Proceedings of the Royal Society of Edinburgh, **150**, 2632–2641, *2020*
DOI:10.1017/prm.2019.36

Reconstructing directed graphs from generalized gauge actions on their Toeplitz algebras

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(MS Received 1 May 2019; accepted 1 May 2019)

We show how to reconstruct a finite directed graph E from its Toeplitz algebra, its gauge action, and the canonical finite-dimensional abelian subalgebra generated by the vertex projections. We also show that if E has no sinks, then we can recover E from its Toeplitz algebra and the generalized gauge action that has, for each vertex, an independent copy of the circle acting on the generators corresponding to edges emanating from that vertex. We show by example that it is not possible to recover E from its Toeplitz algebra and gauge action alone.

Keywords: Graph C^*-algebra; Toeplitz algebra; KMS state

2010 *Mathematics subject classification:* Primary: 46L05

1. Introduction

In recent years, there has been an enormous amount of work, led by Eilers and his collaborators (see e.g., [**2**–**7**, **17**]) on determining which moves on finite directed graphs generate the equivalence relations determined by various types of isomorphism of the associated C^*-algebras. One spectacular example of this is [**5**, theorem 3.1]: if E and F are graphs with finitely many vertices, then the graph C^*-algebras $C^*(E)$ and $C^*(F)$ are stably isomorphic if and only if E can be transformed into F using a finite sequence of in-splittings, out-splittings, reductions, additions of sinks, Cuntz splices, Pulelehua moves, and the inverses of these moves.

By contrast, relatively little attention has been paid to the Toeplitz algebras of directed graphs, until the recent interest in KMS-theory (see e.g., [**1**, **10**, **11**, **13**, **18**]) brought them to the fore. It has been known for some time [**12**, **16**] that the *non-selfadjoint* Toeplitz algebra (also called the tensor algebra or the quiver operator algebra) of a directed graph E contains all of the information about E—if E and F are directed graphs with isomorphic tensor algebras, then they are themselves isomorphic. But there are no results in this direction for the Toeplitz C^*-algebras of directed graphs.

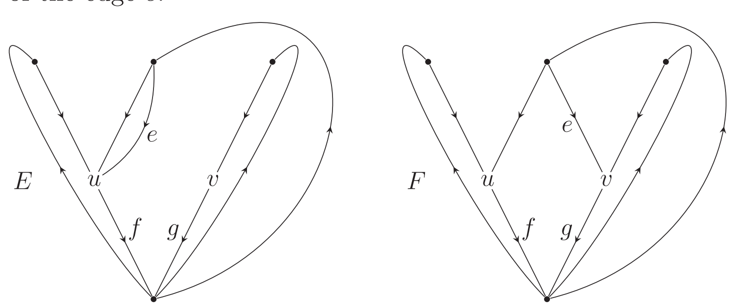
Here we consider the extent to which a finite directed graph can be recovered from its Toeplitz algebra and gauge action. We show that at least one additional piece of information is needed (see examples 2.1 and 3.10) and identify two pieces of information, either of which suffices for finite graphs with no sinks. Our key tool is the KMS structure of $\mathcal{T}C^*(E)$ for the dynamics arising from its gauge action; we show that using this we can recover the rank-one projections in $\mathcal{T}C^*(E)$ that correspond to the vertices of E. From this, using the spectral subspaces of the gauge action, it is straightforward to count the number of edges (indeed, the number of paths of length n for any n) emanating from a given vertex. However, additional information is required to determine which of these paths have the same ranges. We show that the subalgebra $M_E = \operatorname{span}\{q_v : v \in E^0\} \subseteq \mathcal{T}C^*(E)$ generated by the vertex projections is sufficient to recover this information, and that if E has no sinks then the action κ^E of the torus $\mathbb{T}^{E^0}$ such that $\kappa^E_z(t_e) = z_{s(e)}t_e$ for each $e \in E^1$ also suffices.

Conventions. We use the conventions of [**15**] for graphs and their C^*-algebras; so the Toeplitz algebra of a directed graph E is the universal C^*-algebra generated by projections $\{q_v : v \in E^0\}$ and partial isometries $\{t_e : e \in E^1\}$ such that $t_e^*t_e = q_{s(e)}$ and $q_v \geqslant \sum_{r(e)=v} t_e t_e^*$. We use the notational convention in which, for example, $vE^1 = r^{-1}(v)$ and $E^1v = s^{-1}(v)$.

2. An example

We started this project by asking whether it is possible to recover a directed graph E from its Toeplitz algebra and gauge action. The following example shows that the answer is no, even for the particularly well-behaved class of strongly connected finite graphs in which every cycle has an entrance. We thank Søren Eilers for very helpful conversations that led to the construction of this example. For a simpler example involving graphs that are not strongly connected, and have sinks and sources, see example 3.10.

Example 2.1. Consider the following directed graphs E and F, that differ only in the range of the edge e:



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Let (t, q) be the universal generating Toeplitz–Cuntz–Krieger F-family in $\mathcal{T}C^*(F)$. Define elements $\{Q_w : w \in E^0\}$ and $\{T_h : h \in E^1\}$ in $\mathcal{T}C^*(F)$ as follows:

$$Q_u = q_u + t_e t_e^*, \quad Q_v = q_v - t_e t_e^*, \quad T_f = t_f + t_g t_e t_e^*, \quad T_g = t_g(q_v - t_e t_e^*), \quad \text{and}$$
$$Q_w = q_w \text{ for } w \in E^0 \setminus \{u, v\} \quad \text{and} \quad T_h = t_h \text{ for } h \in E^1 \setminus \{f, g\}.$$

It is routine to check that (Q, T) is a Toeplitz–Cuntz–Krieger E-family that generates $\mathcal{T}C^*(F)$, and that the elements $Q_w - \sum_{h \in wE^1} T_h T_h^*$ are all nonzero. So the universal property of $\mathcal{T}C^*(E)$ yields a surjective homomorphism $\pi_{Q,T} : \mathcal{T}C^*(E) \to \mathcal{T}C^*(F)$ such that $\pi_{Q,T}(q_w) = Q_w$ and $\pi_{Q,T}(t_h) = T_h$, and [**8**, theorem 4.1] implies that $\pi_{Q,T}$ is injective. It is immediate from the definitions of the T_h and Q_w that $\pi_{Q,T}$ is gauge-equivariant. So $(\mathcal{T}C^*(E), \gamma^E) \cong (\mathcal{T}C^*(F), \gamma^F)$, but there is no graph-isomorphism from E to F because, for example, E has a pair of parallel edges, whereas F does not.

In fact, since the canonical diagonals $D_E = \overline{\operatorname{span}}\{t_\mu t_\mu^* : \mu \in E^*\}$ and $D_F = \overline{\operatorname{span}}\{t_\mu t_\mu^* : \mu \in F^*\}$ are maximal abelian in $\mathcal{T}C^*(E)$ and $\mathcal{T}C^*(F)$, we see that $\pi_{Q,T}(D_E)$ is a maximal abelian subalgebra of $\mathcal{T}C^*(F)$. Since this maximal abelian subalgebra is contained in the maximal abelian subalgebra D_F of $\mathcal{T}C^*(F)$, we deduce that $\pi_{Q,T}(D_E) = D_F$. So the triples $(\mathcal{T}C^*(E), \gamma^E, D_E)$ and $(\mathcal{T}C^*(F), \gamma^F, D_F)$ are isomorphic even though E and F are not.

3. The main theorem

Example 2.1 shows that recovering a directed graph from its Toeplitz algebra requires more information than just the gauge action. Our main result identifies two additional bits of data, either one of which bridges the gap. The first one is the C*-subalgebra generated by the vertex projections inside the Toeplitz algebra. The second one is a higher dimensional generalization of the gauge action.

Definition 3.1. When E is a directed graph, the *generalized gauge action* on $\mathcal{T}C^*(E)$ is the action $\kappa^E : \mathbb{T}^{E^0} \to \operatorname{Aut} \mathcal{T}C^*(E)$ determined by $\kappa^E_z(t_e) = z_{s(e)} t_e$ for all $e \in E^1$ and $z \in \mathbb{T}^{E^0}$. When E and F are two directed graphs, we say that an isomorphism $\rho : \mathcal{T}C^*(E) \to \mathcal{T}C^*(F)$ *intertwines the generalized gauge actions* κ^E and κ^F if there is a bijection $\varphi : E^0 \to F^0$ such that the induced homomorphism $\varphi^* : \mathbb{T}^{E^0} \to \mathbb{T}^{F^0}$ satisfies $\rho \circ \kappa^E_z = \kappa^F_{\varphi^*(z)} \circ \rho$ for all $z \in \mathbb{T}^{E^0}$.

Theorem 3.2. *Let E and F be finite directed graphs. As before, let γ^E be the gauge action of $\mathbb{T}$ on $\mathcal{T}C^*(E)$, and let $M_E := \operatorname{span}\{q_v : v \in E^0\} \subseteq \mathcal{T}C^*(E)$. Let κ^E be the generalized gauge action of $\mathbb{T}^{E^0}$ on $\mathcal{T}C^*(E)$ given by $\kappa^E_z(t_e) = z_{s(e)} t_e$ for all $e \in E^1$. Denote by γ^F, M_F, and κ^F the corresponding concepts for $\mathcal{T}C^*(F)$.*

(1) *There is an isomorphism $\mathcal{T}C^*(E) \cong \mathcal{T}C^*(F)$ that intertwines γ^E and γ^F and carries M_E to M_F if and only if $E \cong F$.*

(2) *Suppose that E and F have no sinks. Then there is an isomorphism $\mathcal{T}C^*(E) \cong \mathcal{T}C^*(F)$ that intertwines the generalized gauge actions κ^E and κ^F if and only if $E \cong F$.*

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Remark 3.3. In both parts of theorem 3.2, the additional data required beyond the gauge actions include the number of vertices in the graphs. We point out, however, that this number is already available as an isomorphism invariant of the C^*-algebra $\mathcal{T}C^*(E)$ alone: by [**8**, theorem 4.1] combined with [**14**, theorem 4.4], the Toeplitz algebra $\mathcal{T}C^*(E)$ is KK-equivalent to $\mathbb{C}^{E^0}$, and in particular $K_0(\mathcal{T}C^*(E)) \cong \mathbb{Z}E^0$. So if $\mathcal{T}C^*(E) \cong \mathcal{T}C^*(F)$, we already know that $|E^0| = |F^0|$.

The proof of the 'if' implication is easy in both cases. If $\varphi^0 : E^0 \to F^0$ and $\varphi^1 : E^1 \to F^1$ constitute an isomorphism of graphs, then the isomorphism $\rho : \mathcal{T}C^*(E) \to \mathcal{T}C^*(F)$ given by $\rho(t_e) = t_{\varphi^1(e)}$ and $\rho(q_v) = q_{\varphi^0(v)}$ carries M^E to M^F and intertwines κ^E and κ^F (and, by restriction, γ^E and γ^F), via the isomorphism $\mathbb{T}^{E^0} \cong \mathbb{T}^{F^0}$ induced by φ^0.

To prove the reverse implications we shall use the results of [**9**, **10**] on the KMS structure of the Toeplitz algebra $\mathcal{T}C^*(E)$ for the dynamics $\alpha : \mathbb{R} \to \operatorname{Aut}(\mathcal{T}C^*(E))$, where $\alpha_t = \gamma^E_{e^{it}}$ is the lift of the gauge action; that is

$$\alpha_t(q_v) = q_v \quad \text{and } \alpha_t(t_e) = e^{it} t_e \quad \text{for all } v \in E^0, e \in E^1, \quad \text{and } t \in \mathbb{R}. \tag{3.1}$$

We write

$$\operatorname{Ext}_\beta(\alpha) := \{\phi : \phi \text{ is an extremal KMS}_\beta \text{ state of } (\mathcal{T}C^*(E), \alpha)\}.$$

We first need to be able to recognize, using the data $(\mathcal{T}C^*(E), \alpha)$, when a real number β is strictly greater than the natural logarithm of the spectral radius of the adjacency matrix A_E of the directed graph E. For this, as in [**10**], we write $\sim$ for the equivalence relation on E^0 given by $v \sim w$ if both $vE^*w \neq \emptyset$ and $wE^*v \neq \emptyset$. We call the equivalence classes $C \in E^0/{\sim}$ the *strongly connected components* of E. For $C \in E^0/{\sim}$, we write A_C for the $C \times C$ submatrix of A_E, which is the adjacency matrix of the subgraph of E with vertices C and edges CE^1C.

Lemma 3.4. *Let E be a finite directed graph. If $\beta < \log \rho(A_E) < \beta'$, then* $|\operatorname{Ext}_\beta(\alpha)| < |\operatorname{Ext}_{\beta'}(\alpha)|$.

Proof. If E has no cycles, then [**9**, lemma A.1(b)] shows that $\log \rho(A_E) = -\infty$, and so the result is vacuous. So suppose that E has at least one cycle. Then $\rho(A_E) = \max\{\rho(A_C) : C \text{ is a nontrivial strongly connected component of } E\}$, as discussed at the beginning of [**10**, § 4]. Let $H_\beta := \{s(\mu) : \mu \in E^* \text{ and } r(\mu) \in \bigcup_{\log \rho(A_C) > \beta} C\}$. theorem 3.1 of [**9**] shows that $|\operatorname{Ext}_{\beta'}(\alpha)| = |E^0|$, and theorem 5.3 of [**10**] implies that $|\operatorname{Ext}_\beta(\alpha)| \leqslant |E^0 \setminus H_\beta|$. Since $\beta < \log \rho(A_E) = \max\{\log \rho(A_C) : C \in E^0/{\sim}\}$, we have $H_\beta \neq \emptyset$. Hence $|E^0 \setminus H_\beta| < |E^0|$, which proves the result. □

Lemma 3.5. *The interior in $\mathbb{R}$ of the set*

$$\big\{\beta \in (0, \infty) : |\operatorname{Ext}_{\beta'}(\alpha)| = |\operatorname{Ext}_\beta(\alpha)| \textit{ for all } \beta' \geqslant \beta\big\} \tag{3.2}$$

is the open half-line $(\log \rho(A_E), \infty)$.

Proof. Theorem 3.1 of [**9**] shows that if $\beta > \log \rho(A_E)$, then we have $|\operatorname{Ext}_{\beta'}(\alpha)| = |\operatorname{Ext}_\beta(\alpha)|$ for all $\beta' \geqslant \beta$, and lemma 3.4 shows that if $\beta < \log \rho(A_E)$, then $|\operatorname{Ext}_{\beta'}(\alpha)| > |\operatorname{Ext}_\beta(\alpha)|$ for some $\beta' > \beta$. □

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Throughout the rest of this note, we shall let $\pi : \mathcal{T}C^*(E) \to \mathcal{B}(\ell^2(E^*))$ be the canonical (faithful) path-space representation of $\mathcal{T}C^*(E)$. We will need to show that the minimal projections in $\mathcal{T}C^*(E)$ corresponding to vertices of E can be recovered using the gauge action γ^E. For each $\mu \in E^*$, we define

$$\Delta_\mu := t_\mu\Big(q_{s(\mu)} - \sum_{e \in s(\mu)E^1} t_e t_e^*\Big)t_\mu^* \in \mathcal{T}C^*(E). \tag{3.3}$$

The Δ_μ are minimal projections in the canonical copy of $\bigoplus_{v\in E^0} \mathcal{K}(\ell^2(E^*v))$ in $\mathcal{T}C^*(E)$; indeed, each $\pi(\Delta_\mu)$ is the rank-1 projection $\theta_{\delta_\mu,\delta_\mu}$ onto the span of the basis vector $\delta_\mu \in \ell^2(E^*)$.

LEMMA 3.6. *Let E be a finite directed graph. Let α be the dynamics (3.1). Let β be any real number greater than $\max\{0, \log\rho(A_E)\}$. Let ϕ be an extremal KMS$_\beta$ state of $(\mathcal{T}C^*(E), \alpha)$. Let $P_{\min}$ denote the collection of minimal projections on $\mathcal{T}C^*(E)$. There is a unique $p_\phi \in P_{\min}$ such that $\phi(p_\phi) = \max\{\phi(q) : q \in P_{\min}\}$. Moreover, with Δ_v as in (3.3), we have $p_\phi = \Delta_{v_\phi}$ for some $v_\phi \in E^0$.*

Proof. For each $v \in E^0$, let $\varepsilon^v_{(\cdot)}$ denote the measure $(\sum_{\mu\in E^*v} e^{-\beta|\mu|})^{-1}\delta_v(\cdot)$ on E^0. Since $\beta > \log\rho(A_E)$, [9, theorem 3.1] implies that there is a unique $v_\phi \in E^0$ such that ϕ satisfies

$$\phi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta|\mu|}\varepsilon^{v_\phi}_{s(\mu)}, \quad \text{for all } \mu, \nu \in E^*.$$

By the proof of [9, theorem 3.1(b)], we know that ϕ satisfies

$$\phi(a) = \sum_{\mu \in E^* v_\phi} e^{-\beta|\mu|}(\pi(a)\delta_\mu|\delta_\mu)\varepsilon^{v_\phi}_{s(\mu)} \quad \text{for all } a \in \mathcal{T}C^*(E).$$

We have

$$\phi(\Delta_{v_\phi}) = \varepsilon^{v_\phi}_{v_\phi} = \left(\sum\nolimits_{\mu\in E^*v_\phi} e^{-\beta|\mu|}\right)^{-1}.$$

Fix $q \in P_{\min} \setminus \Delta_{v_\phi}$. It suffices to show that $\phi(q) < \phi(\Delta_{v_\phi})$. Let $\pi_{v_\phi} : \mathcal{T}C^*(E) \to \mathcal{B}(\ell^2(E^*v_\phi))$ be the direct summand in π corresponding to v_ϕ. Then ϕ factors through π_{v_ϕ}. If $\phi(q) = 0$ then we certainly have $\phi(q) < \phi(\Delta_{v_\phi})$, so suppose that $\phi(q) \neq 0$. Then $\pi_{v_\phi}(q) \neq 0$, and so $\pi_{v_\phi}(q)$ is a minimal projection in $\pi_{v_\phi}(\mathcal{T}C^*(E))$. Since $\pi_{v_\phi}(\mathcal{T}C^*(E))$ contains all of $\mathcal{K}(\ell^2(E^*v_\phi))$, it follows that $\pi(q)$ is the rank-one projection $\theta_{\xi,\xi}$ corresponding to a unit vector $\xi \in \ell^2(E^*v_\phi)$. Hence

$$\begin{aligned}\phi(q) &= \sum_{\mu\in E^*v_\phi} e^{-\beta|\mu|}(\pi(q)\delta_\mu|\delta_\mu)\varepsilon^{v_\phi}_{s(\mu)} = \sum_{\mu\in E^*v_\phi} e^{-\beta|\mu|}(\theta_{\xi,\xi}(\delta_\mu)|\delta_\mu)\varepsilon^{v_\phi}_{s(\mu)} \\ &= \sum_{\mu\in E^*v_\phi} e^{-\beta|\mu|}((\xi \mid \delta_\mu)\xi|\delta_\mu)\varepsilon^{v_\phi}_{s(\mu)}.\end{aligned}$$

Since $\beta > 0$, we have $e^{-\beta|\mu|} = 1$ when $\mu = v_\phi$ and $e^{-\beta|\mu|} \leqslant e^{-\beta}$ when $\mu \neq v_\phi$, and so we deduce that

$$\phi(q) \leqslant \left(|\xi_{v_\phi}|^2 + e^{-\beta}\sum_{\mu\neq v_\phi}|\xi_\mu|^2\right)\varepsilon^{v_\phi}_{s(\mu)}.$$

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Since $q \neq \Delta_{v_\phi}$, we have $\xi \neq \delta_{v_\phi}$, and so $|\xi_{v_\phi}| < 1$. We have $\sum |\xi_\mu|^2 = \|\xi\|^2 = 1$, and so $e^{-\beta} \sum_{\mu \neq v_\phi} |\xi_\mu|^2 = e^{-\beta}(1 - |\xi_{v_\phi}|^2) < 1 - |\xi_{v_\phi}|^2$. Hence $\phi(q) < \varepsilon^v_{s(\mu)} = \phi(\Delta_{v_\phi})$ as claimed. □

Lemma 3.6 allows us to recover the projections Δ_v of $\mathcal{T}C^*(E)$ from $\mathcal{T}C^*(E)$ together with its simplex of KMS states. Since the KMS states are intrinsic to the pair $(\mathcal{T}C^*(E), \gamma^E)$, it follows that we can recover the Δ_v from the Toeplitz algebra and its gauge action. We show next how to recover the cardinalities of the sets $E^n v$ as well. We start with some notation.

Notation 3.7. *For each $\mu, \nu \in E^*$ with $s(\mu) = s(\nu)$ we define $\Theta_{\mu,\nu} := t_\mu \Delta_{s(\mu)} t_\nu^*$. Recall that the path-space representation π carries each $\Theta_{\mu,\nu}$ to the canonical matrix unit $\theta_{\delta_\mu,\delta_\nu}$. Recall also that for $n \in \mathbb{Z}$, the n^{th} spectral subspace $\mathcal{T}C^*(E)_n$ of $\mathcal{T}C^*(E)$ with respect to γ is*

$$\mathcal{T}C^*(E)_n := \{a \in \mathcal{T}C^*(E) : \gamma^E_z(a) = z^n a \ \ \textit{for all } z \in \mathbb{T}\}.$$

Lemma 3.8. *Let E be a finite directed graph. For $n \geqslant 0$, we have $\mathcal{T}C^*(E)_n \Delta_v = \operatorname{span}\{\Theta_{\mu,v} : \mu \in E^n v\}$; in particular, $|E^n v| = \dim(\mathcal{T}C^*(E)_n \Delta_v)$.*

Proof. It is standard that $\mathcal{T}C^*(E)_n = \overline{\operatorname{span}}\{t_\mu t_\nu^* : \mu, \nu \in E^*, |\mu| - |\nu| = n, s(\mu) = s(\nu)\}$. The path-space representation π carries Δ_v to $\theta_{\delta_v,\delta_v}$, and carries each $t_\mu t_\nu^*$ to the strong-operator sum $\sum_{\lambda \in s(\nu)E^*} \theta_{\delta_{\mu\lambda},\delta_{\nu\lambda}}$. The latter is nonzero at δ_v only if $v = \nu\lambda$ for some $\lambda \in s(\mu)E^*$, which forces $\nu = v = \lambda = s(\mu)$. So if $a \in \mathcal{T}C^*(E)_n$ and $a\Delta_v \neq 0$, then $a\Delta_v \in \operatorname{span}\{t_\mu \Delta_v : \mu \in E^n v\} = \operatorname{span}\{\Theta_{\mu,v} : \mu \in E^n v\}$. Since each $\Theta_{\mu,v} = \Theta_{\mu,v}\Delta_v$, the reverse containment is clear. □

We can now prove the first part of the main theorem.

Proof of theorem 3.2 (1). By lemma 3.5 we may determine the value of $\log \rho(A_E)$ from the KMS state structure of α, and then choose $\beta > \log \rho(A_E)$. For $\phi \in \operatorname{Ext}_\beta(\alpha)$, lemma 3.6 yields a unique minimal projection p_ϕ of $\mathcal{T}C^*(E)$ such that $\phi(p_\phi) = \max\{\phi(q) : q \text{ is a minimal projection of } \mathcal{T}C^*(E)\}$, and we have $p_\phi = \Delta_{v_\phi}$ for some $v_\phi \in E^0$. We have $q_{v_\phi} \geqslant \Delta_{v_\phi}$, and then for $w \neq v_\phi$ in E^0 we have $q_w \Delta_{v_\phi} = q_w q_{v_\phi} \Delta_{v_\phi} = 0$. So there is a unique minimal projection $P_\phi \in M_E$ that dominates p_ϕ, namely $P_\phi = q_{v_\phi}$.

For $\phi, \psi \in \operatorname{Ext}_\beta(\alpha)$, let

$$N(\phi, \psi) := \dim P_\phi \mathcal{T}C^*(E)_1 p_\psi.$$

Let $\widetilde{E}$ be the directed graph with vertices $\operatorname{Ext}_\beta(\alpha)$ and with $|\phi \widetilde{E}^1 \psi| = N(\phi, \psi)$ for all $\phi, \psi \in \operatorname{Ext}_\beta(\alpha)$. By construction, the graph $\widetilde{E}$ is an isomorphism invariant of the triple $(\mathcal{T}C^*(E), \gamma^E, M_E)$. We claim that $\widetilde{E} \cong E$.

We know from lemma 3.6 that $\phi \mapsto v_\phi$ from $\widetilde{E}^0$ to E^0 is a bijection, so it suffices to show that $N(\phi, \psi) = |v_\phi E^1 v_\psi|$ for all ϕ, ψ. Lemma 3.8 shows that $\mathcal{T}C^*(E)_1 p_\psi = \operatorname{span}\{\Theta_{e,v_\psi} : e \in E^1 v_\psi\}$. Since for each $e \in E^1 v_\psi$ we have $\Theta_{e,v_\psi}\Theta^*_{e,v_\psi} = \Theta_{e,e} \leqslant$

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$q_{r(e)}$, it follows that $P_\phi \mathcal{T}C^*(E)_1 p_\psi = \operatorname{span}\{\Theta_{e,v_\psi} : e \in v_\phi E^1 v_\psi\}$. Hence

$$|v_\phi E^1 v_\psi| = \dim P_\phi \mathcal{T}C^*(E)_1 p_\psi = N(\phi,\psi).$$

So $\widetilde{E} \cong E$, as claimed. Applying the process of the preceding three paragraphs to the system $(\mathcal{T}C^*(F), \gamma^F, M_F)$ we obtain a graph $\widetilde{F} \cong F$. Since the systems $(\mathcal{T}C^*(E), \gamma^E, M_E)$ and $(\mathcal{T}C^*(F), \gamma^F, M_F)$ are isomorphic, we see that $\widetilde{E} \cong \widetilde{F}$, and therefore $E \cong F$. □

To prove statement (2) of theorem 3.2 we first show how to determine which coordinate of the generalized gauge action κ^E corresponds to the minimal projection p_ϕ obtained from $\phi \in \operatorname{Ext}_\beta(\alpha)$ as in lemma 3.6.

Lemma 3.9. *Let E be a finite directed graph with no sinks, and let κ^E and α be as in definition 3.1 and (3.1). Fix $\beta > \ln\rho(A_E)$ and let ϕ be an extremal KMS$_\beta$ state of $(\mathcal{T}C^*(E), \alpha)$. Let p_ϕ be the projection of lemma 3.6. Then the vertex v_ϕ such that $p_\phi = \Delta_{v_\phi}$ is the unique vertex such that $\kappa^E_z(a) = z_{v_\phi} a$ for all $a \in \mathcal{T}C^*(E)_1 p_\phi$ and $z \in \mathbb{T}^{E^0}$.*

Proof. For $w \in E^0$, lemma 3.8 gives $\mathcal{T}C^*(E)_1\Delta_w = \operatorname{span}\{\Theta_{e,w} : e \in E^1 w\} = \operatorname{span}\{t_e\Delta_w : e \in E^1\}$, and so it follows from the definition of κ^E that $\kappa^E_z(a) = z_w a$ for all $a \in \mathcal{T}C^*(E)_1\Delta_w$ and $z \in \mathbb{T}^{E^0}$. Since E has no sinks, each $\operatorname{span}\{\Theta_{e,w} : e \in E^1 w\}$ is nontrivial, which proves uniqueness. □

Proof of theorem 3.2 (2). First observe that the dynamics α of $\mathcal{T}C^*(E)$ defined in (3.1) is determined by κ^E via $\alpha_t = \kappa^E_{(e^{it},\dots,e^{it})}$. Using lemma 3.5 as in the proof of theorem 3.2 (1), fix $\beta > \ln\rho(A_E)$. For each extremal KMS$_\beta$ state $\phi \in \operatorname{Ext}_\beta(\alpha)$, lemma 3.6 yields a unique minimal projection p_ϕ of $\mathcal{T}C^*(E)$ such that

$$\phi(p_\phi) = \max\{\phi(q) : q \text{ is a minimal projection of } \mathcal{T}C^*(E)\}.$$

Lemma 3.9 shows that $p_\phi = \Delta_{v_\phi}$ where $v_\phi \in E^0$ is the unique vertex such that $\kappa^E_z(a) = z_{v_\phi} a$ for all $a \in \mathcal{T}C^*(E)_1 p_\phi$.

Suppose that $\phi, \psi \in \operatorname{Ext}_\beta(\alpha)$ are distinct. For $z \in \mathbb{T}$ let $\omega(\phi,\psi,z) \in \mathbb{T}^{E^0}$ be the element such that

$$\omega(\phi,\psi,z)_u = \begin{cases} \overline{z} & \text{if } u = v_\phi \\ z & \text{if } u = v_\psi \\ 1 & \text{otherwise.} \end{cases}$$

Define an action $\gamma^{\phi,\psi} : \mathbb{T} \to \operatorname{Aut}(\mathcal{T}C^*(E))$ by $\gamma^{\phi,\psi}_z = \kappa^E_{\omega(\phi,\psi,z)}$. Note that this action fixes the partial isometry t_{ef} associated to $ef \in E^2 v_\psi$ if and only if $r(f) = s(e) = v_\phi$. Combining the fixed point algebra $\mathcal{T}C^*(E)^{\gamma^{\phi,\psi}}$ of $\gamma^{\phi,\psi}$ with the second spectral

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subspace of the gauge action γ^E, we define

$$N(\phi,\psi) := \dim(\mathcal{T}C^*(E)^{\gamma^{\phi,\psi}} \cap \mathcal{T}C^*(E)_2 p_\psi)/\dim(\mathcal{T}C^*(E)_1 p_\phi).$$

We extend the definition of N to the case $\phi = \psi \in \mathrm{Ext}_\beta(\alpha)$ by setting

$$N(\psi,\psi) := \dim(\mathcal{T}C^*(E)_1 p_\psi) - \sum_{\phi\neq\psi} N(\phi,\psi).$$

We claim that $N(\phi,\psi) \in \mathbb{N}$ for all $\phi,\psi \in \mathrm{Ext}_\beta(\alpha)$, and that E is isomorphic to the directed graph $\widetilde{E}$ with vertices $\widetilde{E}^0 := \mathrm{Ext}_\beta(\alpha)$, and such that $|\phi\widetilde{E}^1\psi| = N(\phi,\psi)$ for all $\phi,\psi \in \mathrm{Ext}_\beta(\alpha)$. Since we already have a bijection $\phi \mapsto v_\phi$ from $\widetilde{E}^0$ to E^0, to prove the claim, we just have to show that $N(\phi,\psi) = |v_\phi E^1 v_\psi|$ for all ϕ,ψ.

For this, fix $\phi,\psi \in \mathrm{Ext}_\beta(\alpha)$ and let $ef \in E^2$. Then

$$\gamma_z^{\phi,\psi}(t_{ef}p_\psi) = \begin{cases} t_{ef}p_\psi & \text{if } f \in v_\phi E^1 v_\psi \\ z^2 t_{ef}p_\psi & \text{if } f \in v_\psi E^1 v_\psi \\ z t_{ef}p_\psi & \text{if } f \in E^1 v_\psi \setminus (v_\phi E^1 v_\psi \cup v_\psi E^1 v_\psi) \\ 0 & \text{if } f \notin E^1 v_\psi. \end{cases}$$

So lemma 3.8 implies that

$$\mathcal{T}C^*(E)^{\gamma^{\phi,\psi}} \cap \mathcal{T}C^*(E)_2 p_\psi = \mathrm{span}\{\Theta_{ef,v_\psi} : ef \in E^1 v_\phi\ E^1 v_\psi\}.$$

Hence, $|E^1 v_\phi| \cdot |v_\phi E^1 v_\psi| = |E^1 v_\phi E^1 v_\psi| = \dim(\mathcal{T}C^*(E)^{\gamma^{\phi,\psi}} \cap \mathcal{T}C^*(E)_2 p_\psi)$. Using lemma 3.8, we see that $|E^1 v_\phi| = \dim(\mathcal{T}C^*(E)_1 p_\phi)$. Since, by hypothesis, E has no sinks, we have $|E^1 v_\phi| \neq 0$, and so we deduce that

$$|v_\phi E^1 v_\psi| = \dim(\mathcal{T}C^*(E)^{\gamma^{\phi,\psi}} \cap \mathcal{T}C^*(E)_2 p_\psi)/\dim(\mathcal{T}C^*(E)_1 p_\phi) = N(\phi,\psi).$$

Now for each $\psi \in \mathrm{Ext}_\beta(\alpha)$, we see that

$$\begin{aligned} |v_\psi E^1 v_\psi| &= |E^1 v_\psi| - \sum_{\phi\neq\psi} |v_\phi E^1 v_\psi| \\ &= \dim(\mathcal{T}C^*(E)_1 p_\psi) \\ &\quad - \sum_{\phi\neq\psi} \dim\left(\mathcal{T}C^*(E)^{\gamma^{\phi,\psi}} \cap \mathcal{T}C^*(E)_2 p_\psi\right)/\dim(\mathcal{T}C^*(E)_1 p_\phi) \\ &= N(\psi,\psi). \end{aligned}$$

This shows that $E \cong \widetilde{E}$ and concludes the proof of the claim.

To finish the proof of the 'only if' assertion in theorem 3.2 (2) assume now there exist an isomorphism $\rho : \mathcal{T}C^*(E) \to \mathcal{T}C^*(F)$ and a bijection $\varphi : E^0 \to F^0$ intertwining the generalized gauge actions κ^E and κ^F. Then $\varphi^* : \mathbb{T}^{E^0} \to \mathbb{T}^{F^0}$ maps constant functions to constant functions. That is, φ^* respects the diagonal embeddings of $\mathbb{T}$. Hence ρ intertwines the gauge actions γ^E and γ^F, and also the dynamics α^E and α^F obtained from them on setting $z = e^{it}$. Passing to extremal KMS$_\beta$ states, we get a bijection $\widetilde{E}^0 := \mathrm{Ext}_\beta(\alpha^E) \cong \mathrm{Ext}_\beta(\alpha^F) =: \widetilde{F}^0$ in which $\phi \mapsto \phi' := \phi \circ \rho^{-1}$. The isomorphism ρ also intertwines the action $\gamma^{\phi,\psi} : \mathbb{T} \to \mathrm{Aut}(\mathcal{T}C^*(E))$ with the action $\gamma^{\phi',\psi'} : \mathbb{T} \to \mathrm{Aut}(\mathcal{T}C^*(F))$ and thus $N(\phi, \psi) = N(\phi', \psi')$. Thus, much like in the final paragraph of the proof of the 'only if' assertion in theorem 3.2 (1), we conclude that $\widetilde{E} \cong \widetilde{F}$ and hence that $E \cong F$. □

Example 3.10. As compared to statement (1), statement (2) of our main theorem has the additional hypothesis that E and F have no sinks. Here we present an example—first shown to the fourth author in the context of Cohn path algebras by Gene Abrams, and then independently by Søren Eilers—that shows that the additional hypothesis in statement (2) is necessary. Consider the graphs

There is an isomorphism $\mathcal{T}C^*(E) \to \mathcal{T}C^*(F)$ that carries q_v to $q_v - t_e t_e^*$, carries q_u to $q_u + t_e t_e^*$ and takes each of the remaining generators of $\mathcal{T}C^*(E)$ to the generator of $\mathcal{T}C^*(F)$ with the same label. This isomorphism intertwines κ^E and κ^F because in both graphs every edge has source w. It does not, however, carry M_E to M_F since, for example, $q_v - t_e t_e^* \notin M_F$.

Acknowledgements

This research was supported by Australian Research Council grant DP180100595. We are grateful to Søren Eilers for very helpful conversations that led to example 2.1.

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