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A family of rational maps with buried Julia components

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Abstract. It is known that the disconnected Julia set of any polynomial map does not contain buried Julia components. But such Julia components may arise for rational maps. The first example is due to Curtis T. McMullen who provided a family of rational maps for which the Julia sets are Cantor of Jordan curves. However, all known examples of buried Julia components, up to now, are points or Jordan curves and comes from rational maps of degree at least five. This paper introduces a family of hyperbolic rational maps with disconnected Julia set whose exchanging dynamics of postcritically separating Julia components is encoded by a weighted dynamical tree. Each of these Julia sets presents buried Julia components of several types: points, Jordan curves, but also Julia components which are neither points nor Jordan curves. Moreover, this family contains some rational maps of degree three with explicit formula that answers a question McMullen raised.

Contents

1. *Introduction*

For any rational map f of degree $d \geqslant 2$ on the Riemann sphere $\widehat{\mathbb{C}}$, we denote by $J(f)$ its Julia set, namely the closure of the set of repelling periodic points. We recall that $J(f)$ is a

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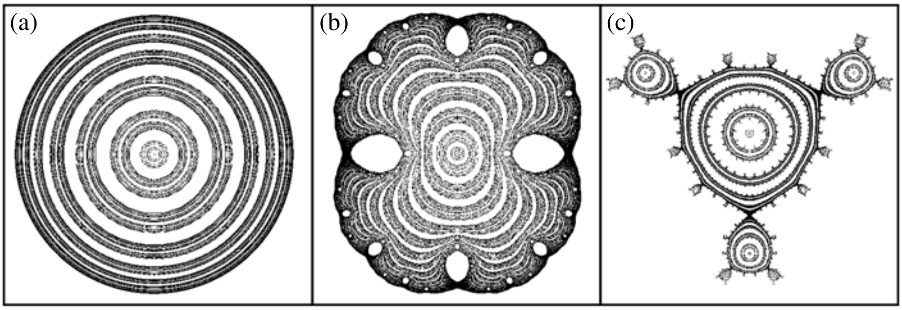

FIGURE 1. Diagrams showing (a) $J(g_{0,\lambda})$ for $d_\infty = 2$, $d_0 = 3$ and $\lambda \approx 10^{-9}$, (b) $J(\widetilde{g_{-1,\lambda}})$ for $d_\infty = 2$, $d_0 = 3$ and $\lambda \approx 10^{-9}$ and (c) $J(g_{c,\lambda})$ for $d_\infty = d_0 = 3$, $c = -i$ and $\lambda \approx 10^{-9}$.

fully invariant non-empty perfect compact set which either is connected or has uncountably many connected components (see [**Bea91**, **CG93**, **Mil06**]). This paper focuses on the disconnected case. Every connected component of $J(f)$ is called a Julia component and every connected component of the Fatou set $\widehat{\mathbb{C}} - J(f)$ is called a Fatou domain.

A Julia component is said to be *buried* if it has no intersection with the boundary of any Fatou domain. In particular, buried Julia components cannot occur in the polynomial case (since the Julia set coincides with the boundary of the unbounded Fatou domain). The same holds if the Julia set is a Cantor set, or more generally if the complementary of every Julia component is connected (since the Fatou set is then connected). That suggests much more sophisticated topological structures for Julia sets with some buried Julia components than those encountered in the polynomial case.

The first example of rational maps with buried Julia components is due to Curtis T. McMullen. Consider the family of rational maps given by

$$g_{c,\lambda} : z \mapsto z^{d_\infty} + c + \frac{\lambda}{z^{d_0}} \quad \text{where } d_\infty, d_0 \geqslant 1 \text{ and } c, \lambda \in \mathbb{C}.$$

The special case $c = 0$ has been studied in [**McM88**] (see also [**DHL08**]), where it is proved that if the following condition is satisfied

$$\frac{1}{d_\infty} + \frac{1}{d_0} < 1 \tag{H0}$$

and if $|\lambda| > 0$ is small enough, then $J(g_{0,\lambda})$ is a Cantor of Jordan curves, namely homeomorphic to the product of a Cantor set with a Jordan curve (see Figure 1(a)). Recall that any Cantor set is homeomorphic to the middle third set on a line segment which contains uncountably many points which are not endpoints of any removing open segment. Each of these points corresponds to a buried Jordan curve in $J(g_{0,\lambda})$.

In [**PT00**], the authors have provided another example by slightly modifying the map $g_{-1,\lambda}$ for $d_\infty = 2$ and $d_0 = 3$ (that satisfies assumption (H0)) in a clever way:

$$\widetilde{g_{-1,\lambda}} : z \mapsto \frac{1}{z} \circ (z^2 - 1) \circ \frac{1}{z} + \frac{\lambda}{z^3} = \frac{z^2}{1 - z^2} + \frac{\lambda}{z^3} \quad \text{where } \lambda \in \mathbb{C}.$$

If $|\lambda| > 0$ is small enough, then $J(\widetilde{g_{-1,\lambda}})$ has the same topological structure than $J(g_{0,\lambda})$ except that one fixed Julia component (which contains the boundary of the unbounded

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Fatou domain and, hence, is not buried) is quasiconformally homeomorphic to the Julia set of $z \mapsto z^2 - 1$. The uncountably many Julia components which are not eventually mapped under iteration onto this fixed Julia component are buried Jordan curves in $J(\widetilde{g_{-1,\lambda}})$ (see Figure 1(b)).

Examples of buried Jordan components which are not Jordan curves have appeared in some works. For instance, in [**BDGR08**] (see also [**GMR13**]), the authors have studied the family $g_{c,\lambda}$ for $d_\infty = d_0 \geqslant 3$ (that satisfies assumption (H0)) and for a fixed parameter c chosen so that for the polynomial $z \mapsto z^{d_\infty} + c$ the critical point zero lies in a cycle of period at least two. In that case, if $|\lambda| > 0$ is small enough, then $J(g_{c,\lambda})$ still has uncountably many Jordan curves as buried Jordan components but also uncountably many points. The remaining Julia components are eventually mapped under iteration onto a fixed Julia component (which coincides with the boundary of the unbounded Fatou domain and, hence, is not buried) quasiconformally homeomorphic to the Julia set of $z \mapsto z^{d_\infty} + c$. Each of these not buried Julia components has infinitely many 'decorations' and every buried point component is accumulated by a nested sequence of such decorations (see Figure 1(c)).

All of the previous examples are rational maps of degree $d_\infty + d_0$ at least five according to assumption (H0). The existence question of buried Julia components for rational maps of degree less than five has been raised in [**McM88**]. In the last decade, a number of papers have appeared that deal with subfamilies of $g_{c,\lambda}$ or some slightly perturbations of it. Some of them present sophisticated Julia sets with buried Julia components, however the degree of these examples is always at least equal to five. Furthermore, the buried Julia components of these examples are points or Jordan curves.

The aim of this paper is to answer the question McMullen has raised by providing a family of rational maps of degree three which does not come from the family $g_{c,\lambda}$ and whose Julia set presents buried Julia components of several types: points, Jordan curves but also Julia components which are neither points nor Jordan curves. One of our main results here is the following.

THEOREM 1. *Consider the family of cubic rational maps given by*

$$f_\lambda : z \mapsto \frac{(1-\lambda)[(1-4\lambda+6\lambda^2-\lambda^3)z - 2\lambda^3]}{(z-1)^2[(1-\lambda-\lambda^2)z - 2\lambda^2(1-\lambda)]} \quad \textit{where } \lambda \in \mathbb{C}.$$

If $|\lambda| > 0$ is small enough, then $J(f_\lambda)$ contains buried Julia components of several types.

(i) Point type: *uncountably many points.*

(ii) Circle type: *uncountably many Jordan curves.*

(iii) Complex type: *countably many preimages of a fixed Julia component which is quasiconformally homeomorphic to the connected Julia set of $f_0 : z \mapsto (1/(z-1)^2)$.*

An example of such a Julia set is depicted in Figure 2. Here $J(f_\lambda)$ is called a 'Persian carpet' because of similarities with sophistications from carpet-weaving art: the Julia set of $f_0 : z \mapsto (1/(z-1)^2)$ appears as a watermark in the central motif of the carpet whose surface is covered by an elaborate pattern of Cantor of Jordan curves, and there are some small Julia components everywhere that looks like dust. These small dusts contain nested sequences of finite coverings of the Persian carpet which accumulate buried point components.

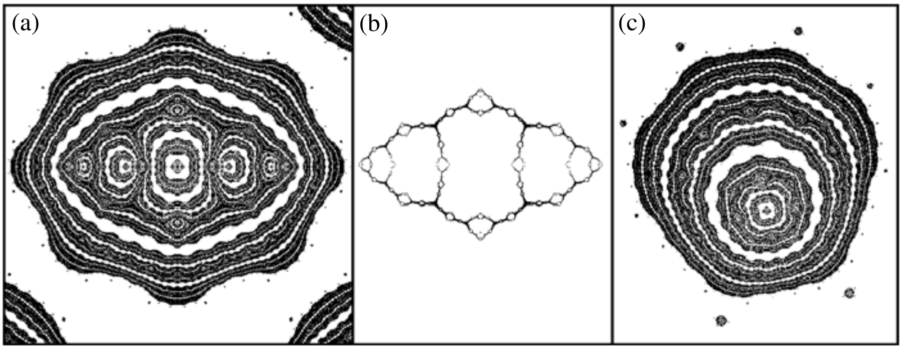


FIGURE 2. (a) A Persian carpet: $J(f_\lambda)$ for $\lambda \approx 10^{-3}$; (b) $J(f_0)$ which appears as a buried Julia component in $J(f_\lambda)$; (c) a magnification about a dust of the Persian carpet.

The Persian carpet example is maximal among rational maps with buried Julia components in the sense that buried Julia components cannot occur for rational maps of degree less than three. Indeed, by a theorem in [**Mil00**], the Julia set of any quadratic rational maps is either connected or a Cantor set.

Furthermore, the Persian carpet example is maximal among geometrically finite rational maps (namely rational maps such that every critical point in the Julia set is preperiodic, in our case f_λ is hyperbolic, namely it has no critical point in $J(f_\lambda)$ for $|\lambda| > 0$ small enough) in the sense that every Julia component (not necessarily buried) of such a map is one of the three types described in Theorem 1. That follows from two results. First, by a theorem in [**McM88**], every periodic Julia component of a rational map is either a point or quasiconformally homeomorphic to the connected Julia set of a rational map. Second, it has been proved in [**PT00**] that every Julia component of a geometrically finite rational map which is not eventually mapped under iteration onto a periodic Julia component is either a point or a Jordan curve.

The underlying idea in the construction of the Persian carpet example is that the sophisticated configuration on $\widehat{\mathbb{C}}$ of Julia components which are not points may be encoded by a tree. Tree structures have appeared in various works on holomorphic dynamics (for instance, Hubbard trees in [**DH84**] to classify postcritically finite polynomial maps). The tree considered here is not embedded in $\widehat{\mathbb{C}}$. It is seen as an abstract object which is very similar to, and actually inspired by, the trees introduced by Mitsuhiro Shishikura in [**Shi89**] which describe the configurations of Herman rings for rational maps.

However, the purpose of this paper is only to introduce a family of rational maps coming from a particular tree which answers the question McMullen has raised, but not to discuss the general existence question of rational maps whose configuration of Julia components is encoded by any given tree (that will be the purpose of future works) even if a general construction may be suggested (especially statements and discussions in §2).

1.1. *Organization of the paper.* Section 2 deals with exchanging dynamics of critically separating Julia components by weighted dynamical tree.

In §2.1, we specify the idea mentioned above by showing that, under assumption (H0), the exchanging dynamics of Julia components for the family $g_{0,\lambda}$ is encoded by a certain weighted dynamical tree $(\mathcal{H}_Q, w)$ (see Theorem 2).

The purpose of §2.2, is then to do the converse: starting from a particular dynamical tree $\mathcal{H}_P$ more sophisticated than $\mathcal{H}_Q$ and a weight function w on its edges, Theorem 3 states the existence of rational maps with disconnected Julia set whose exchanging dynamics of postcritically separating Julia components is encoded by $(\mathcal{H}_P, w)$ if (and, actually, only if) two conditions (H1) and (H2) hold. Theorem 4 shows that the Julia sets of these rational maps own buried Julia components of every expected type.

The main part of the proofs of Theorems 3 and 4, that is the construction by quasiconformal surgery of the required rational maps, is detailed in §3.

In §4, some properties of the rational maps constructed in the previous section are shown. The properties about exchanging dynamics (§4.1) conclude the proof of Theorem 3 while the properties about topology of some Julia components (§4.2) give the proof of Theorem 4.

Section 5 deals with a particular choice of the weight function w for which the two assumptions (H1) and (H2) are satisfied and such that the rational maps in Theorems 3 and 4 have degree three. In this case, an explicit formula is provided that concludes the proof of Theorem 1.

Finally, some technical results used in the construction of §3 are collected in Appendix A with references.

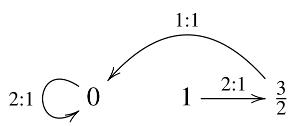
2. *Encoding by weighted dynamical trees*

For any rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, we denote by $\mathcal{J}(f)$ the set of Julia components and we recall that f induces a topological dynamical system on $\mathcal{J}(f)$ endowed with the usual distance between continua on $\widehat{\mathbb{C}}$ equipped with the spherical metric (note that $\mathcal{J}(f)$ is closed for this distance, that is not true in general for the Hausdorff distance). This topological dynamical system is called the *exchanging dynamics of Julia components*.

We recall that the critical points of f are the points where f is not locally injective, and the postcritical points of f are the points of the form $f^n(c)$ for some $n \geqslant 1$ and for some critical point c. A Julia component $J \in \mathcal{J}(f)$ is said to be critically separating if J separates the postcritical set of f, or equivalently if $\widehat{\mathbb{C}} - J$ has at least two connected components containing at least one postcritical point of f each. We denote by $\mathcal{J}_{\text{crit}}(f)$ the subset of critically separating Julia components in $\mathcal{J}(f)$. Note that $\mathcal{J}_{\text{crit}}(f)$ is forward invariant, and thus f induces a topological dynamical system on $\mathcal{J}_{\text{crit}}(f)$.

2.1. *McMullen's example.*

Consider the cubic polynomial $Q : z \mapsto 3z^2(3/2 - z)$. It has two simple critical points: 0 which is fixed, and 1 which is mapped on 0 after two iterations:



Let $\mathcal{H}_Q$ be its Hubbard tree, namely the smallest closed connected infinite union of internal rays which contains the postcritical set $\{0, 3/2\}$ (see [**DH84**]). In fact, $\mathcal{H}_Q$ is the straight real segment $[0, 3/2]$ or more precisely the union of two edges $[0, 1] \cup [1, 3/2]$ while the vertices are 0, 1 and 3/2. Both edges of $\mathcal{H}_Q$ are homeomorphically mapped by Q onto the whole tree (see Figure 3(b)).

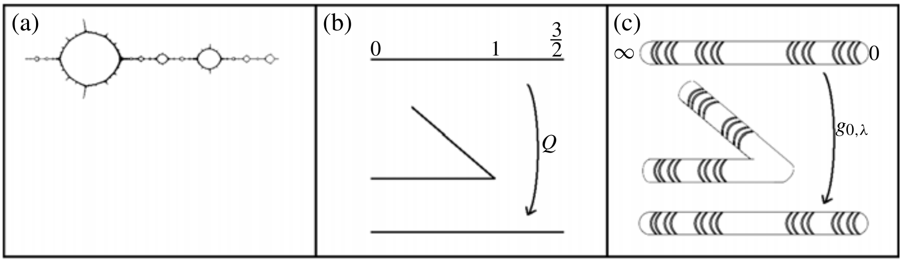


FIGURE 3. (a) The Julia set of the polynomial Q. (b) The action of Q on the Hubbard tree $\mathcal{H}_Q$. (c) The action of $g_{0,\lambda}$ on the set of Julia components $\mathcal{J}(g_{0,\lambda})$.

Denote by $\mathcal{J}(\mathcal{H}_Q)$ the intersection set between the Hubbard tree $\mathcal{H}_Q$ and the Julia set $J(Q)$. Note that $\mathcal{J}(\mathcal{H}_Q)$ is disconnected (actually a Cantor set) and Q induced a dynamical system on it since the Hubbard tree $\mathcal{H}_Q$ and the Julia set $J(Q)$ are both invariant.

Finally, let w be a weight function on the set of edges of $\mathcal{H}_Q$, say $w([0, 1]) = d_\infty$ and $w([1, 3/2]) = d_0$ where d_∞ and d_0 are positive integers.

The result about the family $g_{0,\lambda}$ discussed in the introduction (see §1) may be reformulated as follows.

THEOREM 2. *If the weighted dynamical tree $(\mathcal{H}_Q, w)$ satisfies the following condition*

$$\frac{1}{d_\infty} + \frac{1}{d_0} < 1, \tag{H0}$$

then for every $|\lambda| > 0$ small enough, the exchanging dynamics of Julia component of $g_{0,\lambda}$ is encoded by $(\mathcal{H}_Q, w)$ in the following sense:

(i) *every critical orbit accumulates the super-attracting fixed point ∞;*

(ii) *there exists a homeomorphism $h : \mathcal{J}(g_{0,\lambda}) \to \mathcal{J}(\mathcal{H}_Q)$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{J}(g_{0,\lambda}) & \xrightarrow{\;g_{0,\lambda}\;} & \mathcal{J}(g_{0,\lambda}) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{J}(\mathcal{H}_Q) & \xrightarrow[\;Q\;]{} & \mathcal{J}(\mathcal{H}_Q) \end{array}$$

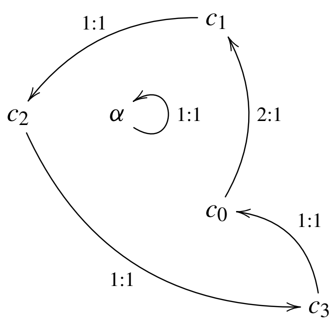
(iii) *for every Julia component $J \in \mathcal{J}(g_{0,\lambda})$, the restriction map $g_{0,\lambda}|_J$ has degree $w(e)$ where e is the edge of $\mathcal{H}_Q$ which contains $h(J)$.*

Note that $\mathcal{J}(g_{0,\lambda}) = \mathcal{J}_{\text{crit}}(g_{0,\lambda})$ for $|\lambda| > 0$ small enough since every Julia component is a Jordan curve which separates the fixed critical point ∞ from some critical values close to 0.

Proof. We only sketch the proof since the main part is done in [**McM88**]. Indeed it is shown that there exists a large annulus A centered at 0 and containing $J(g_{0,\lambda})$ whose preimage consists of two disjoint annuli A_∞, A_0 both nested in A and such that the restriction maps $g_{0,\lambda}|A_\infty : A_\infty \to A$ and $g_{0,\lambda}|A_0 : A_0 \to A$ are coverings of degree d_∞ and d_0, respectively. Using combinatorial reasoning from holomorphic dynamics, it is a classical exercise to prove that the set of connected components of $J(g_{0,\lambda}) = \bigcap_{n\geqslant 0} g_{0,\lambda}^{-n}(A)$ is homeomorphic to the space of all sequences of two digits $\Sigma_2 = \{0, 1\}^{\mathbb{N}}$ (equipped with the product topology making it a Cantor set) and the exchanging dynamics is topologically conjugated to a two-to-one shift map $\sigma : \Sigma_2 \to \Sigma_2$ defined by $\sigma(s_0, s_1, s_2, \ldots) = (s_1, s_2, s_3, \ldots)$. The same holds for the dynamical system induced by Q on $\mathcal{J}(\mathcal{H}_Q)$ since for $\varepsilon > 0$ small enough the real segment $I = [\varepsilon, 3/2 - \varepsilon]$ contains $\mathcal{J}(\mathcal{H}_Q)$ and its preimage consists of two disjoint real segment both included in I (one in each of the two edges of $\mathcal{H}_Q$). □

Heuristically speaking, we may topologically think of the Riemann sphere $\widehat{\mathbb{C}}$ as a smooth neighborhood's boundary of the tree $\mathcal{H}_Q$ embedded in the space $\mathbb{R}^3$. The two points on this topological sphere which correspond to ∞ and 0 should be closed to the corresponding vertices of $\mathcal{H}_Q$ which are 0 and 3/2, respectively. If the neighborhood becomes smaller and smaller, every Jordan curve in $J(g_{0,\lambda})$ is shrunk to a point in $J(\mathcal{H}_Q)$ (see Figure 3(c)).

2.2. *Persian carpet example.* Consider a quadratic polynomial of the form $P : z \mapsto z^2 + c$ where the parameter $c \in \mathbb{C}$ is chosen in order that the critical point 0 is periodic of period four. There are exactly six choices of such a parameter. Let us fix c to be that one with the largest imaginary part, that is $c \approx -0.157 + 1.032i$. The postcritical points are denoted by $c_k = P^k(0)$ for every $k \in \{0, 1, 2, 3\}$:



Let $\mathcal{H}_P$ be the Hubbard tree of P (see Figure 4(b)). As one-dimensional simplicial complex, $\mathcal{H}_P$ may be described by a set of five vertices $\{c_0, c_1, c_2, c_3, \alpha\}$ where α is a fixed point of P and the following four edges:

$$e_0 = [\alpha, c_0]_{\mathcal{H}_P}; \quad e_1 = [\alpha, c_1]_{\mathcal{H}_P}; \quad e_2 = [\alpha, c_2]_{\mathcal{H}_P}; \quad e_3 = [c_0, c_3]_{\mathcal{H}_P}.$$

Here P homeomorphically acts on the edges as follows:

$$\begin{cases} P(e_0) = e_1, \\ P(e_1) = e_2, \\ P(e_2) = e_0 \cup e_3, \\ P(e_3) = e_0 \cup e_1. \end{cases}$$

Denote by $\mathcal{J}(\mathcal{H}_P)$ the intersection set between the Hubbard tree $\mathcal{H}_P$ and the Julia set $J(P)$. Note that $\mathcal{J}(\mathcal{H}_P)$ is disconnected (actually a Cantor set) and P induced a dynamical

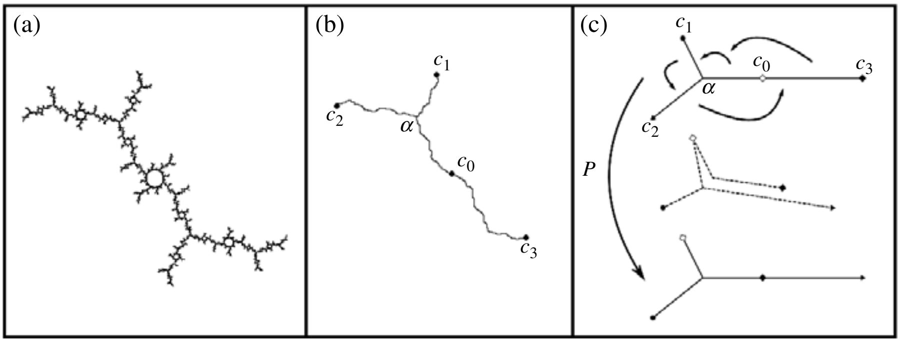


FIGURE 4. (a) The Julia set of the polynomial P. (b) The Hubbard tree $\mathcal{H}_P$. (c) The action of P on a straightened copy of $\mathcal{H}_P$.

system on it. Moreover, the fixed branching point α belongs to $\mathcal{J}(\mathcal{H}_P)$ but not to the boundary of any connected component of $\mathcal{H}_P - \mathcal{J}(\mathcal{H}_P)$. Finally, let w be a weight function on the set of edges of $\mathcal{H}_P$, say $w(e_k) = d_k$ where d_k is a positive integer for every $k \in \{0, 1, 2, 3\}$.

Definition 1. The transition matrix of the weighted dynamical tree $(\mathcal{H}_P, w)$ is the four-by-four matrix $M = (m_{i,j})_{i,j\in\{0,1,2,3\}}$ whose entries are defined as follows:

$$\text{for all } i, j \in \{0, 1, 2, 3\}, \quad m_{i,j} = \begin{cases} \dfrac{1}{w(e_i)} & \text{if } e_j \subset P(e_i), \\ 0 & \text{otherwise.} \end{cases}$$

Since M is a non-negative matrix, it follows from the Perron–Frobenius theorem that the eigenvalue with the largest modulus is real and non-negative. Let us call $\lambda(\mathcal{H}_P, w)$ this leading eigenvalue. The weighted dynamical tree $(\mathcal{H}_P, w)$ is said to be *unobstructed* if $\lambda(\mathcal{H}_P, w) < 1$.

Let us give some remarks about this definition.

(i) This definition is strongly related to obstructions which occur in Thurston characterization of postcritically finite rational maps and all of the related theory (see **[DH93]**).

(ii) When $(\mathcal{H}_P, w)$ is unobstructed, the Perron–Frobenius theorem and continuity of the spectral radius ensure the existence of a vector $V \in \mathbb{R}^4$ with positive entries such that $MV < V$. This remark will be useful later.

(iii) Actually the transition matrix of $(\mathcal{H}_P, w)$ is given by

$$M = \begin{pmatrix} 0 & \dfrac{1}{d_0} & 0 & 0 \\ 0 & 0 & \dfrac{1}{d_1} & 0 \\ \dfrac{1}{d_2} & 0 & 0 & \dfrac{1}{d_2} \\ \dfrac{1}{d_3} & \dfrac{1}{d_3} & 0 & 0 \end{pmatrix}$$

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and an easy computation shows that $\lambda(\mathcal{H}_P, w)$ is the largest root of

$$X^4 - \left(\frac{1}{d_0 d_1 d_2} + \frac{1}{d_1 d_2 d_3}\right) X - \frac{1}{d_0 d_1 d_2 d_3}.$$

Note that if $\lambda(\mathcal{H}_P, w) \geqslant 1$, then $\lambda(\mathcal{H}_P, w) \leqslant 1/(d_0 d_1 d_2) + 1/(d_1 d_2 d_3) + 1/(d_0 d_1 d_2 d_3)$, thus $(\mathcal{H}_P, w)$ is unobstructed as soon as at least three of weights d_0, d_1, d_2, and d_3 are $\geqslant 2$. Conversely, if $(\mathcal{H}_P, w)$ is unobstructed then one can show by exhaustion that at least two of weights d_0, d_1, d_2, and d_3 are $\geqslant 2$.

(iv) For McMullen's example, the transition matrix of $(\mathcal{H}_Q, w)$ may be defined as well and we get

$$M = \begin{pmatrix} \frac{1}{d_\infty} & \frac{1}{d_\infty} \\ \frac{1}{d_0} & \frac{1}{d_0} \end{pmatrix}.$$

An easy computation gives that $\lambda(\mathcal{H}_Q, w) = 1/d_\infty + 1/d_0$. Consequently the weighted dynamical tree $(\mathcal{H}_Q, w)$ is unobstructed if and only if the assumption (H0) holds.

The following result is analogous to Theorem 2.

THEOREM 3. *If the weighted dynamical tree $(\mathcal{H}_P, w)$ satisfies the following conditions*

$$\widehat{d} = \tfrac{1}{2}(d_0 + d_1 + d_2 - 1) \text{ is an integer} \geqslant 2 \quad \text{and} \quad \max\{d_0, d_1, d_2\} \leqslant \widehat{d} \tag{H1}$$

$$(\mathcal{H}_P, w) \text{ is unobstructed,} \tag{H2}$$

then there exists a rational map f of degree $\widehat{d} + d_3$ such that the exchanging dynamics of postcritically separating Julia components of f is encoded by $(\mathcal{H}_P, w)$ in the following sense:

(i) *every critical orbit accumulates a super-attracting cycle $\{z_0, z_1, z_2, z_3\}$ of period four;*

(ii) *there exists a homeomorphism $h : \mathcal{J}_{\text{crit}}(f) \to \mathcal{J}(\mathcal{H}_P)$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{J}_{\text{crit}}(f) & \xrightarrow{\ f\ } & \mathcal{J}_{\text{crit}}(f) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{J}(\mathcal{H}_P) & \xrightarrow[\ P\ ]{} & \mathcal{J}(\mathcal{H}_P) \end{array}$$

(iii) *for every Julia component $J \in \mathcal{J}_{\text{crit}}(f)$ such that $h(J)$ is not eventually mapped under iteration to the fixed branching point α, the restriction map $f|_J$ has degree $w(e_k) = d_k$ where e_k is the edge of $\mathcal{H}_P$ which contains $h(J)$.*

The same heuristic as for Theorem 2 still holds: we may topologically think the Riemann sphere $\widehat{\mathbb{C}}$ as a smooth neighborhood's boundary of the tree $\mathcal{H}_P$ embedded in the space $\mathbb{R}^3$. The action of f on this topological sphere follows that of the dynamical tree $\mathcal{H}_P$. The points on this topological sphere which correspond to the points in the super-attracting periodic cycle $\{z_0, z_1, z_2, z_3\}$ should be close to the corresponding vertices $\{c_0, c_1, c_2, c_3\}$ of $\mathcal{H}_P$, and every Julia component in $\mathcal{J}_{\text{crit}}(f)$ closely surrounds a corresponding point in $\mathcal{J}(\mathcal{H}_P)$.

THEOREM 4. *Under assumptions (H1) and (H2) there exists a rational map f satisfying Theorem 3 and such that $J(f)$ contains buried Julia components of several types.*

(i) Point type: *uncountably many points.*

(ii) Circle type: *uncountably many Jordan curves.*

(iii) Complex type: *countably many preimages of a fixed Julia component lying over the fixed branching point α, say $J_\alpha = h^{-1}(\alpha) \in \mathcal{J}(f)$, which is quasiconformally homeomorphic to the connected Julia set of a rational map $\widehat{f}$.*

Moreover, $\widehat{f}$ has degree $\widehat{d}$ and has only one critical orbit which is a super-attracting cycle $\{\widehat{z_0}, \widehat{z_1}, \widehat{z_2}\}$ of period three such that the local degree of $\widehat{f}$ at $\widehat{z_k}$ is d_k for every $k \in \{0, 1, 2\}$.

Let us give some comments about these results.

(i) The rational map $\widehat{f}$ corresponds to the dynamics of f on the fixed Julia component J_α lying over the fixed branching point α. More precisely, there is a quasiconformal map φ from a neighborhood of $J(\widehat{f})$ onto a neighborhood of J_α such that $\varphi \circ \widehat{f} = f \circ \varphi$ (see the construction of f in §3).

(ii) The rational map $\widehat{f}$ may also be seen as encoded by a weighted dynamical tree. Consider the quadratic polynomial $R : z \mapsto z^2 + \widehat{c}$ where $\widehat{c} \in \mathbb{C}$ is the parameter with the largest imaginary part such that the critical point 0 is periodic of period three, that is $\widehat{c} \approx -0.123 + 0.745i$ ($J(R)$ is known as the Douady's rabbit). The Hubbard tree $\mathcal{H}_R$ of R is described by a set of four vertices $\{\widehat{c_0}, \widehat{c_1}, \widehat{c_2}, \widehat{\alpha}\}$ where $\widehat{c_k} = R^k(0)$ and $\widehat{\alpha}$ is a fixed point of R, and three edges of the form $\widehat{e_k} = [\widehat{\alpha}, \widehat{c_k}]_{\mathcal{H}_R}$ for every $k \in \{0, 1, 2\}$. Consider the weight function w defined by $w(\widehat{e_k}) = d_k$ for every $k \in \{0, 1, 2\}$. Then the weighted dynamical tree $(\mathcal{H}_R, w)$ encodes the action of $\widehat{f}$ in the same setting as in Theorems 2 and 3. Note that the intersection set between $\mathcal{H}_R$ and $J(R)$ is reduced to $\mathcal{J}(\mathcal{H}_R) = \{\widehat{\alpha}\}$, that corresponds to the unique Julia component in $\mathcal{J}(\widehat{f}) = \mathcal{J}_{\text{crit}}(\widehat{f}) = \{J(\widehat{f})\}$. Finally, note that the weighted dynamical tree $(\mathcal{H}_R, w)$ is unobstructed as soon as assumption (H1) holds (actually $\lambda(\mathcal{H}_R, w) = 1/(d_0 d_1 d_2)$).

(iii) The rational map $\widehat{f}$ is unique up to conjugation by a Möbius map or, equivalently, it is unique as soon as its critical orbit $\{\widehat{z_0}, \widehat{z_1}, \widehat{z_2}\}$ is fixed in $\widehat{\mathbb{C}}$ (see Lemma 1). However, the rational map f is not unique since the critical points which do not belong to the super-attracting periodic cycle $\{z_0, z_1, z_2, z_3\}$ (but whose orbits accumulate it) may be perturbed in some neighborhoods without changing the exchanging dynamics and the topology of Julia components.

(iv) The rational map f is not postcritically finite since $J(f)$ is disconnected (but it is hyperbolic from point (i) in Theorem 3). In particular, Thurston characterization of postcritically finite rational maps (see [**DH93**]) cannot be used to prove the existence of f. However, one could use the works of Tan Lei and Cui Guizhen about sub-hyperbolic semi-rational maps in [**CT11**] but this paper presents a more explicit and more constructive method by quasiconformal surgery (see §3).

(v) The assumption (H1) is necessary. Indeed it is the smallest requirement such that there exists a topological model for $\widehat{f}$, that is a branched covering combinatorially equivalent to $\widehat{f}$ (see Lemma A.1 and the proof of Lemma 1).

(vi) The assumption (H2) is necessary. Otherwise we can find a Thurston obstruction, that is to say a multicurve Γ whose transition matrix is equal to M with leading eigenvalue $\lambda(\Gamma) = \lambda(\mathcal{H}_P, w) \geqslant 1$. According to a result of McMullen in [**McM94**]

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it follows that $\lambda(\Gamma) = 1$ and at least one curve in Γ is contained in an union of Fatou domains where f is biholomorphically conjugated to a rotation. That is a contradiction since every critical orbit of f accumulates a super-attracting periodic cycle.

3. *Construction*

The aim of this section is to construct by quasiconformal surgery (we refer readers to [**BF13**] for a comprehensive treatment on this powerful method) a rational map f which satisfies Theorems 3 and 4. The strategy is to start from a rational map $\widehat{f}$ whose Julia set corresponds to the branching point α in $\mathcal{H}_P$ (see Theorem 4) and then to modify this map in order to create a folding corresponding to the critical point c_0.

3.1. *The branching map $\widehat{f}$.*

The first step of the construction is to prove the existence of the rational map $\widehat{f}$ which appears in Theorem 4. This is done by Lemma 1 below.

LEMMA 1. *If assumption (H1) holds, then there exists a rational map $\widehat{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $\widehat{d}$ such that:*

(i) *$\widehat{f}$ has only one critical orbit which is a super-attracting cycle $\{\widehat{z_0}, \widehat{z_1}, \widehat{z_2}\}$ of period three such that the local degree of $\widehat{f}$ at $\widehat{z_k}$ is d_k for every $k \in \{0, 1, 2\}$;*

(ii) *$J(\widehat{f})$ is connected and the Fatou set $\widehat{\mathbb{C}} - J(\widehat{f})$ has infinitely many connected components which are simply connected.*

Moreover, $\widehat{f}$ is unique up to conjugation by a Möbius map.

There are many ways to prove the existence of $\widehat{f}$ (for instance, by 'blowing up' the edges of some triangle invariant by a Möbius map, see [**PT98**]). Here we give a simple proof provided a particular solution of the Hurwitz problem (see Appendix A).

Proof. Up to conjugation by a Möbius map, we may fix three distinct points $\widehat{z_0}$, $\widehat{z_1}$, and $\widehat{z_2}$ in $\widehat{\mathbb{C}}$. Note that if at least one of the integers d_0, d_1, and d_2 is equal to one, say $d_0 = 1$, then assumption (H1) leads to $d_1 = d_2 = \widehat{d}$ and the rational map $\widehat{f} = \varphi \circ (z \mapsto z^{\widehat{d}}) \circ \widetilde{\varphi}^{-1}$ where φ and $\widetilde{\varphi}$ are two Möbius maps such that

$$\begin{aligned} &\widetilde{\varphi}(1) = \widehat{z_0}, \quad \widetilde{\varphi}(0) = \widehat{z_1}, \quad \widetilde{\varphi}(\infty) = \widehat{z_2}, \\ \text{and} \quad &\varphi(1) = \widehat{z_1}, \quad \varphi(0) = \widehat{z_2}, \quad \varphi(\infty) = \widehat{z_2}, \end{aligned}$$

satisfies (i). Consequently, we may assume that d_0, d_1, and d_2 are $\geqslant 2$.

If follows that we may apply Lemma A.1 since assumption (H1) easily implies condition (H1$'$) for the abstract branch data coming from $d = \widehat{d}$, and $d_{i,1} = d_{i-1}$ for every $i \in \{1, 2, 3\}$. We get a degree $\widehat{d}$ branched covering $H : \mathbb{S}^2 \to \mathbb{S}^2$ and three distinct points $x_{1,1}$, $x_{2,1}$, and $x_{3,1}$ in $\mathbb{S}^2$ such that the local degree of H at $x_{i,1}$ is d_{i-1} for every $i \in \{1, 2, 3\}$ and H has no more critical points than $x_{1,1}$, $x_{2,1}$, and $x_{3,1}$. Let $\varphi : \mathbb{S}^2 \to \widehat{\mathbb{C}}$ be any homeomorphism such that $\varphi(H(x_{i,1})) = \widehat{z_i}$ for every $i \in \{1, 2, 3\}$. Note that the map $\varphi \circ H : \mathbb{S}^2 \to \widehat{\mathbb{C}}$ induces a complex structure on $\mathbb{S}^2$. In other words, the uniformization theorem gives a homeomorphism $\widetilde{\varphi} : \mathbb{S}^2 \to \widehat{\mathbb{C}}$ such that the map $\widehat{f} = \varphi \circ H \circ \widetilde{\varphi}^{-1}$ is holomorphic on $\widehat{\mathbb{C}}$ and thus a rational map of degree $\widehat{d}$. Moreover, up to postcomposition with a Möbius map, we may assume that $\widetilde{\varphi}(x_{i,1}) = \widehat{z_{i-1}}$ for every $i \in \{1, 2, 3\}$ so that $\widehat{f}$ satisfies (i).

Now note that for every $k \in \{0, 1, 2\}$, the connected component containing $\widehat{z_k}$ of the super-attracting basin of $\widehat{f}$ is simply connected since it contains at most one critical point. Moreover, any other Fatou component is eventually mapped by homeomorphisms onto one of these simply connected components. It follows that $\widehat{f}$ satisfies (ii).

Finally let $\widehat{g}$ be another rational map of degree $\widehat{d}$ which satisfies (i) and (ii) for the same super-attracting periodic cycle $\{\widehat{z_0}, \widehat{z_1}, \widehat{z_2}\}$. Then $z \mapsto \widehat{f}(z) - \widehat{g}(z)$ is a rational map of degree at most $2\widehat{d}$ for which 0 has at least $d_0 + d_1 + d_2 = 2\widehat{d} + 1$ preimages counted with multiplicity (every $\widehat{z_k}$ is a preimage of 0 with multiplicity d_k). Consequently this map is identically equal to 0, that is $\widehat{f} = \widehat{g}$. □

Note that the previous proof strongly uses the fact that the postcritical set contains only three points. Indeed if the postcritical set contains more than three points, there is still a uniformization map $\widetilde{\varphi}$ for $\mathbb{S}^2$ equipped with the complex structure coming from $\varphi \circ H$, but that may not be possible to postcompose $\widetilde{\varphi}$ with a Möbius map so that $\widehat{f}$ satisfies (i). In fact, we would also need to check that the branched covering H has no Thurston obstructions (see [**DH93**]).

3.2. *Cutting along a system of equipotentials.* Starting with the map $\widehat{f}$ coming from Lemma 1, we need to divide $\widehat{\mathbb{C}}$ into several pieces on which the map f (or more precisely a quasiregular map F) will be piecewisely defined. This partition comes from a certain system of equipotentials of $\widehat{f}$ defined in Lemma 2 below.

For every $k \in \{0, 1, 2\}$, denote by $B(\widehat{z_k})$ the connected component containing $\widehat{z_k}$ of the super-attracting basin of $\widehat{f}$. Recall that each $B(\widehat{z_k})$ is a marked hyperbolic disk. More precisely, Böttcher's theorem provides Riemann mappings $\phi_k : \mathbb{D} \to B(\widehat{z_k})$ (namely biholomorphic maps from the open unit disk $\mathbb{D}$ onto $B(\widehat{z_k})$ such that $\phi_k(0) = \widehat{z_k}$ and the following diagram commutes:

$$\begin{array}{ccc}
B(\widehat{z_0}) & \xleftarrow{\ \phi_0\ } & \mathbb{D} \\
\downarrow \widehat{f} & & \downarrow z \mapsto z^{d_0} \\
B(\widehat{z_1}) & \xleftarrow{\ \phi_1\ } & \mathbb{D} \\
\downarrow \widehat{f} & & \downarrow z \mapsto z^{d_1} \\
B(\widehat{z_2}) & \xleftarrow{\ \phi_2\ } & \mathbb{D} \\
\downarrow \widehat{f} & & \downarrow z \mapsto z^{d_2} \\
B(\widehat{z_0}) & \xleftarrow{\ \phi_0\ } & \mathbb{D}
\end{array}$$

Recall that an equipotential β in any $B(\widehat{z_k})$ is the image by ϕ_k of an Euclidean circle in $\mathbb{D}$ centered at 0. The radius of this circle is called the level of β and is denoted by $L_k(\beta) \in]0, 1[$, in order that $\beta = \{z \in B(\widehat{z_k}) \mid |\phi_k^{-1}(z)| = L_k(\beta)\}$.

Recall that any pair of disjoint continua β, β' in $\widehat{\mathbb{C}}$ uniquely defines an open annulus in $\widehat{\mathbb{C}}$ denoted by $A(\beta, \beta')$. If β, β' contain at least two points each, $A(\beta, \beta')$ is biholomorphic

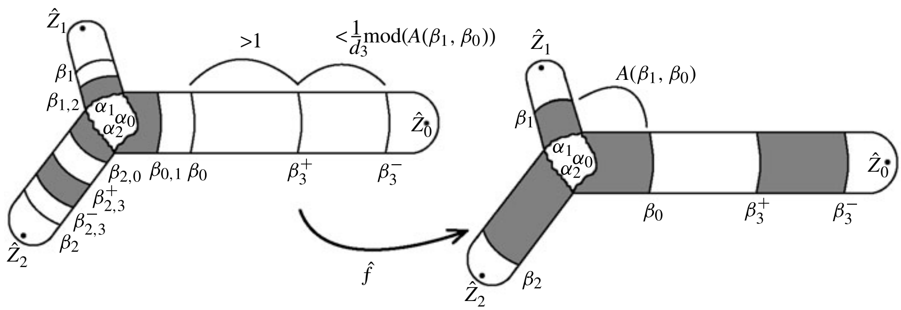

FIGURE 5. The pattern of the equipotentials (and their preimages) coming from Lemma 2 displayed on the Riemann sphere which is topologically distorted to emphasize the three domains $B(\widehat{z_0})$, $B(\widehat{z_1})$, and $B(\widehat{z_2})$ (compare with Figure 4(c)).

to a round annulus of the form $A_r = \{z \in \mathbb{C} \mid r < |z| < 1\}$ where $r \in]0, 1[$ only depends on $A(\beta, \beta')$. The modulus of $A(\beta, \beta')$ is defined to be $\mathrm{mod}(A(\beta, \beta')) = (1/2\pi)\log(1/r)$. In particular, if β, β' are two equipotentials in the same domain $B(\widehat{z_k})$ of levels $L_k(\beta) > L_k(\beta')$, then

$$\mathrm{mod}(A(\beta, \beta')) = \frac{1}{2\pi}\log\left(\frac{L_k(\beta)}{L_k(\beta')}\right).$$

Finally for every $k \in \{0, 1, 2\}$, denote by α_k the compact connected subset of $J(\widehat{f})$ which corresponds to the boundary of $B(\widehat{z_k})$.

LEMMA 2. *If assumption (H2) holds, then there exist three equipotentials β_0 in $B(\widehat{z_0})$, β_1 in $B(\widehat{z_1})$, and β_2 in $B(\widehat{z_2})$, together with two equipotentials β_3^+ and β_3^- in $B(\widehat{z_0})$ such that*

$$L_0(\beta_0) > L_0(\beta_3^+) > L_0(\beta_3^-)$$

and the following linear system of inequalities holds:

$$\begin{cases} \frac{1}{d_0}\,\mathrm{mod}(A(\alpha_1, \beta_1)) < \mathrm{mod}(A(\alpha_0, \beta_0)), \\ \frac{1}{d_1}\,\mathrm{mod}(A(\alpha_2, \beta_2)) < \mathrm{mod}(A(\alpha_1, \beta_1)), \\ \frac{1}{d_2}\,\mathrm{mod}(A(\alpha_0, \beta_0)) + \frac{1}{d_2}\,\mathrm{mod}(A(\beta_3^+, \beta_3^-)) < \mathrm{mod}(A(\alpha_2, \beta_2)), \\ \frac{1}{d_3}\,\mathrm{mod}(A(\beta_1, \beta_0)) < \mathrm{mod}(A(\beta_3^+, \beta_3^-)), \\ \textit{and} \quad \mathrm{mod}(A(\beta_0, \beta_3^+)) > 1. \end{cases} \tag{1}$$

Recall that the modulus is a conformal invariant, or more precisely if there is a holomorphic covering of degree d from an open annulus A onto another one A', then $\mathrm{mod}(A) = 1/d\,\mathrm{mod}(A')$. Hence, the first three inequalities in linear system (1) implies that the preimages under $\widehat{f}$ of these equipotentials are arranged as shown in Figure 5. The fourth inequality will allow us to realize the preimage of the branching point α in $\mathcal{H}_P$ (see Lemma 4) while the last inequality ensures sufficient space to realize the folding corresponding to the critical point c_0 (see Lemma 3).

The key point of the proof needs an inverse Grötzch's inequality due to Cui Guizhen and Tan Lei (see Appendix A).

Proof. Let $C > 0$ be the constant coming from Lemma A.2 for the marked hyperbolic disks $B(\widehat{z_0})$, $B(\widehat{z_1})$. Thus, for every pair of equipotentials β_0 in $B(\widehat{z_0})$ and β_1 in $B(\widehat{z_1})$, we have

$$\frac{1}{d_3}\operatorname{mod}(A(\beta_1, \beta_0)) \leqslant \frac{1}{d_3}(\operatorname{mod}(A(\alpha_0, \beta_0)) + \operatorname{mod}(A(\alpha_1, \beta_1)) + C).$$

Now consider the following linear system of inequations with real unknowns x_0, x_1, x_2, x_3:

$$\begin{cases} \dfrac{1}{d_0}x_1 < x_0, \\ \dfrac{1}{d_1}x_2 < x_1, \\ \dfrac{1}{d_2}x_0 + \dfrac{1}{d_2}x_3 < x_2, \\ \dfrac{1}{d_3}(x_0 + x_1 + C) < x_3. \end{cases} \tag{2}$$

Using the transition matrix M coming from Definition 1, this system is equivalent to

$$MX + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{C}{d_3} \end{pmatrix} < X \quad \text{where } X = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Recall that assumption (H2) states that the leading eigenvalue $\lambda(\mathcal{H}_P, w)$ of M is less than one. The existence of a vector $V \in \mathbb{R}^4$ with positive entries such that $MV < V$ follows from the Perron–Frobenius theorem and the continuity of the spectral radius. Now taking $\mu > 0$ large enough (for instance, $\mu = ((C/d_3) + 1)(v_3 - (1/d_3)v_0 - (1/d_3)v_1)^{-1}$), the vector $X = \mu V$ with positive entries solves the linear system of inequations (2).

The equipotentials $\beta_0, \beta_1, \beta_2$ are uniquely defined by

$$\frac{1}{2\pi}\log\left(\frac{1}{L_k(\beta_k)}\right) = \operatorname{mod}(A(\alpha_k, \beta_k)) = x_k \quad \text{for every } k \in \{0, 1, 2\}.$$

For β_3^+, choose an arbitrary equipotential in $B(\widehat{z_0})$ such that

$$L_0(\beta_0) > L_0(\beta_3^+) \quad \text{and} \quad \frac{1}{2\pi}\log\left(\frac{L_0(\beta_0)}{L_0(\beta_3^+)}\right) = \operatorname{mod}(A(\beta_0, \beta_3^+)) > 1.$$

Then β_3^- is uniquely defined by

$$L_0(\beta_3^+) > L_0(\beta_3^-) \quad \text{and} \quad \frac{1}{2\pi}\log\left(\frac{L_0(\beta_3^+)}{L_0(\beta_3^-)}\right) = \operatorname{mod}(A(\beta_3^+, \beta_3^-)) = x_3.$$

It follows from construction that $\beta_0, \beta_1, \beta_2, \beta_3^+$, and β_3^- satisfy all of the requirements of Lemma 2, the fourth inequality in linear system (1) coming from the last inequality in linear system (2) and Lemma A.2. □

It turns out in the proof above that the lower bound of the last inequality in linear system (1) may be changed for any positive constant (which depends only on the integers

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d_0, d_1, d_2, and d_3). As we will see later in Lemma 3, the lower bound 1 ensures sufficient space to make the surgery in $A(\beta_0, \beta_3^+)$. However, the author guesses that the last inequality in linear system (1) is not necessary (see the discussion after the proof of Lemma A.3).

The system of equipotentials coming from Lemma 2 will be used to divide $\widehat{\mathbb{C}}$ into several pieces on which a quasiregular map F will be piecewisely defined. This map F should be carefully defined in such a way that its dynamics is encoded by the weighted dynamical tree $(\mathcal{H}_P, w)$ (see Theorem 3).

For instance, the first step of the construction which corresponds to the dynamics on $e_1 \cup e_2$ for $\mathcal{H}_P$ is the following. Denote by $\beta_{0,1}$ the preimage of β_1 in $B(\widehat{z_0})$ (see Figure 5). From the first inequality in linear system (1), $\beta_{0,1}$ is an equipotential of level $L_0(\beta_{0,1}) > L_0(\beta_0)$. Denote by $D(\beta_{0,1})$ the open disk bounded by $\beta_{0,1}$ and containing $\{\widehat{z_1}, \widehat{z_2}\}$ (and, hence, $J(\widehat{f}) \cup B(\widehat{z_1}) \cup B(\widehat{z_2})$ as well). Then F is defined to be the rational map $\widehat{f}$ on $D(\beta_{0,1})$. Note that $F|_{D(\beta_{0,1})}$ continuously extends to $\beta_{0,1}$ by a degree d_0 covering denoted by $F|_{\beta_{0,1}} : \beta_{0,1} \to \beta_1$.

3.3. *Folding with an annulus-disk surgery.* The aim of this part of the construction is to realize the folding corresponding to the critical point c_0 in $\mathcal{H}_P$. More precisely F should holomorphically map a small annulus (corresponding to a neighborhood of c_0 in $\mathcal{H}_P$) onto a disk (corresponding to a neighborhood of c_1 in $\mathcal{H}_P$) with respect to the degrees d_0, d_3.

Let γ_1 be an arbitrary equipotential in $B(\widehat{z_1})$ such that $L_1(\gamma_1) < L_1(\beta_1)$. Denote by $D(\gamma_1)$ the open disk bounded by γ_1 and containing $\widehat{z_1}$. In order to follow more easily the construction, we will slightly improve the notation. So let $\gamma_{0,1}$ be the equipotential β_0, keeping in mind that $\gamma_{0,1}$ will be mapped onto γ_1 by a degree d_0 covering. Note that the first inequality in linear system (1) of Lemma 2 implies $L_0(\beta_{0,1}) > L_0(\gamma_{0,1})$. Similarly let $\beta_{3,1}$ be the equipotential β_3^+, keeping in mind that $\beta_{3,1}$ will be mapped onto β_1 by a degree d_3 covering.

LEMMA 3. *There exist an equipotential $\gamma_{3,1}$ in $B(\widehat{z_0})$ and a holomorphic branched covering $F|_{A(\gamma_{0,1},\gamma_{3,1})} : A(\gamma_{0,1}, \gamma_{3,1}) \to D(\gamma_1)$ such that:*
- (i) $L_0(\beta_{0,1}) > L_0(\gamma_{0,1}) > L_0(\gamma_{3,1}) > L_0(\beta_{3,1})$;
- (ii) *$F|_{A(\gamma_{0,1},\gamma_{3,1})}$ has degree $d_0 + d_3$ and has $d_0 + d_3$ critical points counted with multiplicity, which one of them, denoted by c, satisfies $F|_{A(\gamma_{0,1},\gamma_{3,1})}(c) = \widehat{z_1}$;*
- (iii) *$F|_{A(\gamma_{0,1},\gamma_{3,1})}$ continuously extends to $\gamma_{0,1} \cup \gamma_{3,1}$ by a degree d_0 covering $F|_{\gamma_{0,1}} : \gamma_{0,1} \to \gamma_1$ and a degree d_3 covering $F|_{\gamma_{3,1}} : \gamma_{3,1} \to \gamma_1$.*

Proof. Let $G : A(\gamma, \gamma') \to \mathbb{D}$ be a holomorphic branched covering coming from Lemma A.3 for the integers $n = d_0$ and $n' = d_3$. Define the equipotential $\gamma_{3,1}$ by

$$L_0(\gamma_{0,1}) > L_0(\gamma_{3,1}) \quad \text{and} \quad \frac{1}{2\pi} \log\left(\frac{L_0(\gamma_{0,1})}{L_0(\gamma_{3,1})}\right) = \operatorname{mod}(A(\gamma_{0,1}, \gamma_{3,1})) = \operatorname{mod}(A(\gamma, \gamma')).$$

Since $\operatorname{mod}(A(\gamma_{0,1}, \beta_{3,1})) = \operatorname{mod}(A(\beta_0, \beta_3^+)) > 1$ (from the last inequality in linear system equation (1) of Lemma 2) and $\operatorname{mod}(A(\gamma_{0,1}, \gamma_{3,1})) = \operatorname{mod}(A(\gamma, \gamma')) \leqslant 1$ (from the point (iii) in Lemma A.3), it follows that $L_0(\gamma_{3,1}) > L_0(\beta_{3,1})$ and the point (i) holds.

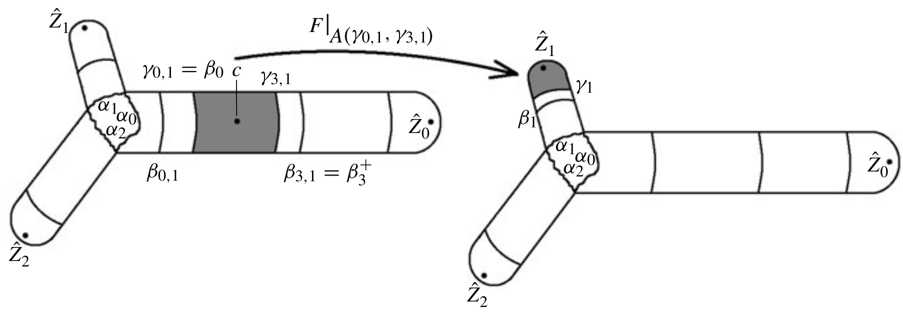

FIGURE 6. The map $F|_{A(\gamma_{0,1},\gamma_{3,1})}$ coming from Lemma 3 displayed on the Riemann sphere which is topologically distorted to emphasize the three domains $B(\widehat{z_0})$, $B(\widehat{z_1})$, and $B(\widehat{z_2})$ (compare with Figure 4(c)).

Now let ψ be any biholomorphic map from $A(\gamma_{0,1}, \gamma_{3,1})$ onto $A(\gamma, \gamma')$. The existence of such a biholomorphic map is ensured by the fact that these two open annuli have same modulus. Since $A(\gamma_{0,1}, \gamma_{3,1})$ and $A(\gamma, \gamma')$ are bounded by quasicircles, ψ may be continuously extended to $\gamma_{0,1} \cup \gamma_{3,1}$ by two homeomorphisms.

Let c be the preimage under ψ of any critical point of G and let $\phi : \mathbb{D} \to D(\gamma_1)$ be any Riemann mapping of $D(\gamma_1)$ such that $\phi(G(\psi(c))) = \widehat{z_1}$. Since $D(\gamma_1)$ is bounded by an equipotential, ϕ may be continuously extended to $\partial\mathbb{D}$ by a homeomorphism.

Then $F|_{A(\gamma_{0,1},\gamma_{3,1})} = \phi \circ G \circ \psi$ is holomorphic on $A(\gamma_{0,1}, \gamma_{3,1})$ and satisfies (ii), and (iii) by construction. □

Figure 6 depicts the map $F|_{A(\gamma_{0,1},\gamma_{3,1})}$ coming from Lemma 3.

3.4. *Preimage of the branching part.* According to the last two sections, the map F is defined up to there on the union of the open disk $D(\beta_{0,1})$ containing $\{\widehat{z_1}, \widehat{z_2}\}$ with the open annulus $A(\gamma_{0,1}, \gamma_{3,1})$ containing c. Moreover, F maps c to $\widehat{z_1}$, $\widehat{z_1}$ to $\widehat{z_2}$ and $\widehat{z_2}$ to $\widehat{z_0}$. Now we need to define F near $\widehat{z_0}$ by sending $\widehat{z_0}$ to c in order to realize a cycle of period four as required in Theorem 3. This should be done carefully so that the quasiconformal surgery may be concluded.

The first problem is that some preimage of $J(\widehat{f})$ (or more precisely of the open annulus $A(\beta_1, \beta_0)$ containing $J(\widehat{f})$) must appear in $B(\widehat{z_0})$ (compare with Figure 4(c) where the edge $e_3 = [c_0, c_3]_{\mathcal{H}_P}$ contains a preimage of the branching point α). This is done in Lemma 4 below which essentially uses the fourth inequality in linear system (1) of Lemma 2.

LEMMA 4. *There exist an equipotential $\beta_{3,0}$ in $B(\widehat{z_0})$ and a holomorphic covering $F|_{A(\beta_{3,1},\beta_{3,0})} : A(\beta_{3,1}, \beta_{3,0}) \to A(\beta_1, \beta_0)$ such that:*
(i) $L_0(\beta_{3,1}) > L_0(\beta_{3,0}) > L_0(\beta_3^-)$;
(ii) $F|_{A(\beta_{3,1},\beta_{3,0})}$ *has degree d_3 and has no critical point;*
(iii) $F|_{A(\beta_{3,1},\beta_{3,0})}$ *continuously extends to $\beta_{3,1} \cup \beta_{3,0}$ by two degree d_3 coverings $F|_{\beta_{3,1}} : \beta_{3,1} \to \beta_1$ and $F|_{\beta_{3,0}} : \beta_{3,0} \to \beta_0$.*

Proof. Define the equipotential $\beta_{3,0}$ by

$$L_0(\beta_{3,1}) > L_0(\beta_{3,0})$$

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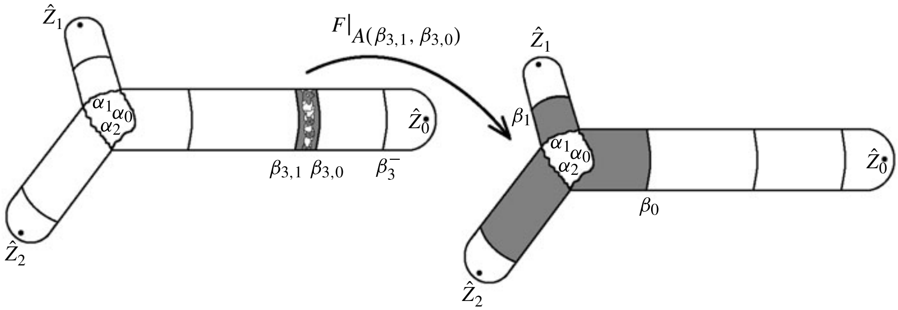

FIGURE 7. The map $F|_{A(\beta_{3,1},\beta_{3,0})}$ coming from Lemma 4 displayed on the Riemann sphere which is topologically distorted to emphasize the three domains $B(\widehat{z_0})$, $B(\widehat{z_1})$, and $B(\widehat{z_2})$ (compare with Figure 4(c)).

and

$$\frac{1}{2\pi}\log\left(\frac{L_0(\beta_{3,1})}{L_0(\beta_{3,0})}\right) = \text{mod}(A(\beta_{3,1},\,\beta_{3,0})) = \frac{1}{d_3}\,\text{mod}(A(\beta_1,\,\beta_0)).$$

Since we have mod $(A(\beta_{3,1},\,\beta_{3,0})) = (1/d_3)\,\text{mod}(A(\beta_1,\beta_0)) < \text{mod}(A(\beta_3^+,\,\beta_3^-)) = \text{mod}(A(\beta_{3,1},\,\beta_3^-))$ (from the fourth inequality in linear system (1) of Lemma 2), it follows that $L_0(\beta_{3,0}) > L_0(\beta_3^-)$ and the point (i) holds.

Now let ψ be any biholomorphic map from $A(\beta_{3,1},\,\beta_{3,0})$ onto a round annulus of the form $A_r = \{z \in \mathbb{C} \mid r < |z| < 1\}$ where r is defined by

$$\frac{1}{2\pi}\log\left(\frac{1}{r}\right) = \text{mod}(A_r) = \text{mod}(A(\beta_{3,1},\,\beta_{3,0})).$$

Since $A(\beta_{3,1},\,\beta_{3,0})$ is bounded by equipotentials, ψ may be continuously extended to $\beta_{3,1} \cup \beta_{3,0}$ by two homeomorphisms which send $\beta_{3,1}$ onto $\{z \in \mathbb{C} \mid |z| = 1\}$ and $\beta_{3,0}$ onto $\{z \in \mathbb{C} \mid |z| = r\}$.

Similarly, let Ψ be any biholomorphic map from the round annulus $A_{r^{d_3}}$ onto $A(\beta_1,\,\beta_0)$. The existence of such a biholomorphic map is ensured by the fact that

$$\text{mod}(A_{r^{d_3}}) = \frac{1}{2\pi}\log\left(\frac{1}{r^{d_3}}\right) = \frac{d_3}{2\pi}\log\left(\frac{1}{r}\right) = d_3\,\text{mod}(A(\beta_{3,1},\,\beta_{3,0})) = \text{mod}(A(\beta_1,\,\beta_0)).$$

Since $A(\beta_1,\,\beta_0)$ is bounded by equipotentials, Ψ may be continuously extended to $\partial A_{r^{d_3}}$ by two homeomorphisms which send $\{z \in \mathbb{C} \mid |z| = 1\}$ onto β_1 and $\{z \in \mathbb{C} \mid |z| = r^{d_3}\}$ onto β_0.

Then $F|_{A(\beta_{3,1},\beta_{3,0})} = \Psi \circ (z \mapsto z^{d_3}) \circ \psi$ is holomorphic on $A(\beta_{3,1},\,\beta_{3,0})$ and satisfies (ii), and (iii) by construction. □

Figure 7 depicts the map $F|_{A(\beta_{3,1},\beta_{3,0})}$ coming from Lemma 4.

3.5. *Achievement of the super-attracting cycle of period* 4. Now we achieve the definition of F near $\widehat{z_0}$. This is done in two parts. First, Lemma 5 realizes a preimage of a neighborhood of $\widehat{z_0}$ in $B(\widehat{z_0})$. Then Lemma 6 defines F near $\widehat{z_0}$ by sending a neighborhood of $\widehat{z_0}$ onto a neighborhood of c (mapping $\widehat{z_0}$ to c).

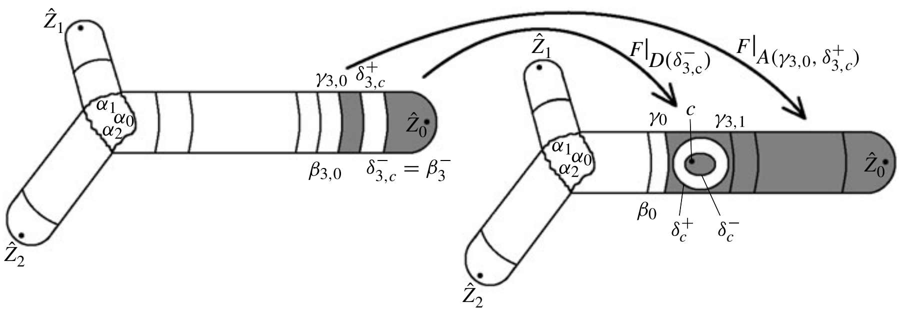

FIGURE 8. The maps $F|_{A(\gamma_{3,0},\delta^+_{3,c})}$ and $F|_{D(\delta^-_{3,c})}$ coming from Lemmas 5 and 6 displayed on the Riemann sphere which is topologically distorted to emphasize the three domains $B(\widehat{z_0})$, $B(\widehat{z_1})$, and $B(\widehat{z_2})$ (compare with Figure 4(c)).

Let γ_0 be an arbitrary equipotential in $B(\widehat{z_0})$ such that $L_0(\beta_0) = L_0(\gamma_{0,1}) > L_0(\gamma_0) > L_0(\gamma_{3,1})$ and $A(\gamma_0, \gamma_{3,1})$ contains the critical point c.

LEMMA 5. *There exist two equipotentials $\gamma_{3,0}$ and $\delta^+_{3,c}$ in $B(\widehat{z_0})$, a quasicircle δ^+_c in $A(\gamma_0, \gamma_{3,1})$ which separates c from $\gamma_0 \cup \gamma_{3,1}$, and a holomorphic covering $F|_{A(\gamma_{3,0},\delta^+_{3,c})} : A(\gamma_{3,0}, \delta^+_{3,c}) \to A(\gamma_0, \delta^+_c)$ such that:*

(i) $L_0(\beta_{3,0}) > L_0(\gamma_{3,0}) > L_0(\delta^+_{3,c}) > L_0(\beta^-_3)$;

(ii) $F|_{A(\gamma_{3,0},\delta^+_{3,c})}$ *has degree d_3 and has no critical point;*

(iii) $F|_{A(\gamma_{3,0},\delta^+_{3,c})}$ *continuously extends to $\gamma_{3,0} \cup \delta^+_{3,c}$ by two degree d_3 coverings $F|_{\gamma_{3,0}} : \gamma_{3,0} \to \gamma_0$ and $F|_{\delta^+_{3,c}} : \delta^+_{3,c} \to \delta^+_c$.*

Proof. Applying Lemma A.4, we get a quasicircle δ^+_c in $A(\gamma_0, \gamma_{3,1})$ which separates c from $\gamma_0 \cup \gamma_{3,1}$ and such that

$$\frac{1}{d_3} \operatorname{mod}(A(\gamma_0, \delta^+_c)) < \operatorname{mod}(A(\beta_{3,0}, \beta^-_3)).$$

Therefore we can find two equipotentials $\gamma_{3,0}$ and $\delta^+_{3,c}$ in $B(\widehat{z_0})$ so that

$$L_0(\beta_{3,0}) > L_0(\gamma_{3,0}) > L_0(\delta^+_{3,c}) > L_0(\beta^-_3)$$

and

$$\frac{1}{2\pi} \log\left(\frac{L_0(\gamma_{3,0})}{L_0(\delta^+_{3,c})}\right) = \operatorname{mod}(A(\gamma_{3,0}, \delta^+_{3,c})) = \frac{1}{d_3} \operatorname{mod}(A(\gamma_0, \delta^+_c)).$$

The point (i) holds by definition. For the two other points, the proof may be achieved in the same way as the proof of Lemma 4. □

Figure 8 depicts the equipotentials involved in Lemma 5 and the map $F|_{A(\gamma_{3,0},\delta^+_{3,c})}$.

It remains to define F near $\widehat{z_0}$. Let δ^-_c be an arbitrary quasicircle which separates c from δ^+_c. We slightly improve the notation by denoting by $\delta^-_{3,c}$ the equipotential β^-_3 keeping in mind that $\delta^-_{3,c}$ will be mapped onto δ^-_c by a degree d_3 covering. Finally, denote by $D(\delta^-_{3,c})$ the open disk bounded by $\delta^-_{3,c}$ and containing $\widehat{z_0}$, and by $D(\delta^-_c)$ the open disk bounded by δ^-_c and containing c.

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TABLE 1. A summary of the definition of F before §3.6.

Domains	Images	Cont. extensions on boundaries	Critical points with multiplicity	Critical values
$D(\beta_{0,1})$	$\widehat{\mathbb{C}}$	$\beta_{0,1} \xrightarrow{d_0:1} \beta_1$	$\widehat{z_1}$ with mult. $d_1 - 1$ $\widehat{z_2}$ with mult. $d_2 - 1$	$F(\widehat{z_1}) = \widehat{z_2}$ $F(\widehat{z_2}) = \widehat{z_0}$
$A(\gamma_{0,1}, \gamma_{3,1})$	$D(\gamma_1)$	$\gamma_{0,1} \xrightarrow{d_0:1} \gamma_1$ $\gamma_{3,1} \xrightarrow{d_3:1} \gamma_1$	$c \in \{d_0 + d_3$ crit. pts counted with mult.$\}$	$F(c) = \widehat{z_1}$ and others
$A(\beta_{3,1}, \beta_{3,0})$	$A(\beta_1, \beta_0)$	$\beta_{3,1} \xrightarrow{d_3:1} \gamma_1$ $\beta_{3,0} \xrightarrow{d_3:1} \beta_0$	$\emptyset$	$\emptyset$
$A(\gamma_{3,0}, \delta^+_{3,c})$	$A(\gamma_0, \delta^+_c)$	$\gamma_{3,0} \xrightarrow{d_3:1} \gamma_0$ $\delta^+_{3,c} \xrightarrow{d_3:1} \delta^+_c$	$\emptyset$	$\emptyset$
$D(\delta^-_{3,c})$	$D(\delta^-_c)$	$\delta^-_{3,c} \xrightarrow{d_3:1} \delta^-_c$	$\widehat{z_0}$ with mult. $d_3 - 1$	$F(\widehat{z_0}) = c$

LEMMA 6. *There exists a holomorphic branched covering* $F|_{D(\delta^-_{3,c})} : D(\delta^-_{3,c}) \to D(\delta^-_c)$ *such that:*

(i) $F|_{D(\delta^-_{3,c})}$ *has degree* d_3 *and has only one critical point which is* $\widehat{z_0}$ *with* $F|_{D(\delta^-_{3,c})}(\widehat{z_0}) = c$;

(ii) $F|_{D(\delta^-_{3,c})}$ *continuously extends to* $\delta^-_{3,c}$ *by a degree* d_3 *covering* $F|_{\delta^-_{3,c}} : \delta^-_{3,c} \to \delta^-_c$.

Proof. Let $\phi : \mathbb{D} \to D(\delta^-_{3,c})$ be any Riemann mapping of $D(\delta^-_{3,c})$ such that $\phi(0) = \widehat{z_0}$, and let $\Phi : \mathbb{D} \to D(\delta^-_c)$ be any Riemann mapping of $D(\delta^-_c)$ such that $\Phi(0) = c$. Since $D(\delta^-_{3,c})$ and $D(\delta^-_c)$ are bounded by quasicircles, ϕ and Φ may be continuously extended to $\partial\mathbb{D}$ by homeomorphisms.

Then $F|_{D(\delta^-_{3,c})} = \Phi \circ (z \mapsto z^{d_3}) \circ \phi^{-1}$ gives the result. □

Figure 8 depicts the map $F|_{D(\delta^-_{3,c})}$ coming from Lemma 6.

3.6. *Uniformization.* At first we sum up in Table 1 the definition of F up to there.

So F is holomorphically defined on $H = D(\beta_{0,1}) \cup A(\gamma_{0,1}, \gamma_{3,1}) \cup A(\beta_{3,1}, \beta_{3,0}) \cup A(\gamma_{3,0}, \delta^+_{3,c}) \cup D(\delta^-_{3,c})$ with continuous extension on the boundary. It remains to define F on the complement $Q = \widehat{\mathbb{C}} - \overline{H} = A(\beta_{0,1}, \gamma_{0,1}) \cup A(\gamma_{3,1}, \beta_{3,1}) \cup A(\beta_{3,0}, \gamma_{3,0}) \cup A(\delta^+_{3,c}, \delta^-_{3,c})$. This is done in the following lemma.

LEMMA 7. *The map* $F|_{\overline{H}} : \overline{H} \to \widehat{\mathbb{C}}$ *extends to a quasiregular map* $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *by quasiconformal coverings defined on each connected component of* $Q = \widehat{\mathbb{C}} - \overline{H}$.

Moreover, there exists an open subset $E \subset H$ *such that* $F(E) \subset E$ *and* $F^2(\overline{Q}) \subset E$.

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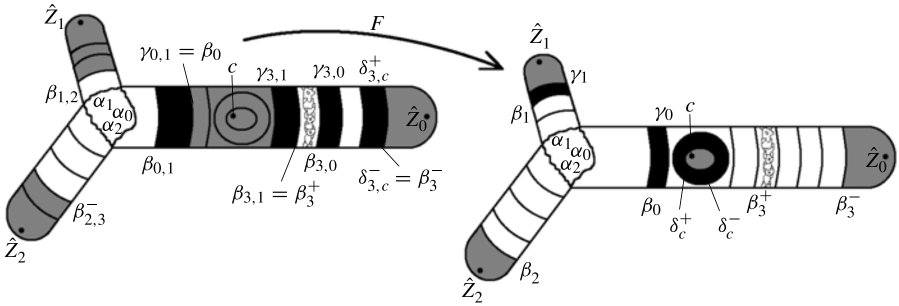


FIGURE 9. The map F coming from Lemma 7. On the left topological sphere, the black area represents Q and the gray area represents E. On the right topological sphere, the black area represents $F(Q)$ and the gray area represents $F(E)$.

In particular, note that the quasiregular map $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ has no more critical points than those coming from the holomorphic restriction $F|_H : H \to \widehat{\mathbb{C}}$.

Proof. Note that every connected component of Q is an open annulus whose boundary is the disjoint union of two quasicircles where F realizes two coverings of the same degree (and same orientation). By interpolation, F may be continuously extended to each connected component of Q by a covering of degree corresponding to that one on the boundary. Since all of the connected components of the boundary of Q, together with their images by F, are quasicircles, each interpolation may be carefully done in such a way that the resulting map is actually quasiconformal on the Riemann sphere. In short, F quasiregularly extends to Q by:

(i) a degree d_0 quasiconformal covering $F|_{A(\beta_{0,1},\gamma_{0,1})} : A(\beta_{0,1}, \gamma_{0,1}) \to A(\beta_1, \gamma_1)$;
(ii) a degree d_3 quasiconformal covering $F|_{A(\gamma_{3,1},\beta_{3,1})} : A(\gamma_{3,1}, \beta_{3,1}) \to A(\gamma_1, \beta_1)$;
(iii) a degree d_3 quasiconformal covering $F|_{A(\beta_{3,0},\gamma_{3,0})} : A(\beta_{3,0}, \gamma_{3,0}) \to A(\beta_0, \gamma_0)$;
(iv) a degree d_3 quasiconformal covering $F|_{A(\delta^+_{3,c},\delta^-_{3,c})} : A(\delta^+_{3,c}, \delta^-_{3,c}) \to A(\delta^+_c, \delta^-_c)$.

In particular, we have $F(Q) = A(\beta_1, \gamma_1) \cup A(\beta_0, \gamma_0) \cup A(\delta^+_c, \delta^-_c)$ (see Figure 9 to follow the continuation of the proof).

Now denote by $\beta_{1,2}$ the preimage of β_2 in $B(\widehat{z_1})$ under F (thus under $\widehat{f}$) and similarly by $\beta^-_{2,3}$ the preimage of β^-_3 in $B(\widehat{z_2})$ (see Figure 5). Moreover, denote by $D(\beta_{1,2})$ the open disk bounded by $\beta_{1,2}$ and containing $\widehat{z_1}$, and by $D(\beta^-_{2,3})$ the open disk bounded by $\beta^-_{2,3}$ and containing $\widehat{z_2}$. Finally, let E be the union $D(\beta_{1,2}) \cup D(\beta^-_{2,3}) \cup D(\delta^-_{3,c}) \cup A(\gamma_{0,1}, \gamma_{3,1})$.

At first note that E is an open subset of $H = D(\beta_{0,1}) \cup A(\gamma_{0,1}, \gamma_{3,1}) \cup A(\beta_{3,1}, \beta_{3,0}) \cup A(\gamma_{3,0}, \delta^+_{3,c}) \cup D(\delta^-_{3,c})$. Indeed we have $\overline{D(\beta_{1,2})} \cup \overline{D(\beta^-_{2,3})} \subset D(\beta_{0,1})$ from the definition of $D(\beta_{0,1})$.

Moreover, it follows from definition of F on H that $F(E) = D(\beta_2) \cup D(\beta^-_3) \cup D(\delta^-_c) \cup D(\gamma_1)$ where $D(\beta_2)$ denotes the open disk bounded by β_2 and containing $\widehat{z_2}$, and $D(\beta^-_3) = D(\delta^-_{3,c})$ is the open disk bounded by $\beta^-_3 = \delta^-_{3,c}$ and containing $\widehat{z_0}$.

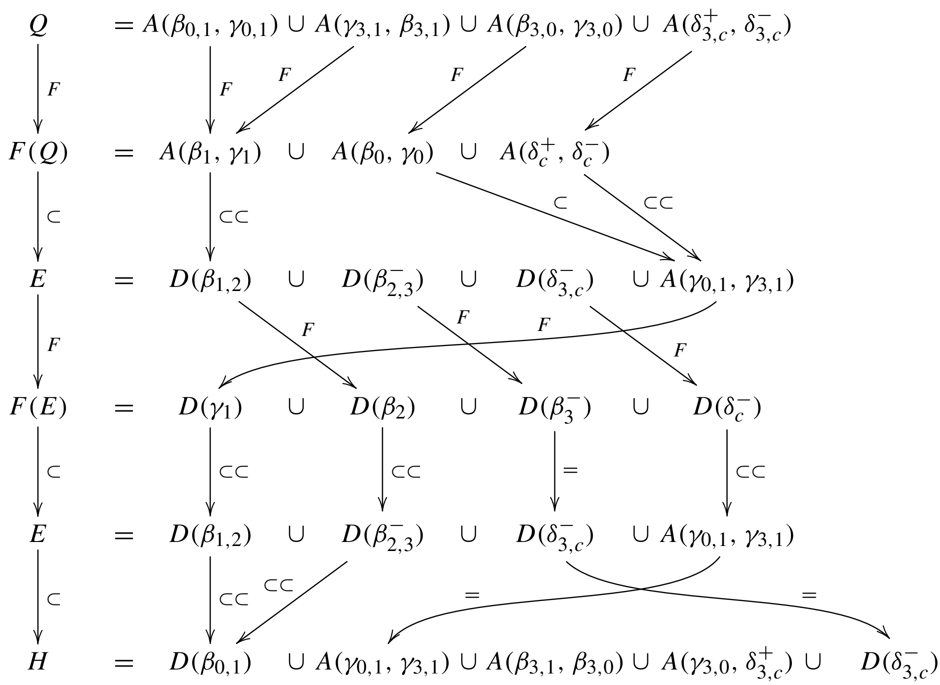
Furthermore, according to the whole construction, we have:

(i) from Lemma 2 and the definition of γ_1, $\overline{A(\beta_1, \gamma_1)} \cup \overline{D(\gamma_1)} \subset D(\beta_{1,2})$ and $\overline{D(\beta_2)} \subset D(\beta^-_{2,3})$;

(ii) from the definition of γ_0 and recalling $\beta_0 = \gamma_{0,1}$, $\overline{A(\beta_0, \gamma_0)} \subset \overline{A(\gamma_{0,1}, \gamma_{3,1})}$;

(iii) from the definitions of δ_c^-, δ_c^+ and γ_0, $\overline{A(\delta_c^+, \delta_c^-) \cup D(\delta_c^-)} \subset A(\gamma_0, \gamma_{3,1}) \subset A(\gamma_{0,1}, \gamma_{3,1})$.

Putting everything together gives the following diagram in which the arrows $\xrightarrow{F}$ denote images under F, $\xrightarrow{\subset}$ denote inclusions, $\xrightarrow{\subset\subset}$ denote compact inclusions (namely $A \xrightarrow{\subset\subset} B$ if and only if $\overline{A} \subset B$) and $\xrightarrow{=}$ denote equality:



In particular, we deduce that $F(Q) \subset E$ and $F(E) \subset E \subset H$. Furthermore, following compact inclusions, it turns out that $F^2(\overline{Q}) \subset E$. □

Now we have a quasiregular map F from the Riemann sphere to itself whose dynamics follows that one of the weighted dynamical tree $(\mathcal{H}_P, w)$ (see Figure 4(c)). We need to find a holomorphic map f conjugated to F so that f follows the same dynamics as well (f should satisfy the requirements of Theorems 3 and 4). To do so, we will apply the Shishikura's fundamental lemma for quasiconformal surgery (stated for the first time in [**Shi87**]) that we recall below.

LEMMA 8. (Shishikura's fundamental lemma for quasiconformal surgery) *Let $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiregular map. Assume there are an open set $E \subset \widehat{\mathbb{C}}$ and an integer $N \geqslant 0$ which satisfy the following conditions:*

(i) *$g(E) \subset E$;*

(ii) *g is holomorphic on E;*

(iii) *g is holomorphic on an open set containing $\widehat{\mathbb{C}} - g^{-N}(E)$.*

Then there exists a quasiconformal map $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that the map $\varphi \circ g \circ \varphi^{-1}$ is holomorphic.

The result stated in [**Shi87**] is a little more general but it easily implies the more explicit statement of Lemma 8 (we refer the reader to [**Shi87**, **BF13**] for a proof and more details).

Here our map F satisfies the three assumptions (indeed F is holomorphic on H, hence on $E \subset H$, and Lemma 7 implies that $\widehat{\mathbb{C}} - F^{-2}(E) \subset \widehat{\mathbb{C}} - \overline{Q} = H$), so applying Lemma 8 gives a holomorphic map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ quasiconformally conjugated to $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as desired.

LEMMA 9. *The rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ obtained above has degree $\widehat{d} + d_3$ and has a super-attracting cycle $\{z_0, z_1, z_2, z_3\}$ of period four which is accumulated by every critical orbit. In particular, f is hyperbolic.*

Proof. Since f is quasiconformally conjugated to F, the critical points of f are images under a quasiconformal map φ of the critical points of F with same the multiplicities. More precisely, the critical points of f are:

(i) $z_1 = \varphi(\widehat{z_1}) \in \varphi(D(\beta_{1,2})) \subset \varphi(E)$ with multiplicity $d_1 - 1$;

(ii) $z_2 = \varphi(\widehat{z_2}) \in \varphi(D(\beta^-_{2,3})) \subset \varphi(E)$ with multiplicity $d_2 - 1$;

(iii) $d_0 + d_3$ critical points counted with multiplicity in $\varphi(A(\gamma_{0,1}, \gamma_{3,1})) \subset \varphi(E)$, one of which is given by $z_0 = \varphi(c)$;

(iv) $z_3 = \varphi(\widehat{z_0}) \in \varphi(D(\delta^-_{3,c})) \subset \varphi(E)$ with multiplicity $d_3 - 1$.

According to the Riemann–Hurwitz formula, it follows that the number of critical points counted with multiplicity is given by

$$2\deg(f) - 2 = (d_1 - 1) + (d_2 - 1) + (d_0 + d_3) + (d_3 - 1)$$

and, hence,

$$\deg(f) = \frac{1}{2}(d_0 + d_1 + d_2 - 1) + d_3 = \widehat{d} + d_3.$$

Note that $\{z_0, z_1, z_2, z_3\}$ forms a super-attracting cycle of period four. Moreover, every critical point of f lies in the forward invariant open set $\varphi(E)$, namely a disjoint union of four open subsets of $\widehat{\mathbb{C}}$ each containing one point of $\{z_0, z_1, z_2, z_3\}$. Consequently, every critical orbit accumulates this super-attracting cycle. □

4. *Properties*

The aim of this section is to achieve the proofs of Theorems 3 and 4. More precisely, we are going to show that the rational map f constructed in the previous section satisfies all of the requirements of these two theorems. Section 4.1 focuses on the dynamical properties of f (stated in Theorem 3), and §4.2 deals with the topological properties of the Julia component of f (stated in Theorem 4).

In order to lighten notation, we forget the quasiconformal map φ provided by Lemma 8 to denote the image under φ of any set introduced in the previous section (equivalently speaking, we act as if the quasiregular map F constructed in the previous section is actually holomorphic).

4.1. *Exchanging dynamics.*

Consider the following pairwise disjoint open annuli (see Figure 10).

$$A_0 = A(\alpha_0, \beta_0), \quad A_1 = A(\alpha_1, \beta_1), \quad A_2 = A(\alpha_2, \beta_2), \quad \text{and} \quad A_3 = A(\beta_3^+, \beta_3^-).$$

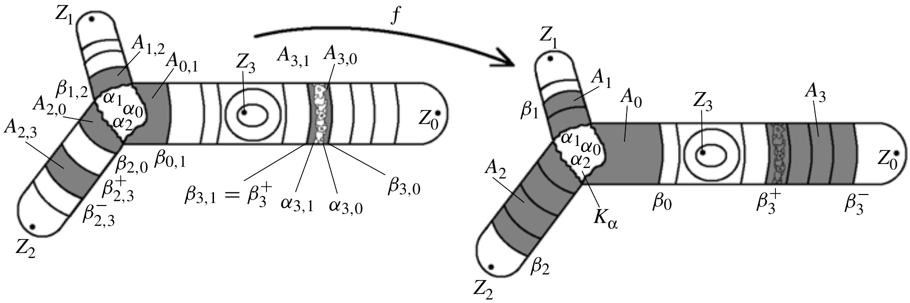


FIGURE 10. The various annuli considered to encode the exchanging dynamics.

Then, consider the connected components of the preimage under f of $A_0 \cup A_1 \cup A_2 \cup A_3$ which are contained as essential subannuli in one of these open annuli, namely:

(i) $A_{0,1} = A(\alpha_0, \beta_{0,1})$;

(ii) $A_{1,2} = A(\alpha_1, \beta_{1,2})$;

(iii) $A_{2,0} = A(\alpha_2, \beta_{2,0})$ where $\beta_{2,0}$ is the preimage of β_0 in $B(\widehat{z_2})$ (see Figure 5);

(iv) $A_{2,3} = A(\beta^+_{2,3}, \beta^-_{2,3})$ where $\beta^+_{2,3}$ is the preimage of β^+_3 in $B(\widehat{z_2})$ (see Figure 5);

(v) $A_{3,0} = A(\alpha_{3,0}, \beta_{3,0})$ where $\alpha_{3,0}$ is the preimage of α_0 in $A(\beta_{3,1}, \beta_{3,0})$ (see Lemma 4);

(vi) $A_{3,1} = A(\beta_{3,1}, \alpha_{3,1})$ where $\alpha_{3,1}$ is the preimage of α_1 in $A(\beta_{3,1}, \beta_{3,0})$ (see Lemma 4).

Note that the notation is chosen so that each $A_{i,j}$ is contained as an essential subannulus in A_i, and $f|_{A_{i,j}} : A_{i,j} \to A_j$ is a degree d_i covering. Note that some connected components of $f^{-1}(A_3)$ are included in A_3 as well (from Lemma 5, see Figure 8), but none of them is contained in A_3 as an essential subannulus.

Denote by $\mathcal{A}$ the collection of all connected components of the non-escaping set induced by $f|_U : U \to A_0 \cup A_1 \cup A_2 \cup A_3$ on the union of subannuli $U = A_{0,1} \cup A_{1,2} \cup A_{2,0} \cup A_{2,3} \cup A_{3,0} \cup A_{3,1}$:

$$\mathcal{A} = \{J \text{ connected component of } \{z \in U \mid \forall n \geqslant 0,\ \ f^n(z) \in U\}\}.$$

Let J_α be the continuum in $\widehat{\mathbb{C}}$ which corresponds to the Julia set $J(\widehat{f})$ of $\widehat{f}$ (more precisely, J_α is the image of $J(\widehat{f})$ under the quasiconformal map φ provided by Lemma 8). Note that J_α is fixed under iteration of f and J_α intersects $\overline{U}$ (along $\alpha_0 \cup \alpha_1 \cup \alpha_2$). Denote by $\mathcal{A}_\alpha$ the collection of all continua which are eventually mapped onto J_α and whose every iterate intersects $\overline{U}$:

$$\mathcal{A}_\alpha = \left\{ J \text{ connected component of } \bigcup_{n\geqslant 0} f^{-n}(J_\alpha) \text{ such that } \forall n \geqslant 0,\ \ f^n(J) \cap \overline{U} \neq \emptyset \right\}.$$

Finally, denote by $\mathcal{A}^\star$ the union $\mathcal{A} \cup \mathcal{A}_\alpha$. As a collection of pairwise disjoint continua, $\mathcal{A}^\star$ is endowed with the topology coming from the usual distance between continua on the Riemann sphere $\widehat{\mathbb{C}}$ (equipped with the spherical metric). It turns out that f induced a topological dynamical system on $\mathcal{A}^\star$. This dynamical system may be encoded by the weighted dynamical tree $(\mathcal{H}_P, w)$ (see §2.2) as is shown in the following lemma.

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LEMMA 10. *There exists a homeomorphism $h : \mathcal{A}^\star \to \mathcal{J}(\mathcal{H}_P)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A}^\star & \xrightarrow{\ f\ } & \mathcal{A}^\star \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{J}(\mathcal{H}_P) & \xrightarrow[\ P\ ]{} & \mathcal{J}(\mathcal{H}_P) \end{array}$$

Moreover, for every $J \in \mathcal{A}$, the restriction map $f|_J$ has degree $w(e_k) = d_k$ where e_k is the edge of $\mathcal{H}_P$ which contains $h(J)$.

Proof. At first, remark there is a subannulus $A_{i,j}$ for some $i, j \in \{0, 1, 2, 3\}$ if and only if the (i, j)-entry of the transition matrix $M = (m_{i,j})_{i,j\in\{0,1,2,3\}}$ is non-zero (see Definition 1). Indeed, recall that the transition matrix is

$$M = \begin{pmatrix} 0 & \dfrac{1}{d_0} & 0 & 0 \\ 0 & 0 & \dfrac{1}{d_1} & 0 \\ \dfrac{1}{d_2} & 0 & 0 & \dfrac{1}{d_2} \\ \dfrac{1}{d_3} & \dfrac{1}{d_3} & 0 & 0 \end{pmatrix}.$$

According to this remark, we introduce the subshift of finite type (Σ, σ) associated to the transition matrix M, namely the restriction of the four-to-one shift map on the subset of all infinite sequences of digits in $\{0, 1, 2, 3\}$ such that every adjacent pair of entries lies in $\{(0, 1), (1, 2), (2, 0), (2, 3), (3, 0), (3, 1)\}$:

$$\Sigma = \{s = (s_0, s_1, s_2, \ldots) \in \{0, 1, 2, 3\}^{\mathbb{N}} \mid \forall k \geqslant 0,\ m_{s_k, s_{k+1}} \neq 0\}$$

$$\sigma : \Sigma \to \Sigma,\ s = (s_0, s_1, s_2, \ldots) \mapsto \sigma(s) = (s_1, s_2, s_3, \ldots).$$

Here Σ is endowed with the topology coming from the following distance, making it a Cantor set:

$$\text{for all } s, s' \in \Sigma, \quad d(s, s') = \sum_{k\geqslant 0} \frac{|s_k - s'_k|}{4^k}.$$

Let S_α be the subset of Σ of three infinite sequences of repeating 0, 1, 2 digits:

$$S_\alpha = \{(0, 1, 2, 0, 1, 2, \ldots),\ (1, 2, 0, 1, 2, 0, \ldots),\ (2, 0, 1, 2, 0, 1, \ldots)\}.$$

We shall identify these three sequences in Σ, and similarly every subset of sequences which are eventually mapped in S_α after the same itinerary under σ. More precisely, let $\sim$ be the equivalence relation on Σ defined by

$$\text{for all } s, s' \in \Sigma, \quad s \sim s' \iff \text{there exists } n \geqslant 0 \left| \begin{cases} \text{for all } k \in \{0, 1, \ldots, n\}, & s_k = s'_k \\ \sigma^n(s), \sigma^n(s') \in S_\alpha \end{cases} \right.$$

and let $\Sigma^\star$ be the topological quotient space $\Sigma/\sim$. Recall that $\Sigma^\star$ is a Cantor set as well for the quotient topology induced by $\sim$. Abusing notation, every equivalence class

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containing only one infinite sequence $s \in \Sigma$ which is not eventually mapped in S_α is still denoted by $s \in \Sigma^\star$, and the map induced by the shift map on $\Sigma^\star$ is still denoted by σ.

We are going to show that $(\mathcal{A}^\star, f)$ is topologically conjugated to $(\Sigma^\star, \sigma)$. To do so, consider the itinerary map $h_1 : \mathcal{A} \to \Sigma^\star$ defined by

$$\text{for all } J \in \mathcal{A}, \quad h_1(J) = (s_0, s_1, s_2, \ldots) \quad \text{with } f^k(J) \subset A_{s_k} \text{ for every } k \geqslant 0.$$

This map is well defined and injective by the definition of $\mathcal{A}$.

To prove that h_1 extends to a homeomorphism from $\mathcal{A}^\star$ to $\Sigma^\star$, we first define by induction for every $s = (s_0, s_1, s_2, \ldots) \in \Sigma$ an infinite sequence of subannuli $(A_{s_0,s_1,\ldots,s_n})_{n\geqslant 0}$ such that for every $n \geqslant 0$, $A_{s_0,s_1,\ldots,s_n}$ is contained in A_{s_0} as essential subannulus, and $f|_{A_{s_0,s_1,\ldots,s_n}} : A_{s_0,s_1,\ldots,s_n} \to A_{s_1,s_2,\ldots,s_n}$ is a degree d_{s_0} covering. Denote by $A_s = A_{s_0,s_1,s_2,\ldots}$ the limit set $\bigcap_{n\geqslant 0} \overline{A_{s_0,s_1,\ldots,s_n}}$ which is a continuum.

If s is not eventually mapped in S_α, then $\overline{A_{s_0,s_1,\ldots,s_n}}$ is contained in $U = A_{0,1} \cup A_{1,2} \cup A_{2,0} \cup A_{2,3} \cup A_{3,0} \cup A_{3,1}$ for every $n \geqslant 0$ large enough and thus A_s is a connected component of the non-escaping set, that is an element of $\mathcal{A}$. Moreover, $h_1(A_s) = s$ holds from definition of the itinerary map h_1.

In contrast, if s is in S_α, then A_s is either α_0, α_1, or α_2, and in particular A_s is contained in J_α. More generally, if s is eventually mapped in S_α, then A_s is contained in a continuum J which is eventually mapped onto J_α, that is an element of $\mathcal{A}_\alpha$. Moreover, for every $s' \in \Sigma$ such that $s' \sim s$, $A_{s'}$ is contained in the same continuum $J \in \mathcal{A}_\alpha$.

Therefore, h_1 extends to a bijective map from $\mathcal{A}^\star$ to $\Sigma^\star$, by associating to $J \in \mathcal{A}_\alpha$ the equivalence class $h_1(J) \in \Sigma^*$ of the itinerary $s = (s_0, s_1, s_2, \ldots) \in \Sigma$ of any subcontinuum in J which is eventually mapped into $\alpha_0 \cup \alpha_1 \cup \alpha_2$. Furthermore, this extension is actually a conjugation between f and σ:

$$\text{for all } J \in \mathcal{A}^\star, \quad h_1(f(J)) = \sigma(h_1(J)).$$

It remains to prove the continuity. Fix $J \in \mathcal{A}^\star$ and let $s = (s_0, s_1, s_2, \ldots) \in \Sigma$ be a class representative of $h_1(J)$. Let J' be another element of $\mathcal{A}^\star$ such that some class representative $s' = (s'_0, s'_1, s'_2, \ldots) \in \Sigma$ of $h_1(J')$ is arbitrary close to s. That implies the first n digits of s and s' coincide for arbitrary large $n \geqslant 0$. In particular, A_s and $A_{s'}$ are contained in $\overline{A_{s_0,s_1,\ldots,s_n}}$. Note that $f^n|_{A_{s_0,s_1,\ldots,s_n}} : A_{s_0,s_1,\ldots,s_n} \to A_{s_n}$ is a covering of degree $d_{s_0} d_{s_1} \ldots d_{s_{n-1}}$ tending to infinity with n (since assumption (H2) implies that at least two of weights d_0, d_1, d_2, and d_3 are $\geqslant 2$, see Definition 1). Therefore A_s and $A_{s'}$ are contained in an open annulus of arbitrary small modulus. Then, using extremal length (see [**Ahl73**]), it follows that $A_s \subset J$ and $A_{s'} \subset J'$ are arbitrary close, hence J and J' are arbitrary close in $\mathcal{A}^\star$. Consequently h_1^{-1} is continuous. The continuity of h_1 follows from a similar argument.

Similarly, we can show that $(\mathcal{J}(\mathcal{H}_P), P)$ is topologically conjugated to $(\Sigma^\star, \sigma)$ by a homeomorphism $h_2 : \mathcal{J}(\mathcal{H}_P) \to \Sigma^\star$. Indeed recall that the dynamical tree $\mathcal{H}_P$ is described by a set of four edges e_0, e_1, e_2, e_3 where P acts as follows (see §2.2):

$$\begin{cases} P(e_0) = e_1, \\ P(e_1) = e_2, \\ P(e_2) = e_0 \cup e_3, \\ P(e_3) = e_0 \cup e_1. \end{cases}$$

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Thus, we may find four connected open subsets I_0, I_1, I_2, and I_3 respectively included in e_0, e_1, e_2, and e_3 together with six connected open subsets $I_{0,1}$, $I_{1,2}$, $I_{2,0}$, $I_{2,3}$, $I_{3,0}$, and $I_{3,1}$ such that:

(i) each $I_{i,j}$ is contained in I_i and $P|_{I_{i,j}} : I_{i,j} \to I_j$ is a homeomorphism;

(ii) and $\mathcal{J}(\mathcal{H}_P) = \{z \in V \mid \forall n \geqslant 0,\ P^n(z) \in V\} \cup \{z \text{ point in } \bigcup_{n\geqslant 0} P^{-n}(\alpha) \cap \overline{V}\}$ where $V = I_{0,1} \cup I_{1,2} \cup I_{2,0} \cup I_{2,3} \cup I_{3,0} \cup I_{3,1}$.

Consequently, we can show as above that the itinerary map $h_2 : \{z \in V \mid \forall n \geqslant 0,\ P^n(z) \in V\} \to \Sigma^\star$ extends to a homeomorphism from $\mathcal{J}(\mathcal{H}_P)$ to $\Sigma^\star$ which conjugates the dynamics of P and σ.

Finally, taking $h = h_2^{-1} \circ h_1$ concludes the proof. □

Note that the proof of Theorem 3 is almost completed. Indeed point (i) comes from Lemma 9 while points (ii) and (iii) follows from Lemma 10 (since $\mathcal{A}$ is, by definition, the set of continua J in $\mathcal{A}^\star$ such that J is not eventually mapped under iteration to the fixed continuum J_α or, equivalently, such that $h(J)$ is not eventually mapped under iteration to the fixed branching point α). It only remains to prove that $\mathcal{A}^\star$ is actually the set $\mathcal{J}_{\text{crit}}(f)$ of all critically separating Julia components of f.

LEMMA 11. *The following equality of sets holds:*

$$\mathcal{A}^\star = \mathcal{J}_{\text{crit}}(f).$$

Proof. Recall that the postcritical set is contained in the forward invariant set $E = D(\beta_{1,2}) \cup D(\beta_{2,3}^-) \cup D(\delta_{3,c}^-) \cup A(\gamma_{0,1}, \gamma_{3,1})$ (see Lemma 7 and Figure 9) and each point of the super-attracting cycle $\{z_0, z_1, z_2, z_3\}$ lies in a different connected component of E. In particular, $J(f)$ is the set of all points whose orbit remains in $\widehat{\mathbb{C}} - E = \overline{A_0} \cup \overline{A_1} \cup \overline{A_2} \cup \overline{A_3} \cup K_\alpha$ where K_α is the complement in $\widehat{\mathbb{C}}$ of $B(\widehat{z_0}) \cup B(\widehat{z_1}) \cup B(\widehat{z_2})$ (see Figure 10).

It follows that every element J in $\mathcal{A}$ is a Julia component. Moreover, J is critically separating as a limit set of nested essential subannuli which separate each super-attracting cycle $\{z_0, z_1, z_2, z_3\}$ (see the proof of Lemma 10). Therefore, $\mathcal{A} \subset \mathcal{J}_{\text{crit}}(f)$.

Similarly, every element J in $\mathcal{A}_\alpha$ is a Julia component. Moreover, recall that J intersects $\overline{U}$ along a limit set of nested essential subannuli which separate each the super-attracting cycle $\{z_0, z_1, z_2, z_3\}$ (see proof of Lemma 10). Therefore, $\mathcal{A}_\alpha \subset \mathcal{J}_{\text{crit}}(f)$ and $\mathcal{A}^\star = \mathcal{A} \cup \mathcal{A}_\alpha \subset \mathcal{J}_{\text{crit}}(f)$.

Conversely, let J be a critically separating Julia component of f. Note that J is not contained in $K_\alpha - J_\alpha$. Indeed, recall that every connected component of $\widehat{\mathbb{C}} - J_\alpha$ is simply connected (see Lemma 1) and that $\partial K_\alpha = \alpha_0 \cup \alpha_1 \cup \alpha_2 \subset J_\alpha$, therefore every connected compact subset of any connected component of $K_\alpha - J_\alpha$ does not separate the postcritical points. Consequently either J is $J_\alpha \in \mathcal{A}_\alpha \subset \mathcal{A}^\star$ or $f^n(J)$ stays in $\overline{A_0} \cup \overline{A_1} \cup \overline{A_2} \cup \overline{A_3}$ for every $n \geqslant 0$. Assume that J is not J_α.

Recall that every connected component of the preimage under f of $\overline{A_0} \cup \overline{A_1} \cup \overline{A_2} \cup \overline{A_3}$, which is contained in this compact union, is contained either in $\overline{U}$ or in some connected components of $\overline{f^{-1}(A_3)}$ included in A_3 (from Lemma 5, see Figure 8), say $\overline{A'_{3,3}}$. However, every $A'_{3,3}$ is not contained in A_3 as essential subannulus, and hence does not separate the postcritical points. In particular, J is not contained in any $\overline{A'_{3,3}}$. Furthermore, J cannot eventually fall in some $\overline{A'_{3,3}}$ after some iterations of f, otherwise $f^n(J)$ would not be postcritically separating for some $n \geqslant 0$ contradicting the fact that J is critically separating. It follows that $f^n(J)$ stays in $\overline{U}$ for every $n \geqslant 0$ and, hence, $J \in \mathcal{A}^\star$, which concludes the proof. □

4.2. *Topology of buried Julia components.* The existence of each of the three types of buried Julia components which occurs in $J(f)$ is shown in this section, which proves Theorem 4.

LEMMA 12. (Point type buried Julia components) *There exist uncountably many buried Julia components in $J(f)$ which are points.*

Proof. Let $A'_{3,3} = A(\beta^+_{3,3}, \beta^-_{3,3})$ be a connected component of $f^{-1}(A_3)$ contained in $A_3 = A(\beta^+_3, \beta^-_3)$ (from Lemma 5, see Figure 8) where $\beta^+_{3,3}$ and $\beta^-_{3,3}$ are preimages of β^+_3 and β^-_3, respectively. Recall that $A'_{3,3}$ is not contained in A_3 as an essential subannulus. In particular, the connected component of $\widehat{\mathbb{C}} - \beta^+_{3,3}$ containing $A'_{3,3}$ is an open disk $D(\beta^+_{3,3})$ contained in A_3 and such that $f|_{D(\beta_{3,3})} : D(\beta^+_{3,3}) \to D$ is a homeomorphism where $D = D(\beta^+_3)$ is the open disk bounded by β^+_3 and containing A_3.

Using notation coming from the proof of Lemma 10, consider the subannulus $A_{3,0,1,2,3}$ contained in A_3 as essential subannulus and such that $f^4|_{A_{3,0,1,2,3}} : A_{3,0,1,2,3} \to A_3$ is a degree $d_3 d_0 d_1 d_2$ covering. Since assumption (H2) implies that at least two of weights d_0, d_1, d_2, and d_3 are $\geqslant 2$ (see Definition 1), it follows that this degree is $\geqslant 2$ and hence, there are at least two disjoint preimages under $f^4|_{A_{3,0,1,2,3}}$ of $D(\beta^+_{3,3})$ in $A_{3,0,1,2,3} \subset A_3 \subset D$, say D_0 and D_1.

Finally we have two disjoint open disks D_0 and D_1 in D such that $f^5|_{D_0} : D_0 \to D$ and $f^5|_{D_1} : D_1 \to D$ are homeomorphisms. It is then a classical exercise to prove that the non-escaping set

$$\mathcal{D} = \{z \in D_0 \cup D_1 \;/\; \forall n \geqslant 0,\ (f^5)^n \in D_0 \cup D_1\}$$

is a Cantor set homeomorphic to the space of all sequences of two digits $\Sigma_2 = \{0, 1\}^{\mathbb{N}}$. In particular, $\mathcal{D}$ contains uncountably many points. Furthermore, every point in $\mathcal{D}$ is a buried point in $J(f)$ since $A_3 \subset D$ contains infinitely many postcritically separating Julia components. □

LEMMA 13. (Circle-type buried Julia components) *There exist uncountably many buried Julia components in $J(f)$ which are wandering Jordan curves.*

Proof. This is mostly a consequence of the main result in [**PT00**] claiming that every wandering Julia component of a geometrically finite rational map is either a point or a Jordan curve. Here our map f is hyperbolic (from Lemma 9), therefore every wandering Julia component in $\mathcal{J}_{\text{crit}}(f)$ must be a Jordan curve (since a point is obviously not critically separating). Moreover, according to the proof of Lemma 10, the set of wandering Julia components in $\mathcal{J}_{\text{crit}}(f)$ exactly corresponds to the set of all of the infinite sequences in $\Sigma^\star$ which are not eventually periodic. In particular, there are uncountably many such Julia components. Finally, uncountably many of them must be buried since the Fatou set only has countably many Fatou domains and each of them only has countably many Jordan curves as connected components of its boundary. □

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LEMMA 14. (Complex-type buried Julia components) *The Julia component J_α and all of its countably many preimages, are buried Julia components in $J(f)$.*

Proof. Coming back to the proof of Lemma 10, recall that every infinite sequence in S_α is not isolated in Σ. Therefore, α_k has no intersection with the boundary of any Fatou domain contained in $B(\widehat{z_k})$ for every $k \in \{0, 1, 2\}$. It remains to show that J_α has no intersection with the boundary of any Fatou domain in $K_\alpha = \widehat{\mathbb{C}} - (B(\widehat{z_0}) \cup B(\widehat{z_1}) \cup B(\widehat{z_2}))$. Recall that every connected component of $K_\alpha - J_\alpha$, that is a connected component of $\widehat{\mathbb{C}} - J_\alpha$, is eventually mapped under iteration onto $B(\widehat{z_k})$ for some $k \in \{0, 1, 2\}$ (since f is defined to be $\widehat{f}$ on $K_\alpha \subset D(\beta_{0,1})$). By the continuity of f, it follows that J_α has no intersection with the boundary of any Fatou domain contained in any connected component of $K_\alpha - J_\alpha$. Consequently J_α is buried. The same holds as well for every preimage of J_α by continuity of f. □

5. *Explicit formula in the cubic case*

In this section, we prove Theorem 1 stated in the introduction (see §1). First, we show that a particular choice of the weight function w gives a rational map of degree three (in Lemma 15). Then we compute an explicit formula for this particular example.

LEMMA 15. *The following weight function on the set of edges of $\mathcal{H}_P$*

$$(d_0, d_1, d_2, d_3) = (1, 2, 2, 1)$$

satisfies assumptions (H1) and (H2) from Theorems 3 and 4. In particular, there are some rational maps of degree three whose Julia set contains buried Julia components of several types:

(i) Point type: *uncountably many points.*
(ii) Circle type: *uncountably many Jordan curves.*
(iii) Complex type: *countably many preimages of a fixed Julia component which is quasiconformally homeomorphic to the connected Julia set of $\widehat{f} : z \mapsto (1/(z-1)^2)$.*

Proof. Assumption (H1) is obviously satisfied, indeed

$$\widehat{d} = \tfrac{1}{2}(d_0 + d_1 + d_2 - 1) = \tfrac{1}{2}(1 + 2 + 2 - 1) = 2 = \max\{d_0, d_1, d_2\}.$$

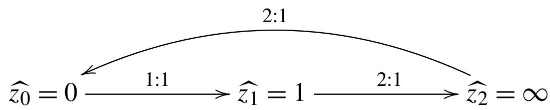
For assumption (H2), the transition matrix (see Definition 1) for this choice of weight function is given by

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

and an easy computation shows that $\lambda(\mathcal{H}_P, w)$ is the largest root of $X^4 - 1/2X - 1/4$ that is $\lambda(\mathcal{H}_P, w) \approx 0.918 < 1$.

Applying Theorems 3 and 4 gives a rational map of degree $\widehat{d} + d_3 = 2 + 1 = 3$.

Furthermore, recall that the rational map $\widehat{f}$ which appears in Theorem 4 has degree $\widehat{d} = 2$ and has only one critical orbit which is a super-attracting cycle $\{\widehat{z_0}, \widehat{z_1}, \widehat{z_2}\}$ of period three such that the local degrees of $\widehat{f}$ at $\widehat{z_0}$, $\widehat{z_1}$ and $\widehat{z_2}$ are $d_0 = 1$, $d_1 = 2$ and $d_2 = 2$, respectively. Up to conjugation by a Möbius map, we may assume that $\widehat{z_0} = 0$, $\widehat{z_1} = 1$, and $\widehat{z_2} = \infty$. It turns out that there is then only one such quadratic rational map which is $\widehat{f} : z \mapsto (1/(z-1)^2)$:



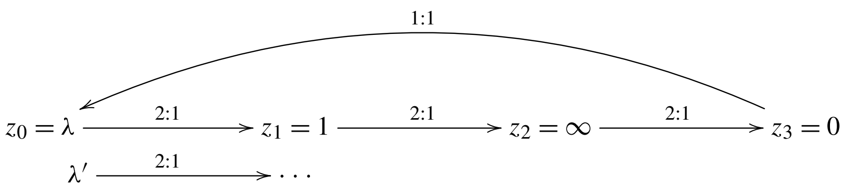
□

Note that this choice of weight function is the only one which gives a degree three and which satisfies assumptions (H1) and (H2).

The construction by quasiconformal surgery detailed in §3 does not provide an algebraic formula for the rational map f in Theorems 3 and 4. Furthermore the degree $\widehat{d} + d_3$ of f increases quickly with the weight function w so the algebraic relations behind are complicated to study. However, the particular rational map of degree three coming from Lemma 15 is simple enough to allow a computation by hand of an algebraic formula.

Let f be a rational map coming from the construction detailed in §3 for the particular choice of weight function in Lemma 15. Recall that the local degrees of f at z_1, z_2, and z_3 are $d_1 = 2$, $d_2 = 2$, and $d_3 = 1$, respectively. In particular, z_1 and z_2 are simple critical points. There remain $d_0 + d_3 = 1 + 1 = 2$ critical points counted with multiplicity coming from definition of f near z_0 (see Lemma 3), namely two simple critical points, one is z_0 by construction and the orbit of the other one accumulates the super-attracting cycle $\{z_0, z_1, z_2, z_3\}$.

Up to conjugation by a Möbius map, we assume that $z_1 = 1$, $z_2 = \infty$ and $z_3 = 0$. So 1 and ∞ are critical points whereas 0 is a singular point. In order to simplify notation, denote by λ the critical point z_0 (λ will be the parameter of our family) and by λ' the last critical point:



Since f has degree three, it is of the form

$$f : z \mapsto \frac{a_3 z^3 + a_2 z^2 + a_1 z + a_0}{b_3 z^3 + b_2 z^2 + b_1 z + b_0}.$$

Since $z_1 = 1$ is mapped to $z_2 = \infty$ with a local degree two, the denominator may factor as

$$f : z \mapsto \frac{a_3 z^3 + a_2 z^2 + a_1 z + a_0}{(z-1)^2 (b_1' z + b_0')}.$$

We do likewise for $z_2 = \infty$ which is mapped to $z_3 = 0$ with a local degree two:

$$f : z \mapsto \frac{a_1 z + a_0}{(z-1)^2 (b_1' z + b_0')}.$$

Now use the fact that $z_3 = 0$ is mapped to $z_0 = \lambda$ to get

$$f : z \mapsto \frac{a_1 z + \lambda}{(z-1)^2 (b_1' z + 1)}. \tag{3}$$

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There remain two pieces of information coming from the fact that $z_0 = \lambda$ is mapped to $z_1 = 1$ with a local degree two. Namely $f(\lambda) = 1$ and $f'(\lambda) = 0$ which lead to the two following equations satisfied by a_1 and b'_1.

$$\begin{cases} (\lambda-1)^2(\lambda b'_1+1) = \lambda(a_1+1), \\ a_1(\lambda-1)^2(\lambda b'_1+1) = \lambda(a_1+1)[(3\lambda^2-4\lambda+1)b'_1+2(\lambda-1)]. \end{cases}$$

Note that we may easily simplify the second equation by using the first (luckily)

$$\begin{cases} (\lambda-1)^2(\lambda b'_1+1) = \lambda(a_1+1), \\ a_1 = (3\lambda^2-4\lambda+1)b'_1+2(\lambda-1), \end{cases}$$

or, equivalently,

$$\begin{cases} \lambda a_1 - \lambda(1-\lambda)^2 b'_1 = 1-3\lambda+\lambda^2, \\ a_1 - (1-\lambda)(1-3\lambda)b'_1 = -2+2\lambda, \end{cases}$$

and solving this linear system of two equations gives

$$\begin{cases} a_1 = \dfrac{(1-3\lambda)(1-3\lambda+\lambda^2)-\lambda(1-\lambda)(-2+2\lambda)}{\lambda(1-3\lambda)-\lambda(1-\lambda)} = \dfrac{1-4\lambda+6\lambda^2-\lambda^3}{-2\lambda^2}, \\ b'_1 = \dfrac{(1-3\lambda+\lambda^2)-\lambda(-2+2\lambda)}{-\lambda(1-\lambda)^2+\lambda(1-\lambda)(1-3\lambda)} = \dfrac{1-\lambda-\lambda^2}{-2\lambda^2(1-\lambda)}. \end{cases}$$

Finally, putting these expressions in expression (3) leads to the following formula for f which depends on the parameter λ:

$$\boxed{f_\lambda : z \mapsto \frac{(1-\lambda)[(1-4\lambda+6\lambda^2-\lambda^3)z-2\lambda^3]}{(z-1)^2[(1-\lambda-\lambda^2)z-2\lambda^2(1-\lambda)]}}.$$

Note that $f_\lambda(z) = (1/(z-1)^2)(1-4\lambda+O_{\lambda\to 0}(\lambda^2))$ for every complex number z, thus f_λ is actually a particular perturbation of $f_0 = \widehat{f} : z \mapsto (1/(z-1)^2)$.

Some more computations provide an algebraic formula for the critical point λ', namely

$$\lambda' = -\frac{\lambda(1-6\lambda+11\lambda^2-10\lambda^3+5\lambda^4)}{(1-\lambda-\lambda^2)(1-4\lambda+6\lambda^2-\lambda^3)} = -\lambda + \underset{\lambda\to 0}{O}(\lambda^2).$$

According to the construction detailed in §3, there exist some choices of λ such that f_λ satisfies Theorem 1. Recall that the two critical points $z_0 = \lambda$ and $\lambda' \sim_{\lambda\to 0} -\lambda$ should lie in $B(\widehat{z_0})$ (see §3), and hence near $\widehat{z_0}$ which corresponds to $z_3 = 0$. Indeed, we can roughly prove for every $|\lambda| > 0$ small enough that:

(i) $f_\lambda(\lambda')$ lies in a disk centered at $z_1 = 1$ and of radius of order $|\lambda|$;

(ii) the image under f_λ of a disk centered at $z_1 = 1$ and of radius of order $|\lambda|$ is contained in the complement of a disk centered at 0 (thus containing $z_2 = \infty$) and of radius of order $|\lambda|^{-2}$;

(iii) the image under f_λ of the complement of a disk centered at 0 (thus containing $z_2 = \infty$) and of radius of order $|\lambda|^{-2}$ is contained in a disk centered at $z_3 = 0$ and of radius of order $|\lambda|^4$;

(iv) the image under f_λ of a disk centered at $z_3 = 0$ and of radius of order $|\lambda|^4$ is contained in a disk centered at $z_0 = \lambda$ and of radius of order $|\lambda|^2$;

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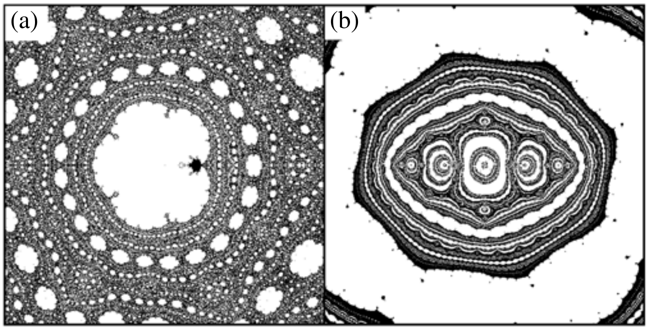

FIGURE 11. (a) The parameter plane of f_λ for $|\lambda| \lesssim 10^{-2}$, that includes the bifurcation locus (in black) and hyperbolic parameters (in white); 0 is at the center of the picture and the big hyperbolic component around corresponds to the Persian carpets. (b) The dynamical plane for $\lambda \approx 10^{-3}$, that includes the Persian carpet $J(f_\lambda)$ (in black) and the Fatou set (in white).

(v) the image under f_λ of a disk centered at $z_0 = \lambda$ and of radius of order $|\lambda|^2$ is contained in a disk centered at $z_1 = 1$ and of radius of order $|\lambda|^3$.

It turns out that the orbit of the critical point λ' accumulates the super-attracting cycle $\{z_0, z_1, z_2, z_3\}$ for every $|\lambda| > 0$ small enough. Consequently, we may encode the exchanging dynamics of Julia components of f_λ as it is explained in §4, proving that f_λ satisfies Theorem 1 for every $|\lambda| > 0$ small enough.

Numerically, picking any parameter λ in the big hyperbolic component surrounding 0 of the parameter space of the family f_λ (see Figure 11(a)) provides a Persian Carpet example in the dynamical plane (see Figure 11(b)).

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A. *Appendix*

In this section, we collect some technical results used in the construction of §3.

A.1. *A particular solution of the Hurwitz problem.*

The first result of this section deals with the Hurwitz problem on the topological sphere $\mathbb{S}^2$. Namely given an abstract branch data of degree $d \geqslant 2$, that is a table of positive integers $\mathcal{D} = (d_{i,j})_{(i,j)\in\mathcal{I}}$ where $\mathcal{I} = \{(i, j) \mid i \in \{1, 2, \ldots, n\} \text{ and } j \in \{1, 2, \ldots, k_i\}\}$ for some positive integers $n, k_1, k_2, \ldots, k_n$ and such that for every $i \in \{1, 2, \ldots, n\}$:

$$d_{i,j} \geqslant 2 \quad \text{for some } j \in \{1, 2, \ldots, k_i\}, \quad \text{and} \quad \sum_{j=1}^{k_i} d_{i,j} = d, \tag{A1}$$

we consider the question on realizability of this abstract branch data by a branched covering on $\mathbb{S}^2$, that is the existence of a degree d branched covering $H : \mathbb{S}^2 \to \mathbb{S}^2$ and a finite collection of distinct points $X = \{x_{i,j} \mid (i, j) \in \mathcal{I}\}$ in $\mathbb{S}^2$ such that:

(i) for all $(i, j) \in \mathcal{I}$, $H(x_{i,j}) = y_i$ for some $y_i \in \mathbb{S}^2$;
(ii) $H|_{\mathbb{S}^2 - X} : \mathbb{S}^2 - X \to \mathbb{S}^2 - \{y_i \mid i \in \{1, 2, \ldots, n\}\}$ is a degree d covering;
(iii) for all $(i, j) \in \mathcal{I}$, the local degree of H at $x_{i,j}$ is $d_{i,j}$.

Adolf Hurwitz has provided (see [**Hur91**]) a necessary and sufficient condition in terms of symmetric group (see also [**Bar01**] for another approach). In particular, the following lemma gives the solution in a very specific case involved in Lemma 1.

LEMMA A.1. (Hurwitz solution) *Let $\mathcal{D}$ be an abstract branch data of degree $d \geqslant 2$ such that $n = 3$ and $d_{i,j} = 1$ for every $i \in \{1, 2, 3\}$ and $j \geqslant 2$. Then $\mathcal{D}$ is realizable if and only if the following condition is satisfied:*

$$d = \tfrac{1}{2}(d_{1,1} + d_{2,1} + d_{3,1} - 1). \qquad \text{(H1}'\text{)}$$

Note that in this special case the abstract branch data $\mathcal{D}$ is uniquely determined by a degree $d \geqslant 2$ together with three positive integers $d_{1,1}$, $d_{2,1}$, and $d_{3,1}$ such that $2 \leqslant d_{i,1} \leqslant d$ for every $i \in \{1, 2, 3\}$.

A.2. *An inverse Grötzsch's inequality.* The following useful result is due to Cui Guizhen and Tan Lei [**CT11**]. It is the key ingredient of the proof of Lemma 2.

LEMMA A.2. (Inverse Grötzsch's inequality) *Let D, D' be two disjoint marked hyperbolic disks in $\widehat{\mathbb{C}}$ whose boundaries (not necessarily disjoint) are respectively denoted by α, α'. Then there exists a positive constant $C > 0$ such that for every pair of equipotentials β in D and β' in D' the following inequalities hold:*

$$\begin{aligned}\operatorname{mod}(A(\alpha, \beta)) + \operatorname{mod}(A(\alpha', \beta')) &\leqslant \operatorname{mod}(A(\beta, \beta'))\\ &\leqslant \operatorname{mod}(A(\alpha, \beta)) + \operatorname{mod}(A(\alpha', \beta')) + C.\end{aligned}$$

The left-hand side is the classical Grötzsch's inequality. The right-hand side is a consequence of the Koebe 1/4 theorem. We refer the reader to [**CT11**] for a complete proof.

A.3. *An annulus-disk holomorphic map.* The following lemma is a technical ingredient in the construction of §3 needed to holomorphically map an annulus onto a disk (see Lemma 3). It is very similar to the key lemma in [**PT99**] (see also [**BF13**]) about an annulus-disk branched covering. However, our annulus-disk map here is required to be holomorphic (see Lemmas 7 and 8).

LEMMA A.3. (Annulus-disk holomorphic map) *Let n, n' be two positive integers. Then there exists a holomorphic branched covering $G : A(\gamma, \gamma') \to \mathbb{D}$ from an open annulus in $\widehat{\mathbb{C}}$ bounded by a pair of disjoint quasicircles γ, γ' onto the open unit disk $\mathbb{D}$ such that:*
(i) *G has degree $n + n'$ and has $n + n'$ critical points counted with multiplicity;*
(ii) *G continuously extends to $\gamma \cup \gamma'$ by a degree n covering $G|_\gamma : \gamma \to \partial\mathbb{D}$ and a degree n' covering $G|_{\gamma'} : \gamma' \to \partial\mathbb{D}$;*
(iii) $\operatorname{mod}(A(\gamma, \gamma')) \leqslant 1$.

There are many ways to prove the existence of such a map. One of the simplest is to use the properties of McMullen's family

$$g_{0,\lambda} : z \mapsto z^n + \frac{\lambda}{z^{n'}}$$

for $|\lambda| > 0$ small enough (see [**McM88**, **DHL08**] for a complete study of this family). Recall that $g_{0,\lambda}$ has degree $n + n'$, and has $n + n'$ simple critical points which are mapped near 0. It is straightforward to prove that the preimage of the open unit disk $\mathbb{D}$, namely $A = g_{0,\lambda}^{-1}(\mathbb{D})$ is an open annulus separating 0 and ∞, and that A is contained as an essential subannulus in a round annulus of modulus 1.

The constant 1 is obviously not the optimal upper bound for $\mathrm{mod}(A)$. The author guesses that this modulus is arbitrarily small when λ is close to 0. But one can prove that the modulus of the smallest round annulus containing A as an essential subannulus is bounded by below by a positive constant which does not depend on λ. The same happens if the open unit disk $\mathbb{D}$ is replaced by any Euclidean open disk centered at 0 and containing the critical values. However, we do not need a sharper estimation than (iii) in this paper (see Lemma 2 and the proof of Lemma 3).

A.4. *A separating quasicircle.* The following lemma is used to define the quasicircle δ_c^+ in the construction of §3 (see the proof of Lemma 5).

LEMMA A.4. (Separating quasicircle) *Let $A(\gamma, \gamma')$ be an open annulus in $\widehat{\mathbb{C}}$ bounded by a pair of disjoint quasicircles γ, γ', and let a be a point in $A(\gamma, \gamma')$. Then there exists a quasicircle δ in $A(\gamma, \gamma')$ which separates a from $\gamma \cup \gamma'$ such that* $\mathrm{mod}(A(\gamma, \delta))$ *is arbitrarily small.*

The main idea is merely to define a quasicircle δ close enough to the boundary γ, and to use the definition of the modulus by extremal length (see [**Ahl73**]).

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