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AC-KBO revisited †*

AKIHISA YAMADA
Research Institute for Secure Systems, AIST, Amagasaki, Japan

SARAH WINKLER
Institute of Computer Science, University of Innsbruck, Innsbruck, Austria

NAO HIROKAWA
School of Information Science, JAIST, Nomi, Japan

AART MIDDELDORP
Institute of Computer Science, University of Innsbruck, Innsbruck, Austria
(*e-mail:* aart.middeldorp@uibk.ac.at)

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Abstract

Equational theories that contain axioms expressing associativity and commutativity (AC) of certain operators are ubiquitous. Theorem proving methods in such theories rely on well-founded orders that are compatible with the AC axioms. In this paper, we consider various definitions of AC-compatible Knuth-Bendix orders. The orders of Steinbach and of Korovin and Voronkov are revisited. The former is enhanced to a more powerful version, and we modify the latter to amend its lack of monotonicity on non-ground terms. We further present new complexity results. An extension reflecting the recent proposal of subterm coefficients in standard Knuth-Bendix orders is also given. The various orders are compared on problems in termination and completion.

KEYWORDS: Term rewriting, termination, associative-commutative theory, Knuth-Bendix order

1 Introduction

Associative and commutative (AC) operators appear in many applications, e.g. in automated reasoning with respect to algebraic structures such as commutative

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groups or rings. We are interested in proving termination of term rewrite systems with AC symbols. AC termination is important when deciding validity in equational theories with AC operators by means of completion.

Several termination methods for plain rewriting have been extended to deal with AC symbols. Ben Cherifa and Lescanne (1987) presented a characterization of polynomial interpretations that ensures compatibility with the AC axioms. There have been numerous papers on extending the recursive path order (RPO) of Dershowitz (1982) to deal with AC symbols, starting with the associative path order of Bachmair and Plaisted (1985) and culminating in the fully syntactic AC-RPO of Rubio (2002). Several authors (Giesl and Kapur 2001; Kusakari and Toyama 2001; Marché and Urbain 2004; Alarcón *et al.* 2010) adapted the influential dependency pair method of Arts and Giesl (2000) to AC rewriting.

We are aware of only two papers on AC extensions of the order (KBO) of Knuth and Bendix (1970). In this paper, we revisit these orders and present yet another AC-compatible KBO. Steinbach (1990) presented a first version, which comes with the restriction that AC symbols are minimal in the precedence. By incorporating ideas of Rubio (2002), Korovin and Voronkov (2003a) presented a version without this restriction. Actually, they present two versions. One is defined on ground terms and another one on arbitrary terms. For (automatically) proving AC termination of rewrite systems, an AC-compatible order on arbitrary terms is required[1]. We show that the second order of Korovin and Voronkov lacks the monotonicity property which is required by the definition of simplification orders. Nevertheless, we prove that the order is sound for proving termination by extending it to an AC-compatible simplification order. We furthermore present a simpler variant of this latter order which properly extends the order of Steinbach (1990). In particular, Steinbach's order is a correct AC-compatible simplification order, contrary to what is claimed in Korovin and Voronkov (2003a). We also present new complexity results which confirm that AC rewriting is much more involved than plain rewriting. Apart from these theoretical contributions, we implemented the various AC-compatible KBOs to compare them also experimentally.

The remainder of this paper is organized as follows. After recalling basic concepts of rewriting modulo AC and orders, we revisit Steinbach's order in Section 3. Section 4 is devoted to the two orders of Korovin and Voronkov. We present a first version of our AC-compatible KBO in Section 5, also giving the nontrivial proof that it has the required properties. (The proofs in Korovin and Voronkov (2003a) are limited to the order on ground terms.) In Section 6, we consider the complexity of the membership and orientation decision problems for the various orders. In Section 7, we compare AC-KBO with AC-RPO. In Section 8, our order is strengthened with subterm coefficients. In order to show effectiveness of these orders experimental data is provided in Section 9. The paper is concluded in Section 10.

[1] Any AC-compatible reduction order $>_{\mathrm{g}}$ on ground terms can trivially be extended to arbitrary terms by defining $s > t$ if and only if $s\sigma >_{\mathrm{g}} t\sigma$ for all grounding substitutions σ. This is, however, only of (mild) theoretical interest.

This article is an updated and extended version of Yamada *et al.* (2014). Our earlier results on complexity are extended by showing that the orientability problems for different versions of AC-KBO are in NP. Moreover, we include a comparison with AC-RPO, which we present in a slightly simplified manner compared to (Rubio 2002). Due to space limitations, some proofs can be found in the online appendix.

2 Preliminaries

We assume familiarity with rewriting and termination. Throughout this paper, we deal with rewrite systems over a set $\mathscr{V}$ of variables and a *finite* signature $\mathscr{F}$ together with a designated subset $\mathscr{F}_{\mathsf{AC}}$ of binary AC symbols. The congruence relation induced by the equations $f(x,y) \approx f(y,x)$ and $f(f(x,y),z) \approx f(x,f(y,z))$ for all $f \in \mathscr{F}_{\mathsf{AC}}$ is denoted by $=_{\mathsf{AC}}$. A term rewrite system (TRS for short) $\mathscr{R}$ is AC terminating if the relation $=_{\mathsf{AC}} \cdot \rightarrow_{\mathscr{R}} \cdot =_{\mathsf{AC}}$ is well-founded. In this paper, AC termination is established by *AC-compatible simplification orders* $\succ$, which are strict orders (i.e., irreflexive and transitive relations) closed under contexts and substitutions that have the subterm property $f(t_1,\ldots,t_n) \succ t_i$ for all $1 \leqslant i \leqslant n$ and satisfy $=_{\mathsf{AC}} \cdot \succ \cdot =_{\mathsf{AC}} \subseteq \succ$. A strict order $\succ$ is *AC-total* if $s \succ t$, $t \succ s$ or $s =_{\mathsf{AC}} t$, for all ground terms s and t. A pair $(\succsim, \succ)$ consisting of a preorder $\succsim$ and a strict order $\succ$ is said to be an *order pair* if the *compatibility* condition $\succsim \cdot \succ \cdot \succsim \subseteq \succ$ holds.

Definition 2.1
Let $\succ$ be a strict order and $\succsim$ be a preorder on a set A. The *lexicographic extensions* $\succ^{\mathsf{lex}}$ and $\succsim^{\mathsf{lex}}$ are defined as follows:

- $\vec{x} \succsim^{\mathsf{lex}} \vec{y}$ if $\vec{x} \sqsupset_k^{\mathsf{lex}} \vec{y}$ for some $1 \leqslant k \leqslant n$,
- $\vec{x} \succ^{\mathsf{lex}} \vec{y}$ if $\vec{x} \sqsupset_k^{\mathsf{lex}} \vec{y}$ for some $1 \leqslant k < n$.

Here $\vec{x} = (x_1,\ldots,x_n)$, $\vec{y} = (y_1,\ldots,y_n)$, and $\vec{x} \sqsupset_k^{\mathsf{lex}} \vec{y}$ denotes the following condition: $x_i \succsim y_i$ for all $i \leqslant k$ and either $k < n$ and $x_{k+1} \succ y_{k+1}$ or $k = n$. The *multiset extensions* $\succ^{\mathsf{mul}}$ and $\succsim^{\mathsf{mul}}$ are defined as follows:

- $M \succsim^{\mathsf{mul}} N$ if $M \sqsupset_k^{\mathsf{mul}} N$ for some $0 \leqslant k \leqslant \min(m,n)$,
- $M \succ^{\mathsf{mul}} N$ if $M \sqsupset_k^{\mathsf{mul}} N$ for some $0 \leqslant k \leqslant \min(m-1,n)$.

Here $M \sqsupset_k^{\mathsf{mul}} N$ if M and N consist of $x_1,\ldots,x_m$ and $y_1,\ldots,y_n$, respectively such that $x_j \succsim y_j$ for all $j \leqslant k$, and for every $k < j \leqslant n$ there is some $k < i \leqslant m$ with $x_i \succ y_j$.

Note that these extended relations depend on both $\succsim$ and $\succ$. The following result is folklore; a recent formalization of multiset extensions in Isabelle/HOL is presented in Thiemann *et al.* (2012).

Theorem 2.2
If $(\succsim, \succ)$ is an order pair then $(\succsim^{\mathsf{lex}}, \succ^{\mathsf{lex}})$ and $(\succsim^{\mathsf{mul}}, \succ^{\mathsf{mul}})$ are order pairs. □

3 Steinbach's order

In this section, we recall the AC-compatible KBO $>_{\mathsf{S}}$ of Steinbach (1990), which reduces to the standard KBO if AC symbols are absent.[2] The order $>_{\mathsf{S}}$ depends on a precedence and an admissible weight function. A *precedence* $>$ is a strict order on $\mathscr{F}$. A *weight function* (w, w_0) for a signature $\mathscr{F}$ consists of a mapping $w : \mathscr{F} \to \mathbb{N}$ and a constant $w_0 > 0$ such that $w(c) \geqslant w_0$ for every constant $c \in \mathscr{F}$. The *weight* of a term t is recursively computed as follows:

$$w(t) = \begin{cases} w_0 & \text{if } t \in \mathscr{V} \\ w(f) + \sum_{1 \leqslant i \leqslant n} w(t_i) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

A weight function (w, w_0) is *admissible* for $>$ if every unary f with $w(f) = 0$ satisfies $f > g$ for all function symbols g different from f. Throughout this paper we assume admissibility.

The *top-flattening* (Rubio 2002) of a term t with respect to an AC symbol f is the multiset $\nabla_f(t)$ defined inductively as follows:

$$\nabla_f(t) = \begin{cases} \{t\} & \text{if } \mathsf{root}(t) \neq f \\ \nabla_f(t_1) \uplus \nabla_f(t_2) & \text{if } t = f(t_1, t_2) \end{cases}$$

Definition 3.1
Let $>$ be a precedence and (w, w_0) a weight function. The order $>_{\mathsf{S}}$ is inductively defined as follows: $s >_{\mathsf{S}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathscr{V}$ and either $w(s) > w(t)$, or $w(s) = w(t)$ and one of the following alternatives holds:

0. $s = f^k(t)$ and $t \in \mathscr{V}$ for some $k > 0$,
1. $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, and $f > g$,
2. $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $f \notin \mathscr{F}_{\mathsf{AC}}$, $(s_1, \ldots, s_n) >_{\mathsf{S}}^{\mathsf{lex}} (t_1, \ldots, t_n)$,
3. $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathscr{F}_{\mathsf{AC}}$, and $\nabla_f(s) >_{\mathsf{S}}^{\mathsf{mul}} \nabla_f(t)$.

The relation $=_{\mathsf{AC}}$ is used as preorder in $>_{\mathsf{S}}^{\mathsf{lex}}$ and $>_{\mathsf{S}}^{\mathsf{mul}}$.

Cases 0–2 are the same as in the standard Knuth–Bendix order. In case 3 terms rooted by the same AC symbol f are treated by comparing their top-flattenings in the multiset extension of $>_{\mathsf{S}}$.

Example 3.2
Consider the signature $\mathscr{F} = \{\mathsf{a}, \mathsf{f}, +\}$ with $+ \in \mathscr{F}_{\mathsf{AC}}$, precedence $\mathsf{f} > \mathsf{a} > +$ and admissible weight function (w, w_0) with $w(\mathsf{f}) = w(+) = 0$ and $w_0 = w(\mathsf{a}) = 1$. Let $\mathscr{R}_1$ be the following ground TRS:

$$\mathsf{f}(\mathsf{a} + \mathsf{a}) \to \mathsf{f}(\mathsf{a}) + \mathsf{f}(\mathsf{a}) \qquad (1) \qquad\qquad \mathsf{a} + \mathsf{f}(\mathsf{f}(\mathsf{a})) \to \mathsf{f}(\mathsf{a}) + \mathsf{f}(\mathsf{a}) \qquad (2)$$

For $1 \leqslant i \leqslant 2$, let ℓ_i and r_i be the left- and right-hand side of rule (i), $S_i = \nabla_+(\ell_i)$ and $T_i = \nabla_+(r_i)$. Both rules vacuously satisfy the variable condition. We have

[2] The version in Steinbach (1990) is slightly more general, since non-AC function symbols can have arbitrary status. To simplify the discussion, we do not consider status in this paper.

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$w(\ell_1) = 2 = w(r_1)$ and $\mathsf{f} > +$, so $\ell_1 >_{\mathsf{S}} r_1$ holds by case 1. We have $w(\ell_2) = 2 = w(r_2)$, $S_2 = \{\mathsf{a}, \mathsf{f}(\mathsf{f}(\mathsf{a}))\}$, and $T_2 = \{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{a})\}$. Since $\mathsf{f}(\mathsf{a}) >_{\mathsf{S}} \mathsf{a}$ holds by case 1, $\mathsf{f}(\mathsf{f}(\mathsf{a})) >_{\mathsf{S}} \mathsf{f}(\mathsf{a})$ holds by case 2, and therefore $\ell_2 >_{\mathsf{S}} r_2$ by case 3.

Theorem 3.3 (*Steinbach 1990*)
If every symbol in $\mathscr{F}_{\mathsf{AC}}$ is minimal with respect to $>$ then $>_{\mathsf{S}}$ is an AC-compatible simplification order.[3]

In Section 5 we reprove[4] Theorem 3.3 by showing that $>_{\mathsf{S}}$ is a special case of our new AC-compatible Knuth–Bendix order.

4 Korovin and Voronkov's orders

In this section, we recall the orders of Korovin and Voronkov (2003a). The first one is defined on ground terms. The difference with $>_{\mathsf{S}}$ is that in case 3 of the definition a further case analysis is performed based on terms in S and T whose root symbols are not smaller than f in the precedence. Rather than recursively comparing these terms with the order being defined, a lighter non-recursive version is used in which the weights and root symbols are considered. This is formally defined below.

Given a multiset T of terms, a function symbol f, and a binary relation R on function symbols, we define the following submultisets of T:

$$T\restriction_{\mathscr{V}} = \{x \in T \mid x \in \mathscr{V}\} \qquad T\restriction_f^R = \{t \in T \setminus \mathscr{V} \mid \mathsf{root}(t)\ R\ f\}$$

Definition 4.1
Let $>$ be a precedence and (w, w_0) a weight function.[5] First we define the auxiliary relations $=_{\mathsf{kv}}$ and $>_{\mathsf{kv}}$ on ground terms as follows:

- $s =_{\mathsf{kv}} t$ if $w(s) = w(t)$ and $\mathsf{root}(s) = \mathsf{root}(t)$,
- $s >_{\mathsf{kv}} t$ if either $w(s) > w(t)$ or both $w(s) = w(t)$ and $\mathsf{root}(s) > \mathsf{root}(t)$.

The order $>_{\mathsf{KV}}$ is inductively defined on ground terms as follows: $s >_{\mathsf{KV}} t$ if either $w(s) > w(t)$, or $w(s) = w(t)$ and one of the following alternatives holds:

1. $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, and $f > g$,
2. $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $f \notin \mathscr{F}_{\mathsf{AC}}$, $(s_1, \ldots, s_n) >_{\mathsf{KV}}^{\mathsf{lex}} (t_1, \ldots, t_n)$,
3. $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathscr{F}_{\mathsf{AC}}$, and for $S = \nabla_f(s)$ and $T = \nabla_f(t)$
   (a) $S\restriction_f^{\not<} >_{\mathsf{kv}}^{\mathsf{mul}} T\restriction_f^{\not<}$, or
   (b) $S\restriction_f^{\not<} =_{\mathsf{kv}}^{\mathsf{mul}} T\restriction_f^{\not<}$ and $|S| > |T|$, or
   (c) $S\restriction_f^{\not<} =_{\mathsf{kv}}^{\mathsf{mul}} T\restriction_f^{\not<}$, $|S| = |T|$, and $S >_{\mathsf{KV}}^{\mathsf{mul}} T$.

Here $=_{\mathsf{AC}}$ is used as preorder in $>_{\mathsf{KV}}^{\mathsf{lex}}$ and $>_{\mathsf{KV}}^{\mathsf{mul}}$, whereas $=_{\mathsf{kv}}$ is used in $>_{\mathsf{kv}}^{\mathsf{mul}}$.

[3] In Steinbach (1990) AC symbols are further required to have weight 0 because terms are flattened. Our version of $>_{\mathsf{S}}$ does not impose this restriction due to the use of top-flattening.
[4] The counterexample in Korovin and Voronkov (2003a) against the monotonicity of $>_{\mathsf{S}}$ is invalid as the condition that AC symbols are *minimal* in the precedence is not satisfied.
[5] Here we do not impose totality on precedences, cf. Korovin and Voronkov (2003a). See also Example 5.11.

Only in cases 2 and 3(c) the order $>_{\mathsf{KV}}$ is used recursively. In case 3 terms rooted by the same AC symbol f are compared by extracting from the top-flattenings S and T the multisets $S\restriction_f^{\not<}$ and $T\restriction_f^{\not<}$ consisting of all terms rooted by a function symbol not smaller than f in the precedence. If $S\restriction_f^{\not<}$ is larger than $T\restriction_f^{\not<}$ in the multiset extension of $>_{\mathsf{kv}}$, we conclude in case 3(a). Otherwise, the multisets must be equal (with respect to $=_{\mathsf{kv}}^{\mathsf{mul}}$). If S has more terms than T, we conclude in case 3(b). In the final case 3(c), S and T have the same number of terms and we compare S and T in the multiset extension of $>_{\mathsf{KV}}$.

Theorem 4.2 (*Korovin and Voronkov 2003a*)
The order $>_{\mathsf{KV}}$ is an AC-compatible simplification order on ground terms. If $>$ is total then $>_{\mathsf{KV}}$ is AC-total on ground terms.

The two orders $>_{\mathsf{KV}}$ and $>_{\mathsf{S}}$ are incomparable on ground TRSs.

Example 4.3
Consider again the ground TRS $\mathscr{R}_1$ of Example 3.2. To orient rule (1) with $>_{\mathsf{KV}}$, the weight of the unary function symbol f must be 0 and admissibility demands $\mathsf{f} > \mathsf{a}$ and $\mathsf{f} > +$. Hence rule (1) is handled by case 1 of the definition. For rule (2), the multisets $S = \{\mathsf{a}, \mathsf{f}(\mathsf{f}(\mathsf{a}))\}$ and $T = \{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{a})\}$ are compared in case 3. We have $S\restriction_+^{\not<} = \{\mathsf{f}(\mathsf{f}(\mathsf{a}))\}$ if $+ > \mathsf{a}$ and $S\restriction_+^{\not<} = S$ otherwise. In both cases, we have $T\restriction_+^{\not<} = T$. Note that neither $\mathsf{a} >_{\mathsf{kv}} \mathsf{f}(\mathsf{a})$ nor $\mathsf{f}(\mathsf{f}(\mathsf{a})) >_{\mathsf{kv}} \mathsf{f}(\mathsf{a})$ holds. Hence, case 3(a) does not apply. But also cases 3(b) and 3(c) are not applicable as $\mathsf{f}(\mathsf{f}(\mathsf{a})) =_{\mathsf{kv}} \mathsf{f}(\mathsf{a})$ and $\mathsf{a} \neq_{\mathsf{kv}} \mathsf{f}(\mathsf{a})$. Hence, independent of the choice of $>$, $\mathscr{R}_1$ cannot be proved terminating by $>_{\mathsf{KV}}$. Conversely, the TRS $\mathscr{R}_2$ resulting from reversing rule (2) in $\mathscr{R}_1$ can be proved terminating by $>_{\mathsf{KV}}$ but not by $>_{\mathsf{S}}$.

Next we present the second order of Korovin and Voronkov (2003a), the extension of $>_{\mathsf{KV}}$ to non-ground terms. Since it coincides with $>_{\mathsf{KV}}$ on ground terms, we use the same notation for the order.

In case 3 of the following definition, also variables appearing in the top-flattenings S and T are taken into account in the first multiset comparison. Given a relation R on terms, we write $S\ R^f\ T$ for

$$S\restriction_f^{\not<}\ R^{\mathsf{mul}}\ T\restriction_f^{\not<} \uplus T\restriction_{\mathscr{V}} - S\restriction_{\mathscr{V}}$$

Note that R^f depends on a precedence $>$. Whenever we use R^f, $>$ is defined.

Definition 4.4
Let $>$ be a precedence and (w, w_0) a weight function. The orders $=_{\mathsf{kv}}$ and $>_{\mathsf{kv}}$ are extended to non-ground terms as follows:

- $s =_{\mathsf{kv}} t$ if $|s|_x = |t|_x$ for all $x \in \mathscr{V}$, $w(s) = w(t)$ and $\mathsf{root}(s) = \mathsf{root}(t)$,
- $s >_{\mathsf{kv}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathscr{V}$ and either $w(s) > w(t)$ or both $w(s) = w(t)$ and $\mathsf{root}(s) > \mathsf{root}(t)$.

Some tricky features of the relations $=_{\mathsf{kv}}$ and $>_{\mathsf{kv}}$ are illustrated below.

Example 4.5

Let c be a constant and f a unary symbol. We have $\mathsf{f}(\mathsf{c}) >_{\mathsf{kv}} \mathsf{c}$ whenever admissibility is assumed: If $w(\mathsf{f}) > 0$ then $w(\mathsf{f}(\mathsf{c})) > w(\mathsf{c})$, and if $w(\mathsf{f}) = 0$ then admissibility imposes $\mathsf{f} > \mathsf{c}$. On the other hand, $\mathsf{f}(x) >_{\mathsf{kv}} x$ holds only if $w(\mathsf{f}) > 0$, since $\mathsf{f} \not> x$. Furthermore, $\mathsf{f}(x) =_{\mathsf{kv}} x$ does not hold as $\mathsf{f} \neq x$.

Example 4.6

Let c be a constant with $w(\mathsf{c}) = w_0$, f a unary symbol, and g a non-AC binary symbol. We do not have $\ell = \mathsf{g}(\mathsf{f}(\mathsf{c}), x) >_{\mathsf{kv}} \mathsf{g}(\mathsf{c}, \mathsf{f}(\mathsf{c})) = r$ since $w(\ell) = w(r)$ and $\mathsf{root}(\ell) = \mathsf{root}(r) = \mathsf{g}$. On the other hand, $\ell =_{\mathsf{kv}} r$ also does not hold since the condition "$|s|_x = |t|_x$ for all $x \in \mathscr{V}$" is not satisfied.

Now the non-ground version of $>_{\mathsf{KV}}$ is defined as follows.

Definition 4.7

Let $>$ be a precedence and (w, w_0) a weight function. The order $>_{\mathsf{KV}}$ is inductively defined as follows: $s >_{\mathsf{KV}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathscr{V}$ and either $w(s) > w(t)$, or $w(s) = w(t)$ and one of the following alternatives holds:

0. $s = f^k(t)$ and $t \in \mathscr{V}$ for some $k > 0$,
1. $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, and $f > g$,
2. $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $f \notin \mathscr{F}_{\mathsf{AC}}$, $(s_1, \ldots, s_n) >^{\mathsf{lex}}_{\mathsf{KV}} (t_1, \ldots, t_n)$,
3. $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathscr{F}_{\mathsf{AC}}$, and for $S = \nabla_f(s)$ and $T = \nabla_f(t)$
   (a) $S >^f_{\mathsf{kv}} T$, or
   (b) $S =^f_{\mathsf{kv}} T$ and $|S| > |T|$, or
   (c) $S =^f_{\mathsf{kv}} T$, $|S| = |T|$, and $S >^{\mathsf{mul}}_{\mathsf{KV}} T$.

Here $=_{\mathsf{AC}}$ is used as preorder in $>^{\mathsf{lex}}_{\mathsf{KV}}$ and $>^{\mathsf{mul}}_{\mathsf{KV}}$ whereas $=_{\mathsf{kv}}$ is used in $>^{\mathsf{mul}}_{\mathsf{kv}}$.

Contrary to what is claimed in Korovin and Voronkov (2003a), the order $>_{\mathsf{KV}}$ of Definition 4.7 is not a simplification order because it lacks the monotonicity property (i.e., $>_{\mathsf{KV}}$ is not closed under contexts), as shown in the following examples.

Example 4.8

We continue Example 4.5 by adding an AC symbol $+$. We obviously have $\mathsf{f}(x) >_{\mathsf{KV}} x$. However, $\mathsf{f}(x) + y >_{\mathsf{KV}} x + y$ does not hold if $w(\mathsf{f}) = 0$. Let

$$S = \nabla_+(s) = \{\mathsf{f}(x), y\} \qquad T = \nabla_+(t) = \{x, y\}$$

We have $S{\restriction}^{\not<}_+ = \{\mathsf{f}(x)\}$, and $T{\restriction}^{\not<}_+ \cup T{\restriction}_{\mathscr{V}} - S{\restriction}_{\mathscr{V}} = \{x\}$. As shown in Example 4.5, neither $\mathsf{f}(x) >_{\mathsf{kv}} x$ nor $\mathsf{f}(x) =_{\mathsf{kv}} x$ holds. Hence none of the cases 3(a,b,c) of Definition 4.7 can be applied.

Note that the use of a unary function of weight 0 is not crucial. The following example illustrates that the non-ground version of $>_{\mathsf{KV}}$ need not be closed under contexts, even if there is no unary symbol of weight zero.

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Example 4.9
We continue Example 4.6 by adding an AC symbol + with $\mathsf{g} > + > \mathsf{c}$. We have

$$\ell = \mathsf{g}(\mathsf{f}(\mathsf{c}), x) >_{\mathsf{KV}} \mathsf{g}(\mathsf{c}, \mathsf{f}(\mathsf{c})) = r$$

by case 2. However, $s = \ell + \mathsf{c} >_{\mathsf{KV}} r + \mathsf{c} = t$ does not hold. Let

$$S = \nabla_+(s) = \{\ell, \mathsf{c}\} \qquad T = \nabla_+(t) = \{r, \mathsf{c}\}$$

We have $S\upharpoonright_+^{\not<} = \{\ell\}$, $T\upharpoonright_+^{\not<} = \{r\}$, and $S\upharpoonright_{\mathscr{V}} = T\upharpoonright_{\mathscr{V}} = \varnothing$. As shown in Example 4.6, $\ell >_{\mathsf{kv}} r$ does not hold. Hence case 3(a) in Definition 4.7 does not apply. But also $\ell =_{\mathsf{kv}} r$ does not hold, excluding 3(b) and 3(c).

These examples do not refute the soundness of $>_{\mathsf{KV}}$ for proving AC termination; note that e.g. in Example 4.8 also $x + y >_{\mathsf{KV}} \mathsf{f}(x) + y$ does not hold. We prove soundness by extending $>_{\mathsf{KV}}$ to $>_{\mathsf{KV'}}$ which has all desired properties.

Definition 4.10
The order $>_{\mathsf{KV'}}$ is obtained as in Definition 4.7 after replacing $=^f_{\mathsf{kv}}$ by $\geqslant^f_{\mathsf{kv'}}$ in cases 3(b) and 3(c), and using $\geqslant_{\mathsf{kv'}}$ as preorder in $>^{\mathsf{mul}}_{\mathsf{kv}}$ in case 3(a). Here the relation $\geqslant_{\mathsf{kv'}}$ is defined as follows:

- $s \geqslant_{\mathsf{kv'}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathscr{V}$ and either $w(s) > w(t)$, or $w(s) = w(t)$ and either $\mathsf{root}(s) \geqslant \mathsf{root}(t)$ or $t \in \mathscr{V}$.

Note that $\geqslant_{\mathsf{kv'}}$ is a preorder that contains $=_{\mathsf{AC}}$.

Example 4.11
Consider again Example 4.8. We have $\mathsf{f}(x) \geqslant_{\mathsf{kv'}} x$ due to the new possibility "$t \in \mathscr{V}$". We have $\mathsf{f}(x) + y >_{\mathsf{KV'}} x + y$ because now case 3(c) applies: $S\upharpoonright_+^{\not<} = \{\mathsf{f}(x)\} \geqslant^{\mathsf{mul}}_{\mathsf{kv'}} \{x\} = T\upharpoonright_+^{\not<} \uplus T\upharpoonright_{\mathscr{V}} - S\upharpoonright_{\mathscr{V}}$, $|S| = 2 = |T|$, and $S = \{\mathsf{f}(x), y\} >^{\mathsf{mul}}_{\mathsf{KV'}} \{x, y\} = T$ because $\mathsf{f}(x) >_{\mathsf{KV'}} x$. Analogously, we have $\ell + \mathsf{c} >_{\mathsf{KV'}} r + \mathsf{c}$ for Example 4.9.

The proof of the following result can be found in the online appendix.

Theorem 4.12
The order $>_{\mathsf{KV'}}$ is an AC-compatible simplification order.

Since the inclusion $>_{\mathsf{KV}} \subseteq >_{\mathsf{KV'}}$ obviously holds, it follows that $>_{\mathsf{KV}}$ is a sound method for establishing AC termination, despite the lack of monotonicity.

5 AC-KBO

In this section, we present another AC-compatible simplification order. In contrast to $>_{\mathsf{KV'}}$, our new order $>_{\mathsf{ACKBO}}$ contains $>_{\mathsf{S}}$. Moreover, its definition is simpler than $>_{\mathsf{KV'}}$ since we avoid the use of an auxiliary order in case 3. In the next section, we show that $>_{\mathsf{ACKBO}}$ is decidable in polynomial-time, whereas the membership decision problem for $>_{\mathsf{KV'}}$ is NP-complete. Hence it will be used as the basis for the extension discussed in Section 8.

Definition 5.1
Let $>$ be a precedence and (w, w_0) a weight function. We define $>_{\mathsf{ACKBO}}$ inductively as follows: $s >_{\mathsf{ACKBO}} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathscr{V}$ and either $w(s) > w(t)$, or $w(s) = w(t)$ and one of the following alternatives holds:

0. $s = f^k(t)$ and $t \in \mathscr{V}$ for some $k > 0$,
1. $s = f(s_1, \ldots, s_n)$, $t = g(t_1, \ldots, t_m)$, and $f > g$,
2. $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $f \notin \mathscr{F}_{\mathsf{AC}}$, $(s_1, \ldots, s_n) >^{\mathsf{lex}}_{\mathsf{ACKBO}} (t_1, \ldots, t_n)$,
3. $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathscr{F}_{\mathsf{AC}}$, and for $S = \nabla_f(s)$ and $T = \nabla_f(t)$
   (a) $S >^f_{\mathsf{ACKBO}} T$, or
   (b) $S =^f_{\mathsf{AC}} T$, and $|S| > |T|$, or
   (c) $S =^f_{\mathsf{AC}} T$, $|S| = |T|$, and $S{\restriction}^{<}_f >^{\mathsf{mul}}_{\mathsf{ACKBO}} T{\restriction}^{<}_f$.

The relation $=_{\mathsf{AC}}$ is used as preorder in $>^{\mathsf{lex}}_{\mathsf{ACKBO}}$ and $>^{\mathsf{mul}}_{\mathsf{ACKBO}}$.

Note that, in contrast to $>_{\mathsf{KV}}$, in case 3(c) we compare the multisets $S{\restriction}^{<}_f$ and $T{\restriction}^{<}_f$ rather than S and T in the multiset extension of $>_{\mathsf{ACKBO}}$.

Steinbach's order is a special case of the order defined above.

Theorem 5.2
If every AC symbol has minimal precedence then $>_{\mathsf{S}} = >_{\mathsf{ACKBO}}$.

Proof
Suppose that every function symbol in $\mathscr{F}_{\mathsf{AC}}$ is minimal with respect to $>$. We show that $s >_{\mathsf{S}} t$ if and only if $s >_{\mathsf{ACKBO}} t$ by induction on s. It is clearly sufficient to consider case 3 in Definition 3.1 and cases 3(a,b,c) in Definition 5.1. So let $s = f(s_1, s_2)$ and $t = f(t_1, t_2)$ such that $w(s) = w(t)$ and $f \in \mathscr{F}_{\mathsf{AC}}$. Let $S = \nabla_f(s)$ and $T = \nabla_f(t)$.

- Let $s >_{\mathsf{S}} t$ by case 3. We have $S >^{\mathsf{mul}}_{\mathsf{S}} T$. Since $S >^{\mathsf{mul}}_{\mathsf{S}} T$ involves only comparisons $s' >_{\mathsf{S}} t'$ for subterms s' of s, the induction hypothesis yields $S >^{\mathsf{mul}}_{\mathsf{ACKBO}} T$. Because f is minimal in $>$, $S = S{\restriction}^{\not<}_f \uplus S{\restriction}_{\mathscr{V}}$ and $T = T{\restriction}^{\not<}_f \uplus T{\restriction}_{\mathscr{V}}$. For no elements $u \in S{\restriction}_{\mathscr{V}}$ and $v \in T{\restriction}^{\not<}_f$, $u >_{\mathsf{ACKBO}} v$ or $u =_{\mathsf{AC}} v$ holds. Hence $S >^{\mathsf{mul}}_{\mathsf{ACKBO}} T$ implies $S >^f_{\mathsf{ACKBO}} T$ or both $S =^f_{\mathsf{AC}} T$ and $S{\restriction}_{\mathscr{V}} \supsetneqq T{\restriction}_{\mathscr{V}}$. In the former case $s >_{\mathsf{ACKBO}} t$ is due to case 3(a) in Definition 5.1. In the latter case we have $|S| > |T|$ and $s >_{\mathsf{ACKBO}} t$ follows by case 3(b).
- Let $s >_{\mathsf{ACKBO}} t$ by applying one of the cases 3(a,b,c) in Definition 5.1.
  — Suppose 3(a) applies. Then we have $S >^f_{\mathsf{ACKBO}} T$. Since f is minimal in $>$, $S{\restriction}^{\not<}_f = S - S{\restriction}_{\mathscr{V}}$ and $T{\restriction}^{\not<}_f \uplus T{\restriction}_{\mathscr{V}} = T$. Hence $S >^{\mathsf{mul}}_{\mathsf{ACKBO}} (T - S{\restriction}_{\mathscr{V}}) \uplus S{\restriction}_{\mathscr{V}} \supseteq T$. We obtain $S >^{\mathsf{mul}}_{\mathsf{S}} T$ from the induction hypothesis and thus case 3 in Definition 3.1 applies.
  — Suppose 3(b) applies. Analogous to the previous case, the inclusion $S =^{\mathsf{mul}}_{\mathsf{AC}} (T - S{\restriction}_{\mathscr{V}}) \uplus S{\restriction}_{\mathscr{V}} \supseteq T$ holds. Since $|S| > |T|$, $S =^{\mathsf{mul}}_{\mathsf{AC}} T$ is not possible. Thus, $(T - S{\restriction}_{\mathscr{V}}) \uplus S{\restriction}_{\mathscr{V}} \supsetneqq T$ and hence $S >^{\mathsf{mul}}_{\mathsf{S}} T$.
  — If case 3(c) applies then $S{\restriction}^{<}_f >^{\mathsf{mul}}_{\mathsf{ACKBO}} T{\restriction}^{<}_f$. This is impossible since both sides are empty as f is minimal in $>$. □

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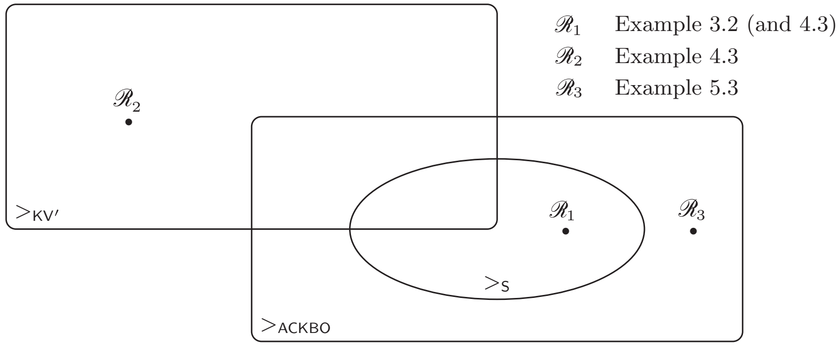

Fig. 1. Comparison.

The following example shows that $>_{\mathsf{ACKBO}}$ is a proper extension of $>_{\mathsf{S}}$ and incomparable with $>_{\mathsf{KV}'}$.

Example 5.3
Consider the TRS $\mathscr{R}_3$ consisting of the rules

$$\mathsf{f}(x+y) \to \mathsf{f}(x)+y \qquad \mathsf{h}(\mathsf{a},\mathsf{b}) \to \mathsf{h}(\mathsf{b},\mathsf{a}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{a}),\mathsf{a}) \to \mathsf{h}(\mathsf{a},\mathsf{g}(\mathsf{b}))$$
$$\mathsf{g}(x)+y \to \mathsf{g}(x+y) \qquad \mathsf{h}(\mathsf{a},\mathsf{g}(\mathsf{g}(\mathsf{a}))) \to \mathsf{h}(\mathsf{g}(\mathsf{a}),\mathsf{f}(\mathsf{a})) \qquad \mathsf{h}(\mathsf{g}(\mathsf{a}),\mathsf{b}) \to \mathsf{h}(\mathsf{a},\mathsf{g}(\mathsf{a}))$$
$$\mathsf{f}(\mathsf{a})+\mathsf{g}(\mathsf{b}) \to \mathsf{f}(\mathsf{b})+\mathsf{g}(\mathsf{a})$$

over the signature $\{+,\mathsf{f},\mathsf{g},\mathsf{h},\mathsf{a},\mathsf{b}\}$ with $+ \in \mathscr{F}_{\mathsf{AC}}$. Consider the precedence

$$\mathsf{f} > + > \mathsf{g} > \mathsf{a} > \mathsf{b} > \mathsf{h}$$

together with the admissible weight function (w, w_0) with

$$w(+) = w(\mathsf{h}) = 0 \qquad w(\mathsf{f}) = w(\mathsf{a}) = w(\mathsf{b}) = w_0 = 1 \qquad w(\mathsf{g}) = 2$$

The interesting rule is $\mathsf{f}(\mathsf{a})+\mathsf{g}(\mathsf{b}) \to \mathsf{f}(\mathsf{b})+\mathsf{g}(\mathsf{a})$. For $S = \nabla_+(\mathsf{f}(\mathsf{a})+\mathsf{g}(\mathsf{b}))$ and $T = \nabla_+(\mathsf{f}(\mathsf{b})+\mathsf{g}(\mathsf{a}))$ the multisets $S' = S\restriction_+^{\nless} = \{\mathsf{f}(\mathsf{a})\}$ and $T' = T\restriction_+^{\nless} \uplus T\restriction_{\mathscr{V}} - S\restriction_{\mathscr{V}} = \{\mathsf{f}(\mathsf{b})\}$ satisfy $S' >_{\mathsf{ACKBO}}^{\mathsf{mul}} T'$ as $\mathsf{f}(\mathsf{a}) >_{\mathsf{ACKBO}} \mathsf{f}(\mathsf{b})$, so that case 3(a) of Definition 5.1 applies. All other rules are oriented from left to right by both $>_{\mathsf{KV}'}$ and $>_{\mathsf{ACKBO}}$, and they enforce a precedence and weight function which are identical (or very similar) to the one given above. Since $>_{\mathsf{KV}'}$ orients the rule $\mathsf{f}(\mathsf{a})+\mathsf{g}(\mathsf{b}) \to \mathsf{f}(\mathsf{b})+\mathsf{g}(\mathsf{a})$ from right to left, $\mathscr{R}_3$ cannot be compatible with $>_{\mathsf{KV}'}$. It is easy to see that the rule $\mathsf{g}(x)+y \to \mathsf{g}(x+y)$ requires $+ > \mathsf{g}$, and hence $>_{\mathsf{S}}$ cannot be applied.

Figure 1 summarizes the relationships between the orders introduced so far. In the following, we show that $>_{\mathsf{ACKBO}}$ is an AC-compatible simplification order. As a consequence, correctness of $>_{\mathsf{S}}$ (i.e., Theorem 3.3) is concluded by Theorem 5.2.

In the online appendix we prove the following property.

Lemma 5.4
The pair $(=_{\mathsf{AC}}, >_{\mathsf{ACKBO}})$ is an order pair.

The subterm property is an easy consequence of transitivity and admissibility.

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Lemma 5.5
The order $>_{\mathsf{ACKBO}}$ has the subterm property. □

Next we prove that $>_{\mathsf{ACKBO}}$ is closed under contexts. The following lemma is an auxiliary result needed for its proof. In order to reuse this lemma for the correctness proof of $>_{\mathsf{KV'}}$ in the online appendix, we prove it in an abstract setting.

Lemma 5.6
Let $(\gtrsim, \succ)$ be an order pair and $f \in \mathscr{F}_{\mathsf{AC}}$ with $f(u,v) \succ u, v$ for all terms u and v. If $s \gtrsim t$ then $\{s\} \gtrsim^{\mathsf{mul}} \nabla_f(t)$ or $\{s\} \succ^{\mathsf{mul}} \nabla_f(t)$. If $s \succ t$ then $\{s\} \succ^{\mathsf{mul}} \nabla_f(t)$.

Proof
Let $\nabla_f(t) = \{t_1, \ldots, t_m\}$. If $m = 1$ then $\nabla_f(t) = \{t\}$ and the lemma holds trivially. Otherwise we get $t \succ t_j$ for all $1 \leqslant j \leqslant m$ by recursively applying the assumption. Hence $s \succ t_j$ by the transitivity of $\succ$ or the compatibility of $\succ$ and $\gtrsim$. We conclude that $\{s\} \succ^{\mathsf{mul}} \nabla_f(t)$. □

In the following proof of closure under contexts, admissibility is essential. This is in contrast to the corresponding result for standard KBO.

Lemma 5.7
If (w, w_0) is admissible for $>$ then $>_{\mathsf{ACKBO}}$ is closed under contexts.

Proof
Suppose $s >_{\mathsf{ACKBO}} t$. We consider the context $h(\Box, u)$ with $h \in \mathscr{F}_{\mathsf{AC}}$ and u an arbitrary term, and prove that $s' = h(s,u) >_{\mathsf{ACKBO}} h(t,u) = t'$. Closure under contexts of $>_{\mathsf{ACKBO}}$ follows then by induction; contexts rooted by a non-AC symbol are handled as in the proof for standard KBO.

If $w(s) > w(t)$ then obviously $w(s') > w(t')$. So we assume $w(s) = w(t)$. Let $S = \nabla_h(s)$, $T = \nabla_h(t)$, and $U = \nabla_h(u)$. Note that $\nabla_h(s') = S \uplus U$ and $\nabla_h(t') = T \uplus U$. Because $>^{\mathsf{mul}}_{\mathsf{ACKBO}}$ is closed under multiset sum, it suffices to show that one of the cases 3(a,b,c) of Definition 5.1 holds for S and T. Let $f = \mathsf{root}(s)$ and $g = \mathsf{root}(t)$. We distinguish the following cases.

- Suppose $f \not< h$. We have $S = S\restriction_h^{\not<} = \{s\}$, and from Lemmata 5.5 and 5.6 we obtain $S >^{\mathsf{mul}}_{\mathsf{ACKBO}} T$. Since T is a superset of $T\restriction_h^{\not<} \uplus T\restriction_{\mathscr{V}} - S\restriction_{\mathscr{V}}$, 3(a) applies.
- Suppose $f = h > g$. We have $T\restriction_h^{\not<} \uplus T\restriction_{\mathscr{V}} = \varnothing$. If $S\restriction_h^{\not<} \neq \varnothing$, then 3(a) applies. Otherwise, since AC symbols are binary and $T = \{t\}$, $|S| \geqslant 2 > 1 = |T|$. Hence 3(b) applies.
- If $f = g = h$ then $s >_{\mathsf{ACKBO}} t$ must be derived by one of the cases 3(a,b,c) for S and T.
- Suppose $f, g < h$. We have $S\restriction_h^{\not<} = T\restriction_h^{\not<} \uplus T\restriction_{\mathscr{V}} = \varnothing$, $|S| = |T| = 1$, and $S\restriction_h^{<} = \{s\} >^{\mathsf{mul}}_{\mathsf{ACKBO}} \{t\} = T\restriction_h^{<}$. Hence 3(c) holds.

Note that $f \geqslant g$ since $w(s) = w(t)$ and $s >_{\mathsf{ACKBO}} t$. Moreover, if $t \in \mathscr{V}$ then $s = f^k(t)$ for some $k > 0$ with $w(f) = 0$, which entails $f > h$ due to the admissibility assumption. □

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Closure under substitutions is the trickiest part since by substituting AC-rooted terms for variables that appear in the top-flattening of a term, the structure of the term changes. In the proof, the multisets $\{t \in T \mid t \notin \mathscr{V}\}$, $\{t\sigma \mid t \in T\}$, and $\{\nabla_f(t) \mid t \in T\}$ are denoted by $T{\restriction_{\mathscr{F}}}$, $T\sigma$, and $\nabla_f(T)$, respectively.

Lemma 5.8
Let $>$ be a precedence, $f \in \mathscr{F}_{\mathsf{AC}}$, and $(\gtrsim, \succ)$ an order pair on terms such that $\gtrsim$ and $\succ$ are closed under substitutions and $f(x,y) \succ x, y$. Consider terms s and t such that $S = \nabla_f(s)$, $T = \nabla_f(t)$, $S' = \nabla_f(s\sigma)$, and $T' = \nabla_f(t\sigma)$.

1. If $S \succ^f T$ then $S' \succ^f T'$.
2. If $S \gtrsim^f T$ then $S' \succ^f T'$ or $S' \gtrsim^f T'$. In the latter case $|S| - |T| \leqslant |S'| - |T'|$ and $S'{\restriction_f^<} \succ^{\mathsf{mul}} T'{\restriction_f^<}$ whenever $S{\restriction_f^<} \succ^{\mathsf{mul}} T{\restriction_f^<}$.

Proof
Let v be an arbitrary term. By the assumption on $\succ$ we have either $\{v\} = \nabla_f(v)$ or both $\{v\} \succ^{\mathsf{mul}} \nabla_f(v)$ and $1 < |\nabla_f(v)|$. Hence, for any set V of terms, either $V = \nabla_f(V)$ or both $V \succ^{\mathsf{mul}} \nabla_f(V)$ and $|V| < |\nabla_f(V)|$. Moreover, for $V = \nabla_f(v)$, the following equalities hold:

$$\nabla_f(v\sigma){\restriction_f^{\not<}} = V{\restriction_f^{\not<}}\sigma \uplus \nabla_f(V{\restriction_{\mathscr{V}}}\sigma){\restriction_f^{\not<}} \qquad \nabla_f(v\sigma){\restriction_{\mathscr{V}}} = \nabla_f(V{\restriction_{\mathscr{V}}}\sigma){\restriction_{\mathscr{V}}}$$

To prove the lemma, assume $S \mathrel{R^f} T$ for $R \in \{\gtrsim, \succ\}$. We have $S{\restriction_f^{\not<}} \mathrel{R^{\mathsf{mul}}} T{\restriction_f^{\not<}} \uplus U$ where $U = (T - S){\restriction_{\mathscr{V}}}$. Since multiset extensions preserve closure under substitutions, $S{\restriction_f^{\not<}}\sigma \mathrel{R^{\mathsf{mul}}} T{\restriction_f^{\not<}}\sigma \uplus U\sigma$ follows. Using the above (in)equalities, we obtain

$$\begin{aligned}
S'{\restriction_f^{\not<}} = {} & S{\restriction_f^{\not<}}\sigma \uplus \nabla_f(S{\restriction_{\mathscr{V}}}\sigma){\restriction_f^{\not<}} \\
R^{\mathsf{mul}} \; & T{\restriction_f^{\not<}}\sigma \uplus \nabla_f(S{\restriction_{\mathscr{V}}}\sigma){\restriction_f^{\not<}} \uplus U\sigma \\
O \; & T{\restriction_f^{\not<}}\sigma \uplus \nabla_f(S{\restriction_{\mathscr{V}}}\sigma){\restriction_f^{\not<}} \uplus \nabla_f(U\sigma) \\
= {} & T{\restriction_f^{\not<}}\sigma \uplus \nabla_f(S{\restriction_{\mathscr{V}}}\sigma){\restriction_f^{\not<}} \uplus \nabla_f(U\sigma){\restriction_{\mathscr{V}}} \uplus \nabla_f(U\sigma){\restriction_f^{\not<}} \uplus \nabla_f(U\sigma){\restriction_f^{<}} \\
P \; & T{\restriction_f^{\not<}}\sigma \uplus \nabla_f(T{\restriction_{\mathscr{V}}}\sigma){\restriction_f^{\not<}} \uplus \nabla_f(U\sigma){\restriction_{\mathscr{V}}} \\
= {} & T{\restriction_f^{\not<}}\sigma \uplus \nabla_f(T{\restriction_{\mathscr{V}}}\sigma){\restriction_f^{\not<}} \uplus \nabla_f(T{\restriction_{\mathscr{V}}}\sigma){\restriction_{\mathscr{V}}} - \nabla_f(S{\restriction_{\mathscr{V}}}\sigma){\restriction_{\mathscr{V}}} \\
= {} & T'{\restriction_f^{\not<}} \uplus T'{\restriction_{\mathscr{V}}} - S'{\restriction_{\mathscr{V}}}
\end{aligned}$$

Here O denotes $=$ if $U\sigma = \nabla_f(U\sigma)$ and $\succ^{\mathsf{mul}}$ if $|U\sigma| < |\nabla_f(U\sigma)|$, while P denotes $=$ if $U\sigma{\restriction_f^<} = \varnothing$ and $\supsetneq$ otherwise. Since $(\gtrsim^{\mathsf{mul}}, \succ^{\mathsf{mul}})$ is an order pair with $\supseteq \subseteq \gtrsim^{\mathsf{mul}}$ and $\supsetneq \subseteq \succ^{\mathsf{mul}}$, we obtain $S' \mathrel{R^f} T'$.

It remains to show 2. If $S' \not\succ^f T'$ then O and P are both $=$ and thus $U\sigma = \nabla_f(U\sigma)$ and $U\sigma{\restriction_f^<} = \varnothing$. Let $X = S{\restriction_{\mathscr{V}}} \cap T{\restriction_{\mathscr{V}}}$. We have $U = T{\restriction_{\mathscr{V}}} - X$.

- Since $|W{\restriction_{\mathscr{F}}}\sigma| = |W{\restriction_{\mathscr{F}}}|$ and $|W| \leqslant |\nabla_f(W)|$ for an arbitrary set W of terms, we have $|S'| \geqslant |S| - |X| + |\nabla_f(X\sigma)|$. From $|U\sigma| = |U| = |T{\restriction_{\mathscr{V}}}| - |X|$ we obtain

  $$|T'| = |T{\restriction_{\mathscr{F}}}\sigma| + |\nabla_f(U\sigma)| + |\nabla_f(X\sigma)| = |T| - |X| + |\nabla_f(X\sigma)|$$

  Hence $|S| - |T| \leqslant |S'| - |T'|$ as desired.
- Suppose $S{\restriction_f^<} \succ^{\mathsf{mul}} T{\restriction_f^<}$. From $U\sigma{\restriction_f^<} = \varnothing$ we infer $T{\restriction_{\mathscr{V}}}\sigma{\restriction_f^<} \subseteq S{\restriction_{\mathscr{V}}}\sigma{\restriction_f^<}$. Because $S'{\restriction_f^<} = S{\restriction_f^<}\sigma \uplus S{\restriction_{\mathscr{V}}}\sigma{\restriction_f^<}$ and $T'{\restriction_f^<} = T{\restriction_f^<}\sigma \uplus T{\restriction_{\mathscr{V}}}\sigma{\restriction_f^<}$, closure under

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substitutions of $\succ^{\mathsf{mul}}$ (which it inherits from $\succ$ and $\succsim$) yields the desired $S' \restriction_f^{<} \succ^{\mathsf{mul}} T' \restriction_f^{<}$. $\square$

Lemma 5.9

$>_{\mathsf{ACKBO}}$ is closed under substitutions.

Proof

If $s >_{\mathsf{ACKBO}} t$ is obtained by cases 0 or 1 in Definition 5.1, the proof for standard KBO goes through. If 3(a) or 3(b) is used to obtain $s >_{\mathsf{ACKBO}} t$, according to Lemma 5.8 one of these cases also applies to $s\sigma >_{\mathsf{ACKBO}} t\sigma$. The final case is 3(c). So $\nabla_f(s) \restriction_f^{<} >_{\mathsf{ACKBO}}^{\mathsf{mul}} \nabla_f(t) \restriction_f^{<}$. Suppose $s\sigma >_{\mathsf{ACKBO}} t\sigma$ cannot be obtained by 3(a) or 3(b). Lemma 5.8(2) yields $|\nabla_f(s\sigma)| = |\nabla_f(t\sigma)|$ and $\nabla_f(s\sigma) \restriction_f^{<} >_{\mathsf{ACKBO}}^{\mathsf{mul}} \nabla_f(t\sigma) \restriction_f^{<}$. Hence case 3(c) is applicable to obtain $s\sigma >_{\mathsf{ACKBO}} t\sigma$. $\square$

We arrive at the main theorem of this section.

Theorem 5.10

The order $>_{\mathsf{ACKBO}}$ is an AC-compatible simplification order. $\square$

Since we deal with finite non-variadic signatures, simplification orders are well-founded. The following example shows that AC-KBO is not *incremental*, i.e., orientability is not necessarily preserved when the precedence is extended. This is in contrast to the AC-RPO of Rubio (2002). However, this is not necessarily a disadvantage; actually, the example shows that by allowing partial precedences more TRSs can be proved to be AC terminating using AC-KBO.

Example 5.11

Consider the TRS $\mathcal{R}$ consisting of the rules

$$\mathsf{a} \circ (\mathsf{b} \bullet \mathsf{c}) \to \mathsf{b} \circ \mathsf{f}(\mathsf{a} \bullet \mathsf{c}) \qquad\qquad \mathsf{a} \bullet (\mathsf{b} \circ \mathsf{c}) \to \mathsf{b} \bullet \mathsf{f}(\mathsf{a} \circ \mathsf{c})$$

over the signature $\mathcal{F} = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{f}, \circ, \bullet\}$ with $\circ, \bullet \in \mathcal{F}_{\mathsf{AC}}$. By taking the precedence $\mathsf{f} > \mathsf{a}, \mathsf{b}, \mathsf{c}, \circ, \bullet$ and admissible weight function (w, w_0) with

$$w(\mathsf{f}) = w(\circ) = w(\bullet) = 0 \qquad w_0 = w(\mathsf{a}) = w(\mathsf{c}) = 1 \qquad\qquad w(\mathsf{b}) = 2$$

the resulting $>_{\mathsf{ACKBO}}$ orients both rules from left to right. It is essential that $\circ$ and $\bullet$ are incomparable in the precedence: We must have $w(\mathsf{f}) = 0$, so $\mathsf{f} > \mathsf{a}, \mathsf{b}, \mathsf{c}, \circ, \bullet$ is enforced by admissibility. If $\circ > \bullet$ then the first rule can only be oriented from left to right if $\mathsf{a} >_{\mathsf{ACKBO}} \mathsf{f}(\mathsf{a} \bullet \mathsf{c})$ holds, which contradicts the subterm property. If $\bullet > \circ$ then we use the second rule to obtain the impossible $\mathsf{a} >_{\mathsf{ACKBO}} \mathsf{f}(\mathsf{a} \circ \mathsf{c})$. Similarly, $\mathcal{R}$ is also orientable by $>_{\mathsf{KV'}}$ but we must adopt a non-total precedence.

The easy proof of the final theorem in this section can be found in the online appendix.

Theorem 5.12

If $>$ is total then $>_{\mathsf{ACKBO}}$ is AC-total on ground terms.

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6 Complexity

In this section, we discuss complexity issues for the orders defined in the preceding sections. We start with the membership problem: Given two terms s and t, a weight function, and a precedence, does $s > t$ hold? For plain KBO this problem is known to be decidable in linear time (Löchner 2006). For $>_{\mathsf{S}}$, $>_{\mathsf{KV}}$, and $>_{\mathsf{ACKBO}}$, we show the problem to be decidable in polynomial time, but we start with the unexpected result that $>_{\mathsf{KV}'}$ membership is NP-complete. For NP-hardness we use the reduction technique of Thiemann *et al.* (2012, Theorem 4.2).

Theorem 6.1
The decision problem for $>_{\mathsf{KV}'}$ is NP-complete.

Proof
We start with NP-hardness. It is sufficient to show NP-hardness of deciding $S >^{\mathsf{mul}}_{\mathsf{kv}'} T$, since we can easily construct terms s and t such that $S >^{\mathsf{mul}}_{\mathsf{kv}'} T$ if and only if $s >_{\mathsf{KV}'} t$. To wit, for $S = \{s_1, \ldots, s_n\}$ and $T = \{t_1, \ldots, t_m\}$ we introduce an AC symbol $\circ$ and constants c and d such that $\circ > \mathsf{c}, \mathsf{d}$ and define

$$s = s_1 \circ \cdots \circ s_n \circ \mathsf{c} \qquad t = t_1 \circ \cdots \circ t_m \circ \mathsf{d} \circ \mathsf{d}$$

The weights of c and d should be chosen so that $w(s) = w(t)$. If $S >^{\mathsf{mul}}_{\mathsf{kv}'} T$ then case 3(a) applies for $s >_{\mathsf{KV}'} t$. Otherwise, $S \geqslant^{\mathsf{mul}}_{\mathsf{kv}'} T$ implies $n = m$ and thus $|\nabla_\circ(s)| < |\nabla_\circ(t)|$. Hence neither case 3(b) nor 3(c) applies.

We reduce a non-empty SAT problem $\phi = \{C_1, \ldots, C_m\}$ in conjunctive normal form (CNF) over propositional variables $x_1, \ldots, x_n$ to the decision problem $S_\phi >^{\mathsf{mul}}_{\mathsf{kv}'} T_\phi$. The multisets S_ϕ and T_ϕ will consist of terms in $\mathscr{T}(\{\mathsf{a}, \mathsf{f}\}, \{x_1, \ldots, x_n, y_1, \ldots, y_m\})$, where a is a constant with $w(\mathsf{a}) = w_0$ and f has arity $m+1$. For each $1 \leqslant j \leqslant m$ and literal l, we define

$$s_j(l) = \begin{cases} y_j & \text{if } l \in C_j \\ \mathsf{a} & \text{otherwise} \end{cases}$$

Moreover, for each $1 \leqslant i \leqslant n$ we define

$$t_i^+ = \mathsf{f}(x_i, s_1(x_i), \ldots, s_m(x_i)) \qquad t_i^- = \mathsf{f}(x_i, s_1(\neg x_i), \ldots, s_m(\neg x_i))$$

and $t_i = \mathsf{f}(x_i, \mathsf{a}, \ldots, \mathsf{a})$. Note that $w(t_i^+) = w(t_i^-) = w(t_i) > w(y_j)$ for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Finally, we define

$$S_\phi = \{t_1^+, t_1^-, \ldots, t_n^+, t_n^-\} \qquad\qquad T_\phi = \{t_1, \ldots, t_n, y_1, \ldots, y_m\}$$

Note that for every $1 \leqslant i \leqslant n$ there is no $s \in S_\phi$ such that $s >_{\mathsf{kv}} t_i$. Hence, $S_\phi >^{\mathsf{mul}}_{\mathsf{kv}'} T_\phi$ if and only if S_ϕ can be written as $\{s_1, \ldots, s_n, s'_1, \ldots, s'_n\}$ such that $s_i \geqslant_{\mathsf{kv}'} t_i$ for all $1 \leqslant i \leqslant n$, and for all $1 \leqslant j \leqslant m$ there exists an $1 \leqslant i \leqslant n$ such that $s'_i >_{\mathsf{kv}} y_j$. It is easy to see that the only candidates for s_i are t_i^+ and t_i^-.

Now suppose $S_\phi >^{\mathsf{mul}}_{\mathsf{kv}'} T_\phi$ with S_ϕ written as above. Consider the assignment α defined as follows: $\alpha(x_i)$ is true if and only if $s_i = t_i^-$. We claim that α satisfies every $C_j \in \phi$. We know that there exists $1 \leqslant i \leqslant n$ such that $s'_i >_{\mathsf{kv}} y_j$ and thus also $y_j \in \mathscr{V}\mathrm{ar}(s'_i)$. This is only possible if $x_i \in C_j$ (when $s'_i = t_i^+$) or $\neg x_i \in C_j$ (when $s'_i = t_i^-$). Hence, by construction of α, α satisfies C_j.

Conversely, suppose α satisfies ϕ. Let $s_i' = t_i^+$ and $s_i = t_i^-$ if $\alpha(x_i)$ is true and $s_i' = t_i^-$ and $s_i = t_i^+$ if $\alpha(x_i)$ is false. We trivially have $s_i \geqslant_{\mathsf{kv}'} t_i$ for all $1 \leqslant i \leqslant n$. Moreover, for each $1 \leqslant j \leqslant m$, C_j contains a literal $l = (\neg)x_i$ such that $\alpha(l)$ is true. By construction, $y_j \in \mathscr{V}\mathrm{ar}(s_i')$ and thus $s_i' >_{\mathsf{kv}} y_j$. Since ϕ is non-empty, $m > 0$ and hence $S_\phi >_{\mathsf{kv}'}^{\mathsf{mul}} T_\phi$ as desired.

To obtain NP-completeness we need to show membership in NP, which is easy; one just guesses how the terms in the various multisets relate to each other in order to satisfy the multiset comparisons in the definition of $>_{\mathsf{KV}'}$. $\square$

Next we show that the complexity of deciding $>_{\mathsf{KV}}$ and $>_{\mathsf{ACKBO}}$ for given weights and precedence is decidable in polynomial time. Given a sequence $S = s_1, \ldots, s_n$ and an index $1 \leqslant i \leqslant n$, we denote by $S[t]_i$ the sequence obtained by replacing s_i with t in S, and by $S[\,]_i$ the sequence obtained by removing s_i from S. Moreover, we write $\{S\}$ as a shorthand for the multiset $\{s_1, \ldots, s_n\}$.

Lemma 6.2

Let $(\succsim, \succ)$ be an order pair such that $\sim\ := \ \succsim \setminus \succ$ is symmetric. If $s \sim t$ then $M \uplus \{s\} \succ^{\mathsf{mul}} N \uplus \{t\}$ and $M \succ^{\mathsf{mul}} N$ are equivalent.

Proof

We only show that $M \uplus \{s\} \succ^{\mathsf{mul}} N \uplus \{t\}$ implies $M \succ^{\mathsf{mul}} N$, since the other direction is trivial. So suppose $M \uplus \{s\} \sqsupset_k^{\mathsf{mul}} N \uplus \{t\}$, where sequences $S = s_1, \ldots, s_m$ and $T = t_1, \ldots, t_n$ satisfy the conditions for $\sqsupset_k^{\mathsf{mul}}$ in Definition 2.1. Because we have $\{S\} = M \uplus \{s\}$ and $\{T\} = N \uplus \{t\}$, there are indices i and j such that $s = s_i$ and $t = t_j$. In order to establish $M \succ^{\mathsf{mul}} N$ we distinguish four cases.

- If $i, j \leqslant k$ then $s_j \succsim t_j = t \sim s = s_i \succsim t_i$ and thus $\{S[s_j]_i[\,]_j\} \sqsupset_{k-1}^{\mathsf{mul}} \{T[\,]_j\}$.
- If $i \leqslant k < j$ then there exists some $l > k$ such that $s_l \succ t_j = t \sim s = s_i \succsim t_i$. Therefore, $\{S[\,]_i\} \sqsupset_{k-1}^{\mathsf{mul}} \{T[t_i]_j[\,]_i\}$.
- If $j \leqslant k < i$ then $s_j \succsim t_j = t \sim s = s_i$ and thus $s_j \succ t_l$ for every $l > k$ such that $s_i \succ t_l$. Hence $\{S[s_j]_i[\,]_j\} \sqsupset_{k-1}^{\mathsf{mul}} \{T[\,]_j\}$.
- The remaining case $k < i, j$ is analogous to the previous case, and we obtain $\{S[\,]_i\} \sqsupset_k^{\mathsf{mul}} \{T[\,]_j\}$.

Because $\{S[s_j]_i[\,]_j\} = \{S[\,]_i\} = M$ and $\{T[t_i]_j[\,]_i\} = \{T[\,]_j\} = N$ hold, in all cases $M \succ^{\mathsf{mul}} N$ is concluded. $\square$

Lemma 6.3

Let $(\succsim, \succ)$ be an order pair such that $\sim\ := \ \succsim \setminus \succ$ is symmetric and the decision problems for $\succsim$ and $\succ$ are in P. Then the decision problem for $\succ^{\mathsf{mul}}$ is in P.

Proof

Suppose we want to decide whether two multisets S and T satisfy $S \succ^{\mathsf{mul}} T$. We first check if there exists a pair $(s, t) \in S \times T$ such that $s \sim t$, which can be done by testing $s \succsim t$ and $s \not\succ t$ at most $|S| \times |T|$ times. If such a pair is found then according to Lemma 6.2, the problem is reduced to $S - \{s\} \succ^{\mathsf{mul}} T - \{t\}$. Otherwise, we check for each $t \in T$ whether there exists $s \in S$ such that $s \succ t$, which can be done by testing $s \succ t$ at most $|S| \times |T|$ times. $\square$

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Using the above lemma, we obtain the following result by a straightforward induction argument.

Corollary 6.4
The decision problems for $>_{\mathsf{ACKBO}}$, $>_{\mathsf{KV}}$, and $>_{\mathsf{S}}$ belong to P. □

Next we address the complexity of the important orientability problem: Given a TRS $\mathscr{R}$, do there exist a weight function and a precedence such that the rules of $\mathscr{R}$ are oriented from left to right with respect to the order under consideration? It is well known (Korovin and Voronkov 2003b) that KBO orientability is decidable in polynomial time. We show that $>_{\mathsf{KV}}$ and $>_{\mathsf{ACKBO}}$ orientability are NP-complete even for ground TRSs. First, we show NP-hardness of $>_{\mathsf{KV}}$ orientability by a reduction from SAT.

Let $\phi = \{C_1, \ldots, C_n\}$ be a CNF SAT problem over propositional variables $p_1, \ldots, p_m$. We consider the signature $\mathscr{F}_\phi$ consisting of an AC symbol $+$, constants c and $\mathsf{d}_1, \ldots, \mathsf{d}_n$, and unary function symbols $p_1, \ldots, p_m$, a, b, and e_i^j for all $i \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, m\}$. We define a ground TRS $\mathscr{R}_\phi$ on $\mathscr{T}(\mathscr{F}_\phi)$ such that $>_{\mathsf{KV}}$ orients $\mathscr{R}_\phi$ if and only if ϕ is satisfiable. The TRS $\mathscr{R}_\phi$ will contain the following base system $\mathscr{R}_0$ that enforces certain constraints on the precedence and the weight function:

$$\mathsf{a}(\mathsf{c}+\mathsf{c}) \to \mathsf{a}(\mathsf{c})+\mathsf{c} \qquad \mathsf{b}(\mathsf{c})+\mathsf{c} \to \mathsf{b}(\mathsf{c}+\mathsf{c}) \qquad \mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{c}))) \to \mathsf{b}(\mathsf{a}(\mathsf{a}(\mathsf{c})))$$
$$\mathsf{a}(p_1(\mathsf{c})) \to \mathsf{b}(p_2(\mathsf{c})) \qquad \cdots \qquad \mathsf{a}(p_m(\mathsf{c})) \to \mathsf{b}(\mathsf{a}(\mathsf{c})) \qquad \mathsf{a}(\mathsf{a}(\mathsf{c})) \to \mathsf{b}(p_1(\mathsf{c}))$$

Lemma 6.5
The order $>_{\mathsf{KV}}$ is compatible with $\mathscr{R}_0$ if and only if $\mathsf{a} > + > \mathsf{b}$ and $w(\mathsf{a}) = w(\mathsf{b}) = w(p_j)$ for all $1 \leqslant j \leqslant m$. □

Consider the clause C_i of the form $\{p_1', \ldots, p_k', \neg p_1'', \ldots, \neg p_l''\}$. Let U, U', V, and W denote the following multisets:

$$U = \{p_1'(\mathsf{b}(\mathsf{d}_i)), \ldots, p_k'(\mathsf{b}(\mathsf{d}_i))\} \qquad V = \{p_0''(\mathsf{e}_i^{0,1}), \ldots, p_{l-1}''(\mathsf{e}_i^{l-1,l}), p_l''(\mathsf{e}_i^{l,0})\}$$
$$U' = \{\mathsf{b}(p_1'(\mathsf{d}_i)), \ldots, \mathsf{b}(p_k'(\mathsf{d}_i))\} \qquad W = \{p_0''(\mathsf{e}_i^{0,0}), \ldots, p_l''(\mathsf{e}_i^{l,l})\}$$

where we write p_0'' for a and $\mathsf{e}_i^{j,k}$ for $\mathsf{e}_i^j(\mathsf{e}_i^k(\mathsf{c}))$. The TRS $\mathscr{R}_\phi$ is defined as the union of $\mathscr{R}_0$ and $\{\ell_i \to r_i \mid 1 \leqslant i \leqslant n\}$ with

$$\ell_i = \mathsf{b}(\mathsf{b}(\mathsf{c}+\mathsf{c})) + \sum U + \sum V \qquad r_i = \mathsf{b}(\mathsf{c}) + \mathsf{b}(\mathsf{c}) + \sum U' + \sum W$$

Note that the symbols d_i and $\mathsf{e}_i^0, \ldots, \mathsf{e}_i^l$ are specific to the rule $\ell_i \to r_i$.

Example 6.6
Consider a clause $C_1 = \{x, \neg y, \neg z\}$. We have

$$\ell_1 = \mathsf{b}(\mathsf{b}(\mathsf{c}+\mathsf{c})) + x(\mathsf{b}(\mathsf{d}_i)) + \mathsf{a}(\mathsf{e}_1^0(\mathsf{e}_1^1(\mathsf{c}))) + y(\mathsf{e}_1^1(\mathsf{e}_1^2(\mathsf{c}))) + z(\mathsf{e}_1^2(\mathsf{e}_1^0(\mathsf{c})))$$
$$r_1 = \mathsf{b}(\mathsf{c}) + \mathsf{b}(\mathsf{c}) + \mathsf{b}(x(\mathsf{d}_i)) + \mathsf{a}(\mathsf{e}_1^0(\mathsf{e}_1^0(\mathsf{c}))) + y(\mathsf{e}_1^1(\mathsf{e}_1^1(\mathsf{c}))) + z(\mathsf{e}_1^2(\mathsf{e}_1^2(\mathsf{c})))$$

Note that x, y, and z are unary function symbols. We have $w(\ell_1) = w(r_1)$ for any weight function w. Suppose $\mathsf{a} > + > \mathsf{b}$ and $w(\mathsf{a}) = w(\mathsf{b}) = w(x) = w(y) = w(z)$.

We consider a number of cases, depending on the order of x, y, z, and $+$ in the precedence. If $x, y, z > +$ (i.e., x, y, and z are assigned true) then $\ell_1 >_{\mathsf{KV}} r_1$ can

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be satisfied by choosing $w(\mathsf{d}_1)$ large enough such that $w(x(\mathsf{b}(\mathsf{d}_1))) > w(t)$ for all $t \in \nabla_+(r_1)\restriction^{>}_{+}$, where

$$\begin{aligned}\nabla_+(\ell_1)\restriction^{>}_{+} &= \{x(\mathsf{b}(\mathsf{d}_1)), \mathsf{a}(\mathsf{e}_1^0(\mathsf{e}_1^1(\mathsf{c}))), y(\mathsf{e}_1^1(\mathsf{e}_1^2(\mathsf{c}))), z(\mathsf{e}_1^2(\mathsf{e}_1^0(\mathsf{c})))\}\\ \nabla_+(r_1)\restriction^{>}_{+} &= \{\qquad \mathsf{a}(\mathsf{e}_1^0(\mathsf{e}_1^0(\mathsf{c}))), y(\mathsf{e}_1^1(\mathsf{e}_1^1(\mathsf{c}))), z(\mathsf{e}_1^2(\mathsf{e}_1^2(\mathsf{c})))\}\end{aligned}$$

On the other hand, if $y, z > + > x$ (i.e., x is falsified) then $\ell_1 >_{\mathsf{KV}} r_1$ is not satisfiable; no matter how we assign weights to e_1^0, e_1^1, and e_1^2, a term in $\nabla_+(r_1)$ has the maximum weight, where

$$\begin{aligned}\nabla_+(\ell_1)\restriction^{>}_{+} &= \{\mathsf{a}(\mathsf{e}_1^0(\mathsf{e}_1^1(\mathsf{c}))), y(\mathsf{e}_1^1(\mathsf{e}_1^2(\mathsf{c}))), z(\mathsf{e}_1^2(\mathsf{e}_1^0(\mathsf{c})))\}\\ \nabla_+(r_1)\restriction^{>}_{+} &= \{\mathsf{a}(\mathsf{e}_1^0(\mathsf{e}_1^0(\mathsf{c}))), y(\mathsf{e}_1^1(\mathsf{e}_1^1(\mathsf{c}))), z(\mathsf{e}_1^2(\mathsf{e}_1^2(\mathsf{c})))\}\end{aligned}$$

However, if $y > + > x, z$ (i.e., z is falsified) then $\ell_1 >_{\mathsf{KV}} r_1$ can be satisfied by choosing $w(\mathsf{e}_1^2)$ large enough, where

$$\begin{aligned}\nabla_+(\ell_1)\restriction^{>}_{+} &= \{\mathsf{a}(\mathsf{e}_1^0(\mathsf{e}_1^1(\mathsf{c}))), y(\mathsf{e}_1^1(\mathsf{e}_1^2(\mathsf{c})))\}\\ \nabla_+(r_1)\restriction^{>}_{+} &= \{\mathsf{a}(\mathsf{e}_1^0(\mathsf{e}_1^0(\mathsf{c}))), y(\mathsf{e}_1^1(\mathsf{e}_1^1(\mathsf{c})))\}\end{aligned}$$

Similarly, if $+ > x, y, z$ then $\ell_1 >_{\mathsf{KV}} r_1$ can be satisfied by choosing $w(\mathsf{e}_1^1)$ large enough, where

$$\begin{aligned}\nabla_+(\ell_1)\restriction^{>}_{+} &= \{\mathsf{a}(\mathsf{e}_1^0(\mathsf{e}_1^1(\mathsf{c})))\}\\ \nabla_+(r_1)\restriction^{>}_{+} &= \{\mathsf{a}(\mathsf{e}_1^0(\mathsf{e}_1^0(\mathsf{c})))\}\end{aligned}$$

Lemma 6.7
Let $\mathsf{a} > + > \mathsf{b}$. Then, $\mathscr{R}_\phi \subseteq >_{\mathsf{KV}}$ for some (w, w_0) if and only if for every i there is some p such that $p \in C_i$ with $p \not< +$ or $\neg p \in C_i$ with $+ > p$.

Proof
For the "if" direction we reason as follows. Consider a (partial) weight function w such that $w(\mathsf{a}) = w(\mathsf{b}) = w(p_j)$ for all $1 \leqslant j \leqslant m$. We obtain $\mathscr{R}_0 \subseteq >_{\mathsf{KV}}$ from Lemma 6.5. Furthermore, consider $C_i = \{p'_1, \ldots, p'_k, \neg p''_1, \ldots, \neg p''_l\}$ and ℓ_i, r_i, U, V and W defined above. Let $L = \nabla_+(\ell_i)$ and $R = \nabla_+(r_i)$. We clearly have $L\restriction^{\not<}_{+} = U\restriction^{\not<}_{+} \cup V\restriction^{\not<}_{+}$ and $R\restriction^{\not<}_{+} = W\restriction^{\not<}_{+}$. It is easy to show that $w(\ell_i) = w(r_i)$. We show $\ell_i >_{\mathsf{KV}} r_i$ by distinguishing two cases.

1. First suppose that $p'_j \not< +$ for some $1 \leqslant j \leqslant k$. We have $p'_j(\mathsf{b}(\mathsf{d}_i)) \in U\restriction^{\not<}_{+}$. Extend the weight function w such that

$$w(\mathsf{d}_i) = 1 + 2 \cdot \max\{w(\mathsf{e}_i^0), \ldots, w(\mathsf{e}_i^l)\}$$

   Then $p'_j(\mathsf{b}(\mathsf{d}_i)) >_{\mathsf{kv}} t$ for all terms $t \in W$ and hence $L\restriction^{\not<}_{+} >^{\mathsf{mul}}_{\mathsf{kv}} R\restriction^{\not<}_{+}$. Therefore, $\ell_i >_{\mathsf{KV}} r_i$ by case 3(a).
2. Otherwise, $U\restriction^{\not<}_{+} = \varnothing$ holds. By assumption $+ > p''_j$ for some $1 \leqslant j \leqslant l$. Consider the smallest m such that $+ > p''_m$. Extend the weight function w such that

$$w(\mathsf{e}_i^m) = 1 + 2 \cdot \max\{w(\mathsf{e}_i^j) \mid j \neq m\}$$

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Then $w(p''_{m-1}(\mathsf{e}_i^{m-1,m})) > w(p''_j(\mathsf{e}_i^{j,j}))$ for all $j \neq m$. From $p''_{m-1} > +$ we infer $p''_{m-1}(\mathsf{e}_i^{m-1,m}) \in V{\restriction}^{\not<}_{+}$. (Note that $p''_{m-1} = \mathsf{a} > +$ if $m = 1$.) By definition of m, $p''_m(\mathsf{e}_i^{m,m}) \notin W{\restriction}^{\not<}_{+}$. It follows that $L{\restriction}^{\not<}_{+} >^{\mathsf{mul}}_{\mathsf{kv}} R{\restriction}^{\not<}_{+}$ and thus $\ell_i >_{\mathsf{KV}} r_i$ by case 3(a).

Next we prove the "only if" direction. So suppose there exists a weight function w such that $\mathscr{R}_\phi \subseteq >_{\mathsf{KV}}$. We obtain $w(\mathsf{a}) = w(\mathsf{b}) = w(p_j)$ for all $1 \leqslant j \leqslant m$ from Lemma 6.5. It follows that $w(\ell_i) = w(r_i)$ for every $C_i \in \phi$. Suppose for a proof by contradiction that there exists $C_i \in \phi$ such that $+ > p$ for all $p \in C_i$ and $p \not< +$ whenever $\neg p \in C_i$. So $L{\restriction}^{\not<}_{+} = V$ and $R{\restriction}^{\not<}_{+} = W$. Since $|R| = |L| + 1$, we must have $\ell_i >_{\mathsf{KV}} r_i$ by case 3(a) and thus $V >_{\mathsf{kv}} W$. Let s be a term in V of maximal weight. We must have $w(s) \geqslant w(t)$ for all terms $t \in W$. By construction of the terms in V and W, this is only possible if all symbols e_i^j have the same weight. It follows that all terms in V and W have the same weight. Since $|V| = |W|$ and for every term $s' \in V$ there exists a unique term $t' \in W$ with $\mathsf{root}(s') = \mathsf{root}(t')$, we conclude $V =_{\mathsf{kv}} W$, which provides the desired contradiction. □

After these preliminaries we are ready to prove NP-hardness.

Theorem 6.8

The (ground) orientability problem for $>_{\mathsf{KV}}$ is NP-hard.

Proof

It is sufficient to prove that a CNF formula $\phi = \{C_1, \ldots, C_n\}$ is satisfiable if and only if the corresponding $\mathscr{R}_\phi$ is orientable by $>_{\mathsf{KV}}$. Note that the size of $\mathscr{R}_\phi$ is linear in the size of ϕ. First suppose that ϕ is satisfiable. Let α be a satisfying assignment for the atoms $p_1, \ldots, p_m$. Define the precedence $>$ as follows: $\mathsf{a} > + > \mathsf{b}$ and $p_j > +$ if $\alpha(p_j)$ is true and $+ > p_j$ if $\alpha(p_j)$ is false. Then $\mathscr{R}_\phi \subseteq >_{\mathsf{KV}}$ follows from Lemma 6.7. Conversely, if $\mathscr{R}_\phi$ is compatible with $>_{\mathsf{KV}}$ then we define an assignment α for the atoms in ϕ as follows: $\alpha(p)$ is true if $p \not< +$ and $\alpha(p)$ is false if $+ > p$. We claim that α satisfies ϕ. Let C_i be a clause in ϕ. According to Lemma 6.7, $p \not< +$ for one of the atoms p in C_i or $+ > p$ for one of the negative literals $\neg p$ in C_i. Hence α satisfies C_i by definition. □

We can show NP-hardness of $>_{\mathsf{ACKBO}}$ by adapting the above construction accordingly, as shown in Appendix **??**.

Theorem 6.9

The (ground) orientability problem for $>_{\mathsf{ACKBO}}$ is NP-hard. □

The NP-hardness results of Theorems 6.8 and 6.9 can be strengthened to NP-completeness. This is not entirely trivial because there are infinitely many different weight functions to consider.

Lemma 6.10

The orientability problems for $>_{\mathsf{ACKBO}}$ and $>_{\mathsf{KV}}$ belong to NP.

Proof (sketch)
We sketch the proof for $>_{\mathsf{ACKBO}}$. With minor modifications the result for $>_{\mathsf{KV}}$ is obtained.

For each rule $\ell \to r$ of a given TRS $\mathscr{R}$ we guess which choices are made in the definition of $>_{\mathsf{ACKBO}}$ when evaluating $\ell >_{\mathsf{ACKBO}} r$. In particular, we do not guess the weight function, but rather the comparison ($=$ or $>$) of the weights of certain subterms of ℓ and r. These comparisons are transformed into constraints on the weight function by symbolically evaluating the weight expressions. We add the constraints stemming from the definition of the weight function. The resulting problem is a conjunction of linear constraints over unknowns (the weights of the function symbols and w_0) over the integers. It is well known (Schrijver 1986, Section 10.3) that solving such a *linear program* over the rationals can be done in polynomial time. If there is a solution we check the admissibility condition and well-foundedness of the precedence. (If an integer valued weight function is desired, one can simply multiply the weights by the least common multiple of their denominators. This induces the same weight order on terms and does not affect the admissibility condition.)

Since there are polynomially (in the size of the compared terms) many choices in the definition of $>_{\mathsf{ACKBO}}$ and each choice can be checked for correctness in polynomial time, membership in NP follows. □

Corollary 6.11
The orientability problems for $>_{\mathsf{ACKBO}}$ and $>_{\mathsf{KV}}$ are NP-complete. □

The NP-hardness proofs of $>_{\mathsf{KV}}$ and $>_{\mathsf{ACKBO}}$ orientability given earlier do not extend to $>_{\mathsf{S}}$ since the latter requires that AC symbols are minimal in the precedence.

We conjecture that the orientability problem for $>_{\mathsf{S}}$ belongs to P.

7 AC-RPO

In this section, we compare AC-KBO with AC-RPO (Rubio 2002). Since the latter is incremental (Rubio 2002, Lemma 22), we restrict the discussion to total precedences.

Definition 7.1
Let $>$ be a precedence and $t = f(u, v)$ such that $f \in \mathscr{F}_{\mathsf{AC}}$ and $\nabla_f(t) = \{t_1, \dots, t_n\}$. We write $t \rhd^f_{\mathsf{emb}} u$ for all terms u such that $\nabla_f(u) = \{t_1, \dots, t_{i-1}, s_j, t_{i+1}, \dots, t_n\}$ for some $t_i = g(s_1, \dots, s_m)$ with $f > g$ and $1 \leqslant j \leqslant m$.

Using previously introduced notations, AC-RPO can be defined as follows.

Definition 7.2
Let $>$ be a precedence and let $\mathscr{F} \setminus \mathscr{F}_{\mathsf{AC}} = \mathscr{F}_{\mathsf{mul}} \uplus \mathscr{F}_{\mathsf{lex}}$. We define $>_{\mathsf{ACRPO}}$ inductively as follows: $s >_{\mathsf{ACRPO}} t$ if one of the following conditions holds:

0. $s = f(s_1, \dots, s_n)$ and $s_i \geqslant_{\mathsf{ACRPO}} t$ for some $1 \leqslant i \leqslant n$,
1. $s = f(s_1, \dots, s_n)$, $t = g(t_1, \dots, t_m)$, $f > g$, and $s >_{\mathsf{ACRPO}} t_j$ for all $1 \leqslant j \leqslant m$,
2. $s = f(s_1, \dots, s_n)$, $t = f(t_1, \dots, t_n)$, $f \notin \mathscr{F}_{\mathsf{AC}}$, $s >_{\mathsf{ACRPO}} t_j$ for all $1 \leqslant j \leqslant n$, and either

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(a) $f \in \mathscr{F}_{\mathsf{lex}}$ and $(s_1, \ldots, s_n) >^{\mathsf{lex}}_{\mathsf{ACRPO}} (t_1, \ldots, t_n)$, or
(b) $f \in \mathscr{F}_{\mathsf{mul}}$ and $\{s_1, \ldots, s_n\} >^{\mathsf{mul}}_{\mathsf{ACRPO}} \{t_1, \ldots, t_n\}$,

3. $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathscr{F}_{\mathsf{AC}}$, and $s' \geqslant_{\mathsf{ACRPO}} t$ for some s' such that $s \rhd^f_{\mathsf{emb}} s'$,
4. $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathscr{F}_{\mathsf{AC}}$, $s >_{\mathsf{ACRPO}} t'$ for all t' such that $t \rhd^f_{\mathsf{emb}} t'$, and for $S = \nabla_f(s)$ and $T = \nabla_f(t)$
   (a) $S >^f_{\mathsf{ACRPO}} T$,
   (b) $S =^f_{\mathsf{AC}} T$ and $|S| > |T|$, or
   (c) $S =^f_{\mathsf{AC}} T$, $|S| = |T|$, and $S \restriction^<_f >^{\mathsf{mul}}_{\mathsf{ACRPO}} T \restriction^<_f$.

The relation $=_{\mathsf{AC}}$ is used as preorder in $>^{\mathsf{lex}}_{\mathsf{ACRPO}}$ and $>^{\mathsf{mul}}_{\mathsf{ACRPO}}$, and as equivalence relation in $\geqslant_{\mathsf{ACRPO}}$.

Example 7.3
Consider the TRS $\mathscr{R}$ consisting of the rules

$$\mathsf{f}(x) + \mathsf{g}(x) \to \mathsf{g}(x) + (\mathsf{g}(x) + \mathsf{g}(x)) \qquad \mathsf{f}(x) \to \mathsf{g}(x) + \mathsf{a}$$

over the signature $\mathscr{F} = \{\mathsf{f}, \mathsf{g}, +, \mathsf{a}\}$ with $+ \in \mathscr{F}_{\mathsf{AC}}$. Let $\mathscr{R}'$ be the TRS obtained from $\mathscr{R}$ by reverting the first rule. When using AC-RPO with precedence $\mathsf{f} > + > \mathsf{g} > \mathsf{a}$, both rules in $\mathscr{R}$ can be oriented from left to right. Since the second rule requires $\mathsf{f} > +$ and $\mathsf{f} > \mathsf{g}$, termination of $\mathscr{R}'$ cannot be shown with AC-RPO.

In contrast, AC-KBO cannot orient $\mathscr{R}$ due to the variable condition. But the precedence $\mathsf{g} > + > \mathsf{f} > \mathsf{a}$ and admissible weight function (w, w_0) with $w(+) = 0$, $w_0 = w(\mathsf{g}) = w(\mathsf{a}) = 1$ and $w(\mathsf{f}) = 3$ allows the resulting $>_{\mathsf{ACKBO}}$ to orient both rules of $\mathscr{R}'$.

Case 4 in Definition 7.2 differs from the original version in Rubio (2002) in that we used notions introduced for AC-KBO. We now recall the original definition and prove the two versions equivalent in Lemma 7.5.

Definition 7.4
For $S = \{s_1, \ldots, s_n\}$ let $\#(S) = \#(s_1) + \cdots + \#(s_n)$ where $\#(s_i) = s_i$ for $s_i \in \mathscr{V}$ and $\#(s_i) = 1$ otherwise. Then $\#(S) > \#(T)$ ($\#(S) \geqslant \#(T)$) is defined via comparison of linear polynomials over the positive integers.

Let $>$ be a total precedence. The order $>_{\mathsf{ACRPO}'}$ is inductively defined as in Definition 7.2, but with case 4 as follows:

4′. $s = f(s_1, s_2)$, $t = f(t_1, t_2)$, $f \in \mathscr{F}_{\mathsf{AC}}$, $s >_{\mathsf{ACRPO}'} t'$ for all t' such that $t \rhd^f_{\mathsf{emb}} t'$, $S \restriction^>_f \uplus S \restriction_{\mathscr{V}} \geqslant^{\mathsf{mul}}_{\mathsf{ACRPO}'} T \restriction^>_f \uplus T \restriction_{\mathscr{V}}$ for $S = \nabla_f(s)$ and $T = \nabla_f(t)$, and
   (a) $S \restriction^>_f >^{\mathsf{mul}}_{\mathsf{ACRPO}'} T \restriction^>_f$, or
   (b) $\#(S) > \#(T)$, or
   (c) $\#(S) \geqslant \#(T)$, and $S >^{\mathsf{mul}}_{\mathsf{ACRPO}'} T$.

The proof of the following correspondence can be found in the online appendix.

Lemma 7.5
Let $>$ be a total precedence. We have $s >_{\mathsf{ACRPO}} t$ if and only if $s >_{\mathsf{ACRPO'}} t$.

It is known that both orientability and membership are NP-hard for the multiset path order (Krishnamoorthy and Narendran 1985). It is not hard to adapt these proofs to the lexicographic path order (LPO), and NP-hardness for the case of RPO is an easy consequence.

In contrast to AC-KBO, a straightforward application of the definition of AC-RPO (in particular case 4 of Definition 7.2) may generate an exponential number of subproblems, as illustrated by the following example.

Example 7.6
Consider the signature $\mathscr{F} = \{\mathsf{f}, \mathsf{g}, \mathsf{h}, \circ\}$ with $\circ \in \mathscr{F}_{\mathsf{AC}}$ and precedence $\mathsf{f} > \circ > \mathsf{g} > \mathsf{h}$. Let $t = x \circ y$ and $t_n = t\sigma^n$ for the substitution $\sigma = \{x \mapsto \mathsf{g}(x) \circ \mathsf{h}(y), y \mapsto \mathsf{h}(y)\}$. The size of t_n is quadratic in n but the number of terms u that satisfy $t_n\ (\rhd^{\circ}_{\mathsf{emb}})^+\ u$ is exponential in n. Now suppose one wants to decide whether $\mathsf{f}(x) \circ \mathsf{f}(y) >_{\mathsf{ACRPO}} t_n$ holds. Only case 4(a) is applicable but in order to conclude orientability, case 4(a) needs to be applied recursively in order to verify $\mathsf{f}(x) \circ \mathsf{f}(x) >_{\mathsf{ACRPO}} u$ for the exponentially many terms u such that $t_n\ (\rhd^{\circ}_{\mathsf{emb}})^+\ u$.

8 Subterm coefficients

Subterm coefficients were introduced in Ludwig and Waldmann (2007) in order to cope with rewrite rules like $\mathsf{f}(x) \to \mathsf{g}(x, x)$ which violate the variable condition. A *subterm coefficient function* is a partial mapping $sc : \mathscr{F} \times \mathbb{N} \to \mathbb{N}$ such that for a function symbol f of arity n we have $sc(f, i) > 0$ for all $1 \leqslant i \leqslant n$. Given a weight function (w, w_0) and a subterm coefficient function sc, the weight of a term is inductively defined as follows:

$$w(t) = \begin{cases} w_0 & \text{if } t \in \mathscr{V} \\ w(f) + \displaystyle\sum_{1 \leqslant i \leqslant n} sc(f, i) \cdot w(t_i) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

The *variable coefficient* $\mathsf{vc}(x, t)$ of a variable x in a term t is inductively defined as follows:

$$\mathsf{vc}(x, t) = \begin{cases} 1 & \text{if } t = x \\ 0 & \text{if } t \in \mathscr{V} \setminus \{x\} \\ \displaystyle\sum_{1 \leqslant i \leqslant n} sc(f, i) \cdot \mathsf{vc}(x, t_i) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

Definition 8.1
The order $>^{sc}_{\mathsf{ACKBO}}$ is obtained from Definition 5.1 by replacing the condition "$|s|_x \geqslant |t|_x$ for all $x \in \mathscr{V}$" with "$\mathsf{vc}(x, s) \geqslant \mathsf{vc}(x, t)$ for all $x \in \mathscr{V}$" and using the modified weight function introduced above.

In order to guarantee AC compatibility of $>^{sc}_{\mathsf{ACKBO}}$, the subterm coefficient function sc has to assign the value 1 to arguments of AC symbols. This follows by considering

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the terms $t \circ (u \circ v)$ and $(t \circ u) \circ v$ for an AC symbol $\circ$ with $sc(\circ, 1) = m$ and $sc(\circ, 2) = n$. We have

$$w(t \circ (u \circ v)) = 2 \cdot w(\circ) + m \cdot w(t) + mn \cdot w(u) + n^2 \cdot w(v)$$
$$w((t \circ u) \circ v) = 2 \cdot w(\circ) + m^2 \cdot w(t) + mn \cdot w(u) + n \cdot w(v)$$

Since $w(t \circ (u \circ v)) = w((t \circ u) \circ v)$ must hold for all possible terms t, u, and v, it follows that $m = m^2$ and $n^2 = n$, implying $m = n = 1$.[6] The proof of the following theorem is very similar to the one of Theorem 5.10 and hence omitted.

Theorem 8.2
If $sc(f, 1) = sc(f, 2) = 1$ for every function symbol $f \in \mathscr{F}_{\mathsf{AC}}$ then $>^{sc}_{\mathsf{ACKBO}}$ is an AC-compatible simplification order. □

Subterm coefficients can be viewed as linear interpretations. Lankford (1979) suggested to use polynomial interpretations for the weight function of KBO. A general framework for the use of arbitrary well-founded algebras in connection with KBO is described in Middeldorp and Zantema (1997). These developments can be lifted to the AC setting with little effort.

Example 8.3
Consider the following TRS $\mathscr{R}$ with $\circ \in \mathscr{F}_{\mathsf{AC}}$:

$$\mathsf{f}(\mathsf{0}, x \circ x) \to x \quad (1) \qquad \mathsf{f}(\mathsf{s}(x), y) \to \mathsf{f}(x \circ y, \mathsf{0}) \quad (3)$$
$$\mathsf{f}(x, \mathsf{s}(y)) \to \mathsf{f}(x \circ y, \mathsf{0}) \quad (2) \qquad \mathsf{f}(x \circ y, \mathsf{0}) \to \mathsf{f}(x, \mathsf{0}) \circ \mathsf{f}(y, \mathsf{0}) \quad (4)$$

Termination of $\mathscr{R}$ was shown using AC dependency pairs in Kusakari (2000, Example 4.2.30). Consider a precedence $\mathsf{f} > \circ > \mathsf{s} > \mathsf{0}$, and weights and subterm coefficients given by $w_0 = 1$ and the following interpretation $\mathscr{A}$, mapping function symbols in $\mathscr{F}$ to linear polynomials over $\mathbb{N}$:

$$\mathsf{s}_{\mathscr{A}}(x) = x + 6 \qquad \mathsf{f}_{\mathscr{A}}(x, y) = 4x + 4y + 5 \qquad x \circ_{\mathscr{A}} y = x + y + 3 \qquad \mathsf{0}_{\mathscr{A}} = 1$$

It is easy to check that the first three rules result in a weight decrease. The left- and right-hand side of rule (4) are both interpreted as $4x + 4y + 21$, so both terms have weight 29, but since $\mathsf{f} > \circ$ we conclude termination of $\mathscr{R}$ from case 1 in Definition 5.1 (8.1). Note that termination of $\mathscr{R}$ cannot be shown by AC-RPO or any of the previously considered versions of AC-KBO.

9 Experiments

We ran experiments on a server equipped with eight dual-core AMD Opteron® processors 885 running at a clock rate of 2.6 GHz with 64 GB of main memory. The different versions of AC-KBO considered in this paper as well as AC-RPO (Rubio 2002) were implemented on top of $\mathsf{T_TT_2}$ using encodings in SAT/SMT. These encodings resemble those for standard KBO (Zankl *et al.* 2009) and transfinite

[6] This condition is also obtained by restricting Ben Cherifa and Lescanne (1987, Proposition 4) to linear polynomials.

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Table 1. *Experiments on 145 termination and 67 completion problems*

method	orientability			AC-DP			completion		
	yes	time	∞	yes	time	∞	yes	time	∞
AC-KBO	32	1.7	0	66	463.1	3	25	2278.6	37
Steinbach	23	1.6	0	50	463.2	2	24	2235.4	36
Korovin & Voronkov	30	2.0	0	66	474.3	4	25	2279.4	37
KV′	30	2.1	0	66	472.4	3	25	2279.6	37
subterm coefficients	37	47.1	0	68	464.7	2	28	1724.7	26
AC-RPO	63	2.8	0	79	501.5	4	28	1701.6	26
total	72			94			31		

KBO (Winkler *et al.* 2012). The encoding of multiset extensions of order pairs are based on Codish *et al.* (2012), but careful modifications were required to deal with submultisets induced by the precedence.

For termination experiments, our test set comprises all AC problems in the *Termination Problem Data Base 9.0*,[7] all examples in this paper, some further problems harvested from the literature, and constraint systems produced by the completion tool mkbtt (Winkler 2013) (145 TRSs in total). The timeout was set to 60 seconds. The results are summarized in Table 1, where we list for each order the number of successful termination proofs, the total time, and the number of timeouts (column ∞). The 'orientability' column directly applies the order to orient all the rules. Although AC-RPO succeeds on more input problems, termination of 9 TRSs could only be established by (variants of) AC-KBO. We found that our definition of AC-KBO is about equally powerful as Korovin and Voronkov's order, but both are considerably more useful than Steinbach's version. When it comes to proving termination, we did not observe a difference between Definitions 4.7 and 4.10. Subterm coefficients clearly increase the success rate, although efficiency is affected. In all settings partial precedences were allowed.

The "AC-DP" column applies the order in the AC-dependency pair framework of Alarcón *et al.* (2010), in combination with *argument filterings* and *usable rules*. Here AC symbols in dependency pairs are *unmarked*, as proposed in Marché and Urbain (2004). In this setting the variants of AC-KBO become considerably more powerful and competitive to AC-RPO, since argument filterings relax the variable condition, as pointed out in Zankl *et al.* (2009).

For completion experiments, we ran the normalized completion tool mkbtt with AC-RPO and the variants of AC-KBO for termination checks on 67 equational systems collected from the literature. The overall timeout was set to 60 seconds, the timeout for each termination check to 1.5 seconds. The "completion" column in Table 1 summarizes our results, listing for each order the number of successful completions, the total time, and the number of timeouts. It should be noted that the

[7] http://termination-portal.org/wiki/TPDB

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Table 2. *Complexity results (KV is the ground version of $>_{\mathsf{KV}}$)*

problem	KBO	S	AC-KBO	KV	KV′	AC-RPO
membership	P	P	P	P	NP-complete	NP-hard
orientability	P	?	NP-complete	NP-complete	NP-complete	NP-hard

results do not change if the overall timeout is increased to 600 seconds. For several of these input problems it is actually unknown whether an AC-convergent system exists.

All experimental details, source code, and $\mathsf{T_TT_2}$ binaries are available online.[8]

The following example can be completed using AC-KBO, whereas AC-RPO does not succeed.

Example 9.1
Consider the following TRS $\mathscr{R}$ (Marché and Urbain 2004) for addition of binary numbers:

$$\# + \mathsf{0} \to \# \qquad x\mathsf{0} + y\mathsf{0} \to (x+y)\mathsf{0} \qquad x\mathsf{1} + y\mathsf{1} \to (x+y+\#\mathsf{1})\mathsf{0}$$
$$x + \# \to x \qquad x\mathsf{0} + y\mathsf{1} \to (x+y)\mathsf{1}$$

Here $+ \in \mathscr{F}_{\mathsf{AC}}$, $\mathsf{0}$ and $\mathsf{1}$ are unary operators in postfix notation, and $\#$ denotes the empty bit sequence. For example, $\#\mathsf{100}$ represents the number 4. This TRS is not compatible with AC-RPO but AC termination can easily be shown by AC-KBO, for instance with the weight function (w, w_0) with $w(+) = 0$, $w_0 = w(\mathsf{0}) = w(\#) = 1$, and $w(\mathsf{1}) = 3$. It can be completed into an AC-convergent TRS using AC-KBO.

10 Conclusion

We revisited the two variants of AC-compatible extensions of KBO. We extended the first version $>_{\mathsf{S}}$ introduced by Steinbach (1990) to a new version $>_{\mathsf{ACKBO}}$, and presented a rigorous correctness proof. By this we conclude correctness of $>_{\mathsf{S}}$, which had been put in doubt in Korovin and Voronkov (2003a). We also modified the order $>_{\mathsf{KV}}$ by Korovin and Voronkov to a new version $>_{\mathsf{KV'}}$ which is monotone on non-ground terms, in contrast to $>_{\mathsf{KV}}$. We further presented several complexity results regarding these variants (see Table 2). While a polynomial time algorithm is known for the orientability problem of standard KBO (Korovin and Voronkov 2003b), the problem becomes NP-complete even for the ground version of $>_{\mathsf{KV}}$, as well as for our $>_{\mathsf{ACKBO}}$. Somewhat unexpectedly, even deciding $>_{\mathsf{KV'}}$ is NP-complete while deciding standard KBO is linear (Löchner 2006). In contrast, the membership problem is polynomial-time decidable for our $>_{\mathsf{ACKBO}}$. Finally, we implemented these variants of AC-compatible KBO as well as the AC-dependency pair framework of

[8] http://cl-informatik.uibk.ac.at/software/ackbo

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Alarcón *et al.* (2010). We presented full experimental results both for termination proving and normalized completion.

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Supplementary material

To view supplementary material for this article, please visit http://dx.doi.org/10.1017/S1471068415000083.

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