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Towards a variational theory of phase transitions involving curvature

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An anisotropic area functional is often used as a model for the free energy of a crystal surface. For models of faceting, the anisotropy is typically such that the functional becomes non-convex, and then it may be appropriate to regularize it with an additional term involving curvature. When the weight of the curvature term tends to zero, this gives rise to a singular perturbation problem.

The structure of this problem is comparable to the theory of phase transitions studied first by Modica and Mortola. Their ideas are also useful in this context, but they have to be combined with adequate geometric tools. In particular, a variant of the theory of curvature varifolds, introduced by Hutchinson, is used in this paper. This allows an analysis of the asymptotic behaviour of the energy functionals.

1. Introduction

The shape of crystal surfaces is often studied with variational principles involving an anisotropic area. For example, consider a crystal surface $M \subset \mathbb{R}^3$ with normal vector ν. Let $\mathcal{H}^2$ denote the two-dimensional Hausdorff measure. Then the free energy of the surface may be modelled by an integral of the form

$$\int_M \Psi(\nu)\, \mathrm{d}\mathcal{H}^2$$

for a function $\Psi\colon S^2 \to [0,\infty)$ depending on the crystal structure of the material in question. This approach goes back to Wulff [28].

Unless Ψ is constant, such an energy will favour certain directions of the normal vector. In the extreme case where Ψ has zeros, it may be possible to find polyhedral surfaces with vanishing energy, and such a property may be used to model faceting. From the mathematical point of view, the corresponding variational problems are challenging because of a lack of convexity. For example, finding minimizers of the energy may be easy if we work in a space containing suitable polyhedra, but otherwise minimizers may not exist and minimizing sequences may develop microstructures. If we study a corresponding parabolic equation, then problems are ill-posed in general. To overcome these problems (or for other reasons), various authors, beginning with Herring [18], have suggested a modified surface energy involving the curvature [3, 12, 16, 17, 27]. The model of Gurtin and Jabbour [17] is closest to the problem studied in this paper.

Suppose that M is at least C^2-regular and let $\mathcal{A}$ denote its second fundamental form. Let $\epsilon > 0$ and consider the integral

$$\int_M (\epsilon^2|\mathcal{A}|^2 + \Psi(\nu))\, \mathrm{d}\mathcal{H}^2.$$

If this is regarded as a regularization of the previous free energy, then we will eventually let ϵ tend to 0. Sometimes the curvature term is also justified as a model for physical effects that lead to rounded edges, but then it may still be reasonable to study the limit $\epsilon \to 0$ because ϵ is small. We are interested in the asymptotic behaviour of the energy functional (renormalized by the factor $1/\epsilon$) for this limit. Here we have a structure similar to a type of problem studied first by Modica and Mortola [23] and subsequently by other authors [5, 6, 15, 20, 22, 25, 26], and it is even more reminiscent of the higher order version considered by Conti *et al*. [10]. The question is whether the observations made in these theories carry over to a problem that requires the control of surfaces rather than functions or maps.

More precisely, suppose that we have a family of surfaces $M_\epsilon \subset \mathbb{R}^3$ with normal vectors ν_ϵ and second fundamental forms $\mathcal{A}_\epsilon$, such that

$$\limsup_{\epsilon \searrow 0} \int_{M_\epsilon} \left(\epsilon|\mathcal{A}_\epsilon|^2 + \frac{1}{\epsilon}\Psi(\nu_\epsilon)\right) \mathrm{d}\mathcal{H}^2 < \infty.$$

Is this enough to obtain compactness in an appropriate space and if so, can we derive a limiting energy functional?

The corresponding questions for a similar one-dimensional problem have been answered affirmatively by Braides and Malchiodi [7], and variants of it have been studied as well [8, 9]. This theory is motivated by variational methods used in image processing. It is concerned with the boundary curves of domains $E \subset \mathbb{R}^2$, with normal vector ν and curvature κ, and it involves expressions such as

$$\int_{\partial E} \left(\epsilon\kappa^2 + \frac{1}{\epsilon}\psi(\nu)\right) \mathrm{d}\mathcal{H}^1.$$

Here $\psi\colon S^1 \to [0, \infty)$ is a function with finitely many zeros. Braides and Malchiodi derive a Γ-limit result for this type of functional, which can be summarized roughly as follows. Suppose that $E_\epsilon \subset \mathbb{R}^2$ have boundary curves with normal vectors ν_ϵ and curvature κ_ϵ. If

$$\limsup_{\epsilon \searrow 0} \int_{\partial E_\epsilon} \left(\epsilon\kappa_\epsilon^2 + \frac{1}{\epsilon}\psi(\nu_\epsilon)\right) \mathrm{d}\mathcal{H}^1 < \infty,$$

then there exists a sequence $\epsilon_k \searrow 0$ such that the corresponding boundaries converge to a polygon. The energy concentrates on the vertices in the limit, and the limiting energy can be expressed as a sum over all vertices, the contribution from each vertex depending on the orientations of the adjoining edges. (We ignore the case of coinciding vertices here for simplicity.) The ideas from the Modica–Mortola theory are important for the proofs of these results, especially to calculate the energy contributions of the individual vertices.

The two-dimensional counterparts of polygons are polyhedra. In our situation, if we have convergence of the surfaces to a polyhedron, then we expect the energy to concentrate on the edges. The limiting energy may be a weighted sum of the lengths of all edges, the weight of each edge depending on the normals of the adjoining faces. Indeed, we will see that this description is not so far from the truth. But there are a few differences to the one-dimensional case. First, the set of polyhedra is not closed under the relevant notion of convergence. Thus, in order to obtain compactness, we need to enlarge this space. To this end, we use tools from geometric measure theory. One of the consequences is that the formulation of the results becomes more involved, and we postpone the exact statements until we have the necessary tools available. In the introduction, we give only a non-rigorous version of the main results.

Again we use Modica–Mortola-type arguments to determine the weights of the edges in the limiting energy. But in this case we obtain only a lower bound, which will not be optimal in general. This resembles the situation found by Conti *et al*. [10], and the reasons are similar as well. Since ν must be the normal vector of a surface, it cannot be prescribed arbitrarily. These geometric constraints are not fully accounted for in the theory, and therefore the results sometimes suggest a 'limiting energy' that cannot be achieved. In this case, it must be expected that an optimal approximation of the limiting configuration will develop microstructures near the edges.

From now on, we regard this as a purely geometric problem. Then there is no reason to restrict our attention to surfaces in $\mathbb{R}^3$. Let $m, n \in \mathbb{N}$ with $m < n$ and suppose that $\Omega \subset \mathbb{R}^n$ is open. We consider an m-dimensional oriented submanifold $M \subset \Omega$ without boundary. Let G^0 denote the space of all oriented m-dimensional linear subspaces of $\mathbb{R}^n$. Then we have a continuous map $p\colon M \to G^0$ such that $p(x)$ corresponds to the tangent space $T_x M$ at every point $x \in M$. We now replace Ψ by the square of a continuous function $\Phi\colon G^0 \to [0, \infty)$. Let $\mathcal{A}$ denote the second fundamental form of M. We consider the functionals

$$\mathcal{F}_\epsilon(M) = \frac{1}{2}\int_M \left(\epsilon|\mathcal{A}|^2 + \frac{1}{\epsilon}(\Phi(p))^2\right) \mathrm{d}\mathcal{H}^m$$

for $\epsilon > 0$, where $\mathcal{H}^m$ is the m-dimensional Hausdorff measure. We assume that the subset $\mathcal{Q} = \Phi^{-1}(\{0\})$ of G^0 is finite.

Suppose that we have a family of oriented manifolds $M_\epsilon \subset \Omega$ with $\partial M_\epsilon \cap \Omega = \emptyset$, such that

$$\limsup_{\epsilon \searrow 0} \mathcal{F}_\epsilon(M_\epsilon) < \infty. \tag{1.1}$$

Furthermore, we assume that either each M_ϵ is compact or

$$\limsup_{\epsilon \searrow 0} \mathcal{H}^m(M_\epsilon \cap K) < \infty$$

for every compact set $K \subset \Omega$. We then prove the existence of a sequence $\epsilon_k \searrow 0$ such that M_{ϵ_k} converges in a suitable sense and we study the limit. Let $\mathcal{A}_k$ denote the second fundamental form of M_{ϵ_k} and suppose that its orientation is given by $p_k\colon M_{\epsilon_k} \to G^0$. Then the first observation is that Young's inequality implies

$$\limsup_{k \to \infty} \int_{M_{\epsilon_k}} |\mathcal{A}_k| \Phi(p_k)\, \mathrm{d}\mathcal{H}^m < \infty. \tag{1.2}$$

This inequality is the basis for the first step in the analysis. We prove that a uniform bound of the type (1.2), together with a uniform area bound (that can also be derived from the assumptions above) is sufficient to obtain compactness in the space of integral varifolds. (An integral varifold is a generalized submanifold determined by a countably m-rectifiable subset of Ω and an integer-valued multiplicity function; a precise definition is given in § 3).

Claim 1.1. *Under the above assumptions, there is convergence of a subsequence to an integral varifold V.*

The compactness result that we use here is stated in theorem 7.1, and it is explained at the beginning of § 8 how it is applied in this context.

We then study the limit V. Clearly, in the light of condition (1.1), we expect that $\Phi(p) = 0$ almost everywhere in the limit. We have only a finite set $\mathcal{Q} \subset G^0$ where Φ vanishes, and it turns out that we can decompose V into several parts corresponding to the points of $\mathcal{Q}$.

Claim 1.2. *There exists a decomposition*

$$V = \sum_{q \in \mathcal{Q}} V^q$$

into pieces V^q with a constant orientation $q \in \mathcal{Q}$. Furthermore, each V^q has a countably $(m-1)$-rectifiable boundary.

The expression 'boundary' is to be understood in a measure theoretic sense. The precise statement is given in theorem 8.1. A varifold with this type of decomposition can be interpreted as a generalized polyhedron with faces V^q, and the boundaries of V^q then correspond to the edges.

We derive further properties of the boundaries of V^q in theorem 8.2, but as they are somewhat technical, we mention at this point only that a countably $(m-1)$-rectifiable set E is introduced (which can be thought of as the totality of all the edges), together with a collection of multiplicity functions σ^q, such that E and σ^q represent the boundary of V^q.

Finally, we study the energy concentrated on E. We show that there exists a function $\Theta\colon E \to (0,\infty)$ such that, for all $\eta \in C_0^0(\Omega)$,

$$\int_E \eta\Theta \,\mathrm{d}\mathcal{H}^{m-1} \leqslant \frac{1}{2} \liminf_{k\to\infty} \int_{M_{\epsilon_k}} \eta \biggl(\epsilon_k |\mathcal{A}_k|^2 + \frac{1}{\epsilon_k}(\Phi(p_k))^2\biggr) \,\mathrm{d}\mathcal{H}^m. \tag{1.3}$$

Moreover, we have an estimate for Θ. At this stage, we describe only the case of an edge between exactly two faces oriented by $q_1, q_2 \in \mathcal{Q}$ with $q_1 \neq q_2$. Then we consider the set $\Gamma(q_1, q_2)$ comprising all C^1-paths $\gamma\colon [0,1] \to G^0$ connecting q_1 and q_2.

Claim 1.3. *At $\mathcal{H}^{m-1}$-almost every point x on E with $|\sigma^{q_1}(x)| = |\sigma^{q_2}(x)| = 1$ and $\sigma^q(x) = 0$ for $q \in \mathcal{Q} \setminus \{q_1, q_2\}$, the inequality*

$$\Theta(x) \geqslant \inf_{\gamma \in \Gamma(q_1,q_2)} \int_0^1 \Phi(\gamma(t))|\dot{\gamma}(t)| \,\mathrm{d}t$$

holds.

We formulate a more precise and more complete version of this result in theorem 8.3.

But first, in § 2, we discuss the observations made here for an example with $n = 3$ and $m = 2$ and with a cubic potential function Φ. Before we can derive a rigorous theory, we also need to introduce a few notions from geometric measure theory. This is done in § 3. Among these are, in particular, the concepts of oriented varifolds and currents. Furthermore, we recall a few known results about them in § 4.

Both varifolds and currents are generalizations of submanifolds (and polyhedra) that have good compactness properties. We use them simultaneously because, for the problem studied here, they complement each other nicely. Varifolds are particularly suitable for describing the limiting behaviour of the functionals $\mathcal{F}_\epsilon$. But, in order to obtain compactness in the appropriate space of varifolds with the standard methods, we need some control of the curvature, which $\mathcal{F}_\epsilon$ does not provide when $p(x)$ is close to $\mathcal{Q}$ (which will mostly be the case). Currents are much easier to control here. So we use currents near $\mathcal{Q}$ and varifolds away from $\mathcal{Q}$. A variant of the notion of curvature varifolds of Hutchinson [19], together with a localization argument of Mantegazza [21], will allow a separation of the two parts. These tools are discussed in §§ 5 and 6, respectively. With this approach, we obtain a compactness result in § 7 that requires control of the curvature only away from $\mathcal{Q}$. Finally, we have all the tools that we need to analyse the actual problem in § 8.

2. An example

Suppose that $n = 3$ and $m = 2$. Then we may replace G^0 by the sphere S^2 again. Consider the function

$$\Psi(\nu) = ((\nu^1)^2 + (\nu^2)^2)((\nu^1)^2 + (\nu^3)^2)((\nu^2)^2 + (\nu^3)^2), \quad \nu = (\nu^1, \nu^2, \nu^3) \in S^2,$$

and $\Phi = \sqrt{\Psi}$. The corresponding energy

$$\frac{1}{2}\int_M \left(\epsilon|\mathcal{A}|^2 + \frac{1}{\epsilon}\Psi(p)\right) \mathrm{d}\mathcal{H}^2$$

may be used as a model for crystal surfaces with a cubic structure. In this case, the set $\mathcal{Q}$ consists of the six unit vectors parallel or antiparallel to the coordinate axes.

A cube, or, more generally, a rectangular parallelepiped, is a possible limit of surfaces M_ϵ with uniformly bounded energy. (In this context, 'cube' refers to a two-dimensional object, i.e. the union of the faces of the corresponding solid.) Indeed, a sequence of smooth surfaces converging to the cube may be constructed by rounding the edges. This can be done with modifications entirely in an ϵ-neighbourhood of the edges, and such that the second fundamental form is bounded by a constant of order $1/\epsilon$. The asymptotic energy as $\epsilon \searrow 0$ is then proportional to the total length of all edges.

If we have a collection of cubes C_i, for $i \in \mathbb{N}$, with side lengths s_i, such that $\sum_{i=1}^\infty s_i < \infty$, then the union $\bigcup_{i=1}^\infty C_i$ is another limit that can be achieved with finite asymptotic energy. This union may be a rather irregular set (e.g. it may be dense in Ω), and thus it is clear that we need a suitable notion of generalized polyhedra.

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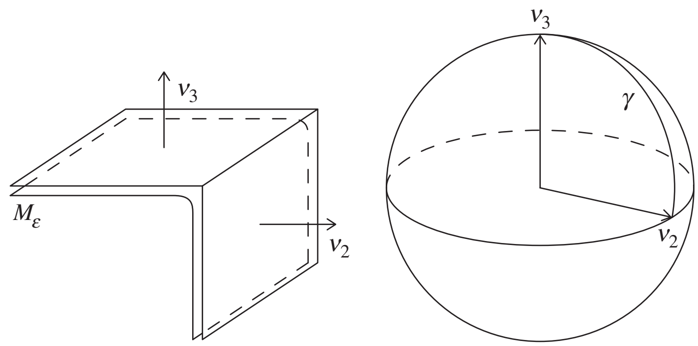

Figure 1. A transition between ν_2 and ν_3 and the corresponding path γ in S^2.

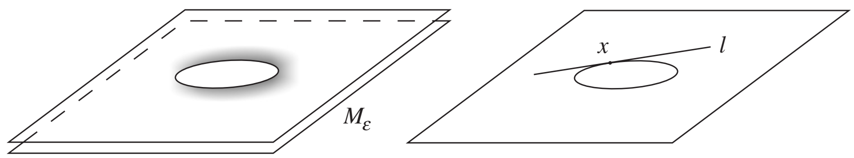

Figure 2. A double layer with an edge that is not a polygon.

Coming back to a single cube, we examine the minimum energy concentrated on one of its edges, say between the faces with normal vectors $\nu_2 = (0, 1, 0)$ and $\nu_3 = (0, 0, 1)$. Let Γ denote the set of all C^1-paths in S^2 between ν_2 and ν_3. Then the lower bound in claim 1.3 is

$$\inf_{\gamma \in \Gamma} \int_0^1 \Phi(\gamma(t))|\dot{\gamma}(t)| \, \mathrm{d}t. \tag{2.1}$$

Owing to the symmetry of Φ, it is easy to see that the infimum is achieved at the curve γ in S^2 that describes a quadrant between ν_2 and ν_3 (see figure 1). We then calculate $\Theta(x) \geqslant \frac{1}{2}$ on the corresponding edges. Indeed, by symmetry, we obtain the same estimate on all edges of this type.

In this situation, the optimal transition between ν_1 and ν_2 is essentially one-dimensional. Thus, a careful construction of M_ϵ, using the method of Braides and Malchiodi [7], will yield exactly this energy density on the edges.

On the other hand, the same potential Φ also gives rise to situations that are considerably more challenging. Consider a transition between faces with normal vectors $\nu_3 = (0, 0, 1)$ and $-\nu_3$. This can happen along any sufficiently regular curve c in a plane perpendicular to ν_3.

Let $x \in c$ and suppose that l is the line through x tangential to c (see figure 2). Then we expect that the optimal lower bound for the energy density at x will depend on l. But a formula such as (2.1) gives a number that depends only on the function Φ and the end points of the curves considered (in this case, ν_3 and $-\nu_3$). The theory developed in this paper is therefore insufficient to fully understand this and similar situations. Note, however, that we obtain a non-trivial lower bound for the energy even in this case, namely $\Theta(x) \geqslant 1$.

3. Notation and terminology

The purpose of this section is mostly to fix the notation and explain the terminology that we use. It is not intended to be self-contained. The necessary background information can be found, for example, in [13, 24].

Let $j \in \{0, \dots, n\}$. Consider the Grassmann manifolds $G(n, j)$, comprising all j-dimensional linear subspaces of $\mathbb{R}^n$, and $G^0(n, j)$, comprising all oriented j-dimensional linear subspaces of $\mathbb{R}^n$. There exists a natural twofold covering

$$\Pi_j : G^0(n, j) \to G(n, j).$$

If $p \in G^0(n, j)$, then we write $-p$ for the other point in the same fibre of Π_j. We can also identify each element of $G^0(n, j)$ with a simple unit j-vector in $\Lambda_j \mathbb{R}^n$. Thus, we obtain an embedding $\Xi_j : G^0(n, j) \to \Lambda_j \mathbb{R}^n$.

We are interested mostly in the case $j = m$, and therefore we use the abbreviations $G = G(n, m)$, $G^0 = G^0(n, m)$, $\Pi = \Pi_m$ and $\Xi = \Xi_m$. As $\Lambda_m \mathbb{R}^n$ is naturally equipped with an inner product, the embedding Ξ induces a Riemannian metric g on G^0. The distance function with respect to this metric is denoted by dist. Let $r > 0$ and $p \in G^0$. Then $B_r^0(p)$ is the open ball in G^0 with radius r and centre p. In contrast, an open ball in $\mathbb{R}^n$ with radius r and centre x is denoted by $B_r(x)$.

We fix a finite subset $\mathcal{Q}$ of G^0 and we write $G^0_{\mathcal{Q}} = G^0 \setminus \mathcal{Q}$.

When we work with multi-vectors or differential forms, it is convenient to use a multi-index notation. Let

$$I(n, j) = \{(\alpha_1, \dots, \alpha_j) \in \mathbb{N}^j : 1 \leqslant \alpha_1 < \cdots < \alpha_j \leqslant n\}.$$

For $v_1, \dots, v_n \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \dots, \alpha_j) \in I(n, j)$, we use the notation $v_\alpha = v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_j}$. We write $e_1, \dots, e_n$ for the standard basis vectors in $\mathbb{R}^n$, so that we obtain the standard basis $(e_\alpha)_{\alpha \in I(n,j)}$ of $\Lambda_j \mathbb{R}^n$. Similarly, we write

$$\mathrm{d}x^\alpha = \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_j}$$

for the standard basis vectors of $\Lambda^j \mathbb{R}^n$. We use the notation $\langle \cdot, \cdot \rangle$ for the pairing of $\Lambda_j \mathbb{R}^n$ and $\Lambda^j \mathbb{R}^n$. In other words, this is the bilinear extension of

$$\langle e_\alpha, \mathrm{d}x^\beta \rangle = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider an open set $\Omega \subset \mathbb{R}^n$. We define $G^0(\Omega) = \Omega \times G^0$ and $G^0_{\mathcal{Q}}(\Omega) = \Omega \times G^0_{\mathcal{Q}}$. If we have a map $\omega \in C^1(G^0(\Omega); \Lambda^j \mathbb{R}^n)$, then for any fixed $p \in G^0$ we can interpret $\omega(\cdot, p)$ as a differential j-form in Ω. We then define $\mathrm{d}\omega(\cdot, p)$ as the exterior derivative of this; that is, if

$$\omega = \sum_{\alpha \in I(n,j)} \omega_\alpha \, \mathrm{d}x^\alpha,$$

then

$$\mathrm{d}\omega = \sum_{\alpha \in I(n,j)} \sum_{i=1}^n \frac{\partial \omega_\alpha}{\partial x^i} \, \mathrm{d}x^i \wedge \mathrm{d}x^\alpha.$$

On the other hand, for a fixed $x \in \Omega$, we obtain a differentiable map $\omega(x, \cdot)$ on G^0. We write $\operatorname{grad} \omega$ for its gradient with respect to the Riemannian metric g.

Suppose that X is a smooth manifold and $\varpi\colon Y \to X$ is a vector bundle over X with bundle metric γ. Then a Radon measure A on X with values in Y is a pair (μ, F), where μ is a Radon measure on X and F is a μ-measurable unit section of Y (in other words, a μ-measurable map $F\colon X \to Y$ with $\varpi(F(x)) = x$ and $\gamma(F(x), F(x)) = 1$ for μ-almost every $x \in X$). For a continuous section ψ of Y with compact support, we then write

$$\int_X \gamma(\psi, \mathrm{d}A) = \int_X \gamma(\psi, F)\,\mathrm{d}\mu.$$

We also use the notation $|A| = \mu$. We will use this concept above all for the vector bundles $\Omega \times TG^0_{\mathcal{Q}}$ and $T^*\Omega \times TG^0_{\mathcal{Q}}$ over $G^0_{\mathcal{Q}}(\Omega)$ (with fibre T_pG^0 and $T^*_x\Omega \times T_pG^0$, respectively, at $(x, p) \in G^0_{\mathcal{Q}}(\Omega)$).

We write $\mathcal{H}^j$ for the j-dimensional Hausdorff measure in Ω.

An oriented j-varifold in Ω is a Radon measure on $\Omega \times G^0(n, j)$; for $j = m$, recall that we have the abbreviation $G^0(\Omega)$ for this manifold. There is a special type of oriented varifolds, represented by

- a countably j-rectifiable and $\mathcal{H}^j$-measurable set $M \subset \Omega$,
- two locally $\mathcal{H}^j$-integrable functions $\theta_+, \theta_-\colon M \to \mathbb{N}_0$ and
- an $\mathcal{H}^j$-measurable function

  $$p\colon M \to G^0(n, j)$$

  such that $\Pi_j(p(x))$ is the approximate tangent space of M at $\mathcal{H}^j$-almost every $x \in M$.

The oriented j-varifold V, defined by

$$\int_{\Omega \times G^0(n,j)} \phi\,\mathrm{d}V = \int_M (\phi(x, p)\theta_+ + \phi(x, -p)\theta_-)\,\mathrm{d}\mathcal{H}^j,$$

is called an oriented integral j-varifold in Ω. We write $V = \mathrm{vf}(M, \theta_+, \theta_-, p)$ and the set consisting of all varifolds of this type is denoted by $\mathrm{IV}^0_j(\Omega)$.

Let $P_j\colon G^0(n, j) \to \mathbb{R}^{n \times n}$ be the map such that $P_j(p)$ is the matrix describing the orthogonal projection onto $\Pi_j(p)$ for every $p \in G^0(n, j)$. Then the first variation of an oriented j-varifold V in Ω is the linear functional δV on $C^1_0(\Omega; \mathbb{R}^n)$ given by

$$\delta V(\psi) = \int_{\Omega \times G^0(n,j)} \operatorname{tr}(P_j \nabla\psi)\,\mathrm{d}V.$$

Let $\pi_j\colon \Omega \times G^0(n, j) \to \Omega$ be the projection. Then every oriented j-varifold in Ω induces a Radon measure $\|V\| = (\pi_j)_\# V$ on Ω. If $U \subset \Omega$ is open, then we also define

$$\|\delta V\|(U) = \sup\Big\{\delta V(\psi)\colon \psi \in C^1_0(U; \mathbb{R}^n) \text{ with } \sup_U |\psi| \leqslant 1\Big\}.$$

For every other Borel set $B \subset \Omega$,

$$\|\delta V\|(B) = \inf\{\|\delta V\|(U)\colon U \subset \Omega \text{ is open and } B \subset U\}.$$

A j-current in Ω is a continuous linear functional on $C_0^\infty(\Omega; \Lambda^j \mathbb{R}^n)$, the space of smooth j-forms in Ω with compact support. If we have

- a countably j-rectifiable and $\mathcal{H}^j$-measurable set $M \subset \Omega$,
- a locally $\mathcal{H}^j$-integrable function $\theta\colon M \to \mathbb{N}_0$ and
- an $\mathcal{H}^j$-measurable map $\xi\colon M \to \Lambda_j \mathbb{R}^n$ such that $\xi(x)$ is a simple unit j-vector and $\Pi_j(\Xi_j^{-1}(\xi(x)))$ is the approximate tangent space of M at $\mathcal{H}^j$-almost every $x \in M$,

then we obtain a j-current T with

$$T(\omega) = \int_M \langle \xi, \omega \rangle \theta \,\mathrm{d}\mathcal{H}^j, \quad \omega \in C_0^\infty(\Omega; \Lambda^j \mathbb{R}^n).$$

If T can be represented this way, then we call it an integer rectifiable j-current. We write $T = \mathrm{ct}(M, \theta, \xi)$, and the set of all integer rectifiable j-currents is denoted by $\mathrm{IC}_j(\Omega)$.

If T is a j-current and $j \geqslant 1$, then the boundary ∂T of T is the $(j-1)$-current defined by

$$\partial T(\omega) = T(\mathrm{d}\omega), \quad \omega \in C_0^\infty(\Omega; \Lambda^{j-1} \mathbb{R}^n).$$

For every open set $U \subset \Omega$, we define

$$\|T\|(U) = \sup\Big\{T(\omega)\colon \omega \in C_0^\infty(U; \Lambda^j \mathbb{R}^n) \text{ with } \sup_U |\omega| \leqslant 1\Big\}.$$

Furthermore,

$$\|T\|(B) = \inf\{\|T\|(U)\colon U \subset \Omega \text{ is open and } B \subset U\}$$

for every Borel set $B \subset \Omega$. If $\|T\|(K) < \infty$ for every compact set $K \subset \Omega$, then $\|T\|$ is a Radon measure on Ω. In this case there exists a locally $\|T\|$-integrable map $\xi\colon \Omega \to \Lambda_j \mathbb{R}^n$ such that

$$T(\omega) = \int_\Omega \langle \xi, \omega \rangle \,\mathrm{d}\|T\| \quad \text{for all } \omega \in C_0^\infty(\Omega; \Lambda^j \mathbb{R}^n).$$

The expression $T(\omega)$ then makes sense for all $\omega \in C_0^0(\Omega; \Lambda^j \mathbb{R}^n)$.

Every oriented j-varifold V gives rise to a j-current $T = \mathcal{T}(V)$ with

$$T(\omega) = \int_{\Omega \times G^0(n,j)} \langle \Xi_j, \omega \rangle \,\mathrm{d}V.$$

If $V \in \mathrm{IV}_j^0(\Omega)$, then $\mathcal{T}(V) \in \mathrm{IC}_j(\Omega)$. Conversely, if $T = \mathrm{ct}(M, \theta, \xi)$ is an integer rectifiable j-current, then we also have a corresponding oriented integral j-varifold $V = \mathrm{vf}(M, \theta, 0, \Xi_j^{-1}(\xi))$. This is denoted by $V = \mathcal{V}(T)$. We always have $\mathcal{T}(\mathcal{V}(T)) = T$, but the varifold $\mathcal{V}(\mathcal{T}(V))$ may differ from V, even if $V \in \mathrm{IV}_j^0(\Omega)$.

A j-varifold can be regarded as an element of the dual space of $C_0^0(\Omega \times G^0(n,j))$. A j-current is in the dual space of $C_0^\infty(\Omega; \Lambda^j \mathbb{R}^n)$ by definition. When we speak of convergence of varifolds or currents, then we always mean weak* convergence in these spaces. We use the notation $V_\ell \overset{*}{\rightharpoonup} V$ or $T_\ell \overset{*}{\rightharpoonup} T$ for such convergence. A similar notation is also used for other Radon measures.

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4. Some known results

In this section we state a few well-known results from geometric measure theory that we use in this paper. The first is a version of a compactness result by Allard [1] for varifolds, which has been extended to oriented varifolds by Hutchinson [19].

THEOREM 4.1 (the Allard–Hutchinson compactness theorem). *Suppose that $V_k \in \mathrm{IV}^0_m(\Omega)$, $k \in \mathbb{N}$, such that, for every compact set $K \subset \Omega$,*

$$\sup_{k\in\mathbb{N}}(\|V_k\|(K) + \|\delta V_k\|(K) + \|\partial\mathcal{T}(V_k)\|(K)) < \infty.$$

Then there exist a subsequence $k_\ell \to \infty$ and a varifold $V \in \mathrm{IV}^0_m(\Omega)$ such that $V_{k_\ell} \overset{*}{\rightharpoonup} V$.

The other two results stated in this section concern currents. Both are due to Federer and Fleming [14], but we use a formulation that is closer to the corresponding statements in a book by Simon [24].

THEOREM 4.2 (the Federer–Fleming compactness theorem). *For $k \in \mathbb{N}$, let $T_k \in \mathrm{IC}_m(\Omega)$ such that, for every compact set $K \subset \Omega$,*

$$\sup_{k\in\mathbb{N}}(\|T_k\|(K) + \|\partial T_k\|(K)) < \infty.$$

Then there exist a subsequence $k_\ell \to \infty$ and a current $T \in \mathrm{IC}_m(\Omega)$ such that $T_k \overset{*}{\rightharpoonup} T$.

THEOREM 4.3 (boundary rectifiability theorem). *If the current $T \in \mathrm{IC}_m(\Omega)$ satisfies $\|T\|(K) + \|\partial T\|(K) < \infty$ for all compact sets $K \subset \Omega$, then $\partial T \in \mathrm{IC}_{m-1}(\Omega)$.*

5. Curvature varifolds

One of the main tools in this paper is a variant of the notion of curvature varifolds, which was introduced by Hutchinson [19]. Mantegazza [21] extended the concept to include the possibility of a boundary. A refined version for oriented varifolds was defined by Delladio and Scianna [11]. All of these are based on the generalization of the same integration-by-parts formula on manifolds. In order to understand the underlying ideas, it is useful to consider a smooth m-dimensional submanifold $M \subset \Omega$ first, possibly with a smooth boundary ∂M. Let H denote its mean curvature vector and ν the outer normal vector on the boundary. Furthermore, for $x \in M$, let $P(x)$ denote the $(n \times n)$-matrix belonging to the orthogonal projection onto the tangent space $T_x M$. Then, for any $\eta \in C^1_0(\Omega)$, we have

$$\int_M \left(\sum_{j=1}^n P_{ij}\frac{\partial\eta}{\partial x^j} + \eta H_i\right) \mathrm{d}\mathcal{H}^m = \int_{\partial M} \eta\nu_i \,\mathrm{d}\mathcal{H}^{m-1}, \quad i = 1,\dots,n. \tag{5.1}$$

Let $\phi \in C^1_0(\Omega \times \mathbb{R}^{n\times n})$ and apply the formula to $\eta(x) = \phi(x, P(x))$. This yields

$$\int_M \left(\sum_{j=1}^n P_{ij}\left(\frac{\partial\phi}{\partial x^j} + \sum_{k,\ell=1}^n \frac{\partial\phi}{\partial y_{k\ell}}\frac{\partial P_{k\ell}}{\partial x^j}\right) + \phi H_i\right) \mathrm{d}\mathcal{H}^m = \int_{\partial M} \phi\nu_i \,\mathrm{d}\mathcal{H}^{m-1} \tag{5.2}$$

for $i = 1, \ldots, n$. Here the functions $P_{k\ell}$ are extended smoothly to Ω so that we can differentiate them with respect to x^j. The quantities

$$\sum_{j=1}^{n} P_{ij} \frac{\partial P_{k\ell}}{\partial x^j}$$

are independent of the extension, and they determine the second fundamental form of M. Furthermore, (5.2) has a counterpart for varifolds, which can be used to define a notion of curvature for varifolds.

We modify these ideas in three ways. First, while the curvature is represented by functions in the aforementioned works, we have to work with a curvature described by Radon measures. This generalization is similar to the step from Sobolev functions to functions of bounded variation. Second, we need to restrict everything to $G^0_{\mathcal{Q}}$, because our variational problem does not control the curvature near $\mathcal{Q}$. Third, we want to avoid the expression

$$\int_M \phi H_i \, \mathrm{d}\mathcal{H}^m$$

(or rather, its counterpart for varifolds) in our definition. The mean curvature corresponds to the first variation of a varifold, and we do not have sufficient control of this either. For this reason, we replace (5.1) and (5.2) by other formulae. In the case of a smooth oriented submanifold of Ω, it is simply Stokes's formula

$$\int_M \mathrm{d}\sigma = \int_{\partial M} \sigma \tag{5.3}$$

for $\sigma \in C^1_0(\Omega; \Lambda^{m-1}\mathbb{R}^n)$. It has been shown by Anzellotti *et al*. [4] that a generalization of this can be used to define functions of bounded variation over a current. A connection between this concept and curvature varifolds has been established by Delladio and Scianna [11]. The following definition is partially inspired by these works.

If $M \subset \Omega$ is a smooth oriented submanifold with smooth boundary and $p : M \to G^0$ is the function that assigns to a point $x \in M$ its oriented tangent space, then we can write (5.3) in the form

$$\int_M \langle \Xi(p), \mathrm{d}\sigma \rangle \, \mathrm{d}\mathcal{H}^m = \int_{\partial M} \sigma.$$

Now suppose that

$$\omega = \sum_{\alpha \in I(n,m-1)} \omega_\alpha \, \mathrm{d}x^\alpha \in C^1_0(G^0(\Omega); \Lambda^{m-1}\mathbb{R}^n)$$

and $\sigma(x) = \omega(x, p(x))$. Then

$$\mathrm{d}\sigma(x) = \mathrm{d}\omega(x, p(x)) + \sum_{i=1}^{n} \sum_{\alpha \in I(n,m-1)} g\left(\operatorname{grad} \omega_\alpha(x, p(x)), \frac{\partial p}{\partial x^i}(x) \right) \mathrm{d}x^i \wedge \mathrm{d}x^\alpha.$$

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Thus,

$$\int_M \langle \Xi(p), \mathrm{d}\omega(x,p)\rangle \,\mathrm{d}\mathcal{H}^m + \sum_{i=1}^n \int_M g\biggl(\langle \Xi(p), \mathrm{d}x^i \wedge \operatorname{grad}\omega(x,p)\rangle, \frac{\partial p}{\partial x^i}\biggr) \,\mathrm{d}\mathcal{H}^m = \int_{\partial M} \omega(x,p). \quad (5.4)$$

The derivatives $\partial p/\partial x^i$ also characterize the second fundamental form; thus, we can use this formula instead of (5.2) to generalize the notion to varifolds.

DEFINITION 5.1. Suppose that $V \in \mathrm{IV}^0_m(\Omega)$. Then $\mathcal{C}_{\mathcal{Q}}V$ is the set of all Radon measures

$$A = \sum_{i=1}^n A_i \,\mathrm{d}x^i$$

on $G^0_{\mathcal{Q}}$ with values in the vector bundle $T^*\Omega \times TG^0_{\mathcal{Q}}$ over $G^0_{\mathcal{Q}}(\Omega)$ with the following property: for every open, precompact set $U \Subset \Omega$ there exists a constant C such that, for all $\omega \in C^1_0(G^0(U); \Lambda^{m-1}\mathbb{R}^n)$ with $\operatorname{supp}(\operatorname{grad}\omega) \subset G^0_{\mathcal{Q}}(U)$,

$$\int_{G^0(U)} \langle \Xi, \mathrm{d}\omega\rangle \,\mathrm{d}V + \sum_{i=1}^n \int_{G^0_{\mathcal{Q}}(U)} g(\langle \Xi, \mathrm{d}x^i \wedge \operatorname{grad}\omega\rangle, \mathrm{d}A_i) \leqslant C \sup_{G^0(U)} |\omega|. \quad (5.5)$$

If $A \in \mathcal{C}_{\mathcal{Q}}V$ exists, then it follows that the left-hand side of (5.5) is represented by a Radon measure on $G^0(\Omega)$ with values in $\Lambda_{m-1}\mathbb{R}^n$, denoted by $\partial_A V$. Thus, we have

$$\int_{G^0(\Omega)} \langle \Xi, \mathrm{d}\omega\rangle \,\mathrm{d}V + \sum_{i=1}^n \int_{G^0_{\mathcal{Q}}(\Omega)} g(\langle \Xi, \mathrm{d}x^i \wedge \operatorname{grad}\omega\rangle, \mathrm{d}A_i) = \int_{G^0(\Omega)} \langle \mathrm{d}\partial_A V, \omega\rangle. \quad (5.6)$$

We interpret this as the counterpart of (5.4) for varifolds, and then $\partial_A V$ corresponds to the boundary of V (hence the notation). Indeed, for $T = \mathcal{T}(V)$, we obtain

$$\partial T(\omega) = \int_{G^0(\Omega)} \langle \mathrm{d}\partial_A V, \omega\rangle$$

for all $\omega \in C^\infty_0(\Omega; \Lambda^{m-1}\mathbb{R}^n)$.

Note that we do not have uniqueness of the measure $A \in \mathcal{C}_{\mathcal{Q}}V$. In this respect, the notion of definition 5.1 is different from the curvature varifolds of Hutchinson and Mantegazza. This is not a consequence of using Stokes's formula instead of (5.1), but rather of dropping the condition that $|A|$ is absolutely continuous with respect to V. Indeed, if $\operatorname{supp} V \subset G^0_{\mathcal{Q}}(\Omega)$ and there is an $A \in \mathcal{C}_{\mathcal{Q}}V$ that is absolutely continuous with respect to V (and thus represented by a function), then it can be shown that V is an oriented version of a curvature varifold with boundary in the sense of Mantegazza [21]. We leave it to the reader to verify this. For the purpose of this paper, only the following weaker statement is important.

PROPOSITION 5.2. *For every $R > 0$ there exists a constant C such that the following holds true. Suppose that $V \in \mathrm{IV}^0_m(\Omega)$ and $A \in \mathcal{C}_{\mathcal{Q}}V$. If* $\operatorname{supp} V \cap (\Omega \times B^0_{2R}(q)) = \emptyset$

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for every $q \in \mathcal{Q}$, then

$$\delta V(\psi) \leqslant C \int_{\Omega \times (G^0 \setminus \bigcup_{q \in \mathcal{Q}} B_R^0(q))} |\psi| \, \mathrm{d}|A| + C \int_{G^0(\Omega)} |\psi| \, \mathrm{d}|\partial_A V|$$

for every $\psi \in C_0^1(\Omega; \mathbb{R}^n)$. In particular the first variation of V is represented by a Radon measure.

Proof. Fix a point $p_0 \in G^0$. Then we can find an open neighbourhood $U \subset G^0$ of p_0 such that there exist smooth maps $\varepsilon_1, \dots, \varepsilon_m \colon U \to \mathbb{R}^n$ with the property that $(\varepsilon_1(p), \dots, \varepsilon_m(p))$ is an orthonormal basis of $\Pi(p)$ and $\Xi(p) = \varepsilon_1(p) \wedge \cdots \wedge \varepsilon_m(p)$ for every $p \in U$. Suppose that

$$\varepsilon_i = \sum_{k=1}^{n} \varepsilon_i^k e_k,$$

and define

$$\gamma_i = \sum_{k=1}^{n} \varepsilon_i^k \, \mathrm{d}x^k, \quad i = 1, \dots, m.$$

Let

$$\tilde{f}_\ell = \sum_{j=1}^{m} (-1)^j \varepsilon_j^\ell \gamma_1 \wedge \cdots \wedge \gamma_{j-1} \wedge \gamma_{j+1} \wedge \cdots \wedge \gamma_m, \quad \ell = 1, \dots, n,$$

and note that

$$\langle \Xi(p), \mathrm{d}x^k \wedge \tilde{f}_\ell(p) \rangle = \sum_{j=1}^{m} \varepsilon_j^k(p) \varepsilon_j^\ell(p)$$

for every $p \in U$. Thus, for $\psi \in C_0^1(\Omega; \mathbb{R}^n)$, we have

$$\operatorname{tr}(P_m(p) \nabla \psi(x)) = \sum_{\ell=1}^{n} \langle \Xi(p), d(\psi_\ell(x) \tilde{f}_\ell(p)) \rangle$$

for $x \in \Omega$ and $p \in U$. Using a partition of unity on G^0, we can construct smooth functions $f_1, \dots, f_n \colon G^0 \to \Lambda^{m-1} \mathbb{R}^n$ such that

$$\operatorname{tr}(P_m \nabla \psi) = \sum_{\ell=1}^{n} \langle \Xi, d(\psi_\ell f_\ell) \rangle.$$

We choose a cut-off function $\chi \in C_0^\infty(G^0)$ with $\chi \equiv 0$ in $B_R^0(q)$ for every $q \in \mathcal{Q}$ and $\chi \equiv 1$ in $G^0 \setminus \bigcup_{q \in \mathcal{Q}} B_{2R}(q)$. Then we still have

$$\delta V(\psi) = \int_{G^0(\Omega)} \operatorname{tr}(P_m \nabla \psi) \, \mathrm{d}V = \sum_{\ell=1}^{n} \int_{G^0(\Omega)} \langle \Xi, d(\psi_\ell \chi f_\ell) \rangle \, \mathrm{d}V.$$

If we test (5.6) with $\omega(x, p) = \psi_\ell(x) \chi(p) f_\ell(p)$, then we immediately obtain the required inequality. □

We also note that the notion introduced in this section is consistent with the second fundamental form of a smooth manifold. Part of this fact is already encapsulated in formula (5.4), but we need a more precise statement about the relationship between A and the second fundamental form.

Proposition 5.3. *Let $M \subset \Omega$ be an oriented m-dimensional submanifold of class C^2 with a boundary ∂M of class C^1. Suppose that $p\colon M \to G^0$ is a continuous map with $\Pi(p(x)) = T_x M$ for every $x \in M$. Furthermore, let $\mathcal{A}$ denote the second fundamental form of M. Consider the oriented m-varifold $V = \mathrm{vf}(M, 1, 0, p)$ in Ω. Then there exists an $A \in \mathcal{C}_\emptyset V$ such that, for all $\phi \in C_0^0(G^0(\Omega))$,*

$$\int_{G^0(\Omega)} \phi \,\mathrm{d}|A| = \int_M \phi(x, p(x)) |\mathcal{A}(x)| \,\mathrm{d}\mathcal{H}^m(x).$$

If $\partial M \cap \Omega = \emptyset$, then $\partial_A V = 0$.

Proof. First extend p to Ω in a way such that $\nu \cdot \nabla p(x) = 0$ for $x \in M$ and $\nu \perp T_x M$. We define $A = \sum_{i=1}^n A_i \,\mathrm{d}x^i$ by the condition that

$$\int_{G^0(\Omega)} g(\psi, \mathrm{d}A_i) = \int_M g\left(\psi(x, p(x)), \frac{\partial p}{\partial x^i}(x)\right) \mathrm{d}\mathcal{H}^m(x)$$

for all continuous sections ψ of $\Omega \times TG^0$ with compact support. Then A belongs to $\mathcal{C}_\emptyset V$ by (5.4), and we also see that $\partial_A V = 0$ if $\partial M \cap \Omega = \emptyset$. Hence, it suffices to show that $|\mathcal{A}| = |\mathrm{d}p|$.

To this end, we recall that Ξ is an isometry between G^0 and a subset of $\Lambda_m \mathbb{R}^n$ by definition. Thus, for $\xi = \Xi \circ p$, we have $|\mathrm{d}p| = |\mathrm{d}\xi|$. Locally on M, we can choose orthonormal vector fields $\varepsilon_1, \dots, \varepsilon_m$ such that $\xi = \varepsilon_1 \wedge \cdots \wedge \varepsilon_m$. Now

$$\frac{\partial \xi}{\partial x^i} = \frac{\partial \varepsilon_1}{\partial x^i} \wedge \varepsilon_2 \wedge \cdots \wedge \varepsilon_m + \cdots + \varepsilon_1 \wedge \cdots \wedge \varepsilon_{m-1} \wedge \frac{\partial \varepsilon_m}{\partial x^i}.$$

Note that $\partial \varepsilon_j / \partial x^i$ is perpendicular to ε_j. Hence, all of the terms in this sum are perpendicular to one another and we have

$$\left|\frac{\partial \xi}{\partial x^i}\right|^2 = \left|\frac{\partial \varepsilon_1}{\partial x^i} \wedge \varepsilon_2 \wedge \cdots \wedge \varepsilon_m\right|^2 + \cdots + \left|\varepsilon_1 \wedge \cdots \wedge \varepsilon_{m-1} \wedge \frac{\partial \varepsilon_m}{\partial x^i}\right|^2.$$

If $(\cdot)^\perp$ denotes the orthogonal projection onto the normal space, then we also see that

$$\left|\frac{\partial \xi}{\partial x^i}\right|^2 = \left|\left(\frac{\partial \varepsilon_1}{\partial x^i}\right)^\perp\right|^2 + \cdots + \left|\left(\frac{\partial \varepsilon_m}{\partial x^i}\right)^\perp\right|^2.$$

Summing over i, we obtain the required identity. □

6. Localization

Mantegazza [21] proved that his notion of curvature varifolds is stable under localization in Ω as well as in G^0. For the concept from definition 5.1, we have a similar property.

LEMMA 6.1. *For every $R > 0$ there exists a constant C with the following properties. Suppose that $V \in \mathrm{IV}^0_m(\Omega)$ and $A \in \mathcal{C}_{\mathcal{Q}} V$.*

(i) *If $x_0 \in \Omega$ such that $B_{2R}(x_0) \subset \Omega$, then there exists a radius $r \in (R, 2R)$ such that the varifolds*

$$V^1 = V \llcorner (B_r(x_0) \times G^0) \quad \textit{and} \quad V^2 = V - V^1$$

and the measures

$$A^1 = A \llcorner (B_r(x_0) \times G^0_{\mathcal{Q}}) \quad \textit{and} \quad A^1 = A - A^2$$

satisfy $A^1 \in \mathcal{C}_{\mathcal{Q}} V^1$ and $A^2 \in \mathcal{C}_{\mathcal{Q}} V^2$. Furthermore,

$$|\partial_{A^1} V^1|(G^0(\Omega)) + |\partial_{A^2} V^2|(G^0(\Omega)) \leqslant C\|V\|(B_{2R}(x_0) \setminus B_R(x_0)) + |\partial_A V|(G^0(\Omega)).$$

(ii) *If $p_0 \in G^0$ such that $B^0_{2R}(p_0) \setminus B^0_R(p_0) \subset G^0_{\mathcal{Q}}$, then there exists a radius $r \in (R, 2R)$ such that the varifolds*

$$V^1 = V \llcorner (\Omega \times B^0_r(p_0)) \quad \textit{and} \quad V^2 = V - V^1$$

and the measures

$$A^1 = A \llcorner (\Omega \times (B^0_r(p_0) \setminus \mathcal{Q})) \quad \textit{and} \quad A^2 = A - A^1$$

satisfy $A^1 \in \mathcal{C}_{\mathcal{Q}} V^1$ and $A^2 \in \mathcal{C}_{\mathcal{Q}} V^2$. Furthermore,

$$|\partial_{A^1} V^1|(G^0(\Omega)) + |\partial_{A^2} V^2|(G^0(\Omega)) \leqslant C|A|(\Omega \times (B^0_{2R}(p_0) \setminus B^0_R(p_0))) + |\partial_A V|(G^0(\Omega)).$$

Proof. The localization in Ω works the same way as the localization in G^0, and both use the same method as in Mantegazza's paper. As the former is carried out in detail there, we concentrate on part (ii).

Define $H = B^0_{2R}(p_0) \setminus B^0_R(p_0)$. We first consider the case $|A|(\Omega \times H) < \infty$.

Let $h \in C^\infty(\mathbb{R})$ with $h \equiv 1$ in $(-\infty, -1]$, $h \equiv 0$ in $[0, \infty)$ and $-2 \leqslant h' \leqslant 0$. Fix $r \in (R, 2R)$ and define

$$h_\ell(t) = h(\ell(t - r)), \quad t \in \mathbb{R}, \quad \ell \in \mathbb{N},$$

and

$$\chi_\ell(p) = h_\ell(\mathrm{dist}(p, p_0)), \quad p \in G^0.$$

Let $\omega \in C^1_0(G^0(\Omega); \Lambda^{m-1}\mathbb{R}^n)$ with $\mathrm{supp}(\mathrm{grad}\,\omega) \subset G^0_{\mathcal{Q}}(\Omega)$. We test (5.6) with

$$\omega_\ell(x, p) = \chi_\ell(p)\omega(x, p).$$

We obtain

$$\int_{G^0(\Omega)} \chi_\ell \langle \Xi, \mathrm{d}\omega \rangle \,\mathrm{d}V + \sum_{i=1}^n \int_{G^0_{\mathcal{Q}}(\Omega)} \chi_\ell g(\langle \Xi, \mathrm{d}x^i \wedge \mathrm{grad}\,\omega \rangle, \mathrm{d}A_i) + \sum_{i=1}^n \int_{G^0_{\mathcal{Q}}(\Omega)} \langle \Xi, \mathrm{d}x^i \wedge \omega \rangle g(\mathrm{grad}\,\chi_\ell, \mathrm{d}A_i) = \int_{G^0(\Omega)} \chi_\ell \langle \mathrm{d}\partial_A V, \omega \rangle.$$

Setting $W_\ell = V \llcorner \chi_\ell$ and $B_\ell = A \llcorner \chi_\ell$, we see that $B_\ell \in \mathcal{C}_{\mathcal{Q}} W_\ell$ and $\partial_{B_\ell} W_\ell$ is given by

$$\int_{G^0(\Omega)} \langle \mathrm{d}\partial_{B_\ell} W_\ell, \omega \rangle = \int_{G^0(\Omega)} \chi_\ell \langle \mathrm{d}\partial_A V, \omega \rangle - \sum_{i=1}^n \int_{G^0_{\mathcal{Q}}(\Omega)} \langle \Xi, \mathrm{d}x^i \wedge \omega \rangle g(\operatorname{grad} \chi_\ell, \mathrm{d}A_i).$$

We have $|\operatorname{grad} \chi_\ell| \leqslant 2\ell$. Thus, if we define

$$f(\rho) = |A|(\Omega \times (B_\rho(p_0) \cap H)),$$

then we obtain

$$\left| \int_{G^0_{\mathcal{Q}}(\Omega)} \langle \Xi, \mathrm{d}x^i \wedge \omega \rangle g(\operatorname{grad} \chi_\ell, \mathrm{d}A_i) \right| \leqslant 2\ell \left(f(r) - f\left(r - \frac{1}{\ell} \right) \right) \sup_{G^0(\Omega)} |\omega|.$$

As f is monotone, it is differentiable at almost every $\rho \in (R, 2R)$. In particular, we can choose $r \in (R, 2R)$ such that

$$f'(r) \leqslant \frac{2}{R}(f(2R) - f(R)),$$

and, at the same time,

$$V(\Omega \times \partial B^0_r(p_0)) = 0 \quad \text{and} \quad |A|(\Omega \times \partial B^0_r(p_0)) = 0.$$

Then we have $W_\ell \overset{*}{\rightharpoonup} V^1$ and $B_\ell \overset{*}{\rightharpoonup} A^1$ as $\ell \to \infty$. Furthermore,

$$\limsup_{\ell \to \infty} |\partial_{B_\ell} W_\ell|(G^0(\Omega)) \leqslant \frac{4}{R} |A|(\Omega \times H) + |\partial_A V|(G^0(\Omega)).$$

Letting $\ell \to \infty$, we derive the required properties of V^1 and A^1, and the arguments are essentially the same for V^2 and A^2.

If $|A|(\Omega \times H) = \infty$, then the inequality becomes trivial and we merely have to show that $A^1 \in \mathcal{C}_{\mathcal{Q}} V^1$ and $A^2 \in \mathcal{C}_{\mathcal{Q}} V^2$. To this end, we choose precompact, open sets $\Omega_j \Subset \Omega$, $j \in \mathbb{N}$, such that

$$\Omega = \bigcup_{j=1}^\infty \Omega_j.$$

We use the same arguments as above for the restriction of V to Ω_j, but we choose r such that the corresponding inequality holds simultaneously for all $j \in \mathbb{N}$ (albeit with different constants). This then implies the claim. □

We will use this lemma at several stages in this paper. The first consequence is an estimate for $\|V\|(\Omega)$ in the case of a compactly supported V.

Proposition 6.2. *Suppose that $\Psi \colon G^0_{\mathcal{Q}} \to (0, \infty)$ is a continuous function. Then for every $c > 0$ there exists a constant C with the following property: suppose that $V \in \mathrm{IV}^0_m(\Omega)$ has compact support in $G^0(\Omega)$ and $A \in \mathcal{C}_{\mathcal{Q}} V$. If*

$$\int_{G^0_{\mathcal{Q}}(\Omega)} \Psi \, \mathrm{d}V + \int_{G^0_{\mathcal{Q}}(\Omega)} \Psi \, \mathrm{d}|A| + |\partial_A V|(G^0(\Omega)) \leqslant c,$$

then $\|V\|(\Omega) \leqslant C$.

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Proof. Choose $R > 0$ such that $B_{2R}(q_1) \cap B_{2R}(q_2) = \emptyset$ for $q_1, q_2 \in \mathcal{Q}$ with $q_1 \neq q_2$. Using lemma 6.1, we can decompose V into

$$V = \tilde{V} + \sum_{q \in \mathcal{Q}} V^q,$$

where $\operatorname{supp} V^q \subset \Omega \times B_{2R}(q)$ and $\operatorname{supp} \tilde{V} \cap (\Omega \times B_R(q)) = \emptyset$ for every $q \in \mathcal{Q}$, and furthermore, we have an $A^q \in \mathcal{C}_{\mathcal{Q}} V^q$ with

$$|\partial_{A^q} V^q|(G^0(\Omega)) \leqslant C_1$$

for a constant C_1 that is independent of V or A.

The inequality

$$\int_{G^0_{\mathcal{Q}}(\Omega)} \Psi \, \mathrm{d}V \leqslant c$$

immediately gives a suitable bound for $\|\tilde{V}\|(\Omega)$. Now fix $q \in \mathcal{Q}$. In order to estimate $\|V^q\|(\Omega)$ as well, we consider the current $T^q = \mathcal{T}(V^q)$. Let $Q = \Pi(q)$ and suppose that $\varpi^q \colon \mathbb{R}^n \to Q$ is the orthogonal projection. Define $S^q = \varpi^q_\# T^q$. If R is chosen sufficiently small, then we have

$$2\|S^q\|(Q) \geqslant \|T^q\|(\Omega) = \|V^q\|(\Omega).$$

Moreover,

$$\|\partial S^q\|(Q) \leqslant |\partial_{A^q} V^q|(G^0(\Omega)) \leqslant C_1.$$

Note that S^q is of the form $S^q = \operatorname{ct}(Q, \theta^q, \Xi(q))$ for a function $\theta^q \colon Q \to \mathbb{N}_0$ with compact support. In fact, since $\|\partial S\|(Q) < \infty$, the function θ^q has bounded variation in Q and S^q satisfies the isoperimetric inequality [2, theorem 3.46]

$$\|S\|(Q) \leqslant C_2(\|\partial S^q\|(Q))^{m/(m-1)}$$

for a constant C_2 that depends only on m. Now the desired inequality follows. □

7. Compactness

The purpose of this section is to prove compactness of bounded sets of oriented integral m-varifolds with a uniform bound for the curvature away from $\mathcal{Q}$. In the case $\mathcal{Q} = \emptyset$, such a property follows from theorem 4.1, because the first variation is then controlled by proposition 5.2. In the case $\mathcal{Q} \neq \emptyset$, the main task is to control the varifolds near the points of $\mathcal{Q}$. The idea is to decompose a given varifold into a part with a good control of the first variation and several parts with nearly constant tangent spaces, using lemma 6.1. The first part can then be controlled with the Allard–Hutchinson compactness theorem again, and the other parts with the Federer–Fleming compactness result for integer rectifiable currents (theorem 4.2).

Theorem 7.1. *For $k \in \mathbb{N}$, let $V_k \in \mathrm{IV}^0_m(\Omega)$ and $A_k \in \mathcal{C}_{\mathcal{Q}} V_k$. Suppose that, for all compact sets $K \subset \Omega$ and $L \subset G^0_{\mathcal{Q}}$,*

$$\sup_{k \in \mathbb{N}} (\|V_k\|(K) + |A_k|(K \times L) + |\partial_{A_k} V_k|(G^0(K))) < \infty.$$

Then there exist a subsequence $k_\ell \to \infty$, a varifold $V \in \mathrm{IV}^0_m(\Omega)$, and a measure $A \in \mathcal{C}_{\mathcal{Q}} V$, such that $V_{k_\ell} \overset{*}{\rightharpoonup} V$ in $G^0(\Omega)$ and $A_{k_\ell} \overset{*}{\rightharpoonup} A$ in $G^0_{\mathcal{Q}}(\Omega)$.

For the proof we need the following lemma.

Lemma 7.2. *Suppose that V is an oriented m-varifold in Ω such that for every $k \in \mathbb{N}$ there exists a $W_k \in \mathrm{IV}^0_m(\Omega)$ satisfying*

$$\left| \int_{G^0(\Omega)} \phi \, \mathrm{d}V - \int_{G^0(\Omega)} \phi \, \mathrm{d}W_k \right| \leqslant 2^{-k} \int_\Omega \sup_{\{x\}\times G^0} (|\phi| + |\operatorname{grad} \phi|) \, \mathrm{d}\|V\|(x)$$

for all $\phi \in C^1_0(G^0(\Omega))$. Then $V \in \mathrm{IV}^0_m(\Omega)$.

Proof. First we see that $\frac{1}{2}\|V\| \leqslant \|W_k\| \leqslant 2\|V\|$ for every k. Hence, the measure

$$\mu = \sum_{k=1}^\infty 2^{-k} \|W_k\|$$

is a Radon measure on Ω. Since each W_k is an integral m-varifold, there exists a countably rectifiable and $\mathcal{H}^m$-measurable set $M \subset \Omega$ such that μ is absolutely continuous with respect to $\mathcal{H}^m \llcorner M$. Since $\|V\|$ is absolutely continuous with respect to μ, the varifold V has a representation of the form

$$\int_{G^0(\Omega)} \phi \, \mathrm{d}V = \int_M \int_{G^0} \phi(x,p) \, \mathrm{d}V^{(x)}(p) \, \mathrm{d}\mathcal{H}^m(x),$$

where $x \mapsto V^{(x)}$ is a locally $\mathcal{H}^m$-integrable map from M to the space of Radon measures on G^0. Similarly, we have representations

$$\int_{G^0(\Omega)} \phi \, \mathrm{d}W_k = \int_M \int_{G^0} \phi(x,p) \, \mathrm{d}W_k^{(x)}(p) \, \mathrm{d}\mathcal{H}^m(x), \quad k \in \mathbb{N},$$

of the same type.

Let $p_+, p_- \colon M \to G^0$ be two $\mathcal{H}^m$-measurable maps such that at $\mathcal{H}^m$-almost every $x \in M$, we have $\Pi(p_+(x)) = \Pi(p_-(x)) = T_xM$ and $p_+(x) = -p_-(x)$. For $p \in G^0$, let δ_p denote the Dirac measure centred at p. We have to show that, for $\mathcal{H}^m$-almost every $x_0 \in M$, we have

$$V^{(x_0)} = \theta_+ \delta_{p_+(x_0)} + \theta_- \delta_{p_-(x_0)} \tag{7.1}$$

for two numbers $\theta_+, \theta_- \in \mathbb{N}_0$. We already know that $W_k^{(x_0)}$ has this form for $\mathcal{H}^m$-almost every $x_0 \in M$.

Choose $h \in C^\infty(\mathbb{R})$ with $0 \leqslant h \leqslant 1$, $h \equiv 1$ in $(-\infty, \frac{1}{2}]$ and $h \equiv 0$ in $[1,\infty)$. For $\ell \in \mathbb{N}$ and $x_0 \in \Omega$, define $\chi_{\ell,x_0}(x,p) = \ell^m h(\ell|x - x_0|)$. Let

$$C = \int_{\mathbb{R}^m \times \{0\}} \chi_{1,0} \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m.$$

Then, for $\mathcal{H}^m$-almost all $x_0 \in M$ and all $\eta \in C^0(G^0)$, we have

$$\int_{G^0(\Omega)} \chi_{\ell,x_0}(x)\eta(p) \, \mathrm{d}V(x,p) \to C \int_{G^0} \eta \, \mathrm{d}V^{(x_0)}$$

and

$$\int_{G^0(\Omega)} \chi_{\ell,x_0}(x)\eta(p) \, \mathrm{d}W_k(x,p) \to C \int_{G^0} \eta \, \mathrm{d}W_k^{(x_0)}, \quad k \in \mathbb{N},$$

as $\ell \to 0$. We fix a point $x_0 \in M$ such that we have these limits and, in addition, each $W_k^{(x_0)}$ is of the form (7.1).

If $V^{(x_0)}$ did not have a representation as in (7.1), then there would be a function $\eta \in C^1(G^0)$ and a number $\alpha > 0$ such that

$$\left| \int_{G^0} \eta \, \mathrm{d}V^{(x_0)} - \int_{G^0} \eta \, \mathrm{d}U \right| \geqslant \alpha$$

for every measure U on G^0 of the form (7.1). For this η, consider the numbers

$$\gamma_{k\ell} = \int_{G^0} \chi_{\ell,x_0} \eta \, \mathrm{d}V - \int_{G^0} \chi_{\ell,x_0} \eta \, \mathrm{d}W_k.$$

On the one hand, for any fixed k, we have

$$|\gamma_{k\ell}| \to C \left| \int_{G^0} \eta \, \mathrm{d}V^{(x_0)} - \int_{G^0} \eta \, \mathrm{d}W_k^{(x_0)} \right| \geqslant C\alpha \quad \text{as } \ell \to \infty.$$

On the other hand,

$$|\gamma_{k\ell}| \leqslant 2^{-k} \sup_{G^0} (|\eta| + |\operatorname{grad} \eta|) \int_{\Omega} \chi_{\ell,x_0} \, \mathrm{d}\|V\| \to 0 \quad \text{as } k \to \infty$$

uniformly in ℓ. Thus, if k is sufficiently large, then we have a contradiction. □

Proof of theorem 7.1. It is clear that there exist an oriented m-varifold V and a Radon measure A such that we have the required convergence for a suitable subsequence. Because of the uniform bounds for $\partial_{A_k} V_k$, it also immediately follows that (5.5) is satisfied. The most difficult part of the proof is to show that $V \in \mathrm{IV}_m^0(\Omega)$. As it suffices to prove this locally, we may assume without loss of generality that

$$\sup_{k \in \mathbb{N}} (\|V_k\|(\Omega) + |A_k|(\Omega \times L) + |\partial_{A_k} V_k|(G^0(\Omega))) < \infty$$

for every compact set $L \subset G^0_{\mathcal{Q}}$.

Fix $R > 0$ such that $B^0_{2R}(q_1) \cap B^0_{2R}(q_2) = \emptyset$ for $q_1, q_2 \in \mathcal{Q}$ with $q_1 \neq q_2$. We apply lemma 6.1 to the ball $B^0_{2R}(q)$ for each $q \in \mathcal{Q}$. Thus, we obtain a decomposition

$$V_k = \tilde{V}_k + \sum_{q \in Q} V_k^q,$$

with corresponding measures $\tilde{A}_k \in \mathcal{C}_{\mathcal{Q}} \tilde{V}_k$ and $A_k^q \in \mathcal{C}_{\mathcal{Q}} V_k^q$, satisfying

$$\operatorname{supp} \tilde{V}_k \subset \bar{\Omega} \times \left(G^0 \setminus \bigcup_{q \in \mathcal{Q}} B^0_R(q) \right)$$

and

$$\operatorname{supp} V_k^q \subset \bar{\Omega} \times B^0_{2R}(q).$$

Furthermore,

$$\sup_{k \in \mathbb{N}} (\|\tilde{V}_k\|(\Omega) + |\tilde{A}_k|(G^0(\Omega)) + |\partial_{\tilde{A}_k} \tilde{V}_k|(G^0(\Omega))) < \infty$$

and

$$\sup_{k\in\mathbb{N}}(\|V_k^q\|(\Omega) + |\partial_{A_k^q} V_k^q|(G^0(\Omega))) < \infty.$$

Using proposition 5.2, we see that the varifolds $\tilde{V}_k$ satisfy the conditions of theorem 4.1. Thus, we may assume that $\tilde{V}_k \overset{*}{\rightharpoonup} \tilde{V}$ for some $\tilde{V} \in \mathrm{IV}_m^0(\Omega)$.

Now fix $q \in \mathcal{Q}$ and consider V_k^q and the corresponding integer rectifiable current $T_k^q = \mathcal{T}(V_k^q)$. We may assume that $V_k^q \overset{*}{\rightharpoonup} V^q$ for a varifold V^q.

Let $\omega \in C_0^\infty(\Omega; \Lambda^{m-1}\mathbb{R}^n)$. Then we have

$$T_k^q(\mathrm{d}\omega) = \int_{G^0(\Omega)} \langle \Xi, \mathrm{d}\omega\rangle \,\mathrm{d}V_k^q = \int_{G^0(\Omega)} \langle \mathrm{d}\partial_{A_k^q} V_k^q, \omega\rangle.$$

Hence,

$$\sup_{k\in\mathbb{N}}(\|T_k^q\|(\Omega) + \|\partial T_k^q\|(\Omega)) < \infty.$$

By theorem 4.2, we may assume that there exists an integer rectifiable m-current T^q in Ω such that $T_k^q \overset{*}{\rightharpoonup} T^q$ as $k \to \infty$. Let $W^q = \mathcal{V}(T^q)$.

We do not necessarily have equality of V^q and W^q, because convergence of currents allows cancellation, whereas convergence of varifolds does not. But since all m-vectors associated to T_k^q are in a ball of radius $2R$ about $\Xi(q)$, cancellation only happens to a limited degree. Therefore, the difference of V^q and W^q is small if R is small. We now want to make this observation more precise.

For $\|V^q\|$-almost every $x \in \Omega$, a fibre measure $V^{q,(x)}$ exists on G^0, such that, for all $\phi \in C_0^0(G^0(\Omega))$,

$$\int_{G^0(\Omega)} \phi \,\mathrm{d}V^q = \int_\Omega \int_{G^0} \phi(x,p)\,\mathrm{d}V^{q,(x)}(p)\,\mathrm{d}\|V\|(x).$$

Consider a function $\xi^q \colon \Omega \to \Lambda_m\mathbb{R}^n$ with

$$\xi^q(x) = \int_{G^0} \Xi \,\mathrm{d}V^{q,(x)}$$

for all $x \in \Omega$ such that this is well-defined. Then we have

$$T^q(\omega) = \int_\Omega \langle \xi^q, \omega\rangle \,\mathrm{d}\|V^q\|$$

for all $\omega \in C_0^\infty(\Omega; \Lambda^m\mathbb{R}^n)$. We know that $T^q \in \mathrm{IC}_m(\Omega)$. Thus, $\xi^q(x)$ is a simple m-vector almost everywhere and

$$\int_{G^0(\Omega)} \phi \,\mathrm{d}W^q = \int_\Omega |\xi^q|\phi\biggl(x, \Xi^{-1}\biggl(\frac{\xi^q}{|\xi^q|}\biggr)\biggr)\,\mathrm{d}\|V^q\|$$

for every $\phi \in C_0^0(G^0(\Omega))$. But since $\operatorname{supp} V^{q,(x)} \subset B_{2R}^0(q)$ for $\|V^q\|$-almost every $x \in \Omega$, we have

$$\biggl|\int_{G^0} \phi(x,\cdot)\,\mathrm{d}V^{q,(x)} - |\xi^q(x)|\phi\biggl(x, \Xi^{-1}\biggl(\frac{\xi^q(x)}{|\xi^q(x)|}\biggr)\biggr)\biggr| \leqslant 4R \sup_{\{x\}\times G^0}(|\phi| + |\operatorname{grad}\phi|)$$

for every such x. Hence,

$$\left|\int_{G^0(\Omega)} \phi\,\mathrm{d}V^q - \int_{G^0(\Omega)} \phi\,\mathrm{d}W^q\right| \leqslant 4R \int_\Omega \sup_{\{x\}\times G^0} (|\phi| + |\mathrm{grad}\,\phi|)\,\mathrm{d}\|V^q\|(x).$$

Obviously,

$$V = \tilde{V} + \sum_{q\in\mathcal{Q}} V^q.$$

Set

$$W = \tilde{V} + \sum_{q\in\mathcal{Q}} W^q.$$

This is an oriented integral m-varifold in Ω. We know that

$$\left|\int_{G^0(\Omega)} \phi\,\mathrm{d}V - \int_{G^0(\Omega)} \phi\,\mathrm{d}W\right| \leqslant 4R \int_\Omega \sup_{\{x\}\times G^0} (|\phi| + |\mathrm{grad}\,\phi|)\,\mathrm{d}\|V\|(x).$$

Since R was chosen arbitrarily, we conclude that V satisfies the hypothesis of lemma 7.2. Thus, $V \in \mathrm{IV}_m^0(\Omega)$. □

8. Analysis of the limiting configuration

Now we consider the continuous function $\Phi\colon G^0 \to [0,\infty)$ with $\Phi^{-1}(\{0\}) = \mathcal{Q}$ again that gives rise to the functionals $\mathcal{F}_\epsilon$ in the introduction. We study a sequence of m-dimensional oriented submanifolds $M_k \subset \Omega$ of class C^2 with $\partial M_k \cap \Omega = \emptyset$ such that there exists a sequence $\epsilon_k \searrow 0$ with

$$\limsup_{k\to\infty} \mathcal{F}_{\epsilon_k}(M_k) < \infty. \tag{8.1}$$

Furthermore, we assume that either each M_k is compact or

$$\limsup_{k\to\infty} \mathcal{H}^m(M_k \cap K) < \infty$$

for every compact set $K \subset \Omega$.

Let $\mathcal{A}_k$ denote the second fundamental form of M_k, and let $p_k\colon M_k \to G^0$ be the maps that give the orientations of M_k. We want to determine the asymptotic behaviour of the manifolds and of the energy densities

$$\frac{1}{2}\left(\epsilon_k|\mathcal{A}_k|^2 + \frac{1}{\epsilon_k}(\Phi(p_k))^2\right).$$

By Young's inequality, we have

$$\limsup_{k\to\infty} \int_{M_k} \Phi(p_k)|\mathcal{A}_k|\,\mathrm{d}\mathcal{H}^m \leqslant \limsup_{k\to\infty} \mathcal{F}_{\epsilon_k}(M_k) < \infty.$$

Thus, if $V_k = \mathrm{vf}(M_k, 1, 0, p_k)$ is the varifold belonging to M_k, then according to proposition 5.3 there exists an $A_k \in \mathcal{C}_\mathcal{Q} V_k$ with $\partial_{A_k} V_k = 0$, such that

$$\limsup_{k\to\infty} \int_{G^0_\mathcal{Q}(\Omega)} \Phi\,\mathrm{d}|A_k| < \infty. \tag{8.2}$$

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We have local uniform bounds for $\|V_k\|$, either directly from our assumptions, or by proposition 6.2. Hence, we may choose a subsequence such that we have weak* convergence of the varifolds and their curvatures. By theorem 7.1, the limits are an oriented integral m-varifold $V \in \mathrm{IV}^0_m(\Omega)$ and a Radon measure $A \in \mathcal{C}_\mathcal{Q} V$. For simplicity, we assume that we have convergence of the whole sequence, that is, $V_k \overset{*}{\rightharpoonup} V$ in $G^0(\Omega)$ and $A_k \overset{*}{\rightharpoonup} A$ in $G^0_\mathcal{Q}(\Omega)$.

THEOREM 8.1. *There exist integer rectifiable m-currents $T^q = \mathrm{ct}(F^q, \theta^q, \Xi(q))$ for $q \in \mathcal{Q}$, such that*

$$V = \sum_{q \in \mathcal{Q}} \mathcal{V}(T^q). \tag{8.3}$$

Furthermore, each boundary ∂T^q is an integer rectifiable $(m-1)$-current.

Proof. It follows from (8.1) that

$$\int_{G^0(\Omega)} \Phi^2 \, \mathrm{d}V = 0.$$

Thus, $\operatorname{supp} V \subset \bar{\Omega} \times \mathcal{Q}$. If we localize about every $q \in \mathcal{Q}$ with the help of Lemma 6.1 (as in the proofs of proposition 6.2 and theorem 7.1), then we obtain a decomposition

$$V = \sum_{q \in \mathcal{Q}} V^q,$$

where $\operatorname{supp} V^q \subset \bar{\Omega} \times \{q\}$. Furthermore, there exists an $A^q \in \mathcal{C}_\mathcal{Q} V^q$. Setting $T^q = \mathcal{T}(V^q)$, we conclude that $V^q = \mathcal{V}(T^q)$. Clearly, T^q has the required structure. The boundary ∂T^q is given by the projection of $\partial_{A^q} V^q$ onto Ω. So in particular $\|\partial T^q\|(K) < \infty$ for every compact set $K \subset \Omega$. By theorem 4.3, we have $\partial T^q \in \mathrm{IC}_{m-1}(\Omega)$. □

Thus, we can think of V as a generalized polyhedron with faces represented by T^q. Note that (8.3) is stronger than

$$\mathcal{T}(V) = \sum_{q \in \mathcal{Q}} T^q.$$

It implies in particular that the collection of the currents T^q accounts for all of the measure $\|V\|$.

The boundaries ∂T^q play the role of the edges of the generalized polyhedron. The analogy with the edges of an actual polyhedron in limited, however, because the structure of ∂T^q can be more complicated. In particular, if for some $q \in \mathcal{Q}$ we also have $-q \in \mathcal{Q}$, then the common boundary of T^q and T^{-q} can have any $(m-1)$-dimensional subspace of $\Pi(q)$ as a tangent space. On the other hand, if $q_1, q_2 \in \mathcal{Q}$ with $q_1 \neq \pm q_2$, then the tangent spaces of the common boundary of T^{q_1} and T^{q_2} are restricted to $\Pi(q_1) \cap \Pi(q_2)$ almost everywhere. If this is not an $(m-1)$-dimensional space, then the corresponding part of the boundary is negligible.

In order to formulate this more precisely, we introduce the set $\mathcal{R}$, comprising all $r \in G^0(n, m-1)$ such that there exist $q_1, q_2 \in \mathcal{Q}$ with $\Pi_{m-1}(r) = \Pi(q_1) \cap \Pi(q_2)$.

THEOREM 8.2. *There exist a countably $(m-1)$-rectifiable and $\mathcal{H}^{m-1}$-measurable set $E \subset \Omega$, an $\mathcal{H}^{m-1}$-measurable map $\zeta\colon E \to \Xi_{m-1}(G^0(n, m-1))$ and locally $\mathcal{H}^{m-1}$-integrable functions $\sigma^q\colon E \to \mathbb{Z}$ for $q \in \mathcal{Q}$, such that*

$$\partial T^q(\omega) = \int_E \langle \zeta, \omega \rangle \sigma^q \, \mathrm{d}\mathcal{H}^{m-1}, \quad q \in \mathcal{Q},$$

for all $\omega \in C_0^\infty(\Omega; \Lambda^{m-1}\mathbb{R}^n)$. For $\mathcal{H}^{m-1}$-almost every $x \in E$,

$$\sum_{q \in \mathcal{Q}} \sigma^q(x) = 0, \tag{8.4}$$

and if $\zeta(x)$ does not belong to $\Xi_{m-1}(\mathcal{R})$, then there exists a $q_0 \in \mathcal{Q}$ such that $-q_0 \in \mathcal{Q}$ and $\sigma^q(x) = 0$ for all $q \in \mathcal{Q} \setminus \{q_0, -q_0\}$.

Proof. We already know that $\partial T^q \in \mathrm{IC}_{m-1}(\Omega)$. Consider the measure

$$\mu = \sum_{q \in \mathcal{Q}} \|\partial T^q\|.$$

This can be represented in the form

$$\mu = (\mathcal{H}^{m-1} \llcorner E) \llcorner s$$

for a countably $(m-1)$-rectifiable and $\mathcal{H}^{m-1}$ measurable set $E \subset \Omega$ and a locally integrable function $s\colon E \to (0, \infty)$. Choose a map ζ such that $\zeta(x)$ orients the approximate tangent space $T_x E$ at $\mathcal{H}^{m-1}$-almost every $x \in E$. Since $\|\partial T^q\|$ is absolutely continuous with respect to μ, there exists a locally $\mathcal{H}^{m-1}$-integrable function $\sigma^q\colon E \to \mathbb{Z}$ such that

$$\partial T^q(\omega) = \int_E \langle \zeta, \omega \rangle \sigma^q \, \mathrm{d}\mathcal{H}^{m-1}$$

for every $\omega \in C_0^\infty(\Omega; \Lambda^{m-1}\mathbb{R}^n)$. We have $\partial_A V = 0$, which implies

$$\sum_{q \in \mathcal{Q}} \partial T^q = \partial \mathcal{T}(V) = 0.$$

Hence, for $\mathcal{H}^{m-1}$-almost every $x \in E$, we have (8.4).

Because T^q is given in terms of the constant m-vector $\Xi(q)$, the $(m-1)$-vector $\zeta(x)$ must belong to an $(m-1)$-dimensional subspace $R_x \subset \Pi(q)$ for $\|\partial T^q\|$-almost every $x \in E$. But for $\mathcal{H}^{m-1}$-almost all $x \in E$, there must be at least two distinct points $q_0, q_1 \in \mathcal{Q}$ such that $R_x \subset \Pi(q_0) \cap \Pi(q_1)$, due to (8.4). This can only be the case if either $R_x \in \Pi_{m-1}(\mathcal{R})$ or $q_1 = -q_0$. Furthermore, if $R_x \notin \Pi_{m-1}(\mathcal{R})$, then it cannot be a subspace of $\Pi(q)$ for any $q \in \mathcal{Q} \setminus \{q_0, -q_0\}$. □

Next we examine the limiting curvature A and the measure $|A| \llcorner \Phi$ that arises from (8.2) when $k \to \infty$. Let c_{m-1} be the volume of the $(m-1)$-dimensional unit ball. We expect that at least a part of the energy density concentrates on the $(m-1)$-dimensional set E; therefore, we are interested in the $(m-1)$-density

$$\Theta(x) = \frac{1}{c_{m-1}} \liminf_{\rho \searrow 0} \left(\rho^{1-m} \int_{G^0_{\mathcal{Q}}(B_\rho(x))} \Phi \, \mathrm{d}|A| \right).$$

Then for every $\eta \in C_0^0(\Omega)$ we have

$$\int_E \eta \Theta \, \mathrm{d}\mathcal{H}^{m-1} \leqslant \int_{G^0_{\mathcal{Q}}(\Omega)} \eta \Phi \, \mathrm{d}|A| \leqslant \frac{1}{2} \liminf_{k\to\infty} \int_{M_k} \eta \biggl(\epsilon_k |\mathcal{A}_k|^2 + \frac{1}{\epsilon_k} \Phi(p_k) \biggr) \, \mathrm{d}\mathcal{H}^m$$

by proposition 5.3 and Young's inequality. So in order to obtain a lower estimate for the limiting energy, it suffices to estimate Θ.

To this end, we first consider a new metric on G^0. Let $g_\Phi = \Phi^2 g$ on $G^0_{\mathcal{Q}}$. That is, we consider the Riemannian metric conformally equivalent to g with conformal factor Φ^2. As a Riemannian metric, this does not extend to G^0, because it becomes degenerate on $\mathcal{Q}$. But it still induces a metric (in the sense of metric spaces) on the whole of G^0. For $p, q \in G^0$, let $\Gamma(p,q)$ be the space of all paths $\gamma \in C^1([0,1]; G^0)$ with $\gamma(0) = p$ and $\gamma(1) = q$. Then we set

$$\operatorname{dist}_\Phi(p,q) = \inf_{\gamma \in \Gamma(p,q)} \int_0^1 \Phi(\gamma) \sqrt{g(\dot\gamma, \dot\gamma)} \, \mathrm{d}t.$$

Now let $\Delta \subset \mathbb{R}^{\mathcal{Q}}$ be the set of all $(\alpha_q)_{q\in\mathcal{Q}}$ such that

$$|\alpha_p - \alpha_q| \leqslant \operatorname{dist}_\Phi(p,q), \quad p, q \in \mathcal{Q}.$$

Theorem 8.3. *For $\mathcal{H}^{m-1}$-almost every $x \in E$,*

$$\Theta(x) \geqslant \sup \biggl\{ \sum_{q\in\mathcal{Q}} \alpha_q \sigma^q(x) \colon (\alpha_q)_{q\in\mathcal{Q}} \in \Delta \biggr\}.$$

The most typical case is of course when only two faces meet at an edge. If $x \in E$ such that there exist $q_1, q_2 \in \mathcal{Q}$ with $\sigma^q(x) = 0$ for $q \in \mathcal{Q} \setminus \{q_1, q_2\}$, and if x is a point where the conclusions of theorem 8.2 hold, then we necessarily have $\sigma^{q_1}(x) = -\sigma^{q_2}(x)$. We then find

$$\Theta(x) \geqslant |\sigma^{q_1}(x)| \operatorname{dist}_\Phi(q_1, q_2).$$

In other situations, we have a more complicated expression. Its exact form comes above all from the method that we use for the proof and may not be optimal.

In order to prove the theorem, we need the following lemma.

Lemma 8.4. *Let $(\alpha_q)_{q\in\mathcal{Q}} \in \Delta$. Then there exists a function $f \in C^{0,1}(G^0)$ with $|\operatorname{grad} f| \leqslant \Phi$ almost everywhere on G^0, such that $f(q) = \alpha_q$ for all $q \in \mathcal{Q}$.*

Proof. We use induction over the size of $\mathcal{Q}$. The statement is obvious for $|\mathcal{Q}| \leqslant 1$. Now suppose that $|\mathcal{Q}| \geqslant 2$. Choose $q_0 \in \mathcal{Q}$ with

$$\alpha_{q_0} = \min_{q\in\mathcal{Q}} \alpha_q$$

and suppose that there exists a function $h \in C^{0,1}(G^0)$ with $|\operatorname{grad} h| \leqslant \Phi$ almost everywhere and $h(q) = \alpha_q$ for every $q \in \mathcal{Q} \setminus \{q_0\}$.

If $h(q_0) \leqslant \alpha_{q_0}$, then the function $f = \max\{h, \alpha_{q_0}\}$ has the required properties. Otherwise, define

$$f_0(p) = \alpha_{q_0} + \operatorname{dist}_\Phi(p, q_0), \quad p \in G^0.$$

Then we have $|\operatorname{grad} f_0| \leqslant \Phi$ almost everywhere and $f_0(q) \geqslant \alpha_q$ for all $q \in \mathcal{Q}$ because $\alpha_q - \alpha_{q_0} \leqslant \operatorname{dist}_\Phi(q, q_0)$. Hence, we can choose $f = \min\{h, f_0\}$. □

Proof of theorem 8.3. First note that in the situation studied here we can rewrite (5.6) as

$$0 = \sum_{q\in\mathcal{Q}} \partial T^q(\omega(\cdot,q)) + \sum_{i=1}^{n} \int_{G^0_{\mathcal{Q}}(\Omega)} g(\langle \Xi, \mathrm{d}x^i \wedge \operatorname{grad}\omega\rangle, \mathrm{d}A_i),$$

using the representation (8.3) of V and the fact that $\partial_A V = 0$. Consider a point $x_0 \in E$ such that $\Theta(x_0) < \infty$ and such that ζ and σ^q are $\mathcal{H}^{m-1}$-approximately continuous at x_0 and E has an approximate tangent space $T_{x_0}E$. Choose a sequence $\rho_k \searrow 0$ such that

$$\lim_{k\to\infty} \biggl(\rho_k^{1-m} \int_{G^0_{\mathcal{Q}}(B_{\rho_k}(x_0))} \Phi \,\mathrm{d}|A|\biggr) = c_{m-1}\Theta(x_0)$$

and rescale everything as follows.

Define $\Omega_k = \rho_k^{-1}(\Omega - x_0)$ and $E_k = \rho_k^{-1}(E - x_0)$. For $x \in E_k$, define $\zeta_k(x) = \zeta(\rho_k x + x_0)$ and $\sigma^q_k(x) = \sigma^q(\rho_k x + x_0)$, $q \in \mathcal{Q}$. Set $T^q_k = \mathrm{ct}(E_k, \sigma^q_k, \zeta_k)$. Furthermore, consider the measures

$$A^{(k)} = \sum_{i=1}^{n} A^{(k)}_i \,\mathrm{d}x^i$$

on $G^0_{\mathcal{Q}}(\Omega_k)$ satisfying

$$\int_{G^0_{\mathcal{Q}}(\Omega_k)} g(\phi, \mathrm{d}A^{(k)}_i) = \rho_k^{1-m} \int_{G^0_{\mathcal{Q}}(\Omega)} g\biggl(\phi\biggl(\frac{x-x_0}{\rho_k}, p\biggr), \mathrm{d}A_i(x,p)\biggr)$$

for every continuous section ϕ of $\Omega_k \times TG^0_{\mathcal{Q}}$ with compact support. Then we have

$$0 = \sum_{q\in\mathcal{Q}} \partial T^q_k(\omega(\cdot,q)) + \sum_{i=1}^{n} \int_{G^0_{\mathcal{Q}}(\Omega)} g(\langle \Xi, \mathrm{d}x^i \wedge \operatorname{grad}\omega\rangle, \mathrm{d}A^{(k)}_i) \tag{8.5}$$

for all $\omega \in C^1_0(G^0(\Omega_k); \Lambda^{m-1}\mathbb{R}^n)$ with $\operatorname{supp}(\operatorname{grad}\omega) \subset G^0_{\mathcal{Q}}(\Omega_k)$. Moreover,

$$\lim_{k\to\infty} \int_{G^0_{\mathcal{Q}}(B_1(0))} \Phi \,\mathrm{d}|A^{(k)}| = c_{m-1}\Theta(x_0) < \infty.$$

Thus, we may assume that $A^{(k)} \overset{*}{\rightharpoonup} \tilde{A}$ in $G^0_{\mathcal{Q}}(B_1(0))$ for a certain Radon measure $\tilde{A}$ on $G^0_{\mathcal{Q}}(B_1(0))$ with values in $T^*B_1(0) \times TG^0_{\mathcal{Q}}$. Moreover, we have

$$c_{m-1}\Theta(x_0) \geqslant \int_{G^0_{\mathcal{Q}}(B_1(0))} \Phi \,\mathrm{d}|\tilde{A}|.$$

We also know that the currents ∂T^q_k converge to $\mathrm{ct}(T_{x_0}E, \sigma^q(x_0), \zeta(x_0))$. Passing to the limit in (8.5), we find

$$\begin{aligned} 0 = \sum_{q\in\mathcal{Q}} \sigma^q(x_0) \int_{T_{x_0}E} \langle \zeta(x_0), \omega(\cdot,q)\rangle \,\mathrm{d}\mathcal{H}^{m-1} \\ + \sum_{i=1}^{n} \int_{G^0_{\mathcal{Q}}(B_1(0))} g(\langle \Xi, \mathrm{d}x^i \wedge \operatorname{grad}\omega\rangle, \mathrm{d}\tilde{A}_i) \end{aligned} \tag{8.6}$$

for every $\omega \in C^1_0(G^0(B_1(0)); \Lambda^{m-1}\mathbb{R}^n)$ with $\operatorname{supp}(\operatorname{grad}\omega) \subset G^0_{\mathcal{Q}}(B_1(0))$.

For simplicity, we assume that $T_{x_0}E = \mathbb{R}^{m-1} \times \{0\}$ and $\zeta(x_0) = e_1 \wedge \cdots \wedge e_{m-1}$. We choose a function $\chi \in C_0^\infty(B_1(0))$. Let $(\alpha_q)_{q\in\mathcal{Q}} \in \Delta$ and choose a function f that satisfies the conditions of lemma 8.4. Fix $\delta > 0$ and let $F \in C^1(G^0)$ with $\operatorname{supp}(\operatorname{grad} F) \subset G^0_{\mathcal{Q}}$ such that

$$\|f - F\|_{C^0(G^0)} \leqslant \delta \quad \text{and} \quad |\operatorname{grad} F| \leqslant \Phi + \delta.$$

Now we test (8.6) with the $(m-1)$-form

$$\omega(x,p) = F(p)\chi(x)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{m-1}.$$

This yields

$$\begin{aligned} 0 = \sum_{q\in\mathcal{Q}} \sigma^q(x_0)F(q) \int_{\mathbb{R}^{m-1}} \chi(x^1,\dots,x^{m-1},0,\dots,0)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{m-1} \\ + \sum_{i=m}^{n} \int_{G^0_{\mathcal{Q}}(B_1(0))} \langle \Xi, \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \cdots \wedge\ \mathrm{d}x^{m-1}\rangle \chi g(\operatorname{grad} F, \mathrm{d}\tilde{A}_i). \end{aligned}$$

Letting χ tend to the characteristic function of $B_1(0)$, we obtain

$$c_{m-1} \sum_{q\in\mathcal{Q}} \sigma^q(x_0)F(q) \leqslant \int_{G^0_{\mathcal{Q}}(B_1(0))} |\operatorname{grad} F|\,\mathrm{d}|\tilde{A}|.$$

Finally, we let $\delta \to 0$, which gives

$$c_{m-1} \sum_{q\in\mathcal{Q}} \sigma^q(x_0)\alpha_q \leqslant \int_{G^0_{\mathcal{Q}}(B_1(0))} \Phi\,\mathrm{d}|\tilde{A}| \leqslant c_{m-1}\Theta(x_0),$$

as required. □

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