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Analytical forms of the first 14 moments of the cosmic ray Fokker–Planck equation

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The Fokker–Planck transport equation describing the motion of energetic particles through a plasma is explored analytically. The latter equation provides a pitch-angle and position-dependent distribution function of the charged particles. In the current paper the first 14 moments of this equation are computed exactly for an arbitrary initial pitch angle. Such analytical forms are required in nonlinear treatments of perpendicular transport and other scenarios in plasma physics and astrophysics.

Key words: energetic particles, magnetized plasma, turbulence

1. Introduction

To understand the propagation of energetic particles through a plasma is a very fundamental problem with a variety of applications in plasma physics, space science and astrophysics. The motion of such particles is described by transport equations and their solutions provide a particle distribution function. As explained in detail in Schlickeiser (2002) there are different levels at which the transport can be described. The most fundamental description is provided by the relativistic Vlasov equation from which a pitch-angle-dependent Fokker–Planck equation can be derived. The latter equation can contain different transport processes ranging from pitch-angle scattering to perpendicular diffusion, stochastic acceleration and adiabatic focusing. It is usually assumed that if the aforementioned Fokker–Planck equation is averaged over all pitch angles, and if a late time limit is considered, one obtains a diffusive transport equation. Pitch-angle-dependent and averaged cosmic ray transport equations are solved in order to describe the acceleration of particles at shock waves or their motion through the solar system, the interstellar medium or the extra galactic space (see, e.g. Li *et al.* 2012; Ferrand *et al.* 2014; Zank 2014; Engelbrecht & Burger 2015; Miyake, Muraishi & Yanagita 2015; Mulcahy *et al.* 2016; Porth *et al.* 2016; Strauss, Dresing & Engelbrecht 2017).

For some applications one could concentrate on the motion of charged particles along a mean magnetic field. Usually this type of transport is called parallel diffusion and it is assumed that parallel transport is the most important process. For example it controls the life time of cosmic rays in the Milky Way (see, e.g. Swordy *et al.* 1990; Schlickeiser 2002; Shalchi & Schlickeiser 2005). Therefore, one can focus on the analytical and numerical study of the two-dimensional Fokker–Planck equation

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which provides the particle distribution as a function of time, parallel position and pitch angle.[1] This type of distribution function also enters nonlinear theories for perpendicular transport, as shown in Shalchi (2010, 2017).

An exact analytical solution of the Fokker–Planck equation is difficult to find. Although some progress has been made recently (see Malkov 2017), such solutions are either based on approximations or involve numerical calculations. For some applications such as the formulation of nonlinear theories for perpendicular transport, exact and pure analytical solutions are desired.

It is the purpose of the current paper to derive exact analytical forms of the first 14 moments of the pitch-angle-dependent cosmic ray Fokker–Planck equation for an arbitrary initial pitch angle. Such explicit formulas for the moments can be important for different applications as also demonstrated in the present article. In all cases we recover the formulas derived previously for the initial pitch-angle-averaged case (see, e.g. Malkov 2017).

The remainder of this paper is organized as follows. After discussing transport equations and some general properties in § 2, the moments are derived step by step in § 3. In § 4 we consider the characteristic function in nonlinear diffusion theory as an example for the applicability of our findings. In § 4 we summarize and conclude.

2. Transport equations

The general Fokker–Planck equation of cosmic ray transport is complicated and contains several terms such as perpendicular diffusion or stochastic acceleration (see, e.g. Skilling 1975; Schlickeiser 2002; Zank 2014). For some applications one can neglect such terms and consider the limit that both the magnetic field and the plasma flow are weakly non-uniform. In this case one obtains a simpler, two-dimensional version of the Fokker–Planck equation which is valid in the plasma flow frame (see, e.g. Schlickeiser 2002, for a detailed derivation and discussion)

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\left[D_{\mu\mu}(\mu) \frac{\partial f}{\partial \mu} \right], \tag{2.1}$$

where we have used the pitch-angle Fokker–Planck coefficient $D_{\mu\mu}$. The solution of this equation provides the particle distribution function $f = f(\mu, z, t)$ where we have used time t, the particle position along the mean magnetic field z and the pitch-angle cosine μ. It has to be emphasized that the parameter $D_{\mu\mu}$ still depends on the particle speed or momentum. The latter parameters, however, are just parameters which do not change the form of the solution $f = f(\mu, z, t)$. If the latter function is averaged over all values of μ, and if a late time limit is considered, one finds a usual diffusion equation of the form (see, e.g. Schlickeiser 2002; Shalchi 2009; Zank 2014)

$$\frac{\partial M}{\partial t} = \kappa_{\parallel} \frac{\partial^2 M}{\partial z^2}, \tag{2.2}$$

where we have used the pitch-angle-averaged distribution function

$$M(z, t) = \frac{1}{2} \int_{-1}^{+1} \mathrm{d}\mu\, f(\mu, z, t). \tag{2.3}$$

[1]In reality the corresponding Fokker–Planck equation provides a solution which depends on the three spatial coordinates, gyro-phase, pitch angle and particle speed or momentum. For some applications, perpendicular diffusion, gyro-phase diffusion and momentum diffusion can be neglected because those effects are usually weaker than pitch-angle scattering. In such cases particle momentum is just a parameter entering the equation via the pitch-angle Fokker–Planck coefficient.

Equation (2.2) is also known as the heat transport equation. The parallel spatial diffusion coefficient $\kappa_\parallel$ therein is related to the pitch-angle Fokker–Planck coefficient $D_{\mu\mu}$ via the famous relation (see, e.g. Earl (1974) for a systematic derivation)

$$\kappa_\parallel = \frac{v^2}{8}\int_{-1}^{+1} \mathrm{d}\mu\, \frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)}. \tag{2.4}$$

It has to be emphasized that generalizations can be found in the literature in which a telegrapher equation has been derived and discussed in the context of cosmic ray transport (see, e.g. Litvinenko & Schlickeiser 2013; Litvinenko *et al.* 2015; Tautz & Lerche 2016; Malkov 2017).

For sharp initial conditions $M(z, t=0) = \delta(z)$, equation (2.2) has a Gaussian solution of the form

$$M(z, t) = \frac{1}{\sqrt{4\pi\kappa_\parallel t}} \mathrm{e}^{-z^2/(4\kappa_\parallel t)}. \tag{2.5}$$

The characteristic function of a transport equation is defined via

$$\langle \mathrm{e}^{\mathrm{i}kz}\rangle = \frac{1}{2}\int_{-1}^{+1} \mathrm{d}\mu \int_{-\infty}^{+\infty} \mathrm{d}z\, \mathrm{e}^{\mathrm{i}kz} f(\mu, z, t) \equiv \int_{-\infty}^{+\infty} \mathrm{d}z\, \mathrm{e}^{\mathrm{i}kz} M(z, t) \tag{2.6}$$

corresponding to the Fourier transform of the distribution function. The characteristic function of a usual diffusion equation (see (2.2) of the current paper) is given by

$$\langle \mathrm{e}^{\mathrm{i}kz}\rangle = \mathrm{e}^{-\kappa_\parallel k^2 t}. \tag{2.7}$$

The latter function can easily be derived by combining (2.5) and (2.6). The characteristic function will be discussed in more detail in § 4 of this paper.

3. The moments of the Fokker–Planck equation

It is the purpose of the current article to derive analytical forms for the first 14 moments of (2.1). In order to do this we need to specify the scattering parameter $D_{\mu\mu}$. In the following we employ the isotropic model (see, e.g. Shalchi *et al.* (2009) for a justification of this form and more details)

$$D_{\mu\mu}(\mu) = D\left(1-\mu^2\right), \tag{3.1}$$

and the Fokker–Planck equation (2.1) becomes

$$\frac{\partial f}{\partial t} + v\mu\frac{\partial f}{\partial z} = D\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial f}{\partial \mu}\right]. \tag{3.2}$$

The parameter D used here is just a constant in the sense that it does not depend on μ. However, D can depend on particle properties such as momentum as well as magnetic field properties.

Some moments of the Fokker–Planck equation have been computed before (see, e.g. Shalchi 2006). In order to derive a general relation for the moments we follow

Malkov (2017) and multiply (3.2) by $z^n\mu^m$, integrate over all positions and average over all pitch angles. We find

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z^n\mu^m\rangle = nv\langle z^{n-1}\mu^{m+1}\rangle + m(m-1)D\langle z^n\mu^{m-2}\rangle - m(m+1)D\langle z^n\mu^m\rangle, \tag{3.3}$$

where we have used

$$\langle z^n\mu^m\rangle = \frac{1}{2}\int_{-1}^{+1}\mathrm{d}\mu\int_{-\infty}^{+\infty}\mathrm{d}z\, z^n\mu^m f(\mu, z, t) \tag{3.4}$$

and $f(\mu, z = \pm\infty, t) = 0$. If the moments $\langle z^{n-1}\mu^{m+1}\rangle$ and $\langle z^n\mu^{m-2}\rangle$ are known, differential equation (3.3) can be solved analytically. In the following paragraphs we compute the first 14 moments by solving (3.3). Alternatively, the moments could be derived from equation (8) of Malkov (2017). The latter equation provides an inductive algorithm to compute moments by integration of exponential and power functions.

3.1. *The moment* $\langle\mu\rangle$

For $n = 0$ and $m = 1$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\mu\rangle = -2D\langle\mu\rangle. \tag{3.5}$$

We can easily solve the latter ordinary differential equation. By using the initial pitch-angle cosine μ_0 the solution is

$$\langle\mu\rangle = \mu_0\,\mathrm{e}^{-2Dt}. \tag{3.6}$$

If we also average over all initial pitch-angle cosine values, this becomes

$$\langle\langle\mu\rangle\rangle = 0, \tag{3.7}$$

for all times. Here we have used the notation $\langle\langle\ldots\rangle\rangle$ which stands for average over μ and μ_0, i.e.

$$\langle\langle A\rangle\rangle = \frac{1}{2}\int_{-1}^{+1}\mathrm{d}\mu_0\langle A\rangle = \frac{1}{4}\int_{-1}^{+1}\mathrm{d}\mu_0\int_{-1}^{+1}\mathrm{d}\mu\int_{-\infty}^{+\infty}\mathrm{d}z\, A\left(\mu_0, \mu, z, t\right). \tag{3.8}$$

However, the correlation between the initial pitch-angle cosine μ_0 and μ is given by

$$\langle\langle\mu_0\mu\rangle\rangle = \frac{1}{2}\int_{-1}^{+1}\mathrm{d}\mu_0\,\mu_0^2\mathrm{e}^{-2Dt} = \frac{1}{3}\mathrm{e}^{-2Dt}. \tag{3.9}$$

Therefore, the velocity correlation function is

$$\langle\langle v_z(t)v_z(0)\rangle\rangle = \frac{v^2}{3}\mathrm{e}^{-vt/\lambda_\parallel}, \tag{3.10}$$

where we have used the parallel mean free path

$$\lambda_\parallel = v/(2D). \tag{3.11}$$

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Velocity correlation functions of the form (3.10) are often used in theories for perpendicular diffusion (see, e.g. Owens 1974; Matthaeus *et al.* 2003) but this exponential form is only correct for an isotropic $D_{\mu\mu}$ (see Shalchi (2011*a*) for a detailed discussion of this matter).

The so-called Taylor–Green–Kubo formula (see Taylor 1922; Green 1951; Kubo 1957) allows us to compute a running diffusion coefficient via

$$\begin{aligned} d_{\parallel}(\mu_0, t) &= \int_0^t \mathrm{d}\tau \, \langle v_z(\tau) v_z(0) \rangle \\ &= v^2 \int_0^t \mathrm{d}\tau \, \mu_0^2 \mathrm{e}^{-2D\tau} \\ &= \frac{v^2 \mu_0^2}{2D} \left(1 - \mathrm{e}^{-2Dt}\right), \end{aligned} \tag{3.12}$$

where we have employed (3.6) again. In the limit $t \to \infty$ this becomes

$$\kappa_{\parallel}(\mu_0) = \frac{v^2 \mu_0^2}{2D} \tag{3.13}$$

and if we average over all initial pitch-angle cosine values we obtain

$$\kappa_{\parallel} = \frac{v^2}{6D}. \tag{3.14}$$

The latter form can alternatively be obtained by combining the isotropic form (3.1) with Earl's relation (2.4). If we average (3.12) directly over μ_0, we derive

$$d_{\parallel}(t) = \kappa_{\parallel} \left(1 - \mathrm{e}^{-vt/\lambda_{\parallel}}\right). \tag{3.15}$$

For $t \to 0$ we can expand the exponential in (3.12) to derive

$$d_{\parallel}(\mu_0, t) \to v^2 \mu_0^2 t, \tag{3.16}$$

corresponding to ballistic transport. In figure 1 we visualize (3.15) to show the turnover from the initial ballistic regime to the normal diffusive regime.

3.2. *The moment* $\langle \mu^2 \rangle$

For $n = 0$ and $m = 2$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \mu^2 \rangle = 2D \left(1 - 3\langle \mu^2 \rangle\right). \tag{3.17}$$

The latter equation has the homogeneous solution

$$\langle \mu^2 \rangle_h = C\mathrm{e}^{-6Dt}, \tag{3.18}$$

with the constant C which will be determined below. Furthermore, a particular solution is provided by the constant

$$\langle \mu^2 \rangle_p = \tfrac{1}{3}. \tag{3.19}$$

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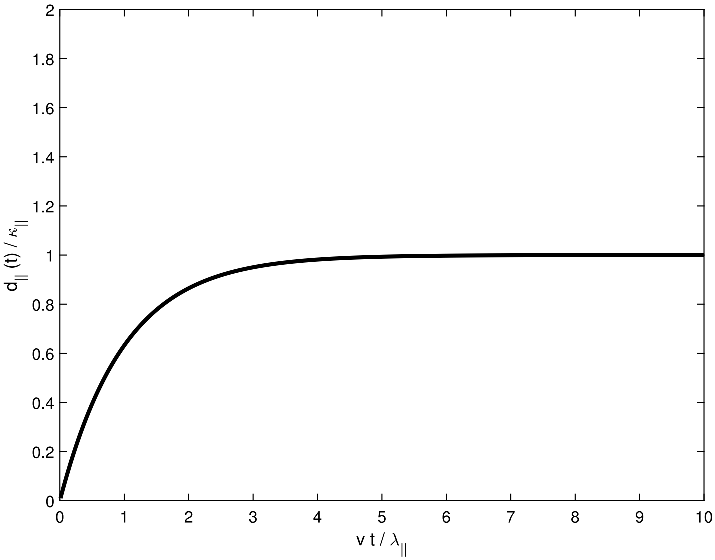

FIGURE 1. The running diffusion coefficient normalized with respect to the late time diffusion coefficient $d_{\parallel}(t)/\kappa_{\parallel}$ versus dimensionless time $vt/\lambda_{\parallel}$.

By using again the initial pitch angle μ_0, we can write the solution of (3.17) as a superposition of homogeneous and particular solutions

$$\langle\mu^2\rangle = \left(\mu_0^2 - \tfrac{1}{3}\right) \mathrm{e}^{-6Dt} + \tfrac{1}{3}. \tag{3.20}$$

For $t \to \infty$ we obtain $\langle\mu^2\rangle \to 1/3$ as expected. This asymptotic limit does not depend on the initial pitch-angle cosine μ_0 due to the pitch-angle isotropization process. If (3.20) is averaged over all μ_0 we derive

$$\langle\langle\mu^2\rangle\rangle = \tfrac{1}{3} \tag{3.21}$$

in agreement with the formula derived in Malkov (2017).

3.3. *The moment* $\langle\mu^3\rangle$

For $n = 0$ and $m = 3$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\mu^3\rangle = 6D\langle\mu\rangle - 12D\langle\mu^3\rangle, \tag{3.22}$$

where the moment $\langle\mu\rangle$ is given by (3.6). The homogeneous solution of (3.22) is

$$\langle\mu^3\rangle_h = C\mathrm{e}^{-12Dt}. \tag{3.23}$$

In order to derive the particular solution, we combine (3.6) and (3.22)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\mu^3\rangle = 6D\mu_0\mathrm{e}^{-2Dt} - 12D\langle\mu^3\rangle, \tag{3.24}$$

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which has the particular solution

$$\langle \mu^3 \rangle_p = A\mathrm{e}^{-2Dt}. \tag{3.25}$$

If the latter *ansatz* is combined with (3.22), we derive

$$A = \tfrac{3}{5}\mu_0 \tag{3.26}$$

and, thus, the total solution is

$$\langle \mu^3 \rangle = C\mathrm{e}^{-12Dt} + \tfrac{3}{5}\mu_0 \mathrm{e}^{-2Dt}. \tag{3.27}$$

The remaining parameter C can be replaced by the initial pitch-angle cosine μ_0 and we finally obtain

$$\langle \mu^3 \rangle = \left(\mu_0^3 - \tfrac{3}{5}\mu_0\right) \mathrm{e}^{-12Dt} + \tfrac{3}{5}\mu_0 \mathrm{e}^{-2Dt}. \tag{3.28}$$

For $t \to \infty$ we find $\langle \mu^3 \rangle \to 0$ due to the pitch-angle isotropization process. Furthermore, we derive $\langle\!\langle \mu^3 \rangle\!\rangle = 0$ as expected due to symmetry.

3.4. *The moment* $\langle \mu^4 \rangle$

For $n = 0$ and $m = 4$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mu^4 \rangle = 12D\langle \mu^2 \rangle - 20D\langle \mu^4 \rangle, \tag{3.29}$$

where the moment $\langle \mu^2 \rangle$ is given by (3.20). The homogeneous solution of (3.29) is

$$\langle \mu^4 \rangle_h = C\mathrm{e}^{-20Dt}. \tag{3.30}$$

To compute the particular solution, we combine (3.29) and (3.20) to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mu^4 \rangle = 12D\left[\left(\mu_0^2 - \frac{1}{3}\right)\mathrm{e}^{-6Dt} + \frac{1}{3}\right] - 20D\langle \mu^4 \rangle. \tag{3.31}$$

In order to determine the particular solution, we employ the *ansatz*

$$\langle \mu^4 \rangle_p = A + B\mathrm{e}^{-6Dt}. \tag{3.32}$$

By using this in (3.31) we find after straightforward algebra

$$A = \tfrac{1}{5} \tag{3.33}$$

and

$$B = \tfrac{6}{7}\left(\mu_0^2 - \tfrac{1}{3}\right). \tag{3.34}$$

Therefore, the total solution of (3.29) is given by

$$\langle \mu^4 \rangle = C\mathrm{e}^{-20Dt} + \tfrac{1}{5} + \tfrac{6}{7}\left(\mu_0^2 - \tfrac{1}{3}\right)\mathrm{e}^{-6Dt}. \tag{3.35}$$

The remaining constant C can be replaced by the initial pitch-angle cosine μ_0. We obtain

$$\langle \mu^4 \rangle = \left[\mu_0^4 - \tfrac{6}{7}\left(\mu_0^2 - \tfrac{1}{3}\right) - \tfrac{1}{5}\right]\mathrm{e}^{-20Dt} + \tfrac{1}{5} + \tfrac{6}{7}\left(\mu_0^2 - \tfrac{1}{3}\right)\mathrm{e}^{-6Dt}. \tag{3.36}$$

In the late time limit $t \to \infty$ we find $\langle \mu^4 \rangle \to 1/5$ as expected. Furthermore, we can average (3.36) over the initial pitch-angle cosine μ_0 to deduce

$$\langle\!\langle \mu^4 \rangle\!\rangle = \tfrac{1}{5}. \tag{3.37}$$

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3.5. *The moment* $\langle z\rangle$

For $n=1$ and $m=0$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z\rangle = v\langle\mu\rangle. \tag{3.38}$$

The quantity at the right-hand side can be replaced by (3.6) and we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z\rangle = v\mu_0 \mathrm{e}^{-2Dt}. \tag{3.39}$$

We can easily integrate the latter formula to derive

$$\langle z\rangle = \frac{v\mu_0}{2D}\left(1-\mathrm{e}^{-2Dt}\right) \equiv \mu_0\lambda_\parallel\left(1-\mathrm{e}^{-vt/\lambda_\parallel}\right). \tag{3.40}$$

For $t\to\infty$ we find for the penetration depth

$$\langle z\rangle_{t\to\infty} = \frac{v\mu_0}{2D} \equiv \mu_0\lambda_\parallel. \tag{3.41}$$

Therefore, the maximal penetration depth is $\pm\lambda_\parallel$. If we average (3.40) over all initial pitch angles we find $\langle\langle z\rangle\rangle=0$. For $t\to 0$, on the other hand, we derive from (3.40)

$$\langle z\rangle \to v\mu_0 t \tag{3.42}$$

corresponding to the unperturbed motion of the particle.

3.6. *The moment* $\langle z\mu\rangle$

For $n=m=1$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z\mu\rangle = v\langle\mu^2\rangle - 2D\langle z\mu\rangle. \tag{3.43}$$

The homogeneous solution is provided by

$$\langle z\mu\rangle_h = C\mathrm{e}^{-2Dt}. \tag{3.44}$$

To obtain the particular solution, we first replace $\langle\mu^2\rangle$ at the right-hand side of (3.43) by (3.20) to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z\mu\rangle + 2D\langle z\mu\rangle = v\left(\mu_0^2-\frac{1}{3}\right)\mathrm{e}^{-6Dt} + \frac{v}{3}. \tag{3.45}$$

The latter equation has the particular solution

$$\langle z\mu\rangle_p = \frac{v}{4D}\left(\frac{1}{3}-\mu_0^2\right)\mathrm{e}^{-6Dt} + \frac{v}{6D}. \tag{3.46}$$

By superposing homogeneous and particular solutions, and by using $\langle z\mu\rangle=0$ for $t=0$, we obtain

$$\langle z\mu\rangle = \frac{v}{4D}\left(\mu_0^2-1\right)\mathrm{e}^{-2Dt} + \frac{v}{4D}\left(\frac{1}{3}-\mu_0^2\right)\mathrm{e}^{-6Dt} + \frac{v}{6D}. \tag{3.47}$$

Again we can consider the limit $t \rightarrow \infty$ to find

$$\langle v_z(t)\Delta z(t)\rangle \equiv v\langle z\mu\rangle = \frac{v^2}{6D} = \kappa_\parallel \tag{3.48}$$

corresponding to the parallel diffusion coefficient. Furthermore, we can average (3.47) also over the initial pitch-angle cosine μ_0 to find

$$v\langle\langle z\mu\rangle\rangle = \frac{v^2}{6D}\left(1-\mathrm{e}^{-2Dt}\right) \equiv \kappa_\parallel\left(1-\mathrm{e}^{-vt/\lambda_\parallel}\right) \tag{3.49}$$

in agreement with (3.15).

3.7. *The moment* $\langle z\mu^2\rangle$

For $n=1$ and $m=2$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z\mu^2\rangle = v\langle\mu^3\rangle + 2D\langle z\rangle - 6D\langle z\mu^2\rangle, \tag{3.50}$$

where the moments $\langle\mu^3\rangle$ and $\langle z\rangle$ are given by (3.28) and (3.40), respectively. The homogeneous solution of (3.50) is

$$\langle z\mu^2\rangle_h = C\mathrm{e}^{-6Dt}. \tag{3.51}$$

In order to compute the particular solution, we combine (3.28) and (3.40) with (3.50) to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z\mu^2\rangle = v\left(\mu_0^3 - \frac{3}{5}\mu_0\right)\mathrm{e}^{-12Dt} - \frac{2}{5}v\mu_0\mathrm{e}^{-2Dt} + v\mu_0 - 6D\langle z\mu^2\rangle. \tag{3.52}$$

For the particular solution we can employ the *ansatz*

$$\langle z\mu^2\rangle_p = A + B\mathrm{e}^{-12Dt} + E\mathrm{e}^{-2Dt}. \tag{3.53}$$

By combining the latter form with (3.50), we derive, after straightforward algebra,

$$A = \frac{v\mu_0}{6D}, \quad B = -\frac{v\mu_0}{6D}\left(\mu_0^2 - \frac{3}{5}\right), \tag{3.54a,b}$$

and

$$E = -\frac{v\mu_0}{10D}. \tag{3.55}$$

Therefore, the particular solution is known. In combination with the homogeneous solution (3.51) we, thus, derive

$$\langle z\mu^2\rangle = C\mathrm{e}^{-6Dt} + \frac{v\mu_0}{6D} - \frac{v\mu_0}{6D}\left(\mu_0^2 - \frac{3}{5}\right)\mathrm{e}^{-12Dt} - \frac{v\mu_0}{10D}\mathrm{e}^{-2Dt}. \tag{3.56}$$

For $t=0$ we need to satisfy $\langle z\mu^2\rangle = 0$. This condition allows us to determine the remaining constant C. One can show that

$$C = \frac{v\mu_0}{6D}\left(\mu_0^2 - 1\right) \tag{3.57}$$

and, thus, we obtain for the solution of (3.50)

$$\langle z\mu^2\rangle = \frac{v\mu_0}{6D}\left(\mu_0^2 - 1\right)\mathrm{e}^{-6Dt} + \frac{v\mu_0}{6D} - \frac{v\mu_0}{6D}\left(\mu_0^2 - \frac{3}{5}\right)\mathrm{e}^{-12Dt} - \frac{v\mu_0}{10D}\mathrm{e}^{-2Dt}. \tag{3.58}$$

One can very easily show that $\langle\langle z\mu^2\rangle\rangle = 0$.

3.8. *The moment* $\langle z\mu^3\rangle$

For $n=1$ and $m=3$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z\mu^3\rangle = v\langle\mu^4\rangle + 6D\langle z\mu\rangle - 12D\langle z\mu^3\rangle, \tag{3.59}$$

where the moments $\langle\mu^4\rangle$ and $\langle z\mu\rangle$ are given by (3.36) and (3.47), respectively. The homogeneous solution of (3.59) is

$$\langle z\mu^3\rangle_h = C\mathrm{e}^{-12Dt}. \tag{3.60}$$

In order to compute the particular solution, we combine (3.36) and (3.47) with (3.59) to find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle z\mu^3\rangle &= v\left[\mu_0^4 - \frac{6}{7}\left(\mu_0^2 - \frac{1}{3}\right) - \frac{1}{5}\right]\mathrm{e}^{-20Dt} - \frac{9v}{14}\left(\mu_0^2 - \frac{1}{3}\right)\mathrm{e}^{-6Dt} \\ &\quad + \frac{3v}{2}\left(\mu_0^2 - 1\right)\mathrm{e}^{-2Dt} + \frac{6v}{5} - 12D\langle z\mu^3\rangle.\end{aligned} \tag{3.61}$$

For the particular solution we can employ the *ansatz*

$$\langle z\mu^3\rangle_p = A + B\mathrm{e}^{-20Dt} + E\mathrm{e}^{-6Dt} + F\mathrm{e}^{-2Dt}. \tag{3.62}$$

We find after lengthy straightforward algebra

$$\left.\begin{aligned} A &= \frac{v}{10D}, \\ B &= -\frac{v}{8D}\left[\mu_0^4 - \frac{6}{7}\left(\mu_0^2 - \frac{1}{3}\right) - \frac{1}{5}\right], \\ E &= \frac{3v}{28D}\left(\frac{1}{3} - \mu_0^2\right), \end{aligned}\right\} \tag{3.63}$$

and

$$F = \frac{3v}{20D}\left(\mu_0^2 - 1\right). \tag{3.64}$$

The total solution of (3.59) can be obtained by superposing the homogeneous solution and the particular solution. The remaining constant C in the homogeneous solution (3.60) can be obtained from the initial condition $\langle z\mu^3\rangle = 0$ for $t=0$. We find

$$C = \frac{v}{40D}\left(5\mu_0^4 - 6\mu_0^2 + 1\right). \tag{3.65}$$

Finally we derive

$$\begin{aligned}\langle z\mu^3\rangle &= \frac{v}{10D} + \frac{3v}{20D}\left(\mu_0^2 - 1\right)\mathrm{e}^{-2Dt} + \frac{3v}{28D}\left(\frac{1}{3} - \mu_0^2\right)\mathrm{e}^{-6Dt} \\ &\quad + \frac{v}{40D}\left(5\mu_0^4 - 6\mu_0^2 + 1\right)\mathrm{e}^{-12Dt} - \frac{v}{8D}\left[\mu_0^4 - \frac{6}{7}\left(\mu_0^2 - \frac{1}{3}\right) - \frac{1}{5}\right]\mathrm{e}^{-20Dt}.\end{aligned} \tag{3.66}$$

If we average over all μ_0, we derive

$$\langle\langle z\mu^3\rangle\rangle = \frac{v}{10D}\left(1 - \mathrm{e}^{-2Dt}\right). \tag{3.67}$$

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3.9. *The moment* $\langle z^2\rangle$

For $n=2$ and $m=0$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z^2\rangle = 2v\langle z\mu\rangle. \tag{3.68}$$

We can directly see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle z^2\rangle = v\langle z\mu\rangle \equiv \langle v_z(t)\Delta z(t)\rangle. \tag{3.69}$$

This relation is well known in diffusion theory (see, e.g. Shalchi (2011*b*) for more details). In order to replace $\langle \mu z\rangle$ therein, we employ (3.47) to derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z^2\rangle = \frac{v^2}{2D}\left(\mu_0^2-1\right)\mathrm{e}^{-2Dt} + \frac{v^2}{2D}\left(\frac{1}{3}-\mu_0^2\right)\mathrm{e}^{-6Dt} + \frac{v^2}{3D}. \tag{3.70}$$

The latter equation can easily be integrated. Together with the initial condition $\langle z^2\rangle = 0$ for $t=0$, we find

$$\langle z^2\rangle = \frac{v^2}{4D^2}\left(\mu_0^2-1\right)\left(1-\mathrm{e}^{-2Dt}\right) + \frac{v^2}{12D^2}\left(\frac{1}{3}-\mu_0^2\right)\left(1-\mathrm{e}^{-6Dt}\right) + \frac{v^2 t}{3D}. \tag{3.71}$$

For early times this becomes

$$\langle z^2\rangle_{t\to 0} \to v^2\mu_0^2 t^2 \tag{3.72}$$

corresponding to the unperturbed motion. For late times we obtain

$$\langle z^2\rangle_{t\to\infty} = \frac{v^2 t}{3D} \equiv 2\kappa_{\parallel} t, \tag{3.73}$$

which does not depend on the initial pitch-angle cosine μ_0. Equation (3.73) corresponds to a normal diffusive motion. If we average (3.71) over all initial pitch angles, we find

$$\langle\langle z^2\rangle\rangle = \frac{v^2 t}{3D} - \frac{v^2}{6D^2}\left(1-\mathrm{e}^{-2Dt}\right) \tag{3.74}$$

in agreement with Malkov (2017).

3.10. *The moment* $\langle z^2\mu\rangle$

For $n=2$ and $m=1$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z^2\mu\rangle = 2v\langle z\mu^2\rangle - 2D\langle z^2\mu\rangle, \tag{3.75}$$

where the moment $\langle z\mu^2\rangle$ is given by (3.58). The homogeneous solution of (3.75) is

$$\langle z^2\mu\rangle_h = C\mathrm{e}^{-2Dt}. \tag{3.76}$$

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In order to compute the particular solution, we combine (3.58) with (3.75) to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z^2\mu\rangle = \frac{v^2\mu_0}{3D}\left(\mu_0^2-1\right)\mathrm{e}^{-6Dt}+\frac{v^2\mu_0}{3D} - \frac{v^2\mu_0}{3D}\left(\mu_0^2-\frac{3}{5}\right)\mathrm{e}^{-12Dt}-\frac{v^2\mu_0}{5D}\mathrm{e}^{-2Dt}-2D\langle\mu z^2\rangle. \tag{3.77}$$

For the particular solution we can employ the *ansatz*

$$\langle z^2\mu\rangle_p = A + B\mathrm{e}^{-12Dt}+E\mathrm{e}^{-6Dt}+F(t)\mathrm{e}^{-2Dt}. \tag{3.78}$$

It has to be pointed out that $F(t)$ needs to be a function of time. A constant coefficient F would not work in this particular case. By combining the latter form with (3.77), we find after straightforward algebra

$$\left.\begin{aligned} A &= \frac{v^2\mu_0}{6D^2},\\ B &= \frac{v^2\mu_0}{30D^2}\left(\mu_0^2-\frac{3}{5}\right),\\ E &= \frac{v^2\mu_0}{12D^2}\left(1-\mu_0^2\right), \end{aligned}\right\} \tag{3.79}$$

and

$$F(t) = -\frac{v^2\mu_0 t}{5D}. \tag{3.80}$$

The total solution is a superposition of homogeneous and particular solutions. The remaining constant C in (3.76) can be determined from the initial condition $\langle z^2\mu\rangle = 0$ for $t=0$. After more lengthy algebra we obtain

$$C = \frac{v^2\mu_0}{20D^2}\left(\mu_0^2-\frac{23}{5}\right). \tag{3.81}$$

Therewith the total solution of (3.75) becomes

$$\langle z^2\mu\rangle = \frac{v^2\mu_0}{20D^2}\left(\mu_0^2-\frac{23}{5}\right)\mathrm{e}^{-2Dt}+\frac{v^2\mu_0}{6D^2} + \frac{v^2\mu_0}{30D^2}\left(\mu_0^2-\frac{3}{5}\right)\mathrm{e}^{-12Dt}+\frac{v^2\mu_0}{12D^2}\left(1-\mu_0^2\right)\mathrm{e}^{-6Dt}-\frac{v^2\mu_0 t}{5D}\mathrm{e}^{-2Dt}. \tag{3.82}$$

We can very easily show that $\langle\langle z^2\mu\rangle\rangle = 0$ as expected.

3.11. *The moment* $\langle z^2\mu^2\rangle$

For $n=2$ and $m=2$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z^2\mu^2\rangle = 2v\langle z\mu^3\rangle + 2D\langle z^2\rangle - 6D\langle z^2\mu^2\rangle, \tag{3.83}$$

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where the moments $\langle z\mu^3\rangle$ and $\langle z^2\rangle$ are given by (3.66) and (3.71), respectively. The homogeneous solution of (3.83) is

$$\langle z^2\mu^2\rangle_h = C\mathrm{e}^{-6Dt}. \tag{3.84}$$

In order to determine the particular solution we need to combine (3.83) with (3.66) and (3.71). We then obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle z^2\mu^2\rangle &= \frac{v^2}{5D} + \frac{3v^2}{10D}\left(\mu_0^2 - 1\right)\mathrm{e}^{-2Dt} + \frac{3v^2}{14D}\left(\frac{1}{3} - \mu_0^2\right)\mathrm{e}^{-6Dt} \\ &\quad + \frac{v^2}{20D}\left(5\mu_0^4 - 6\mu_0^2 + 1\right)\mathrm{e}^{-12Dt} \\ &\quad - \frac{v^2}{4D}\left[\mu_0^4 - \frac{6}{7}\left(\mu_0^2 - \frac{1}{3}\right) - \frac{1}{5}\right]\mathrm{e}^{-20Dt} \\ &\quad + \frac{v^2}{2D}\left(\mu_0^2 - 1\right)\left(1 - \mathrm{e}^{-2Dt}\right) \\ &\quad + \frac{v^2}{6D}\left(\frac{1}{3} - \mu_0^2\right)\left(1 - \mathrm{e}^{-6Dt}\right) + \frac{2v^2t}{3} - 6D\langle z^2\mu^2\rangle.\end{aligned} \tag{3.85}$$

For the particular solution we can employ the *ansatz*

$$\langle z^2\mu^2\rangle_p = A(t) + B\mathrm{e}^{-20Dt} + E\mathrm{e}^{-12Dt} + F(t)\mathrm{e}^{-6Dt} + G\mathrm{e}^{-2Dt}, \tag{3.86}$$

where the factors $A(t)$ and $F(t)$ need to be functions of time. After lengthy straightforward algebra we find for the coefficients and functions therein

$$\left.\begin{aligned} A(t) &= \frac{v^2t}{9D} + \frac{v^2\mu_0^2}{18D^2} - \frac{8v^2}{135D^2}, \\ B &= \frac{v^2}{56D^2}\left[\mu_0^4 - \frac{6}{7}\left(\mu_0^2 - \frac{1}{3}\right) - \frac{1}{5}\right], \\ E &= -\frac{v^2}{120D^2}(5\mu_0^4 - 6\mu_0^2 + 1), \\ F(t) &= \frac{v^2t}{21D}\left(\frac{1}{3} - \mu_0^2\right), \end{aligned}\right\} \tag{3.87}$$

and

$$G = \frac{v^2}{20D^2}\left(1 - \mu_0^2\right). \tag{3.88}$$

The total solution is a superposition of homogeneous and particular solutions. The remaining constant C in (3.84) can be determined from the initial condition $\langle z^2\mu^2\rangle = 0$ for $t = 0$. After lengthy straightforward algebra we obtain

$$\begin{aligned} C &= \frac{8v^2}{135D^2} - \frac{v^2\mu_0^2}{18D^2} - \frac{v^2}{56D^2}\left[\mu_0^4 - \frac{6}{7}\left(\mu_0^2 - \frac{1}{3}\right) - \frac{1}{5}\right] \\ &\quad + \frac{v^2}{120D^2}\left(5\mu_0^4 - 6\mu_0^2 + 1\right) - \frac{v^2}{20D^2}\left(1 - \mu_0^2\right). \end{aligned} \tag{3.89}$$

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The solution of (3.83) is, therefore,

$$\begin{aligned}\langle z^2\mu^2\rangle &= \left\{\frac{8v^2}{135D^2}-\frac{v^2\mu_0^2}{18D^2}-\frac{v^2}{56D^2}\left[\mu_0^4-\frac{6}{7}\left(\mu_0^2-\frac{1}{3}\right)-\frac{1}{5}\right]\right.\\ &\quad\left.+\frac{v^2}{120D^2}\left(5\mu_0^4-6\mu_0^2+1\right)-\frac{v^2}{20D^2}\left(1-\mu_0^2\right)\right\}\mathrm{e}^{-6Dt}\\ &\quad+\frac{v^2t}{9D}+\frac{v^2\mu_0^2}{18D^2}-\frac{8v^2}{135D^2}+\frac{v^2}{56D^2}\left[\mu_0^4-\frac{6}{7}\left(\mu_0^2-\frac{1}{3}\right)-\frac{1}{5}\right]\mathrm{e}^{-20Dt}\\ &\quad-\frac{v^2}{120D^2}\left(5\mu_0^4-6\mu_0^2+1\right)\mathrm{e}^{-12Dt}+\frac{v^2t}{21D}\left(\frac{1}{3}-\mu_0^2\right)\mathrm{e}^{-6Dt}\\ &\quad+\frac{v^2}{20D^2}\left(1-\mu_0^2\right)\mathrm{e}^{-2Dt}.\end{aligned} \tag{3.90}$$

In the late time limit, we can neglect all exponential functions and constants to find

$$\langle z^2\mu^2\rangle_{t\to\infty}\to\frac{v^2t}{9D}. \tag{3.91}$$

If we average (3.90) over the initial pitch-angle cosine μ_0, we find

$$\langle\!\langle z^2\mu^2\rangle\!\rangle=\frac{v^2}{135D^2}\mathrm{e}^{-6Dt}+\frac{v^2}{30D^2}\mathrm{e}^{-2Dt}+\frac{v^2t}{9D}-\frac{11v^2}{270D^2}. \tag{3.92}$$

3.12. *The moment* $\langle z^3\rangle$

For $n=3$ and $m=0$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z^3\rangle=3v\langle z^2\mu\rangle, \tag{3.93}$$

where the moment $\langle z^2\mu\rangle$ is given by (3.82). If we integrate (3.93) and by using $\langle z^3\rangle=0$ for $t=0$, we derive

$$\begin{aligned}\langle z^3\rangle &= \frac{v^3\mu_0}{2D^2}t-\frac{3v^3\mu_0}{20D^3}+\frac{3v^3\mu_0}{40D^3}\left(\mu_0^2-\frac{23}{5}\right)\left(1-\mathrm{e}^{-2Dt}\right)\\ &\quad+\frac{v^3\mu_0}{120D^3}\left(\mu_0^2-\frac{3}{5}\right)\left(1-\mathrm{e}^{-12Dt}\right)+\frac{v^3\mu_0}{24D^3}\left(1-\mu_0^2\right)\left(1-\mathrm{e}^{-6Dt}\right)\\ &\quad+\frac{3v^3\mu_0}{20D^3}(1+2Dt)\,\mathrm{e}^{-2Dt}.\end{aligned} \tag{3.94}$$

We can easily see that if we average (3.94) over the initial pitch-angle cosine μ_0, we find

$$\langle\!\langle z^3\rangle\!\rangle=0. \tag{3.95}$$

This result was also obtained by Malkov (2017). Equation (3.94) provides a generalization of the latter result, because it is valid for an arbitrary initial pitch-angle cosine μ_0.

If we consider the limit $t\to\infty$, equation (3.94) becomes

$$\langle z^3\rangle_{t\to\infty}=\frac{v^3\mu_0}{2D^2}t \tag{3.96}$$

corresponding to a linear increase.

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3.13. *The moment* $\langle z^3\mu\rangle$

For $n=3$ and $m=1$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z^3\mu\rangle = 3v\langle z^2\mu^2\rangle - 2D\langle z^3\mu\rangle, \tag{3.97}$$

where the moment $\langle z^2\mu^2\rangle$ is given by (3.90). The homogeneous solution of (3.97) is

$$\langle z^3\mu\rangle_h = C\mathrm{e}^{-2Dt}. \tag{3.98}$$

In order to compute the particular solution, we combine (3.90) with (3.97) to find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle z^3\mu\rangle &= -\frac{3v^3}{56D^2}\left[\mu_0^4 - \frac{6}{7}\left(\mu_0^2 - \frac{1}{3}\right) - \frac{1}{5}\right]\left(\mathrm{e}^{-6Dt} - \mathrm{e}^{-20Dt}\right)\\ &\quad + \frac{v^3}{40D^2}\left(5\mu_0^4 - 6\mu_0^2 + 1\right)\left(\mathrm{e}^{-6Dt} - \mathrm{e}^{-12Dt}\right)\\ &\quad - \frac{3v^3}{20D^2}\left(1-\mu_0^2\right)\left(\mathrm{e}^{-6Dt} - \mathrm{e}^{-2Dt}\right)\\ &\quad + \frac{v^3 t}{7D}\left(\frac{1}{3} - \mu_0^2\right)\mathrm{e}^{-6Dt} + \frac{8v^3}{45D^2}\left(\mathrm{e}^{-6Dt} - 1\right)\\ &\quad + \frac{v^3\mu_0^2}{6D^2}\left(1 - \mathrm{e}^{-6Dt}\right) + \frac{v^3 t}{3D} - 2D\langle z^3\mu\rangle. \end{aligned} \tag{3.99}$$

For the particular solution we can employ the *ansatz*

$$\langle z^3\mu\rangle_p = A(t) + B\mathrm{e}^{-20Dt} + E\mathrm{e}^{-12Dt} + F(t)\mathrm{e}^{-6Dt} + G(t)\mathrm{e}^{-2Dt}, \tag{3.100}$$

where $A(t)$, $F(t)$ and $G(t)$ need to be functions of time. If this is used in (3.99) we find after long algebra

$$\left.\begin{aligned} A(t) &= \frac{v^3 t}{6D^2} + \frac{v^3}{12D^3}\left(\mu_0^2 - \frac{31}{15}\right)\\ B &= -\frac{v^3}{336D^3}\left[\mu_0^4 - \frac{6}{7}\left(\mu_0^2 - \frac{1}{3}\right) - \frac{1}{5}\right],\\ E &= \frac{v^3}{400D^3}\left(5\mu_0^4 - 6\mu_0^2 + 1\right) \end{aligned}\right\} \tag{3.101}$$

as well as

$$\begin{aligned} F(t) &= -\frac{v^3}{28D^3}\left(\frac{1}{4} + Dt\right)\left(\frac{1}{3} - \mu_0^2\right) - \frac{2v^3}{45D^3} + \frac{v^3\mu_0^2}{24D^3}\\ &\quad + \frac{3v^3}{224D^3}\left[\mu_0^4 - \frac{6}{7}\left(\mu_0^2 - \frac{1}{3}\right) - \frac{1}{5}\right]\\ &\quad - \frac{v^3}{160D^3}\left(5\mu_0^4 - 6\mu_0^2 + 1\right) + \frac{3v^3}{80D^3}\left(1 - \mu_0^2\right) \end{aligned} \tag{3.102}$$

and

$$G(t) = \frac{3v^3 t}{20D^2}\left(1 - \mu_0^2\right). \tag{3.103}$$

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The remaining constant C in (3.98) can be determined from the initial condition $\langle z^3\mu\rangle = 0$ for $t=0$. After lengthy straightforward algebra we obtain

$$\begin{aligned} C = & -\frac{v^3}{12D^3}\left(\mu_0^2-\frac{31}{15}\right)-\frac{v^3}{96D^3}\left[\mu_0^4-\frac{6}{7}\left(\mu_0^2-\frac{1}{3}\right)-\frac{1}{5}\right] \\ & +\frac{3v^3}{800D^3}\left(5\mu_0^4-6\mu_0^2+1\right)+\frac{v^3}{112D^3}\left(\frac{1}{3}-\mu_0^2\right) \\ & +\frac{2v^3}{45D^3}-\frac{v^3\mu_0^2}{24D^3}-\frac{3v^3}{80D^3}\left(1-\mu_0^2\right). \end{aligned} \tag{3.104}$$

Therefore, we find

$$\begin{aligned} \langle z^3\mu\rangle = & \frac{v^3 t}{6D^2}+\frac{v^3}{12D^3}\left(\mu_0^2-\frac{31}{15}\right)-\frac{v^3}{336D^3}\left[\mu_0^4-\frac{6}{7}\left(\mu_0^2-\frac{1}{3}\right)-\frac{1}{5}\right]\mathrm{e}^{-20Dt} \\ & +\frac{v^3}{400D^3}\left(5\mu_0^4-6\mu_0^2+1\right)\mathrm{e}^{-12Dt}+\left\{-\frac{v^3}{28D^3}\left(\frac{1}{4}+Dt\right)\left(\frac{1}{3}-\mu_0^2\right)\right. \\ & -\frac{2v^3}{45D^3}+\frac{v^3\mu_0^2}{24D^3}+\frac{3v^3}{224D^3}\left[\mu_0^4-\frac{6}{7}\left(\mu_0^2-\frac{1}{3}\right)-\frac{1}{5}\right] \\ & \left.-\frac{v^3}{160D^3}\left(5\mu_0^4-6\mu_0^2+1\right)+\frac{3v^3}{80D^3}\left(1-\mu_0^2\right)\right\}\mathrm{e}^{-6Dt} \\ & +\frac{3v^3 t}{20D^2}\left(1-\mu_0^2\right)\mathrm{e}^{-2Dt}+\left\{-\frac{v^3}{12D^3}\left(\mu_0^2-\frac{31}{15}\right)\right. \\ & -\frac{v^3}{96D^3}\left[\mu_0^4-\frac{6}{7}\left(\mu_0^2-\frac{1}{3}\right)-\frac{1}{5}\right]+\frac{3v^3}{800D^3}\left(5\mu_0^4-6\mu_0^2+1\right) \\ & \left.+\frac{v^3}{112D^3}\left(\frac{1}{3}-\mu_0^2\right)+\frac{2v^3}{45D^3}-\frac{v^3\mu_0^2}{24D^3}-\frac{3v^3}{80D^3}\left(1-\mu_0^2\right)\right\}\mathrm{e}^{-2Dt}. \end{aligned} \tag{3.105}$$

In the limit $t\rightarrow\infty$, we omit the terms with exponentials as well as the constant terms and, thus, equation (3.105) becomes

$$\langle z^3\mu\rangle \rightarrow \frac{v^3 t}{6D^2}. \tag{3.106}$$

If we average (3.105) over the initial pitch-angle cosine μ_0, we find

$$\langle\langle z^3\mu\rangle\rangle = \frac{v^3 t}{6D^2}-\frac{13v^3}{90D^3}-\frac{v^3}{180D^3}\mathrm{e}^{-6Dt}+\left(\frac{v^3 t}{10D^2}+\frac{3v^3}{20D^3}\right)\mathrm{e}^{-2Dt}. \tag{3.107}$$

3.14. *The moment* $\langle z^4\rangle$

For $n=4$ and $m=0$, equation (3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z^4\rangle = 4v\langle z^3\mu\rangle, \tag{3.108}$$

where the moment $\langle \mu z^3\rangle$ is given by (3.105). By using the parameters and functions $A(t)$, B, E, $F(t)$, $G(t)$ and C given by (3.101)–(3.104), we can write differential

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equation (3.108) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z^4\rangle = 4vA(t) + 4vB\mathrm{e}^{-20Dt} + 4vE\mathrm{e}^{-12Dt} + 4vF(t)\mathrm{e}^{-6Dt} + 4vG(t)\mathrm{e}^{-2Dt} + 4vC\mathrm{e}^{-2Dt}. \tag{3.109}$$

With $\langle z^4\rangle = 0$ for $t = 0$, we can integrate the latter equation so that

$$\begin{aligned}\langle z^4\rangle &= 4v\int_0^t \mathrm{d}t'\, A(t') + 4vB\int_0^t \mathrm{d}t'\,\mathrm{e}^{-20Dt'} + 4vE\int_0^t \mathrm{d}t'\,\mathrm{e}^{-12Dt'} \\ &\quad + 4v\int_0^t \mathrm{d}t'\, F(t')\mathrm{e}^{-6Dt'} + 4v\int_0^t \mathrm{d}t'\, G(t')\mathrm{e}^{-2Dt'} + 4vC\int_0^t \mathrm{d}t'\,\mathrm{e}^{-2Dt'}.\end{aligned} \tag{3.110}$$

The only integral which is not trivial has the form

$$\int_0^t \mathrm{d}x\, x\mathrm{e}^{-\alpha x} = \frac{1}{\alpha^2}\left[1 - (1+\alpha t)\,\mathrm{e}^{-\alpha t}\right]. \tag{3.111}$$

With this integral we derive

$$\begin{aligned}\langle z^4\rangle &= \frac{v^4 t^2}{3D^2} + \frac{v^4 t}{3D^3}\left(\mu_0^2 - \frac{31}{15}\right) + \frac{vB}{5D}\left(1 - \mathrm{e}^{-20Dt}\right) + \frac{vE}{3D}\left(1 - \mathrm{e}^{-12Dt}\right) \\ &\quad + \frac{2vF_0}{3D}\left(1 - \mathrm{e}^{-6Dt}\right) + \frac{vF_1}{9D^2}\left[1 - (1+6Dt)\,\mathrm{e}^{-6Dt}\right] \\ &\quad + \frac{3v^4}{20D^4}\left(1-\mu_0^2\right)\left[1 - (1+2Dt)\,\mathrm{e}^{-2Dt}\right] + \frac{2vC}{D}\left(1 - \mathrm{e}^{-2Dt}\right),\end{aligned} \tag{3.112}$$

where we have used

$$F_1 = -\frac{v^3}{28D^2}\left(\frac{1}{3} - \mu_0^2\right) \tag{3.113}$$

and

$$\begin{aligned}F_0 &= -\frac{v^3}{112D^3}\left(\frac{1}{3} - \mu_0^2\right) - \frac{2v^3}{45D^3} + \frac{v^3\mu_0^2}{24D^3} + \frac{3v^3}{224D^3}\left[\mu_0^4 - \frac{6}{7}\left(\mu_0^2 - \frac{1}{3}\right) - \frac{1}{5}\right] \\ &\quad - \frac{v^3}{160D^3}\left(5\mu_0^4 - 6\mu_0^2 + 1\right) + \frac{3v^3}{80D^3}\left(1 - \mu_0^2\right).\end{aligned} \tag{3.114}$$

The parameters B, C and E are given by (3.101) and (3.104).

In the limit $t \to \infty$, equation (3.112) becomes

$$\langle z^4\rangle \to \frac{v^4 t^2}{3D^2} = 12\kappa_\parallel^2 t^2. \tag{3.115}$$

Furthermore, we can average (3.112) over the initial pitch-angle cosine μ_0 to deduce

$$\langle\langle z^4\rangle\rangle = \frac{v^4 t^2}{3D^2} - \frac{26v^4 t}{45D^3} + \frac{107v^4}{270D^4} + \frac{v^4}{270D^4}\mathrm{e}^{-6Dt} - \left(\frac{2v^4}{5D^4} + \frac{v^4 t}{5D^3}\right)\mathrm{e}^{-2Dt}. \tag{3.116}$$

The latter formula agrees perfectly with Malkov (2017). We would like to emphasize that (3.112) provides a generalization of (3.116) because it is correct for an arbitrary initial pitch-angle cosine μ_0.

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4. The characteristic function

In the previous section we have computed moments of the pitch-angle-dependent Fokker–Planck equation. Compared to previous work our findings depend on the initial pitch-angle cosine μ_0. In the following we consider an application of such analytical results by exploring the characteristic function and related quantities. Those are important for analytical formulations of nonlinear theories describing the motion of energetic particles across a mean magnetic field.

The characteristic function of a distribution function is defined via (2.6) corresponding to the Fourier transform of that function. After employing a Taylor expansion of the exponential function we derive

$$\left\langle e^{ikz}\right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}(ik)^n \langle z^n \rangle. \tag{4.1}$$

By only taking into account the first four terms therein, we can approximate

$$\left\langle e^{ikz}\right\rangle \approx 1 + ik\langle z\rangle - \tfrac{1}{2}k^2\langle z^2\rangle - \tfrac{1}{6}ik^3\langle z^3\rangle + \tfrac{1}{24}k^4\langle z^4\rangle. \tag{4.2}$$

The characteristic function of the diffusion equation is given by (2.7). A Taylor expansion of the latter function yields

$$\langle e^{ikz}\rangle_D = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\kappa_\parallel k^2 t\right)^n \approx 1 - \kappa_\parallel k^2 t + \frac{1}{2}\kappa_\parallel^2 k^4 t^2 + \cdots \tag{4.3}$$

If we compare (4.2) and (4.3), we find

$$\left.\begin{aligned} \langle z\rangle_D &= 0, \\ \langle z^2\rangle_D &= 2\kappa_\parallel t, \\ \langle z^3\rangle_D &= 0, \end{aligned}\right\} \tag{4.4}$$

and

$$\langle z^4\rangle_D = 12\kappa_\parallel^2 t^2. \tag{4.5}$$

In nonlinear theories for perpendicular diffusion, one needs to know correlations of the form $\langle\langle \mu_0\mu e^{ikz}\rangle\rangle$ (see, e.g. Matthaeus *et al.* 2003; Shalchi 2010). If we expand again as above, we find

$$\langle\langle \mu_0\mu e^{ikz}\rangle\rangle \approx \langle\langle \mu_0\mu\rangle\rangle + ik\langle\langle \mu_0\mu z\rangle\rangle - \tfrac{1}{2}k^2\langle\langle \mu_0\mu z^2\rangle\rangle. \tag{4.6}$$

For the moments we can employ (3.9), (3.47) and (3.82). If we multiply the latter moments by μ_0 and average over all μ_0, we find

$$\left.\begin{aligned} \langle\langle \mu_0\mu\rangle\rangle &= \tfrac{1}{3}e^{-vt/\lambda_\parallel} \\ \langle\langle \mu_0\mu z\rangle\rangle &= 0, \end{aligned}\right\} \tag{4.7}$$

and

$$\langle\langle \mu_0\mu z^2\rangle\rangle = \frac{v^2}{18D^2} + \frac{v^2}{90D^2}e^{-6Dt} - \frac{v^2}{15D^2}(1+Dt)\,e^{-2Dt}. \tag{4.8}$$

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Therewith, equation (4.6) becomes

$$\langle\!\langle \mu_0 \mu \mathrm{e}^{\mathrm{i}kz} \rangle\!\rangle \approx \frac{1}{3}\mathrm{e}^{-vt/\lambda_\parallel} - \frac{1}{2}k^2 \left[\frac{v^2}{18D^2} + \frac{v^2}{90D^2}\mathrm{e}^{-6Dt} - \frac{v^2}{15D^2}(1+Dt)\,\mathrm{e}^{-2Dt} \right]. \quad (4.9)$$

The latter formula can be written as

$$\langle v_z(t) v_z(0) \mathrm{e}^{\mathrm{i}kz} \rangle = \frac{v^2}{3}\mathrm{e}^{-vt/\lambda_\parallel} \left[1 + \frac{v^2 k^2}{10D^2}(1+Dt) \right] - \frac{v^4 k^2}{36D^2} - \frac{v^4 k^2}{180D^2}\mathrm{e}^{-3vt/\lambda_\parallel}. \quad (4.10)$$

In the limit $t \to \infty$, this becomes

$$\langle v_z(t) v_z(0) \mathrm{e}^{\mathrm{i}kz} \rangle_{t\to\infty} = -\frac{v^4 k^2}{36D^2} = -\kappa_\parallel^2 k^2. \quad (4.11)$$

In Matthaeus *et al.* (2003) the following model was used

$$\langle v_z(t) v_z(0) \mathrm{e}^{\mathrm{i}kz} \rangle_M \approx v^2 \langle \mu \mu_0 \rangle \langle \mathrm{e}^{\mathrm{i}kz} \rangle = \frac{v^2}{3}\mathrm{e}^{-vt/\lambda_\parallel}\mathrm{e}^{-\kappa_\parallel k^2 t}. \quad (4.12)$$

If we expand up to second order in k, this becomes

$$\langle v_z(t) v_z(0) \mathrm{e}^{\mathrm{i}kz} \rangle_M \approx \frac{v^2}{3}\mathrm{e}^{-vt/\lambda_\parallel}\left(1 - \kappa_\parallel k^2 t\right), \quad (4.13)$$

which disagrees with (4.12). In Shalchi (2017), on the other hand, the following form was proposed

$$\langle v_z(t) v_z(0) \mathrm{e}^{\mathrm{i}kz} \rangle_S = -\kappa_\parallel^2 k^2 \mathrm{e}^{-\kappa_\parallel k^2 t}. \quad (4.14)$$

In the lowest non-vanishing order in k, this becomes

$$\langle v_z(t) v_z(0) \mathrm{e}^{\mathrm{i}kz} \rangle_S \approx -\kappa_\parallel^2 k^2. \quad (4.15)$$

Therefore, we conclude that the Shalchi (2017) model agrees with the exact calculations presented in the current paper up to second order in k.

As pointed out in Shalchi (2010) one needs to know correlations involving particle positions and velocities in order to formulate advanced nonlinear theories for perpendicular diffusion. In the following we consider the time derivative of the characteristic function

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{e}^{\mathrm{i}kz} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{i}k)^n \frac{\mathrm{d}}{\mathrm{d}t}\langle z^n \rangle, \quad (4.16)$$

where we have used again expansion (4.1). For the time derivative of the moment $\langle z^n \rangle$ we can employ

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle z^n \rangle = nv \langle z^{n-1}\mu \rangle = \langle n z^{n-1} v_z \rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t} z^n \right\rangle. \quad (4.17)$$

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The latter formula is derived from (3.3) by setting $m=0$ therein. Furthermore, we have used again $v_z = v\mu$. With relation (4.17), equation (4.16) becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{e}^{\mathrm{i}kz}\rangle &= \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{i}k)^n \langle n z^{n-1} v_z\rangle \\ &= \mathrm{i}k \sum_{n=1}^{\infty} \frac{1}{(n-1)!}(\mathrm{i}k)^{n-1}\langle z^{n-1} v_z\rangle \\ &= \mathrm{i}k\langle v_z \mathrm{e}^{\mathrm{i}kz}\rangle . \end{aligned} \tag{4.18}$$

This relation was used in Shalchi (2017) in order to derive a time-dependent theory for perpendicular diffusion. In the current paper we confirmed its validity.

5. Summary and conclusion

To solve the pitch-angle-dependent Fokker–Planck equation of energetic particle transport is a difficult task. However, for several applications a pitch-angle and position-dependent particle distribution function is desired. In some cases one is only interested in the late time limit for which a diffusion approximation can be used (see, e.g. Schlickeiser 2002; Shalchi 2009; Zank 2014). In other cases, however, one needs a description of the transport which is valid for early (non-diffusive) times as well.

In the current paper we have derived exact analytical formulas for the first 14 moments of the form $\langle z^n \mu^m\rangle$. The presented results were obtained for an isotropic pitch-angle scattering coefficient but for an arbitrary initial pitch-angle cosine μ_0. Previous formulas derived for the moments (see, e.g. Shalchi 2006; Malkov 2017) can be obtained by considering the initial pitch-angle cosine average of the formulas derived in the present article.

To find moments which depend on the parameter μ_0 is important for applications such as the formulation of nonlinear theories for perpendicular transport (see, e.g. Shalchi 2010, 2017). In the current paper we, therefore, considered characteristic functions as an example and have tested different *ad hoc* assumptions used previously in nonlinear diffusion theories. Further applications of the moments calculated in the present article will be considered in the future.

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