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On cohesive almost zero-dimensional spaces

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Abstract. We investigate C-sets in almost zero-dimensional spaces, showing that closed σC-sets are C-sets. As corollaries, we prove that every rim-σ-compact almost zero-dimensional space is zero-dimensional and that each cohesive almost zero-dimensional space is nowhere rational. To show that these results are sharp, we construct a rim-discrete connected set with an explosion point. We also show that every cohesive almost zero-dimensional subspace of (Cantor set) $\times \mathbb{R}$ is nowhere dense.

1 Introduction

All spaces under consideration are separable and metrizable.

A subset A of a topological space X is called a *C-set in* X if A can be written as an intersection of clopen subsets of X. A σ*C-set* is a countable union of C-sets. A space X is said to be *almost zero-dimensional* provided every point $x \in X$ has a neighborhood basis consisting of C-sets in X.

A space X is *cohesive* if every point $x \in X$ has a neighborhood that contains no non-empty clopen subset of X. Clearly, every cohesive space is nowhere zero-dimensional. The converse is false, even for almost zero-dimensional spaces [10]. Spaces that are both almost zero-dimensional and cohesive include:

$$\textit{Erdős space} \quad \mathfrak{E} = \left\{x \in \ell^2 : x_i \in \mathbb{Q} \text{ for each } i < \omega\right\} \quad \text{and}$$

$$\textit{complete Erdős space} \quad \mathfrak{E}_c = \left\{x \in \ell^2 : x_i \in \{0\} \cup \{1/n : n = 1, 2, 3, \ldots\} \text{ for each } i < \omega\right\},$$

where ℓ^2 stands for the Hilbert space of square summable sequences of real numbers. Other examples include the homeomorphism groups of the Sierpiński carpet and Menger universal curve [8, 33], and various endpoint sets in complex dynamics [2, 31].

Almost zero-dimensionality of $\mathfrak{E}$ and $\mathfrak{E}_c$ follows from the fact that each closed ε-ball in either space is closed in the zero-dimensional topology inherited from $\mathbb{Q}^\omega$, which is weaker than the ℓ^2-norm topology. The spaces are cohesive, because all

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non-empty clopen subsets of $\mathfrak{E}$ and $\mathfrak{E}_c$ are unbounded in the ℓ^2-norm as proved by Erdős [18]. Thus, if we add a point ∞ to ℓ^2 whose neighborhoods are the complements of bounded sets, then we have that $\mathfrak{E} \cup \{\infty\}$ and $\mathfrak{E}_c \cup \{\infty\}$ are connected. The following result is Proposition 5.4 in Dijkstra and van Mill [12].

Proposition 1.1 *Every almost zero-dimensional cohesive space has a one-point connectification. If a space has a one-point connectification, then it is cohesive.*

Actually, open subsets of non-singleton connected spaces are cohesive, because cohesion is open hereditary [12, Remark 5.2]. More information on cohesion and one-point connectifications can be found in [1].

In Section 3, we will show that every cohesive almost zero-dimensional space E is homeomorphic to a dense subset $E' \subset \mathfrak{E}_c$ such that $E' \cup \{\infty\}$ is connected. The result is largely a consequence of earlier work by Dijkstra and van Mill [12, Chapters 4 and 5]. We apply the embedding to show that every cohesive almost zero-dimensional subspace of (Cantor set)$\times\mathbb{R}$ is nowhere dense, and there is a continuous one-to-one image of complete Erdős space that is totally disconnected but not almost zero-dimensional.

In Section 4, we examine C-sets and the rim-type of almost zero-dimensional spaces. We say that X is *rational at* $x \in X$ if x has a neighborhood basis of open sets with countable boundaries. In [32, §6, Example, p. 596], Nishiura and Tymchatyn implicitly proved that D^e, the set of endpoints of Lelek's fan [27, §9], is not rational at any of its points. Results in [5, 6, 23] later established that $D^e \simeq \mathfrak{E}_c$, so $\mathfrak{E}_c$ is nowhere rational. Working in ℓ^2, Banakh [3] recently demonstrated that each bounded open subset of $\mathfrak{E}$ has an uncountable boundary. We generalize these results by proving that each cohesive almost zero-dimensional space is nowhere rational. Moreover, every rim-σ-compact almost zero-dimensional space is zero-dimensional. We also find that in almost zero-dimensional spaces cohesion is preserved if we delete σ-compacta. These results follow from Theorem 4.4, which states that closed σC-sets in almost zero-dimensional spaces are C-sets.

In Section 5, we will construct a rim-discrete connected space τ with an explosion point. The example is partially based on [30, Example 1], which was constructed by the second author to answer a question from the Houston Problem Book [7]. The pulverized complement of the explosion point will be a rim-discrete totally disconnected set that is not zero-dimensional, in contrast with Section 4 results. Additionally, the rim-discrete property guarantees the entire connected set has a rational compactification [19, 20, 35]. We therefore solve [7, Problem 79] in the context of explosion point spaces. Results from Section 4 indicate that this new solution is optimal.

In general, ZD $\Longrightarrow$ AZD $\Longrightarrow$ TD $\Longrightarrow$ HD, where we used abbreviations for zero-dimensional, almost zero-dimensional, totally disconnected, and hereditarily disconnected. In certain contexts, these implications can be reversed. For example,

$$\text{HD} \overset{(1)}{\Longrightarrow} \text{TD} \overset{(2)}{\Longrightarrow} \text{AZD} \overset{(3)}{\Longrightarrow} \text{ZD}$$

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for subsets of hereditarily locally connected continua [24, §50 IV Theorem 9]. As mentioned above, the implication (3) is valid in the larger class of subsets of rational continua. But [30, Example 1] and the example τ in Section 5 show that (1) and (2) are generally false in that context.

2 Preliminaries

A space X is *hereditarily disconnected* if every connected subset of X contains at most one point. A space X is *totally disconnected* if every singleton in X is a C-set. A point x in a connected space X is:

- a *dispersion point* if $X\backslash\{x\}$ is hereditarily disconnected;
- an *explosion point* if $X\backslash\{x\}$ is totally disconnected.

If P is a topological property, then a space X is *rim-P* provided X has a basis of open sets whose boundaries have the property P: *Rational* ≡ rim-countable. *Zero-dimensional* ≡ rim-empty.

For A a subset of a space X, we let A^{o}, $\overline{A}$, and ∂A denote the interior, the closure, and the boundary of A in X, respectively.

Throughout the paper, $\mathfrak{C}$ will denote the middle-third Cantor set in $[0,1]$. The coordinate projections in $\mathbb{R}^2$ are denoted π_0 and π_1; $\pi_0(\langle x,y\rangle) = x$ and $\pi_1(\langle x,y\rangle) = y$. We define $\nabla : [0,1]^2 \to [0,1]^2$ by $\langle x,y\rangle \mapsto \langle xy + \frac{1}{2}(1-y), y\rangle$. The image of ∇ is the region enclosed by the triangle with vertices $\langle 0,1\rangle$, $\langle\frac{1}{2},0\rangle$, and $\langle 1,1\rangle$. Note that $\nabla \restriction [0,1]\times(0,1]$ is a homeomorphism and $\nabla^{-1}(\langle\frac{1}{2},0\rangle) = [0,1]\times\{0\}$. For each $X \subset \mathfrak{C}\times(0,1]$ we put

$$\underset{\cdot}{\nabla} X = \nabla(X) \cup \left\{\left\langle \frac{1}{2}, 0\right\rangle\right\}.$$

The *Cantor fan* is the set $\nabla(\mathfrak{C}\times[0,1]) = \underset{\cdot}{\nabla}(\mathfrak{C}\times(0,1])$, see Figure 1.

Given $X \subset \mathfrak{C}$, a function $\varphi : X \to [0,1]$ is *upper semi-continuous* (abbreviated *USC*) if $\varphi^{-1}[0,t)$ is open in X for every $t \in [0,1]$. Define

$$G_0^{\varphi} = \{\langle x, \varphi(x)\rangle : \varphi(x) > 0\},$$
$$L_0^{\varphi} = \{\langle x, t\rangle : 0 \le t \le \varphi(x)\}.$$

We say φ is a *Lelek function* if φ is USC and G_0^{φ} is dense in L_0^{φ}. Lelek functions with domain $\mathfrak{C}$ exist, and if φ is a Lelek function with domain $\mathfrak{C}$, then ∇L_0^{φ} is a *Lelek fan*; see Figure 2. For example, let $\|\ \|$ be the ℓ^2-norm and identify $\mathfrak{C}$ with the Cantor set $(\{0\}\cup\{1/n : n = 1,2,3,\dots\})^{\omega}$. Define $\eta(x) = 1/(1+\|x\|)$, where $1/\infty = 0$. Then $\mathfrak{E}_{\mathrm{c}}$ is homeomorphic to G_0^{η}, $\eta : \mathfrak{C} \to [0,1]$ is a Lelek function, and ∇L_0^{η} is a Lelek fan; see [34] and the proof of [9, Theorem 3].

3 Embedding into Fans and Complete Erdős Space

Let E be any non-empty cohesive almost zero-dimensional space. Dijkstra and van Mill proved the following: *There is a Lelek function* $\chi : X \to [0,1)$ *such that* E *is homeomorphic to* G_0^{χ}, *and hence* E *admits a dense embedding in* $\mathfrak{E}_{\mathrm{c}}$ [12, Proposition 5.10]. We observe the following theorem.

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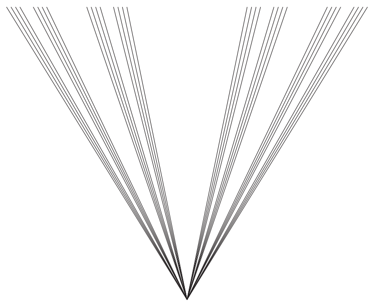

Figure 1: Cantor fan

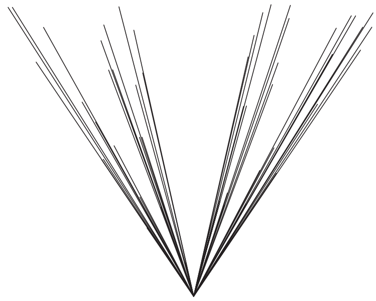

Figure 2: Lelek fan

Theorem 3.1 *For the Lelek function χ constructed in [12], ∇G_0^χ is connected. Thus, there is a dense homeomorphic embedding $\alpha : E \hookrightarrow \mathfrak{E}_c$ such that $\alpha(E) \cup \{\infty\}$ is connected.*

Proof In [12], χ is constructed via two USC functions, φ and ψ, which have the same zero-dimensional domain X. First, φ is given by [12, Lemma 4.11] such that E is homeomorphic to G_0^φ. And then, in the proof of [12, Lemma 5.8], ψ is defined by $\psi(x) = \lim_{\varepsilon\to 0^+} \inf J_\varepsilon(x)$, where

$$\begin{aligned} U_\varepsilon(x) &= \{y \in X : d(x,y) < \varepsilon), \\ J_\varepsilon(x) &= \big\{t \in [0,1) : U_\varepsilon(x) \times (t,1) \cap G_0^\varphi \\ &\qquad \text{contains no non-empty clopen subset of } G_0^\varphi\big\}. \end{aligned}$$

Notice that $J_\varepsilon(x)$ becomes larger as ε decreases, so its infimum decreases. Thus, $\psi(x)$ is well defined. Finally, χ is defined so that $\langle x, \varphi(x)\rangle \mapsto \langle x, \chi(x)\rangle$ is a homeomorphism and $\chi \le \varphi - \psi$ [12, Lemma 4.9].

To prove that $\underset{\cdot}{\nabla} G_0^\chi$ is connected, we let A be any non-empty clopen subset of G_0^χ and show that $0 \in \overline{\pi_1(A)}$. Define $y = \inf\{\varphi(x) : x \in \pi_0(A)\}$ and let $\varepsilon > 0$. Pick an $x \in \pi_0(A)$ with $\varphi(x) < y + \varepsilon$. Since $\{\langle x, \varphi(x)\rangle : x \in \pi_0(A)\}$ is a clopen subset of G_0^φ and X is zero-dimensional, $\psi(x) \ge y$. We have $\langle x, \chi(x)\rangle \in A$ and

$$\pi_1(\langle x, \chi(x)\rangle) = \chi(x) \le \varphi(x) - \psi(x) < (y + \varepsilon) - y = \varepsilon.$$

Since ε was an arbitrary positive number, this shows that $0 \in \overline{\pi_1(A)}$.

We will now construct α. Since χ is Lelek, X is perfect, so we can assume X is dense in $\mathfrak{C}$. Now χ extends to a Lelek function $\overline{\chi} : \mathfrak{C} \to [0,1]$ such that G_0^χ is dense in $G_0^{\overline{\chi}}$ [12, Lemma 4.8]. In particular, $\nabla L_0^{\overline{\chi}}$ is a Lelek fan. By [5, 6], the Lelek fan is unique, so there is a homeomorphism $\Xi : \nabla L_0^{\overline{\chi}} \to \nabla L_0^\eta$ (recall η from Section 2). We observe that $\Xi(\underset{\cdot}{\nabla} G_0^{\overline{\chi}}) = \underset{\cdot}{\nabla} G_0^\eta \simeq \mathfrak{E}_c \cup \{\infty\}$. So there is a homeomorphism $\gamma : \underset{\cdot}{\nabla} G_0^{\overline{\chi}} \to \mathfrak{E}_c \cup \{\infty\}$. We know there is also a homeomorphism $\beta : E \to \nabla G_0^\chi$. Let $\alpha = \gamma \circ \beta$, and notice that $\alpha(E) \cup \{\infty\} = \gamma(\underset{\cdot}{\nabla} G_0^\chi)$ is connected. ∎

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Corollary 3.2 *If Y is a complete space containing E, then there is a complete cohesive almost zero-dimensional space E' such that $E \subset E' \subset Y$.*

Proof Let $\alpha : E \hookrightarrow \mathfrak{E}_c$ be given by Theorem 3.1. Since Y and $\mathfrak{E}_c$ are both complete, Lavrentiev's Theorem [17, Theorem 4.3.21] says α extends to a homeomorphism between G_δ-sets E' and A such that $E \subset E' \subset Y$ and $\alpha(E) \subset A \subset \mathfrak{E}_c$. Since $\alpha(E)$ is dense in $\mathfrak{E}_c$ and $\alpha(E) \cup \{\infty\}$ is connected, $A \cup \{\infty\}$ is connected. So E' is cohesive. ∎

Theorem 3.3 *Every cohesive almost zero-dimensional subset of $\mathfrak{C} \times \mathbb{R}$ is nowhere dense.*

Proof Cohesion is open-hereditary [12, Remark 5.2]. By self-similarity of $\mathfrak{C} \times \mathbb{R}$, it therefore suffices to show there is no dense cohesive almost zero-dimensional subspace of $\mathfrak{C} \times \mathbb{R}$. Suppose on the contrary that E is such a space. By Corollary 3.2, there is a complete cohesive almost zero-dimensional $X \subset \mathfrak{C} \times \mathbb{R}$ such that $E \subset X$. Then X is a dense G_δ-subset of $\mathfrak{C} \times \mathbb{R}$, so by [4, 25], there exists $c \in \mathfrak{C}$ such that $\overline{X \cap (\{c\} \times \mathbb{R})} = \{c\} \times \mathbb{R}$. Let $x = \langle c, r \rangle \in X$. We obtain a contradiction by showing that X is zero-dimensional at x. Let $V \times (a, b)$ be any regular open subset of $\mathfrak{C} \times \mathbb{R}$ that contains x. There exist an $r_1 \in (a, r)$ and an $r_2 \in (r, b)$ such that $x_1 = \langle c, r_1 \rangle$ and $x_2 = \langle c, r_2 \rangle$ are in X. Since X is totally disconnected, there are X-clopen sets W_1 and W_2 such that $x_1 \in W_1, x_2 \in W_2$, and $x \notin W_1 \cup W_2$. Let $U_1, U_2 \subset V$ be $\mathfrak{C}$-clopen sets such that $x_i \in (U_i \times \{r_i\}) \cap X \subset W_i$ for each $i \in \{1, 2\}$. Then $[(U_1 \cap U_2) \times [r_1, r_2] \setminus (W_1 \cup W_2)] \cap X$ is an X-clopen subset of $V \times (a, b)$, which contains x. This shows that X is zero-dimensional at x. ∎

Theorem 3.3 shows that a certain continuous one-to-one image of $\mathfrak{E}_c$ is totally disconnected but not almost zero-dimensional. Define

$$f : \mathfrak{E}_c \longrightarrow \big(\{0\} \cup \{1/n : n = 1, 2, 3, \ldots\}\big)^\omega \times [0, 1]$$

by $f(x) = \big\langle x, \frac{1+\sin\|x\|}{2} \big\rangle$. Let $Y = f(\mathfrak{E}_c)$. Clearly, f is one-to-one and continuous, and Y is totally disconnected. The example Y is essentially the same as [29, Example X_2], and therefore, by [29, Propositions 3 and 5], Y is dense in $\mathfrak{C} \times [0, 1]$ and ∇Y is connected. Thus, Y is cohesive. By Theorem 3.3, Y is not almost zero-dimensional. Both this example and the space τ constructed in Section 5 show that Theorem 3.3 does not extend to totally disconnected spaces.

4 σC-sets and Rim-type

Remark 4.1 If $x \in A^\circ \subset X$ with ∂A a C-set in X, then there is a clopen set C with $x \in C$ and $C \cap \partial A = \varnothing$, and hence $C \cap A^\circ = C \cap \overline{A}$ is also clopen. Consequently, rim-C is equivalent to zero-dimensional.

Lemma 4.2 *For every two disjoint C-sets in a space, there is a clopen set containing one and missing the other.*

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Proof This is identical to the proof of [16, Lemma 1.2.6]. ■

Theorem 4.3 *Let A be a subset of an almost zero-dimensional space X. If there is a σC-set B with $\partial A \subset B \subset \overline{A}$, then $\overline{A}$ is a C-set.*

Proof Suppose $B = \bigcup\{B_i : i < \omega\}$ where each B_i is a C-set, and $\partial A \subset B \subset \overline{A}$. To prove $\overline{A}$ is a C-set, it suffices to show that for every $x \in X \backslash \overline{A}$, there is an X-clopen set C such that $x \in C \subset X \backslash \overline{A}$.

Let $x \in X \backslash \overline{A}$. By the Lindelöf property and almost zero-dimensionality, it is possible to write the open set $X \backslash \overline{A}$ as the union of countably many C-sets in X whose interiors cover $X \backslash \overline{A}$. The property of being a C-set is closed under finite unions, so there is an increasing sequence of C-sets $D_0 \subset D_1 \subset \dots$ with $x \in D_0$ and

$$\bigcup\{D_i : i < \omega\} = \bigcup\{D_i^{\circ} : i < \omega\} = X \backslash \overline{A}.$$

By Lemma 4.2, for each $i < \omega$ there is an X-clopen set C_i such that $D_i \subset C_i \subset X \backslash B_i$. Let $C = \bigcap\{C_i : i < \omega\} \backslash A^{\circ}$. Clearly, C is closed, $x \in C$, and

$$C \subset X \backslash (A^{\circ} \cup B) = X \backslash \overline{A}.$$

Further, if $y \in C$, then there exists $j < \omega$ such that $y \in D_j^{\circ}$. The open set $D_j^{\circ} \cap \bigcap\{C_i : i < j\}$ witnesses that $y \in C^{\circ}$. This shows C is open and thus clopen. ■

Theorem 4.4 *In an almost zero-dimensional space, every closed σC-set is a C-set.*

Proof Given a closed σC-set A, apply Theorem 4.3 with $B = A$. ■

With Remark 4.1 we get the following corollary.

Corollary 4.5 *Every rim-σC almost zero-dimensional space is zero-dimensional.*

Since compacta are C-sets in totally disconnected spaces, we also have the following corollary.

Corollary 4.6 *Every almost zero-dimensional space that is rim-σ-compact or rational is zero-dimensional.*

A space is called *nowhere rim-σC* (*nowhere rim-σ-compact*, resp., *nowhere rational*) if no point has a neighborhood basis consisting of sets that have boundaries that are σC-sets (σ-compact, resp., countable). With Theorem 4.4 and Remark 4.1, we also find the following corollary.

Corollary 4.7 *Cohesive almost zero-dimensional spaces are nowhere rim-σC and hence nowhere rim-σ-compact and nowhere rational.*

Thus, there are no rim-σ-compact or rational connected spaces Y with a point p such that $Y \backslash \{p\}$ is almost zero-dimensional, using Proposition 1.1.

Theorem 4.8 *If X almost zero-dimensional, $Y = X \cup \{p\}$ is connected, and $K \subset X$ is σ-compact, then $Y \backslash K$ is connected.*

Proof Suppose X is almost zero-dimensional, Y is connected, and $K \subset X$ is σ-compact. Striving for a contradiction, suppose $Y \backslash K$ is not connected. Then $Y \backslash K$ is the union of two non-empty relatively closed subsets A and B such that $A \cap B = \varnothing$. We can assume that $p \in B$. The closures of A and B in the open set $Y \backslash (\overline{A} \cap \overline{B})$ are disjoint, so they are contained in disjoint Y-open sets U and V. Note that ∂U in Y is contained in K and is, therefore, σ-compact and hence a σC-set in the totally disconnected space X. By Theorem 4.4, ∂A is a C-set in X. So by Remark 4.1, U contains a nonempty clopen subset C of X. Note that X is open in Y and U is contained in the Y-closed set $Y \backslash B$, so C is also clopen in Y. This violates the assumption that Y is connected. ∎

Since $\mathfrak{E} \cup \{\infty\}$ and $\mathfrak{E}_c \cup \{\infty\}$ are connected we have the following corollary.

Corollary 4.9 *Bounded neighborhoods in $\mathfrak{E}$ and $\mathfrak{E}_c$ do not have σ-compact boundaries.*

Combining Theorem 4.8 with Proposition 1.1 we find the following theorem.

Theorem 4.10 *If X is cohesive and almost zero-dimensional and $K \subset X$ is σ-compact, then $X \backslash K$ is cohesive.*

For the spaces $\mathfrak{E}$, $\mathfrak{E}_c$, and $\mathfrak{E}_c^\omega$ there is a stronger result: in these spaces σ-compacta are negligible; see [11, 13, 23].

A connected space X is *σ-connected* if X cannot be written as the union of ω-many pairwise disjoint non-empty closed subsets. Note that the Sierpiński Theorem [17, Theorem 6.1.27] states that every continuum is σ-connected. Lelek [26, P4] asked whether every connected space with a dispersion point is σ-connected. Duda [15, Example 5] answered this question in the negative.

Theorem 4.11 *If a space X contains an open almost zero-dimensional subspace O with $O \neq \varnothing$ and $X \backslash O \neq \varnothing$, then X is not σ-connected.*

Proof We can assume that X is connected. Since O is almost zero-dimensional and open, we can find for every $x \in O$, a C-set A_x in O that is closed in X and with $x \in A_x^{\mathrm{o}}$. Select a countable subcovering $\{B_i : i < \omega\}$ of $\{A_x : x \in O\}$. Since the union of two C-sets is a C-set, we can arrange that $B_i \subset B_{i+1}$ for each $i < \omega$. Also, we can assume that $B_0 = \varnothing$. Since B_i is a C-set in O, we can find an O-clopen covering $\mathcal{C}_i$ of $O \backslash B_i$. We can assume that $\mathcal{C}_i = \{C_{ij} : j < \omega\}$ is countable. Moreover, by clopenness we can arrange that $\mathcal{C}_i$ is a disjoint collection. Consider the countable closed disjoint covering

$$\mathcal{F} = \big(\{X \backslash O\} \cup \{C_{ij} \cap B_{i+1} : i, j < \omega\}\big) \backslash \{\varnothing\}$$

of X. If $\mathcal{F}$ is finite, then O is closed and hence clopen, violating the connectedness of X. Thus, X is not σ-connected. ∎

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Since every cohesive almost zero-dimensional space has a one-point connectification by Proposition 1.1 it produces an example in answer to Lelek's question. These examples are explosion point spaces rather than just dispersion point spaces. In particular, we have that $\mathfrak{E} \cup \{\infty\}$ and $\mathfrak{E}_c \cup \{\infty\}$ are counterexamples. Note that $\mathfrak{E}_c \cup \{\infty\}$ is complete, which is optimal, because σ-compact dispersion point spaces cannot exist.

5 A Rim-discrete Space with an Explosion Point

Let $\mathfrak{C}$, ∇ and $\underset{\cdot}{\nabla}$ be as defined in Section 2. We will construct a function $\tau : P \to (0,1)$ with domain $P \subset \mathfrak{C}$ such that:

(1) τ is a dense subset of $\mathfrak{C} \times (0,1)$;
(2) $\underset{\cdot}{\nabla}\tau$ is connected;
(3) $\underset{\cdot}{\nabla}\tau$ is rim-discrete.

Here, we identify a function like τ with its graph in the product topology. Clearly, τ will be totally disconnected. Note that τ cannot be almost zero-dimensional by (2), (3), and Corollary 4.6 or (1), (2), and Theorem 3.3.

5.1 Construction of Z

We begin by constructing a rim-discrete connectible set $Z \subset \mathfrak{C} \times \mathbb{R}$ similar to Y in [30, Example 1].

Let E be the set of endpoints of connected components of $\mathbb{R}\backslash\mathfrak{C}$. For each $\sigma \in 2^{<\omega}$, let $n = \operatorname{dom}(\sigma)$ and define

$$B(\sigma) = \left[\sum_{k=0}^{n-1} \frac{2\sigma(k)}{3^{k+1}}, \sum_{k=0}^{n-1} \frac{2\sigma(k)}{3^{k+1}} + \frac{1}{3^n}\right] \cap \mathfrak{C}.$$

Here, $B(\varnothing) = [0,1] \cap \mathfrak{C} = \mathfrak{C}$. The set of all $B(\sigma)$'s is the canonical clopen basis for $\mathfrak{C}$.

Suppose $\sigma \in 2^{<\omega}$, Q is a countable dense subset of $B(\sigma)\backslash E$, and a and b are real numbers with $a < b$. Fix an enumeration $\{q_m : m < \omega\}$ for Q, and define a function

$$f = f_{\langle Q,\sigma,a,b\rangle} : B(\sigma) \longrightarrow [a,b]$$

by the formula

$$f(c) = a + (b-a) \cdot \sum\{2^{-m-1} : m < \omega \text{ and } q_m < c\}.$$

Note that:

- f is well defined and non-decreasing;
- $f \restriction B(\sigma)\backslash E$ is one-to-one;
- f has the same value at consecutive elements of E;
- Q is the set of discontinuities of f; and
- the discontinuity at q_m is caused by a jump of height $(b-a)\cdot 2^{-m-1}$.

Let

$$D = D_{\langle Q,\sigma,a,b\rangle} = f \cup \bigcup\{\{q_m\} \times [f(q_m), f(q_m) + (b-a)\cdot 2^{-m-1}] : m < \omega\}.$$

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Thus, D is equal to (the graph of) f together with vertical arcs corresponding to the jumps in f. Note that $\pi_1(D) = [a, b]$ and D is compact.

Let $\{Q_i^n : n, i < \omega\}$ be a collection of pairwise disjoint countable dense subsets of $\mathfrak{C}\backslash E$. As in [30, Example 1], it is possible to recursively define a sequence $\mathcal{R}_0, \mathcal{R}_1, \ldots$ of finite partial tilings of $\mathfrak{C} \times \mathbb{R}$ so that for each $n < \omega$:

(i) $\mathcal{R}_n$ consists of rectangles $R_i^n = B(\sigma_i^n) \times [a_i^n, b_i^n]$, where $i < |\mathcal{R}_n| < \omega$, $\sigma_i^n \in 2^n$, and $0 < b_i^n - a_i^n \leq \frac{1}{n+1}$ for all $i < |\mathcal{R}_n|$;

(ii) the sets

$$D_i^n = D_{\langle Q_i^n \cap B(\sigma_i^n), \sigma_i^n, a_i^n, b_i^n \rangle}$$

are such that $D_i^n \cap D_j^k = \varnothing$ whenever $k < n$ or $i \neq j$;

(iii) for every arc $I \subset \mathfrak{C} \times [-n, n+1] \backslash \bigcup\{D_i^k : k \leq n \text{ and } i < |\mathcal{R}_k|\}$, there are integers $i < |\mathcal{R}_n|$, $k \leq n$, and $j < |\mathcal{R}_k|$ such that $I \subset R_i^n \cup R_j^k$ and $d(I, D_j^k) \leq \frac{1}{3^n}$, where d is the standard metric on $\mathbb{R}^2$.

Let M_i^n be the (discrete) set of midpoints of the vertical arcs in D_i^n. The key difference between the sets M_i^n and the $T_i^n(M)$ defined in [30, Example 1] is that here we have guaranteed $\pi_0(M_i^n) \cap \pi_0(M_j^k) \subset Q_i^n \cap Q_j^k = \varnothing$ whenever $n \neq k$ or $i \neq j$, whereas a vertical line could intersect multiple $T_i^n(M)$'s.

Let $\{D_n : n < \omega\}$ and $\{M_n : n < \omega\}$ be the sets of all D_i^n's and M_i^n's, respectively. Properties (i) through (iii) guarantee the set $Z = \mathfrak{C} \times \mathbb{R} \backslash \bigcup\{D_n \backslash M_n : n < \omega\}$ is rim-discrete; see [30, Claims 1 and 3]. Essentially, τ will be a subset of Z containing all M_n's, but will be vertically compressed from $\mathfrak{C} \times \mathbb{R}$ into $\mathfrak{C} \times (0, 1)$.

5.2 Construction of $\overline{g}$

We now construct a connected function $\overline{g}$ (*i.e.*, a function with a connected graph) on which τ will be based.

Let $\xi : \mathbb{R} \to (0, 1)$ be a homeomorphism, *e.g.* $\xi = \frac{1}{2} + \frac{1}{\pi}\arctan$. Let $\phi : [0, 1] \to [0, 1]$ be the Cantor function [14], and put $\Phi = \phi \times \xi$. Then each $\Phi(D_n)$ is an arc which resembles the graph of ϕ reflected across the diagonal $x = y$. See Figure 3.

Note that $\phi(E)$ is the set of dyadic rationals in $[0, 1]$. Let

$$g = (\phi(E) \times \{0\}) \cup \bigcup\{\Phi(M_n) : n < \omega\}.$$

Since $\pi_0 \restriction M_n$ is one-to-one and the $\pi_0(M_n)$'s are pairwise disjoint, g is a function. Also,

$$\operatorname{dom}(g) = \phi(E) \cup \bigcup\{\pi_0(\Phi(M_n)) : n < \omega\}$$

is countable and $\operatorname{ran}(g) \subset [0, 1)$. Our goal is to extend g to a connected function $\overline{g} : [0, 1] \to (-1, 1)$. This will be accomplished with the help of two claims. By a *continuum,* we shall mean a compact connected metrizable space with more than one point.

Claim 5.1 *Fix $n < \omega$ and put $D = D_n$ and $M = M_n$. Let $A \subset [0, 1]$ have a dense complement and let $K \subset \Phi(D) \cup (A \times (-1, 1))$ be a continuum. If $|\pi_0(K)| > 1$, then $K \cap \Phi(M) \neq \varnothing$.*

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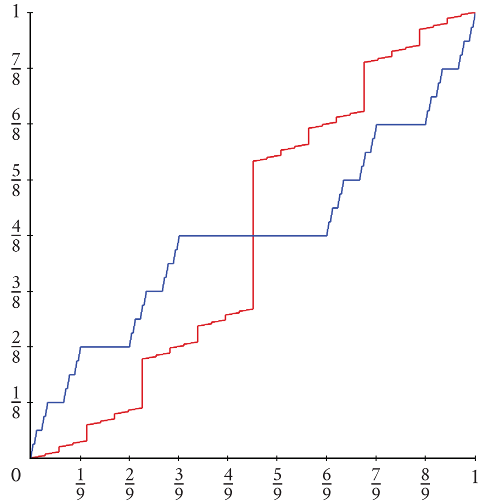

Figure 3: Graph of ϕ (blue) and its "inverse" (red).

Proof Let a and b be two points in K such that $\pi_0(a) < \pi_0(b)$. Since $\pi_0(K)$ is an interval contained in the union of the zero-dimensional set A and the interval $\pi_0(D)$, we have $\pi_0(K) \subset \pi_0(D)$. Noting that $\pi_0(M)$ is dense in $\pi_0(D)$, we find a $p \in \Phi(M)$ such that $\pi_0(p) \in (\pi_0(a), \pi_0(b))$. If $p \notin K$, then we can find $c, d \in [0,1]\backslash A$ such that $U = [c,d] \times \{\pi_1(p)\}$ is disjoint from K and $\pi_0(a) < c < \pi_0(p) < d < \pi_0(b)$. Then $U \cup (\{c\} \times (\pi_1(p), 1)) \cup (\{d\} \times (-1, \pi_1(p))$ separates K with a and b on opposite sides. This contradicts our assumption that K is connected. Therefore, $p \in K$. ∎

Claim 5.2 *Let $A \subset [0,1]$ be any countable set, and let $K \subset \bigcup\{\Phi(D_n) : n < \omega\} \cup (A \times (-1,1))$ be a continuum. If $|\pi_0(K)| > 1$, then $K \cap \Phi(M_n) \neq \varnothing$ for some $n < \omega$.*

Proof For each $x \in [0,1]$, let $K_x = K \cap (\{x\} \times (-1,1))$. Let $\mathcal{K}$ be the decomposition of K consisting of every connected component of every non-empty K_x. Applying [17, Lemma 6.2.21] to the perfect map $\pi_0 \restriction K$, we see that $\mathcal{K}$ is upper semi-continuous. If $q : K \to K'$ is the associated (closed) quotient mapping, then K' is also a continuum. Consider the countable covering $\mathcal{V}$ of K' consisting of the compacta $q(K_x)$ for $x \in A$ and $q(\Phi(D_n) \cap K)$ for all $n < \omega$. By the Baire Category Theorem, there is an element of $\mathcal{V}$ that has nonempty interior in K' and hence contains a (non-degenerate) continuum C' by [17, Theorem 6.1.25]. Each $q(K_x)$ is zero-dimensional by [17, Theorem 6.2.24], so $C' \subset q(\Phi(D_n) \cap K)$ for some $n < \omega$. Since q is a closed monotone map, the pre-image $C = q^{-1}(C')$ is a continuum by [17, Theorem 6.1.29]. Note that $|\pi_0(C)| > 1$, because otherwise C' would be a subset of some zero-dimensional $q(K_x)$. If $x \notin A$, then each connected component of K_x is contained in a single $\Phi(D_i)$ by the Sierpiński Theorem [17, Theorem 6.1.27], because the $\Phi(D_i)$'s are disjoint. Thus, $q(\Phi(D_n) \cap K_x)$ is disjoint from $q(\Phi(D_i) \cap K_x)$ for each $i \neq n$. So $C \subset (A \times (-1,1)) \cup \Phi(D_n)$. By Claim 5.1, we have that $C \cap \Phi(M_n) \neq \varnothing$. ∎

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Now let $\{x_\alpha : \alpha < \mathfrak{c}\}$ enumerate the set $[0,1] \setminus \mathrm{dom}(g)$. Let $\{K_\gamma : \gamma < \mathfrak{c}\}$ be the set of continua in $[0,1] \times (-1,1)$ such that:

- K_γ is not contained in any vertical line;
- $K_\gamma \cap \Phi(M_n) = \varnothing$ for all $n < \omega$.

For each $\alpha < \mathfrak{c}$, let $l_\alpha = (\{x_\alpha\} \times (-1,1)) \setminus \bigcup\{\Phi(D_n) : n < \omega\}$. By transfinite induction, we define for each $\alpha < \mathfrak{c}$ an ordinal

$$\gamma(\alpha) = \min\{\gamma < \mathfrak{c} : l_\alpha \cap K_\gamma \neq \varnothing \text{ and } \gamma \neq \gamma(\beta) \text{ for any } \beta < \alpha\}.$$

We verify that the one-to one function $\gamma : \mathfrak{c} \to \mathfrak{c}$ is well defined. Let $\alpha < \mathfrak{c}$, so $x_\alpha \notin \mathrm{dom}(g)$ and $x_\alpha \notin \pi_0(\Phi(M_n))$ for each n. Since M_n contains the midpoints of all vertical intervals in D_n, we have that $\{x_\alpha\} \times (-1,1)$ contains at most one point of $\Phi(D_n)$. Let A be the countable set

$$\bigcup_{n<\omega} \pi_1(\{x_\alpha\} \times (-1,1)) \cap \Phi(D_n)) \cup \Phi(M_n)).$$

If $a \in (-1,1) \setminus A$, then $K = [0,1] \times \{a\}$ misses every $\Phi(M_n)$, so $K = K_\beta$ for some $\beta < \mathfrak{c}$. Also, we have $l_\alpha \cap K_\beta \neq \varnothing$. Since $|(-1,1) \setminus A| = \mathfrak{c}$, we have that γ is well defined.

For every $\alpha < \mathfrak{c}$, choose a $y_\alpha \in \pi_1(l_\alpha \cap K_{\gamma(\alpha)})$. Define

$$\overline{g} = g \cup \{\langle x_\alpha, y_\alpha \rangle : \alpha < \mathfrak{c}\}$$

and note that $\overline{g} : [0,1] \to (-1,1)$ is a function. To prove that the graph of $\overline{g}$ is connected, let K be a continuum in $[0,1] \times (-1,1)$ such that $|\pi_0(K)| > 1$. We show that $K \cap \overline{g} \neq \varnothing$. The set K intersects some $\Phi(M_n)$, which is a subset of $\overline{g}$, or $K = K_\alpha$ for some $\alpha < \mathfrak{c}$. By the contraposition of Claim 5.2, the projection $A = \pi_0(K_\alpha \setminus \bigcup\{\Phi(D_n) : n < \omega\})$ is uncountable. A is a continuous image of a Polish space, so, in fact, it has cardinality $\mathfrak{c}$ by [21, Corollary 11.20]. Since $[0,1] \setminus \{x_\beta : \beta < \mathfrak{c}\} = \mathrm{dom}(g)$ is countable, this means $B = \{\beta < \mathfrak{c} : l_\beta \cap K_\alpha \neq \varnothing\}$ has cardinality $\mathfrak{c}$. Assuming that $K_\alpha \cap \overline{g} = \varnothing$ we find that α cannot be in the range of $\overline{g}$. If $\beta \in B$, then $l_\beta \cap K_\alpha \neq \varnothing$, so by the definition of γ, we have $\gamma(\beta) < \alpha$. Thus, $\gamma \restriction B$ is a one-to-one function from B into $\{\delta : \delta < \alpha\}$, and we have the desired contradiction. So (the graph of) $\overline{g}$ intersects each continuum in $[0,1] \times (-1,1)$ not lying wholly in a vertical line. By [22, Theorem 2], $\overline{g}$ is connected.

5.3 Definition and Properties of τ

Observe that $\overline{g} \circ \phi \subset (\mathfrak{C} \times (-1,1)) \cup ([0,1] \times \{0\})$. Let

$$\tau = (\overline{g} \circ \phi) \cap (0,1)^2.$$

The domain of τ is the set $P = \pi_0(\tau) \subset \mathfrak{C}$.

Let $X = \nabla(\overline{g} \cap ((0,1) \times [0,1)))$. If A is any clopen subset of X with $\langle \frac{1}{2}, 0 \rangle \in A$, then $A = X$. Otherwise, $\nabla^{-1}(X \setminus A)$ would be a non-empty proper clopen subset of $\overline{g}$, contrary to the fact that $\overline{g}$ is connected. Therefore, X is connected. Note that $\underset{\cdot}{\nabla}\tau \simeq X$, so $\underset{\cdot}{\nabla}\tau$ is also connected. Finally, let $\Xi = \mathrm{id}_{\mathfrak{C}} \times \xi$. By [30, Claims 3 and 4] and the construction of Z, $\underset{\cdot}{\nabla}\Xi(Z)$ is rim-discrete. We have $\underset{\cdot}{\nabla}\tau \subset \underset{\cdot}{\nabla}\Xi(Z)$, so $\underset{\cdot}{\nabla}\tau$ is rim-discrete.

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5.4 Two Questions

A continuum is *Suslinian* if it contains no uncountable collection of pairwise disjoint (non-degenerate) subcontinua [28]. The class of Suslinian continua is slightly larger than the class of rational continua.

Question 5.3 Can $\mathfrak{E}_c$ be embedded into a Suslinian continuum?

Question 5.4 Can $\mathfrak{E}_c$ be densely embedded into the plane $\mathbb{R}^2$?

Added July 2020

E. D. Tymchatyn informed the authors that Question 5.3 has a positive answer. There is, in fact, a Suslinian dendroid that homeomorphically contains the set of endpoints of the Lelek fan. The example is due to Tymchatyn and P. Minc.

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