (a_{2m-1} \setminus F) \cup \{a_{2m-1}\}$ where F is a finite subset of N (for each $m \in N$ such that $2m - 1 \leqslant k$)$\}$.

Set $\mathscr{S} = \{[X \times G] \cup [(X \setminus G) \times X] \mid G \in \mathscr{B}\}$. Then $\mathscr{S}$ is a subbase for a quasi-uniformity $\mathscr{U}$ on X. Consider the topology $\mathscr{T}(\mathscr{U}^*)$ on X where $\mathscr{U}^*$ denotes the uniformity generated by $\{V \cap V^{-1} \mid V \in \mathscr{U}\}$ on X. One easily checks that the points of the $\mathscr{T}(\mathscr{U}^*)$-open subspace ψ of X have their usual neighbourhoods. Moreover, each point of $X \setminus \psi$ is isolated in X. Hence $\mathscr{T}(\mathscr{U}^*)$ is not pseudocompact. Since $\mathscr{T}(\mathscr{U}^*)$ is a Hausdorff topology, $(X, \mathscr{T}(\mathscr{U}^*), \cap \mathscr{U})$ is a uniform ordered space [2, pages 81, 84]. Let $\mathscr{W}$ be a convex uniformity compatible with $(X, \mathscr{T}(\mathscr{U}^*), \cap \mathscr{U})$. Let $Z \in \mathscr{W}$. Since $\mathscr{W}$ is convex, there is a $Z' \in \mathscr{W}$ such that $Z' \subset Z$ and $Z'(x)$ is convex in X for each $x \in X$. Since ψ is a pseudocompact subspace of X, there is a finite subset F of ψ such that $\psi \subset Z'(F)$. Hence there is an $x \in F$ such that $Z'(x)$ contains infinitely many points of $D \setminus A$. Note that, if

$k, n \in N$ and $k \leqslant n$, then $(a_n, a_k) \in \cap \mathscr{U}$. Since $Z'(x)$ is convex in X, $Z'(x)$ contains all but finitely many points of D. We conclude that $\mathscr{W}$ is totally bounded.

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