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Non-existence of conformally flat real hypersurfaces in both the complex quadric and the complex hyperbolic quadric

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Abstract. In this paper, by applying for a new approach of the so-called Tsinghua principle, we prove the nonexistence of locally conformally flat real hypersurfaces in both the m-dimensional complex quadric Q^m and the complex hyperbolic quadric Q^{m*} for $m \geq 3$.

1 Introduction

In this paper, we study real hypersurfaces of the complex quadric Q^m and the complex hyperbolic quadric Q^{m*} for $m \geq 3$. Due to their different background, we consider these two ambient spaces separately.

First, recall that the complex quadric of dimension m is defined by $Q^m := \{[(z_0, z_1, \ldots, z_{m+1})] \in \mathbb{C}P^{m+1} : z_0^2 + z_1^2 + \cdots + z_{m+1}^2 = 0\}$, where $\mathbb{C}P^{m+1}$ is the $(m+1)$-dimensional complex projective space equipped with the Fubini–Study metric g of constant holomorphic sectional curvature 4, and $z_0, \ldots, z_{m+1}$ are the homogeneous coordinates on $\mathbb{C}P^{m+1}$. Being as a complex hypersurface of $\mathbb{C}P^{m+1}$ with the induced metric, denoted still by g, Q^m is a compact and Einstein Riemannian symmetric space (see Reckziegel [22], Smyth [23], and Berndt and Suh [3]). Besides the Kähler structure J that is induced from that of $\mathbb{C}P^{m+1}$, Q^m carries another important geometric structure A, being called the *almost product structure* in [16], and they are related by $AJ = -JA$ (see [3, 22]). The complex quadric Q^m is an important Riemannian manifold, so the study of its submanifolds is significant, which has attracted many geometers. To see the results, among others, we refer to Smyth [23], Reckziegel [22], Berndt and Suh [3], and the many literatures citing them, which could be looked up from MathSciNet.

Next, as the noncompact dual of Q^m, the m-dimensional complex hyperbolic quadric Q^{m*} is a simply connected Riemannian symmetric space whose curvature tensor is the negative of that of Q^m. It is known that Q^{m*} cannot be realized as a homogeneous complex hypersurface of the complex hyperbolic space $\mathbb{C}H^{m+1}$ (see [13, 23]), whereas Montiel and Romero [19] proved that it can be isometrically immersed

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indeed into the indefinite complex hyperbolic space $\mathbb{C}H_1^{m+1}(-c)$ ($c > 0$) as a complex Einstein hypersurface. Accordingly, similar to the complex quadric Q^m, the complex hyperbolic quadric Q^{m*} also admits two important geometric structures, namely an almost product structure (also called real structure) and the Kähler structure, which are still denoted by A and J, and they also satisfy $AJ = -JA$. These two structures satisfy the same properties of those two in Q^m. For more details about the complex hyperbolic quadric Q^{m*}, we refer the readers to [13, 19, 25, 27, 28].

Now, we can state the main result of this paper as the following nonexistence theorem.

Theorem 1.1 *There do not exist any locally conformally flat real hypersurfaces in both the complex quadric Q^m and the complex hyperbolic quadric Q^{m*} with $m \geq 3$.*

Recall that an n-dimensional Riemannian manifold (M^n, g) is said to be locally conformally flat if it admits a coordinate covering $\{(U_\alpha, x_\alpha^i); \alpha \in \Lambda\}$ such that $g(\frac{\partial}{\partial x_\alpha^i}, \frac{\partial}{\partial x_\alpha^j}) = e^{f_\alpha}\delta_{ij}$, for $1 \leq i, j \leq n$, and each $\alpha \in \Lambda$, where f_α is a smooth function defined on U_α. For $n \geq 4$, it is well known that (M^n, g) is locally conformally flat if and only if the Weyl curvature tensor of (M^n, g) vanishes identically. The study of (locally) conformally flat manifolds is an important subject in Riemannian geometry, and particularly from the viewpoint of submanifold theory. Our motivation of proving Theorem 1.1 lies in the latter. Historically, there are many researches on the conformally flat hypersurfaces in Riemannian manifolds. When the ambient space is the space forms, such studies are given, e.g., in [4, 9, 20, 21] and many latter references citing them. When the ambient space is the complex space form $M^n(c)$ of constant holomorphic sectional curvature $c \neq 0$, based on a series studies of Kon [14], Cecil and Ryan [5], and Montiel [18], Ki et al. [11] proved that it admits no real hypersurfaces with harmonic Weyl tensor for $n \geq 3$. It follows that there are no conformally flat real hypersurfaces in such a complex space form $M^n(c)$.

Related to Theorem 1.1, it is worth mentioning that there are many researches on the real hypersurfaces of the complex quadric Q^m very recently. For instance, under additional conditions, Suh [24] studied real and Hopf hypersurfaces of Q^m with harmonic curvature, and also, under some special conditions, Suh [26] investigated real and Hopf hypersurfaces of Q^m with commuting Ricci tensor. On the other hand, without the Hopf condition, Loo [17] studied the pseudo-Einstein hypersurfaces of Q^m and obtained meaningful results in some special cases.

To prove Theorem 1.1, we note that for the complex quadric Q^m and the complex hyperbolic quadric Q^{m*}, the curvature tensors are much more complicated than that of the complex space forms and the Gauss equations become very difficult to solve directly. This makes the problem of investigating conformally flat real hypersurfaces of Q^m and Q^{m*} quite different from that of the nonflat complex space forms. In order to get out of this difficulty, our proof of Theorem 1.1 makes use of a new approach, the so-called *Tsinghua principle* due to H. Li, L. Vrancken, and X. Wang (see a statement in Section 3 of [1]). Roughly speaking, this remarkable principle helps us to use the Codazzi equation and the Ricci identity in a new way to obtain some nice linear equations involving the components of the second fundamental form. For readers' better understanding, we will elaborate on the Tsinghua principle in Section 3.

Remark 1.1 The complex quadric Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature, whereas the complex hyperbolic quadric Q^{2*} is isometric to the Riemannian product of two hyperbolic plane H^2 with constant negative curvature (cf. [3, 28]). Therefore, contrary to cases Q^m and Q^{m*} for $m \geq 3$, in both Q^2 and Q^{2*}, there do exist conformally flat hypersurfaces. Indeed, for any curve $\Gamma \hookrightarrow \mathbb{S}^2(1)$ (resp. $\tilde{\Gamma} \hookrightarrow H^2(-1)$), $\Gamma \times \mathbb{S}^2(1)$ (resp. $\tilde{\Gamma} \times H^2(-1)$) is a conformally flat hypersurface of $\mathbb{S}^2(1) \times \mathbb{S}^2(1)$ (resp. $H^2(-1) \times H^2(-1)$). On the other hand, to our knowledge, the problem of classifying conformally flat hypersurfaces in either Q^2 or Q^{2*} is still open.

This paper is organized as follows. In Section 2, we review some basic materials of the complex quadric Q^m, and, particularly, we derive several fundamental equations for real hypersurfaces of Q^m. In Section 3, as crucial steps toward the proof of Theorem 1.1, we introduce the Tsinghua principle by proving two lemmas. In Section 4, we complete the proof of Theorem 1.1 for Q^m. Finally, in Section 5, thanks to the similarities between Q^m and Q^{m*}, we give an outline proof of Theorem 1.1 for Q^{m*}.

2 Preliminaries

In this section, we shall introduce the basic materials only about the complex quadric Q^m and its real hypersurfaces. For those about the complex hyperbolic quadric Q^{m*} that are completely similar, we refer to [13, 25, 27, 28] for details.

2.1 The complex quadric

As described in the introduction section, the complex quadric Q^m is equipped with a canonical Kähler structure, denoted by $\{J, g\}$, induced from that of $\mathbb{C}P^{m+1}$. For the purpose of this paper, we shall assume $m \geq 3$ in the sequel.

For a nonzero vector $z \in \mathbb{C}^{m+2}$, we denote by $[z]$ the complex span of z, that is, $[z] = \{\lambda z | \lambda \in \mathbb{C}^*\}$. For $[z] \in Q^m$, the tangent space $T_{[z]}Q^m$ can be identified canonically with $\mathbb{C}^{m+2} \ominus ([z] \oplus [\bar{z}])$ in $\mathbb{C}^{m+2}$. Note that $\zeta = -\bar{z}$ is a unit normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point $[z]$, and the shape operator A_ζ of Q^m in $\mathbb{C}P^{m+1}$ with respect to ζ is given by $A_\zeta w = \bar{w}$ for all $w \in T_{[z]}Q^m$. Thus, restricted to $T_{[z]}Q^m$, A_ζ is the complex conjugation. Moreover, acting on the complex vector space $T_{[z]}Q^m$, the shape operator A_ζ is an anticommuting involution such that $A_\zeta^2 = Id$ and $AJ = -JA$. Hence, we have $T_{[z]}Q^m = V(A_\zeta) \oplus JV(A_\zeta)$, where $V(A_\zeta) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the $(+1)$-eigenspace and $JV(A_\zeta) = \mathbf{i}\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (-1)-eigenspace of A_ζ, respectively. Because the normal space $\nu_{[z]}Q^m$ of Q^m in $\mathbb{C}P^{m+1}$ at $[z]$ is a complex subspace of $T_{[z]}\mathbb{C}P^{m+1}$ of complex dimension one, every unit normal vector in $\nu_{[z]}Q^m$ can be written as $\lambda\bar{z}$ with some $\lambda \in S^1 \subset \mathbb{C}$, and it holds that $V(A_{\lambda\bar{z}}) = \lambda V(A_{\bar{z}})$. We denote by $\mathfrak{A}$ the set of all shape operators of $Q^m \hookrightarrow \mathbb{C}P^{m+1}$ associated with unit normal vector fields. Then, $\mathfrak{A}$ is an S^1-sub-bundle of the endomorphism bundle $\mathrm{End}(TQ^m)$, consisting of complex conjugations on the tangent spaces of Q^m. In summary, we have

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Lemma 2.1 (cf. [22, 23]) *For each $A \in \mathfrak{A}$, it holds that*

$$A^2 = Id, \quad g(AX,Y) = g(X,AY), \quad AJ = -JA, \quad \forall X, Y \in TQ^m. \tag{2.1}$$

The sub-bundle $\mathfrak{A}$ is commonly called consisting of complex conjugations (cf. [22]). On the other hand, Lemma 2.1 implies, in particular, that $\mathfrak{A}$ is a family of *almost product structures* on Q^m (cf. [16]), and we will use the latter term in this paper. Furthermore, because the second fundamental form of the embedding $Q^m \hookrightarrow \mathbb{C}P^{m+1}$ is parallel, $\mathfrak{A}$ is contained in a parallel sub-bundle of End(TQ^m). It follows that, for each almost product structure $A \in \mathfrak{A}$, there exists a one-form q on Q^m such that it holds the relation (see [23])

$$(\bar{\nabla}_X A)Y = q(X)JAY, \quad \forall X, Y \in TQ^m, \tag{2.2}$$

where $\bar{\nabla}$ denotes the Levi–Civita connection of Q^m.

The Gauss equation for the complex hypersurface $Q^m \hookrightarrow \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be expressed in terms of the Riemannian metric g, the complex structure J, and a generic $A \in \mathfrak{A}$ as follows (cf. [22]):

$$\begin{aligned}\bar{R}(X,Y)Z =& g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY \\ & - 2g(JX,Y)JZ + g(AY,Z)AX - g(AX,Z)AY \\ & + g(JAY,Z)JAX - g(JAX,Z)JAY,\end{aligned} \tag{2.3}$$

where it should be noted that $\bar{R}$ is independent of the special choice of $A \in \mathfrak{A}$.

According to Reckziegel [22], a real 2-dimensional linear subspace σ of $T_{[z]}Q^m$ is called a 2-flat if the curvature tensor $\bar{R}$ of Q^m vanishes identically on σ. A nonzero tangent vector $W \in T_{[z]}Q^m$ is called $\mathfrak{A}$-singular if it is contained in more than one 2-flat. There are two types of $\mathfrak{A}$-singular tangent vectors for Q^m, namely, $\mathfrak{A}$-principal and $\mathfrak{A}$-isotropic tangent vectors, described as follows (cf. [3, 22]):

1. If there exists an almost product structure $A \in \mathfrak{A}_{[z]}$ such that $W \in V(A)$, then W is $\mathfrak{A}$-singular. Such $W \in T_{[z]}Q^m$ is called an $\mathfrak{A}$-principal tangent vector.
2. If there exists an almost product structure $A \in \mathfrak{A}_{[z]}$ and two orthonormal vectors $X, Y \in V(A)$ such that $\frac{W}{\|W\|} = \frac{X+JY}{\sqrt{2}}$, then W is $\mathfrak{A}$-singular. In this case, we have $g(AW, W) = 0$ and $W \in T_{[z]}Q^m$ is called an $\mathfrak{A}$-isotropic tangent vector.

Further results about the complex quadric are referred to [3, 12, 22].

2.2 Hypersurfaces of the complex quadric

We begin with introducing the basic formulas. Let M be a hypersurface of Q^m and N its unit normal vector field. For a generic almost product structure $A \in \mathfrak{A}$ and an arbitrary tangent vector field X of M, we have the decomposition

$$JX = \phi X + \eta(X)N, \quad AX = TX + \mu(X)N, \tag{2.4}$$

where ϕX (resp. TX) and $\eta(X)N$ (resp. $\mu(X)N$) are the tangent and normal parts of JX (resp. AX), respectively. Here, ϕ and T are tensor fields of type $(1,1)$, and η and μ

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are 1-forms over M. Then, by definition, we have the following relations:

$$\begin{cases} \eta(X) = g(X, \xi), \ \ \eta(\phi X) = 0, \ \ \phi^2 X = -X + \eta(X)\xi, \\ g(\phi X, Y) = -g(X, \phi Y), \ \ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \\ g(TX, Y) = g(X, TY), \ \ \mu(X) = g(AX, N), \end{cases} \tag{2.5}$$

where $\xi := -JN$ is called the *Reeb vector field* of M. The equations in (2.5) show that $\{\phi, \xi, \eta\}$ determines an *almost contact structure* on M.

Let ∇ be the induced connection on M with R its Riemannian curvature tensor. The formulas of Gauss and Weingarten state that

$$\bar{\nabla}_X Y = \nabla_X Y + g(SX, Y)N, \quad \bar{\nabla}_X N = -SX, \ \ \forall\, X, Y \in TM, \tag{2.6}$$

where S is the shape operator of $M \hookrightarrow Q^m$.

Next, we calculate the covariant derivatives of the tensors ϕ, η, T, and μ. These basic equations are necessary for our proof of Theorem 1.1.

Lemma 2.2 *The covariant derivatives of the tensors ϕ, T, μ, and η are given by:*

$$\nabla_X \xi = \phi SX, \tag{2.7}$$

$$(\nabla_X \phi)Y = -g(SX, Y)\xi + \eta(Y)SX, \tag{2.8}$$

$$(\nabla_X T)Y = q(X)[\phi TY - \mu(Y)\xi] + g(SX, Y)[-\phi T\xi + \mu(\xi)\xi] + \mu(Y)SX, \tag{2.9}$$

$$(\nabla_X \mu)Y = q(X)g(TY, \xi) - g(SX, TY) - g(SX, Y)g(T\xi, \xi), \tag{2.10}$$

$$(\nabla_X \eta)Y = -g(SX, \phi Y). \tag{2.11}$$

Proof These equations are derived by direct calculations. Specifically, we have the following derivation by using the basic equations in (2.2), (2.5), and (2.6):

$$\begin{aligned} \nabla_X \xi &= -\bar{\nabla}_X JN - g(SX, \xi)N = -J\bar{\nabla}_X N - g(SX, \xi)N \\ &= JSX - g(SX, \xi)N = \phi SX, \end{aligned}$$

$$\begin{aligned} (\nabla_X \phi)Y &= \nabla_X \phi Y - \phi \nabla_X Y = \bar{\nabla}_X \phi Y - g(SX, \phi Y)N - \phi \nabla_X Y \\ &= [\bar{\nabla}_X (JY - \eta(Y)N)]^\top - \phi \nabla_X Y \\ &= [(\bar{\nabla}_X J)Y + J\bar{\nabla}_X Y]^\top - \eta(Y)\bar{\nabla}_X N - \phi \nabla_X Y \\ &= -g(SX, Y)\xi + \eta(Y)SX, \end{aligned}$$

$$\begin{aligned} (\nabla_X T)Y &= \nabla_X TY - T\nabla_X Y = \bar{\nabla}_X TY - g(SX, TY)N - T\nabla_X Y \\ &= [\bar{\nabla}_X (AY - \mu(Y)N)]^\top - T\nabla_X Y \\ &= [(\bar{\nabla}_X A)Y + A\bar{\nabla}_X Y]^\top - \mu(Y)\bar{\nabla}_X N - T\nabla_X Y \\ &= [q(X)JAY + g(SX, Y)AN]^\top + \mu(Y)SX \\ &= q(X)[\phi TY - \mu(Y)\xi] + g(SX, Y)[-\phi T\xi + \mu(\xi)\xi] + \mu(Y)SX, \end{aligned}$$

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$$\begin{aligned}(\nabla_X\mu)Y &= X(\mu(Y)) - \mu(\nabla_X Y)\\ &= g(\bar\nabla_X AY, N) + g(AY, \bar\nabla_X N) - g(A\bar\nabla_X Y - g(SX,Y)AN, N)\\ &= g((\bar\nabla_X A)Y, N) - g(SX, TY) + g(SX,Y)g(AN,N)\\ &= g(q(X)JAY, N) - g(SX,TY) + g(SX,Y)g(AN,N)\\ &= q(X)g(TY,\xi) - g(SX,TY) - g(SX,Y)g(T\xi,\xi),\end{aligned}$$

$$(\nabla_X\eta)Y = X(\eta(Y)) - \eta(\nabla_X Y) = g(Y, \nabla_X\xi) = -g(SX, \phi Y),$$

where $\cdot^{\top}$ denotes the tangential part. ∎

Third, by using the decomposition of A and the expression of the curvature tensor of Q^m, we have the following Gauss and Codazzi equations of M:

$$\begin{aligned}R(X,Y)Z =& g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y\\ &- 2g(\phi X,Y)\phi Z + g(AY,Z)(AX)^{\top} - g(AX,Z)(AY)^{\top}\\ &+ g(JAY,Z)(JAX)^{\top} - g(JAX,Z)(JAY)^{\top}\\ &+ g(SZ,Y)SX - g(SZ,X)SY,\end{aligned}\tag{2.12}$$

$$\begin{aligned}g\big((\nabla_X S)Y - (\nabla_Y S)X, Z\big) =& \eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z)\\ &- 2g(\phi X,Y)\eta(Z) + \mu(X)g(TY,Z) - \mu(Y)g(TX,Z)\\ &- g(TX,\xi)g(TY,\phi Z) - g(TX,\xi)\mu(Y)\eta(Z)\\ &+ g(TY,\xi)g(TX,\phi Z) + g(TY,\xi)\mu(X)\eta(Z).\end{aligned}\tag{2.13}$$

Contracting Y and Z in (2.12), we get the Ricci tensor of M (cf. (4.1) of [24]):

$$\begin{aligned}\mathrm{Ric}\,(X,Y) &= (2m-1)g(X,Y) - 3\eta(X)\eta(Y) - g(AN,N)g(AX,Y)\\ &\quad + g(AX,N)g(AN,Y) - g(A\xi,N)g(JAX,Y)\\ &\quad + g(A\xi,X)g(A\xi,Y) + Hg(SX,Y) - g(S^2X,Y)\\ &= (2m-1)g(X,Y) - 3\eta(X)\eta(Y) + g(T\xi,\xi)g(TX,Y)\\ &\quad + \mu(X)\mu(Y) + \mu(\xi)g(TX,\phi Y) + \mu(\xi)\mu(X)\eta(Y)\\ &\quad + g(TX,\xi)g(TY,\xi) + Hg(SX,Y) - g(S^2X,Y),\end{aligned}\tag{2.14}$$

where $H = \operatorname{tr} S$ denotes the mean curvature of the hypersurface M.

Let $\nabla^2 S$ denote the second covariant derivative of S, defined by:

$$(\nabla^2 S)(X,Y,Z) := \nabla_X[(\nabla_Y S)Z] - (\nabla_{\nabla_X Y}S)Z - (\nabla_Y S)\nabla_X Z.$$

Then, we have the following Ricci identity:

$$\begin{aligned}&g\big((\nabla^2 S)(X,Y,Z), W\big) - g\big((\nabla^2 S)(Y,X,Z), W\big)\\ &\qquad = -g\big(R(X,Y)Z, SW\big) - g\big(R(X,Y)W, SZ\big).\end{aligned}\tag{2.15}$$

Notice that the tangent bundle TM of M splits orthogonally into $TM = \mathcal{C}\oplus\mathcal{F}$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex sub-bundle of TM and $\mathcal{F} = \mathbb{R}\xi$. When

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restricted to $\mathcal{C}$, the structure tensor field ϕ coincides with the complex structure J. Moreover, at each point $[z] \in M$, the set

$$\mathcal{Q}_{[z]} = \{X \in T_{[z]}M \mid AX \in T_{[z]}M \text{ for all } A \in \mathfrak{A}_{[z]}\}$$

defines a maximal $\mathfrak{A}_{[z]}$-invariant subspace of $T_{[z]}M$.

According to Proposition 3 of [22], at each point $[z] \in M$, we can choose $A \in \mathfrak{A}_{[z]}$ and two orthonormal vectors $Z_1, Z_2 \in V(A)$ such that the unit normal vector field N takes the form

$$N = \cos t\, Z_1 + \sin t\, JZ_2, \tag{2.16}$$

where the parameter function t satisfies $0 \le t \le \frac{\pi}{4}$. Using $\xi = -JN$, we further get

$$\begin{aligned} AN &= \cos t\, Z_1 - \sin t\, JZ_2, \\ \xi &= \sin t\, Z_2 - \cos t\, JZ_1, \\ A\xi &= \sin t\, Z_2 + \cos t\, JZ_1. \end{aligned}$$

This implies that $g(A\xi, N) = 0$ and $A\xi \in T_{[z]}M$.

Remark 2.1 From the above facts, we can get the following conclusions:

(1) If $N_{[z]}$ is $\mathfrak{A}$-principal, we have $t = 0$ at $[z]$, $\mathcal{Q}_{[z]} = \mathcal{C}_{[z]}$, and $\dim \mathcal{Q}_{[z]} = 2m - 2$. For each $A \in \mathfrak{A}_{[z]}$, (2.1) implies that $A|_{\mathcal{Q}_{[z]}}$ has two eigenvalues 1 and −1. For $\varepsilon \in \{1, -1\}$, if denoting by $\mathcal{Q}(\varepsilon)$ the eigenspace of $A|_{\mathcal{Q}}$ corresponding to ε, then it holds that $J\mathcal{Q}(1) = \mathcal{Q}(-1)$ and $\dim \mathcal{Q}(1) = \dim \mathcal{Q}(-1) = m - 1$.

(2) If $N_{[z]}$ is not $\mathfrak{A}$-principal, we get $\mathcal{Q}_{[z]} = \{N_{[z]}, \xi_{[z]}, AN_{[z]}, A\xi_{[z]}\}^{\perp}$ for any $A \in \mathfrak{A}_{[z]}$, and $\dim \mathcal{Q}_{[z]} = 2(m-2)$. For each $A \in \mathfrak{A}_{[z]}$, (2.1) implies that $A|_{\mathcal{Q}_{[z]}}$ has two eigenvalues 1 and −1. Moreover, it holds that $J\mathcal{Q}(1) = \mathcal{Q}(-1)$ and $\dim \mathcal{Q}(1) = \dim \mathcal{Q}(-1) = m - 2$.

Finally, we recall the following lemma due to Berndt and Suh [3] (cf. also [17]). This lemma is very helpful for simplifying our latter calculations. Specifically, it allows us to choose locally, under appropriate conditions, the almost product structure $A \in \mathfrak{A}$ such that $A\xi \in TM$ and $g(AN, \xi) = 0$.

Lemma 2.3 (cf. [3] and [17]) *Let M be a hypersurface of Q^m with unit normal vector field N and $\xi = -JN$. Then, we have*:

(a) *If N is $\mathfrak{A}$-principal on an open set $U \subset M$, then there exists an almost product structure $A \in \mathfrak{A}$ on U such that $AN = N$.*

(b) *If N is not $\mathfrak{A}$-principal at $[z] \in M$, then there exist a neighborhood U of $[z]$ and an almost product structure $A \in \mathfrak{A}$ on U such that $A\xi \in TM$.*

3 Key lemmas related to the Tsinghua principle

We begin with establishing the following general lemma about real hypersurfaces of the complex quadric.

Lemma 3.1 *Let M be a real hypersurface in the complex quadric Q^m $(m \geq 3)$. Then, for a generic almost product structure $A \in \mathfrak{A}$ and any tangent vector fields $W, X, Y, Z \in TM$, we have*

$$-\underset{WXY}{\mathfrak{S}}\{g(R(W,X)Y,SZ)+g(R(W,X)Z,SY)\}=\underset{WXY}{\mathfrak{S}}\mathbf{I}(W,X,Y,Z), \tag{3.1}$$

where $\underset{WXY}{\mathfrak{S}}$ denotes the cyclic summation over W, X, and Y, and

$$\begin{aligned}
\mathbf{I}(W,X,Y,Z) &:= -g(SW,\phi X)g(\phi Y,Z)+g(SW,\phi Y)g(\phi X,Z)\\
&+2g(SW,\phi Z)g(\phi X,Y)-3g(SW,Y)\eta(X)\eta(Z)+3g(SW,X)\eta(Y)\eta(Z)\\
&-[g(TX,SW)+\eta(T\xi)g(SW,X)]g(TY,Z)+\mu(X)g(SW,Y)\mu(Z)\\
&+[g(TY,SW)+\eta(T\xi)g(SW,Y)]g(TX,Z)-\mu(Y)g(SW,X)\mu(Z)\\
&-[g(SW,X)\mu(\xi)+\mu(X)\eta(SW)]g(TY,\phi Z)-g(TX,\phi SW)g(TY,\phi Z)\\
&+\eta(TX)g(SW,Y)\eta(TZ)-g(SW,X)\mu(\xi)\mu(Y)\eta(Z)-g(TX,\phi SW)\mu(Y)\eta(Z)\\
&+[g(SW,Y)\mu(\xi)+\mu(Y)\eta(SW)]g(TX,\phi Z)+g(TY,\phi SW)g(TX,\phi Z)\\
&-\eta(TY)g(SW,X)\eta(TZ)+g(SW,Y)\mu(\xi)\mu(X)\eta(Z)+g(TY,\phi SW)\mu(X)\eta(Z).
\end{aligned}$$

Proof We shall calculate the expression of the cyclic summation

$$\mathfrak{B}:=\underset{WXY}{\mathfrak{S}}\{g((\nabla^2S)(W,X,Y),Z)-g((\nabla^2S)(W,Y,X),Z)\} \tag{3.2}$$

in two different ways. On the one hand, taking the covariant derivative of the Codazzi equation (2.13), we can get

$$\begin{aligned}
&g((\nabla^2S)(W,X,Y),Z)-g((\nabla^2S)(W,Y,X),Z)\\
&=(\nabla_W\eta)(X)g(\phi Y,Z)+\eta(X)g((\nabla_W\phi)Y,Z)-(\nabla_W\eta)(Y)g(\phi X,Z)\\
&-\eta(Y)g((\nabla_W\phi)X,Z)-2g((\nabla_W\phi)X,Y)\eta(Z)-2g(\phi X,Y)(\nabla_W\eta)(Z)\\
&+(\nabla_W\mu)(X)g(TY,Z)+\mu(X)g((\nabla_WT)Y,Z)-(\nabla_W\mu)(Y)g(TX,Z)\\
&-\mu(Y)g((\nabla_WT)X,Z)-g((\nabla_WT)X,\xi)g(TY,\phi Z)-g(TX,\nabla_W\xi)g(TY,\phi Z)\\
&-g(TX,\xi)g((\nabla_WT)Y,\phi Z)-g(TX,\xi)g(TY,(\nabla_W\phi)Z)\\
&-g((\nabla_WT)X,\xi)\mu(Y)\eta(Z)-g(TX,\nabla_W\xi)\mu(Y)\eta(Z)-g(TX,\xi)(\nabla_W\mu)(Y)\eta(Z)\\
&-g(TX,\xi)\mu(Y)(\nabla_W\eta)(Z)+g((\nabla_WT)Y,\xi)g(TX,\phi Z)\\
&+g(TY,\nabla_W\xi)g(TX,\phi Z)+g(TY,\xi)g((\nabla_WT)X,\phi Z)\\
&+g(TY,\xi)g(TX,(\nabla_W\phi)Z)+g((\nabla_WT)Y,\xi)\mu(X)\eta(Z)\\
&+g(TY,\nabla_W\xi)\mu(X)\eta(Z)+g(TY,\xi)(\nabla_W\mu)(X)\eta(Z)+g(TY,\xi)\mu(X)(\nabla_W\eta)(Z).
\end{aligned}$$

Then, by straightforward calculations of the RHS, with the use of equations in (2.5) and Lemma 2.2, we can obtain

$$g((\nabla^2S)(W,X,Y),Z)-g((\nabla^2S)(W,Y,X),Z)=\mathbf{I}(W,X,Y,Z). \tag{3.3}$$

Here, we note that the terms involving $q(W)$ are cancelled out with each other. Interestingly, when taking the covariant derivative of the Codazzi equation for a Lagrangian submanifold of Q^m, this phenomenon also occurred in [16].

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Now, from (3.3), we have

$$\mathfrak{B} = \underset{WXY}{\mathfrak{S}} \mathbf{I}(W,X,Y,Z). \tag{3.4}$$

On the other hand, the cyclic summation $\mathfrak{B}$ can be rewritten as

$$\mathfrak{B} = \underset{WXY}{\mathfrak{S}} \big\{g((\nabla^2 S)(W,X,Y),Z) - g((\nabla^2 S)(X,W,Y),Z)\big\}. \tag{3.5}$$

Then, we can apply the Ricci identity (2.15), so that, from (3.5), we obtain

$$\mathfrak{B} = -\underset{WXY}{\mathfrak{S}} \big\{g(R(W,X)Y,SZ) + g(R(W,X)Z,SY)\big\}. \tag{3.6}$$

From (3.4) and (3.6), we get the assertion (3.1). ∎

Remark 3.1 The method used in the proof of Lemma 3.1 is elementary and is called the Tsinghua principle. This remarkable technique has been applied in many different situations since its first successful attempt in [8], see [1, 6, 7, 16, 29] for details. Essentially, it establishes a bridge between the Codazzi equation and the Ricci identity by calculating the cyclic sum of the second covariant derivative of the second fundamental form. The significance of the Tsinghua principle lies in that, under appropriate conditions, it simplifies some higher-order equations of the second fundamental form into linear equations.

Next, for a conformally flat real hypersurface M of Q^m, Lemma 3.1 reduces to the following lemma, which is crucial for the proof of Theorem 1.1.

Lemma 3.2 *Let M be a conformally flat real hypersurface in the complex quadric Q^m $(m \geq 3)$. Then, for a generic almost product structure $A \in \mathfrak{A}$ and any tangent vector fields $W,X,Y,Z \in TM$, we have*

$$\underset{WXY}{\mathfrak{S}} \mathbf{II}(W,X,Y,Z) = \underset{WXY}{\mathfrak{S}} \mathbf{I}(W,X,Y,Z), \tag{3.7}$$

where $\mathbf{I}(W,X,Y,Z)$ is defined as in Lemma 3.1, *and*

$$\begin{aligned}
\mathbf{II}(W,X,Y,Z) := {} & \frac{3}{2m-3}\big[\eta(Y)\eta(SX) - \eta(X)\eta(SY)\big]g(W,Z) \\
& + \frac{1}{2m-3}\Big\{\eta(T\xi)\big[g(TX,SY) - g(TY,SX)\big] + \mu(X)\mu(SY) - \mu(Y)\mu(SX) \\
& - \mu(\xi)\big[g(TY,\phi SX) - g(TX,\phi SY) + \mu(Y)\eta(SX) - \mu(X)\eta(SY)\big] \\
& + \eta(TX)\eta(TSY) - \eta(TY)\eta(TSX)\Big\}g(W,Z).
\end{aligned}$$

Proof The conformally flatness of the real hypersurface M implies that it has vanishing Weyl curvature tensor or, equivalently, its Riemannian curvature tensor R takes the following form:

$$\begin{aligned}
g(R(X,Y)Z,W) = {} & \frac{1}{2m-3}\big\{\mathrm{Ric}(Y,Z)g(X,W) - \mathrm{Ric}(X,Z)g(Y,W) \\
& + \mathrm{Ric}(X,W)g(Y,Z) - \mathrm{Ric}(Y,W)g(X,Z)\big\} \\
& + \frac{r}{(2m-2)(2m-3)}\big\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\big\},
\end{aligned} \tag{3.8}$$

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where r denotes the scalar curvature of M. Substituting (3.8) into (3.6), we obtain

$$\mathfrak{B} = \frac{1}{2m-3} \underset{WXY}{\mathfrak{S}} \{g(W,Z)[\mathrm{Ric}(X,SY) - \mathrm{Ric}(Y,SX)]\}. \tag{3.9}$$

By using (2.14), we finally have that

$$\mathfrak{B} = \underset{WXY}{\mathfrak{S}} \mathbf{II}(W,X,Y,Z). \tag{3.10}$$

By (3.4) and (3.10), the assertion (3.7) follows. ■

4 Proof of Theorem 1.1 for Q^m

In order to complete the proof of Theorem 1.1 for Q^m $(m \geq 3)$, we suppose on the contrary that Q^m admits a conformally flat real hypersurface M. First of all, with the help of Lemma 3.2, we can prove the following lemma.

Lemma 4.1 *Let M be a conformally flat real hypersurface in the complex quadric Q^m $(m \geq 3)$. Then, the unit normal vector field of M must be $\mathfrak{A}$-principal everywhere.*

Proof We argue by contradiction. Assume that at some point $[z] \in M$, the unit normal vector $N_{[z]}$ is not $\mathfrak{A}$-principal. By Lemma 2.3, there exist a neighborhood U around $[z]$ in M and an almost product structure $A \in \mathfrak{A}$ on U such that $A\xi \in TM$ and $\mu(\xi) = g(A\xi, N) = 0$. It follows that there exist a unit tangent vector field $e_1 \in \mathcal{C}$ and functions a, c with $c > 0$ such that

$$AN = aN + ce_1, \quad A\xi = cJe_1 - a\xi, \quad a^2 + c^2 = 1. \tag{4.1}$$

Then, we get $\eta(T\xi) = g(T\xi, \xi) = -a$.

Put $e_2 = Je_1, e_{2m-1} = \xi$. From (2.1) and that $\dim \mathcal{Q} = 2m-4$ on U, we can choose orthogonal unit tangent vector fields $\{e_3, \ldots, e_{2m-2}\}$ such that

$$e_{2p-3} \in \mathcal{Q}(1), \quad e_{2p-2} = Je_{2p-3} \in \mathcal{Q}(-1), \quad 3 \leq p \leq m.$$

Then, $\{e_i\}_{i=1}^{2m-1}$ forms a local orthonormal frame field of M with the following properties:

$$\begin{cases} AN = aN + ce_1, \quad Ae_1 = cN - ae_1, \quad Ae_2 = ce_{2m-1} + ae_2, \\ Ae_{2p-3} = e_{2p-3}, \quad Ae_{2p-2} = -e_{2p-2}, \quad 3 \leq p \leq m, \\ Ae_{2m-1} = ce_2 - ae_{2m-1}. \end{cases} \tag{4.2}$$

Put $Se_i = \sum_{j=1}^{2m-1} a_{ij}e_j, 1 \leq i \leq 2m-1$, where $a_{ij} = a_{ji}$, $1 \leq i, j \leq 2m-1$.

Now, we apply for Lemma 3.2 with $A \in \mathfrak{A}$ being chosen as above.

By choosing appropriate $W = e_i$, $X = e_j$, $Y = e_k$, $Z = e_l$, $1 \leq i, j, k, l \leq 2m-1$, with the use of (2.4), (4.2), $\mu(\xi) = 0$, and $\eta(T\xi) = -a$, we can calculate $\mathbf{I}(W,X,Y,Z)$ and $\mathbf{II}(W,X,Y,Z)$ directly. As the result, by (3.7), we shall obtain a system of linear equations of the components $\{a_{ij}\}$. For instance, for $3 \leq p \leq m$, letting $(W,X,Y,Z) = (e_1, e_2, e_{2m-1}, e_{2p-3})$, $(e_1, e_2, e_{2m-1}, e_{2p-2})$, respectively, by direct

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calculations, we have

$$\begin{cases} \mathbf{I}(e_1, e_2, e_{2m-1}, e_{2p-3}) = \mathbf{I}(e_2, e_{2m-1}, e_1, e_{2p-3}) = 0, \\ \mathbf{I}(e_{2m-1}, e_1, e_2, e_{2p-3}) = 2a_{2p-2,2m-1}; \\ \mathbf{I}(e_1, e_2, e_{2m-1}, e_{2p-2}) = \mathbf{I}(e_2, e_{2m-1}, e_1, e_{2p-2}) = 0, \\ \mathbf{I}(e_{2m-1}, e_1, e_2, e_{2p-2}) = -2a_{2p-3,2m-1}; \\ \mathbf{II}(e_1, e_2, e_{2m-1}, e_{2p-3}) = \mathbf{II}(e_2, e_{2m-1}, e_1, e_{2p-3}) = \mathbf{II}(e_{2m-1}, e_1, e_2, e_{2p-3}) = 0; \\ \mathbf{II}(e_1, e_2, e_{2m-1}, e_{2p-2}) = \mathbf{II}(e_2, e_{2m-1}, e_1, e_{2p-2}) = \mathbf{II}(e_{2m-1}, e_1, e_2, e_{2p-2}) = 0. \end{cases}$$

It then follows from (3.7) that

$$a_{2p-3,2m-1} = a_{2p-2,2m-1} = 0, \quad 3 \le p \le m. \tag{4.3}$$

In the following, we shall carry the above procedures repeatedly, but the detailed calculation process will be omitted. To start, for $3 \le p \le m$, taking in (3.7),

$$(W, X, Y, Z) = (e_1, e_{2p-3}, e_{2p-2}, e_{2m-1}), \ (e_2, e_{2p-3}, e_{2p-2}, e_{2m-1}),$$

respectively, and using the fact $c > 0$, we obtain

$$a_{2p-3,2p-3} = a_{2p-2,2p-2}, \quad a_{2p-3,2p-2} = 0, \quad 3 \le p \le m. \tag{4.4}$$

For $3 \le p \le m$, taking in (3.7),

$$(W, X, Y, Z) = (e_1, e_{2p-3}, e_{2p-2}, e_1), \ (e_1, e_{2p-3}, e_{2p-2}, e_2), \ (e_2, e_{2p-3}, e_{2p-2}, e_1),$$

respectively, with the use of (4.4), we obtain

$$a_{12} = 0, \quad a_{11} = a_{22} = a_{2p-3,2p-3}, \quad 3 \le p \le m. \tag{4.5}$$

For $3 \le p \le m$, taking in (3.7),

$$(W, X, Y, Z) = (e_{2p-3}, e_{2p-2}, e_{2m-1}, e_1), \ (e_{2p-3}, e_{2p-2}, e_{2m-1}, e_2),$$

respectively, with the use of (4.4), we obtain

$$a_{1,2m-1} = a_{2,2m-1} = 0. \tag{4.6}$$

For $3 \le p \le m$, taking in (3.7),

$$(W, X, Y, Z) = (e_1, e_{2p-3}, e_{2m-1}, e_{2p-2}),$$

respectively, together with (4.6) and $c > 0$, we obtain

$$a_{11} = a_{2m-1,2m-1}. \tag{4.7}$$

For $3 \le p \le m$, taking in (3.7),

$$(W, X, Y, Z) = (e_2, e_{2p-3}, e_{2m-1}, e_2), \ (e_2, e_{2p-2}, e_{2m-1}, e_2),$$

respectively, together with (4.3) and $c > 0$, we obtain

$$\frac{2m-3+2a}{2m-3} a_{2,2p-3} = 0, \quad \frac{2m-3-2a}{2m-3} a_{2,2p-2} = 0, \quad 3 \le p \le m.$$

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Then, as $m \geq 3$ and $-1 < a < 1$, we deduce that

$$a_{2,2p-3} = a_{2,2p-2} = 0, \quad 3 \leq p \leq m. \tag{4.8}$$

For $3 \leq p \leq m$, taking in (3.7),

$$(W, X, Y, Z) = (e_{2p-3}, e_{2p-2}, e_{2m-1}, e_{2p-3}), \ (e_{2p-3}, e_{2p-2}, e_{2m-1}, e_{2p-2}),$$

respectively, together with (4.3), (4.8), and $c > 0$, we obtain

$$a_{1,2p-3} = a_{1,2p-2} = 0, \quad 3 \leq p \leq m. \tag{4.9}$$

If $m = 3$, then the above calculations show that $Se_i = a_{ii}e_i$, $1 \leq i \leq 5$, and $a_{11} = \cdots = a_{55}$. Hence, M is totally umbilical and $S\phi = \phi S$ holds. This is equivalent to that M has isometric Reeb flow. But, Corollary 1.3 of [3] states that there are no real hypersurfaces with isometric Reeb flow in the odd-dimensional complex quadric Q^{2k+1} for each $k \geq 1$. As desired, we get a contradiction.

Next, we continue with the discussions for $m \geq 4$.

For $3 \leq p, s \leq m$ and $p \neq s$, taking in (3.7),

$$\begin{aligned}(W, X, Y, Z) =& (e_{2p-3}, e_1, e_2, e_{2s-3}), \ (e_{2p-3}, e_1, e_2, e_{2s-2}), \\ & (e_{2p-2}, e_1, e_2, e_{2s-3}), \ (e_{2p-2}, e_1, e_2, e_{2s-2}),\end{aligned}$$

respectively, we obtain

$$a_{2p-3,2s-3} = a_{2p-3,2s-2} = a_{2p-2,2s-3} = a_{2p-2,2s-2} = 0, \quad 3 \leq p \neq s \leq m. \tag{4.10}$$

From the above calculations, we see that $Se_i = a_{ii}e_i$, $1 \leq i \leq 2m-1$, and $a_{11} = \cdots = a_{2m-1,2m-1}$. Hence, M is totally umbilical and $S\phi = \phi S$ holds, so M has isometric Reeb flow. But, according to Theorem 1.1 and Proposition 4.1 of Berndt and Suh [3], there are no totally umbilical real hypersurfaces with isometric Reeb flow in the complex quadric Q^m $(m \geq 4)$. This gives again a contradiction.

From the above contradictions, we have completed the proof of Lemma 4.1. ∎

Remark 4.1 Our proof of Lemma 4.1 strongly depends on the results of [3]. In Theorem 1.1 of [3], Berndt and Suh classified real hypersurfaces with isometric Reeb flow in the complex quadrics Q^m $(m \geq 3)$. Moreover, in Proposition 4.1 of [3], the authors calculated the geometric invariants of all real hypersurfaces with isometric Reeb flow in the complex quadrics Q^{2k} $(k \geq 2)$, showing that each of such real hypersurfaces has at least three distinct principal curvatures. Very recently, Hu and Yin [10] obtained new characterizations of the real hypersurfaces with isometric Reeb flow in the complex quadric.

Now, by Lemma 4.1, the unit normal vector field N of the conformally flat real hypersurface M is $\mathfrak{A}$-principal everywhere. According to Lemma 2.3, there exists an almost product structure $A \in \mathfrak{A}$ such that $AN = N$ and $A\xi = -\xi$. Thus, we have $T\xi = -\xi$ and $\mu(X) = g(AX, N) = 0$ for any $X \in TM$.

Put $e_{2m-1} = \xi$. From (2.1) and that $\dim \mathcal{Q} = 2m - 2$, we can choose unit tangent vector fields

$$e_{2p-3} \in \mathcal{Q}(1), \ e_{2p-2} = Je_{2p-3} \in \mathcal{Q}(-1), \quad 2 \leq p \leq m,$$

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so that $\{e_i\}_{i=1}^{2m-1}$ forms a local orthonormal frame field of M with the following properties:

(4.11) $$Ae_{2p-3} = e_{2p-3},\ \ Ae_{2p-2} = -e_{2p-2},\ \ Ae_{2m-1} = -e_{2m-1},\ 2 \le p \le m.$$

Put again $Se_i = \sum_{j=1}^{2m-1} a_{ij}e_j$, where $a_{ij} = a_{ji}, 1 \le i, j \le 2m-1$.

Now, we apply for Lemma 3.2 with $A \in \mathfrak{A}$ being chosen as above.

By choosing appropriate $W = e_i,\ X = e_j,\ Y = e_k,\ Z = e_l,\ 1 \le i, j, k, l \le 2m-1$, with the use of (2.4), (4.11), $T\xi = -\xi$, and that $\mu(X) = 0$, for any $X \in TM$, we can calculate $\mathbf{I}(W, X, Y, Z)$ and $\mathbf{II}(W, X, Y, Z)$ directly. Then, by (3.7), we will obtain a system of linear equations of the components $\{a_{ij}\}$. The procedure is essentially totally similar to that in the proof of Lemma 4.1. Thus, for simplicity, most of the detailed calculations below will be omitted.

For $2 \le p, s \le m$ and $p \ne s$, taking in (3.7),

$$(W, X, Y, Z) = (e_{2p-3}, e_{2p-2}, e_{2m-1}, e_{2s-3}),\ \ (e_{2p-3}, e_{2p-2}, e_{2m-1}, e_{2s-2}),$$

respectively, we can obtain

(4.12) $$a_{2s-3,2m-1} = a_{2s-2,2m-1} = 0,\ \ 2 \le s \le m.$$

It follows that $S\xi = a_{2m-1,2m-1}\xi$, and therefore, M is a Hopf hypersurface such that its unit normal vector field is $\mathfrak{A}$-principal everywhere. Then, we can apply for Theorem 14 of [17] or Theorem 2 of [15] to obtain that M is an open part of a tube around a totally geodesic $Q^{m-1} \hookrightarrow Q^m$. Thus, by Proposition 4.1 of [2], we have

$$Se_{2p-3} = -\frac{2}{\alpha}e_{2p-3},\ \ Se_{2p-2} = 0,\ \ S\xi = \alpha\xi,\ \ \text{for all } 2 \le p \le m.$$

Finally, letting $(W, X, Y, Z) = (e_3, e_2, e_4, e_1)$, by direct calculations, we have

$$\begin{cases} \mathbf{I}(e_3, e_2, e_4, e_1) = -\frac{4}{\alpha},\ \ \mathbf{I}(e_2, e_4, e_3, e_1) = \mathbf{I}(e_4, e_3, e_2, e_1) = 0; \\ \mathbf{II}(e_3, e_2, e_4, e_1) = \mathbf{II}(e_2, e_4, e_3, e_1) = \mathbf{II}(e_4, e_3, e_2, e_1) = 0. \end{cases}$$

It then follows from (3.7) that $\frac{4}{\alpha} = 0$. This is a contradiction, by which we have completed the proof of Theorem 1.1 for the complex quadric $Q^m\ (m \ge 3)$.

Corollary 4.1 *The complex quadric $Q^m\ (m \ge 3)$ does not admit real hypersurface of constant sectional curvature.*

Remark 4.2 Corollary 4.1 follows also from Theorem 21 of [17], where it was shown that the complex quadric $Q^m\ (m \ge 3)$ admits no Einstein real hypersurfaces.

5 Proof of Theorem 1.1 for Q^{m*}

To begin, we first note that, like the complex quadric Q^m, the set $\mathfrak{A}$ of all almost product structures of Q^{m*} is also an S^1-sub-bundle of the endomorphism bundle End(TQ^{m*}), and there are also two types of singular tangent vectors for the complex hyperbolic quadric Q^{m*}: $\mathfrak{A}$-principal and $\mathfrak{A}$-isotropic.

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Now, suppose on the contrary that the complex hyperbolic quadric Q^{m*} $(m \geq 3)$ admits a conformally flat real hypersurface $\bar{M}$, with N and S a unit normal vector field and the shape operator, respectively, and g and ∇ be the induced metric and the corresponding Levi–Civita connection on $\bar{M}$, respectively. Then, there is also naturally an *almost contact structure* $\{\phi, \xi, \eta\}$, a type $(1,1)$ tensor field T, and a 1-form μ induced from a generic almost product structure $A \in \mathfrak{A}$ on $\bar{M}$.

In the following, we outline the proof in three steps, which are essentially the same as the proof of Theorem 1.1 for Q^m $(m \geq 3)$.

Step 1. The real hypersurface $\bar{M}$ in Q^{m*} $(m \geq 3)$ satisfies equation (3.7).

As mentioned in the introduction, the almost product structure A and the Kähler structure J of Q^{m*} also satisfy (2.1) and (2.2) (see [13, 25, 27, 28]), and it makes the tensors ϕ, T, μ, and η on $\bar{M}$ also satisfy the equations (2.7)–(2.11) in Lemma 2.2. Because the Riemannian curvature tensor of Q^{m*} is exactly opposite to the Riemannian curvature tensor of Q^m, it makes both the Gauss–Codazzi equations and the Ricci tensor of $\bar{M}$ a slightly different from (2.12)–(2.14): some terms change at most by negative signs. However, when we apply for the Tsinghua principle with $\bar{M}$, following the same discussions as in Section 3, we see that Lemma 3.2 still holds for conformally flat real hypersurfaces in Q^{m*} $(m \geq 3)$.

Step 2. The unit normal vector field of $\bar{M}$ must be $\mathfrak{A}$-principal everywhere.

In fact, if otherwise, the unit normal vector field N is assumed locally not $\mathfrak{A}$-principal. By Lemma 3.2 of [25], there exists an almost product structure $A \in \mathfrak{A}$ in local such that $A\xi \in T\bar{M}$ and $\mu(\xi) = g(A\xi, N) = 0$. Similarly, we can also choose a local frame field $\{e_i\}_{i=1}^{2m-1}$ of $\bar{M}$ satisfying equations (4.1) and (4.2) as in the proof of Lemma 4.1. Then, following the same calculations as in the proof of Lemma 4.1, we can show that $\bar{M}$ is totally umbilical and $S\phi = \phi S$ holds. Thus, $\bar{M}$ has isometric Reeb flow. But, according to Theorem 1.1 and Proposition 4.1 of Suh [25], there are no totally umbilical real hypersurfaces with isometric Reeb flow in the complex hyperbolic quadric Q^{m*} $(m \geq 3)$. This gives a contradiction.

Step 3. The unit normal vector field of $\bar{M}$ cannot be $\mathfrak{A}$-principal everywhere.

In fact, if the unit normal vector field N is $\mathfrak{A}$-principal, there exists an almost product structure $A \in \mathfrak{A}$ such that $AN = N$ and $A\xi = -\xi$. We can find a local frame field $\{e_i\}_{i=1}^{2m-1}$ $(e_{2m-1} = \xi)$ of $\bar{M}$ satisfying the same equations as (4.11). Then, by similar calculations as taking in (3.7),

$$(W, X, Y, Z) = (e_{2p-3}, e_{2p-2}, e_{2m-1}, e_{2s-3}),\ (e_{2p-3}, e_{2p-2}, e_{2m-1}, e_{2s-2}),$$

for $2 \leq p, s \leq m$ and $p \neq s$, respectively, we can show that $S\xi = \alpha\xi$. Therefore, $\bar{M}$ is a Hopf hypersurface such that the unit normal vector field is $\mathfrak{A}$-principal everywhere. Then, according to Lemma 4.2 (ii) of [27], by choosing the almost product structure $A \in \mathfrak{A}$ such that $AN = N$, it holds $ASX = SX$ for any $X \in \mathcal{C}$.

As $\bar{M}$ is a Hopf hypersurface with $\mathfrak{A}$-principal unit normal vector field, the above fact implies that $SAX = SX$ for all $X \in \mathcal{C}$. It follows that $S\mathcal{Q}(-1) = 0$, where $\mathcal{Q}(-1)$ is the eigenspace of $A|_{\mathcal{C}}$ corresponding to the eigenvalue -1. Then, for any $X \in \mathcal{Q}(-1)$, Lemma 3.4 of [27] shows that $\alpha S\phi X = 2\phi X$. It follows that $\alpha \neq 0$ and $SX = \frac{2}{\alpha}X$ for all $X \in \mathcal{Q}(1) = J\mathcal{Q}(-1)$. Thus, we have

$$Se_{2p-3} = \frac{2}{\alpha}e_{2p-3},\quad Se_{2p-2} = 0,\quad S\xi = \alpha\xi,\quad \text{for all } 2 \leq p \leq m.$$

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Finally, taking $(W, X, Y, Z) = (e_3, e_2, e_4, e_1)$ in (3.7), by direct calculations, we have $\frac{4}{\alpha} = 0$. This is a contradiction.

In summary, we have completed the proof of Theorem 1.1 for Q^{m*} $(m \geq 3)$.

Corollary 5.1 *The complex hyperbolic quadric Q^{m*} $(m \geq 3)$ does not admit real hypersurface of constant sectional curvature.*

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