OPTIMAL INSURANCE CONTRACTS UNDER DISTORTION RISK MEASURES WITH AMBIGUITY AVERSION

BY

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Abstract

This paper presents analytical representations for an optimal insurance contract under distortion risk measure and in the presence of model uncertainty. We incorporate ambiguity aversion and distortion risk measure through the model of Robert and Therond [(2014) *ASTIN Bulletin: The Journal of the IAA*, **44**(2), 277–302.] as per the framework of Klibanoff *et al*. [(2005) A smooth model of decision making under ambiguity. *Econometrica*, **73**(6), 1849–1892.]. Explicit optimal insurance indemnity functions are derived when the decision maker (DM) applies Value-at-Risk as risk measure and is ambiguous about the loss distribution. Our results show that: (1) under model uncertainty, ambiguity aversion results in a distorted probability distribution over the set of possible models with a bias in favor of the model which yields a larger risk; (2) a more ambiguity-averse DM would demand more insurance coverage; (3) for a given budget, uncertainties about the loss distribution result in higher risk level for the DM.

Keywords

Optimal insurance, model uncertainty, distortion risk measure, ambiguity aversion, Karush–Kuhn–Tucker multipliers, calculus of variation, suboptimal contract.

1. Introduction

Insurance is an important tool for risk-averse individuals to hedge risks of financial losses. The optimal forms of insurance contracts have been studied extensively in the insurance and economics literature. Classical results include those developed in Borch (1960a,b), which established that stop-loss contracts are optimal for a decision maker (DM), in the sense that the DM's

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risk (measured by the variance of potential losses) is minimized. Arrow (1963) showed that the stop-loss contracts maximize the expected utility (EU) of a risk-averse DM. More recently, it was shown that stop-loss contracts and their variants are in general optimal when the risk of the DM is measured by more modern risk measures, such as the distortion risk measures. See, for instance, Cui *et al.* (2013), Assa (2015), and Boonen *et al.* (2016).

In practice, both the DM and insurer face some ambiguity (uncertainty) about the distribution of their potential losses. It has been well documented that an individual's decision making is affected by such ambiguity. Ellsberg's classic paper (Ellsberg, 1961) revealed that, for events that involve gains, individuals prefer lotteries with well-specified probabilities over ones with ambiguous probabilities unless the probability of winning is very small. Hogarth and Kunreuther (1989) found that if consumers estimate the probability of an accident as low, but are ambiguous about such estimates, they will be willing to buy insurance at prices considerably above the actuarial value. Kunreuther *et al.* (1993, 1995) found that when insurance underwriters and actuaries are ambiguity averse, premiums are significantly higher for risks when there is ambiguity regarding the occurrence probability of loss events and/or uncertainty about the magnitude of the resulting losses.

Several classical models for ambiguity aversion have been proposed in the literature, including the maxmin EU model of Gilboa and Schmeidler (1989), the Choquet integral (of the utility function) with respect to a capacity (Schmeidler, 1989), and the α-maxmin EU model of Ghirardato *et al.* (2004). These models addressed the Ellsberg's paradox from different perspectives and were applied widely in the literature. Recently, Klibanoff *et al.* (2005) proposed the smooth ambiguity model, which is also known as the Klibanoff-Marinacci-Mukerji (KMM) model. The KMM model is very flexible and parsimonious. It captures a full range of ambiguity attitudes and most importantly it achieves a separation between ambiguity and ambiguity attitude (Klibanoff *et al.*, 2005, 2012). By applying the KMM model, Alary *et al.* (2013) concluded that ambiguity aversion raises the demands for self-insurance and insurance coverage, while it reduces the demands for self-protection. Gollier (2014) examined the characteristics of the optimal insurance contract under ambiguous loss distribution and linear transaction costs. He showed that the optimal contract depends upon the nature of the uncertainty. In particular, if the set of possible prior distributions can be ranked according to the monotone likelihood ratio order, the optimal contract contains a disappearing deductible.

This paper considers the scenario of an ambiguity-averse DM seeking the optimal insurance contract that minimizes its risk level according to some distortion risk measure. Asimit *et al.* (2017) considered such a problem from the point of view of robust control. They focused on Value-at-Risk (VaR) and Conditional Value-at-Risk-based risk measures and applied the worst-case scenario and worst-case regret models to determine the "robust" optimal polices. We take a different approach in this paper. Inspired by Robert and Therond (2014), we impose the KMM ambiguity model to distortion risk measures, which enables us to assess risk and ambiguity aversion separately and to incorporate a variety of ambiguity aversion levels, for example, the nonambiguous, ambiguity-neutral, and the worst-case scenarios. In addition, our model can be extended to the case when the two parties have different levels of ambiguity aversion.

The contributions of this paper are three-fold. First, we provide a characterization of the optimal insurance contract for the distortion-risk-measure-based model under ambiguity aversion. The representation demonstrates that ambiguity aversion distorts the DM's prior belief about the probability weights assigned to the candidate distributions, which results in higher demand for the insurance protection. This conclusion concurs with the results in Alary *et al.* (2013) and Gollier (2014).

Second, we derive the explicit indemnity function under VaR when the DM is ambiguous between two candidate loss distributions. To the best of our knowledge, such an explicit representation is novel. More importantly, the results provide important practical insights to the problem.

Third, we analyze and quantify the cost of model uncertainties to a DM. Our results illustrate that: (1) the demand for insurance coverage increases with the level of ambiguity aversion; (2) for a given budget, uncertainties about the distribution of underlying losses result in higher risk level for the DM.

The remainder of this paper is structured as follows: Section 2 sets up the model and main problem of interest. Section 3 solves the optimization problem and discusses in detail the special cases of nonambiguous, ambiguity-neutral, and extreme ambiguity-averse DMs. Section 4 derives the explicit formula for the optimal insurance policy in the highly practical two-state VaR case. The results also give additional insights on our theoretical results in Section 3. Section 5 illustrates the main results of this paper using numerical examples, which reveals the effect of the ambiguity aversion on the optimal insurance design, as well as the resulting risk levels of the DM. Section 6 concludes.

2. Problem setup

Suppose that a DM faces a random loss X with range $[0, M]$. The DM knows that the distribution of X belongs to a family of distributions $\mathcal{M} = \{F_\theta, \theta \in \Theta\}$, where for simplicity it is assumed that $\Theta = \{1, .., n\}$. However, the true distribution, denoted by $F_X = F_{\theta_T}$, where $\theta_T \in \Theta$, is unknown to the DM. Consequently, the DM assigns a subjective probability α_θ to the distribution F_θ with $\sum_{\theta\in\Theta} \alpha_\theta = 1$, where the values of α_θ could be determined by experience or some expert opinions.

In order to hedge the risk X, the DM considers purchasing an insurance contract, which is characterized by the pair $(I(X), \pi\ (I(X)))$, where $I(X)$ is the indemnity function that specifies the indemnification received by the DM, whereas $\pi\ (I(X))$ is the insurance premium charged by the insurer.

As per the vast literature on optimal insurance within the risk minimization framework, the DM's goal is to minimize the risk as measured by a distortion risk measure. For an arbitrary nonnegative random variable Y, the distortion risk measure is defined as:

$$\mathbf{H}_g(Y) = \int_0^M g\left(S_Y(x)\right) dx,$$

where $g(\cdot)$ is a distortion function with $g(0) = 0$, $g(1) = 1$ and $S_Y(\cdot)$ is the survival function of Y.

In our context, the DM does not know the distribution of losses exactly, but has enough evidence to believe that the true distribution of losses is not an actual mixture, that is, $S_m(x) = \sum_{i=1}^n \alpha_i S_i(x)$. Rather, the true distribution is one of the elements in $\mathcal{M}$. However, exactly which one is true remains unknown to the DM. Therefore, the DM assigns weight α_i to the ith possible distribution based on some personal experiences or expert opinions.

We treat ambiguity aversion along the line of Robert and Therond (2014). Specifically, we incorporate the DM's level of ambiguity aversion into the aversion to mean-preserving spreads (see, e.g., Yaari (1987)) via an increasing convex function $\phi(\cdot)$ by considering the following risk measure under uncertainty:

$$\tilde{\mathbf{H}}_{g,\phi}(X) = \phi^{-1}\left(\sum_{i=1}^n \alpha_i \left(\phi\left(\mathbf{H}_{g,F_i}(X)\right)\right)\right), \tag{2.1}$$

where $\mathbf{H}_{g,F_i}(X)$ denote the risk measure of X with distribution F_i. Note that by Jensen's inequality, we have

$$\tilde{\mathbf{H}}_{g,\phi}(X) \geq \sum_{i=1}^n \alpha_i \mathbf{H}_{g,F_i}(X).$$

Formula (2.1) allows us to separate the aversion to mean-preserving spreads that is measured by the distortion risk measure $\mathbf{H}_{g,F_i}(\cdot)$ and the ambiguity aversion which is modeled by the function ϕ. The function ϕ provides a flexible approach to model the level of ambiguity aversion. For example, suppose $\phi(x) = e^{ax}$, where $a > 0$ (Robert and Therond, 2014). Then the parameter a captures the level of ambiguity aversion. Given candidate distributions $\{F_i\}_{i=1,2,\ldots,n} \in \mathcal{M}$ and the corresponding distortion risk measures $H_i = \int_0^M g(S_i(x))dx$, then

$$\lim_{a\to 0} \phi^{-1}\left(\sum_{i=1}^n \alpha_i \phi(H_i)\right) = \lim_{a\to 0} \frac{\log\left(\sum_{i=1}^n \alpha_i e^{aH_i}\right)}{a} = \sum_{i=1}^n \alpha_i H_i,$$

which indicates that the DM is ambiguity-neutral. On the other hand,

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$$\lim_{a\to\infty}\phi^{-1}\left(\sum_{i=1}^{n}\alpha_i\phi(H_i)\right)=\lim_{a\to\infty}\frac{\log\left(\sum_{i=1}^{n}\alpha_i e^{aH_i}\right)}{a}=\max\{H_1,H_2,\ldots,H_n\},$$

which indicates that the DM is extremely ambiguity averse and only considers the most adverse opinion. This extreme ambiguity aversion case was studied in Asimit *et al*. (2017, 2019) and Liu *et al*. (2020).

Generally, the insurer is also uncertain of the loss distribution. However, to keep the presentation simple, we assume that the insurer believes that the distribution of the random loss X is given by $Q(\cdot)$, which may or may not belong to the set $\mathcal{M}$. Further, we assume that the insurer applies the expectation premium principle so that the insurance premium is given by

$$\pi\,(I(X))=(1+\xi)\mathbf{E}_Q(I(X)), \tag{2.2}$$

where $\xi\geq 0$ is the safety loading factor.

In addition, we assume that the set of admissible indemnity functions is given by

$$\mathcal{C}:=\left\{I:[0,M]\to[0,M]\;\middle|\;\begin{array}{c}I(0)=0,\ 0\leq I(x)\leq x \text{ for } x\in[0,M]\\ \text{and } 0\leq I(y)-I(x)\leq y-x \text{ for } y\geq x\end{array}\right\}.$$

With the indemnity function $I\in\mathcal{C}$, both the insured's and insurer's payments are nondecreasing w.r.t x, which alleviates the potential problems arising from the "moral hazard." Any function $I\in\mathcal{C}$ is 1-Lipschitz continuous and admits the following integral representation

$$I(x)=\int_0^x\eta(t)dt,$$

where the function $\eta(t)\in[0,1]$ is called the marginal indemnity function (Zhuang *et al*., 2016). To facilitate our presentation, we define the following set of admissible marginal indemnity functions:

$$\mathcal{C}':=\left\{\eta(\cdot)\;\middle|\;\eta(x)\in[0,1] \text{ is continuous almost everywhere for } x\in[0,M]\right\}.$$

In summary, the DM's objective is to minimize its risk while accounting for ambiguity aversion on the distribution of losses as measured by (2.1) by choosing an optimal insurance indemnity function. We assume that the maximum premium the DM is willing to spend is π_0. Then, the DM's objective can be mathematically formulated as:

Problem 1 (Main problem).

$$\min_{I\in\mathcal{C}}\ \sum_{i=1}^{n}\alpha_i\phi\left(\mathbf{H}_{g,F_i}(X-I(X)+\pi(I(X)))\right), \tag{2.3}$$

$$s.t.\ \pi(I(X))\leq\pi_0. \tag{2.4}$$

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In (2.3), the term $\mathbf{H}_{g,F_i}(X - I(X) + \pi(I(X)))$ denotes the risk of DM's retained loss if the underlying loss X follows the distribution F_i.

In the next section, we provide general solution to Problem 1 and then study some special cases in detail. All proofs in the paper can be found in the appendices.

3. Optimal insurance contracts

We begin with the general case.

3.1. The general solution

Based on the Lemma 2.1 of Cheung and Lo (2017), an integral representation of $\mathbf{H}_{g,F_i}(I(X))$ for arbitrary $I \in \mathcal{C}$ is

$$\mathbf{H}_{g,F_i}(I(X)) = \int_0^M g\left(S_i(x)\right) \eta(x) dx, \tag{3.1}$$

where $\eta(x) = I'(x)$. In addition, the premium can also be written in integral form

$$\pi(I(X)) = \int_0^M (1+\xi) S_Q(x) \eta(x) dx.$$

Utilizing the comonotonic additivity of the distortion risk measure and the integral representation (3.1), the term $\mathbf{H}_{g,F_i}(X - I(X) + \pi(I(X)))$ on the right-hand side of (2.3) becomes

$$\begin{aligned}\mathbf{H}_{g,F_i}(X - I(X) + \pi(I(X))) &= \mathbf{H}_{g,F_i}(X) - \mathbf{H}_{g,F_i}(I(X)) + \pi(I(X)),\\ &= \mathbf{H}_{g,F_i}(X) - \int_0^M g(S_i(x))\eta(x)dx\\ &\quad + (1+\xi)\int_0^M S_Q(x)\eta(x)dx,\\ &= \mathbf{H}_{g,F_i}(X) + \int_0^M h_i(x)\eta(x)dx,\end{aligned}$$

where $h_i(x) = (1+\xi)S_Q(x) - g(S_i(x))$ for $i = 1, 2, \ldots, n$. Roughly speaking, h_i is the marginal net cost (premium minus the risk) of the insurance contract. Then the main problem can be rewritten as follows.

Problem 2 (Second form of Problem 1).

$$\min_{\eta \in \mathcal{C}'} \sum_{i=1}^n \alpha_i \phi\left(\mathbf{H}_{g,F_i}(X) + \int_0^M h_i(x)\eta(x)dx\right), \tag{3.2}$$

$$s.t.\ (1+\xi)\int_0^M S_Q(x)\eta(x)dx \le \pi_0. \tag{3.3}$$

To guarantee the uniqueness of the solution, throughout this paper we assume $\phi''(\cdot) > 0$. The main result for this section is given below.

Proposition 3.1. *The optimal marginal indemnity function η^* that solves Problem 2 is characterized by*

$$\eta^*(x;\lambda) = \mathbb{1}_{D_\lambda}(x) + \gamma(x)\cdot \mathbb{1}_{E_\lambda}(x), \tag{3.4}$$

where $\gamma(x)$ is an arbitrary function such that $\eta^*(x;\lambda)\in\mathcal{C}'$ and the sets D_λ and E_λ are defined as

$$D_\lambda = \Big\{x \mid K(x) < 0\Big\}, \tag{3.5}$$

$$E_\lambda = \Big\{x \mid K(x) = 0\Big\}, \tag{3.6}$$

with

$$K(x) = \lambda(1+\xi)S_Q(x) + \sum_{i=1}^{n}\alpha_i h_i(x)\phi'\left(\mathbf{H}_{g,F_i}(X) + \int_0^M h_i(x)\eta^*(x;\lambda)dx\right). \tag{3.7}$$

In addition, $\lambda \geq 0$ is the so-called Karush–Kuhn–Tucker (KKT) multiplier that satisfies

$$\left((1+\xi)\int_0^M S_Q(x)\eta^*(x;\lambda)dx - \pi_0\right) \leq 0, \tag{3.8}$$

and

$$\lambda\left((1+\xi)\int_0^M S_Q(x)\eta^*(x;\lambda)dx - \pi_0\right) = 0. \tag{3.9}$$

Note that the form of the optimal marginal indemnity function $\eta^*(x;\lambda)$ provided in (3.4) is generally implicit since the sets D_λ and E_λ are both determined by $\eta^*(x;\lambda)$ itself. However, in a very important case when the DM is ambiguity-neutral, for example, when $\phi(x) = x$, (3.4) becomes explicit and can be easily calculated. Due to its importance, we discuss it in detail as follows.

For an ambiguity-neutral DM, $\phi(x) = x$. Then Problem 2 becomes the following.

Problem 3 (An ambiguity-neutral problem).

$$\begin{aligned}
&\min_{\eta\in\mathcal{C}'} \sum_{i=1}^{n}\alpha_i\left(\mathbf{H}_{g,F_i}(X) + \int_0^M h_i(x)\eta(x)dx\right),\\
&s.t.\ (1+\xi)\int_0^M S_Q(x)\eta(x)dx \leq \pi_0.
\end{aligned} \tag{3.10}$$

We have the following.

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Corollary 3.1. *The solution to Problem 3 is given by* $\eta(x;\lambda,\alpha_1,\ldots,\alpha_n)$, *where*

$$\eta(x;\lambda,\alpha_1,\ldots,\alpha_n)=\mathbb{1}_{\tilde{D}_\lambda}(x)+\gamma(x)\cdot\mathbb{1}_{\tilde{E}_\lambda}(x), \tag{3.11}$$

with

$$\tilde{D}_\lambda=\Big\{x \mid \tilde{K}(x)<0\Big\}, \tag{3.12}$$

$$\tilde{E}_\lambda=\Big\{x \mid \tilde{K}(x)=0\Big\}, \tag{3.13}$$

and

$$\tilde{K}(x)=\lambda(1+\xi)S_Q(x)+\sum_{i=1}^{n}\alpha_i h_i(x). \tag{3.14}$$

The parameter λ *is obtained using the KKT conditions. Particularly, if* $\pi_0\geq(1+\xi)\mathbf{E}_Q[X]$*, then the budget constraint is always satisfied and* $\lambda=0$*. If* $(1+\xi)\mathbf{E}_Q[X]>\pi_0\geq 0$*, then for each* $(\alpha_1,\ldots,\alpha_n)\in\mathcal{A}$*, there exists a* $\lambda\in\mathbf{R}$ *such that the budget constraint is satisfied.*

In the next result, we show that Problem 2 for an ambiguity-averse DM can be solved via the solution for Problem 3 for an ambiguity-neutral DM. The proof for it is provided in Appendix C.

Proposition 3.2. *Let* $\eta(x;\tilde{\lambda},\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n)$ *denote the solution to Problem 3 when the subjective probability weights are given by* $(\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n)$*. Then the solution to the infinite-dimension Problem 2 can be obtained by solving the finite-dimension problem:*

$$\min_{(\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n)\in\mathcal{A}} V(\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n), \tag{3.15}$$

where

$$V(\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n)=\sum_{i=1}^{n}\alpha_i\phi\left(\mathbf{H}_{g,F_i}(X)+\int_0^M h_i(x)\eta(x;\tilde{\lambda},\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n)dx\right), \tag{3.16}$$

and

$$\mathcal{A}=\left\{(\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n)\;\middle|\;\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n\geq 0,\ \sum_{i=1}^{n}\tilde{\alpha}_i=1\right\}.$$

Remark 3.1. *Let* $\tilde{\boldsymbol{\alpha}}^*=(\tilde{\alpha}_1^*,\ldots,\tilde{\alpha}_n^*)$ *solve (3.15). Then, the effect of ambiguity aversion function* ϕ *on the DM can be seen as a distortion of the subjective belief from the original weights* $(\alpha_1,\ldots,\alpha_n)$ *to* $(\tilde{\alpha}_1^*,\ldots,\tilde{\alpha}_n^*)$*.*

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Moreover, Equation (C4) *in the proof indicates that*

$$\tilde{\alpha}_i^* > \alpha_i \rightleftarrows \phi'\left(\mathbf{H}_{g,F_i}(X - I^*(X) + \pi(I(X)))\right)$$
$$> \sum_{j=1}^{n} \alpha_j \phi'\left(\mathbf{H}_{g,F_j}(X - I^*(X) + \pi(I(X)))\right).$$

It illustrates that an ambiguity-averse DM would put more subjective weights on less favorable distributions in evaluating insurance policies. This leads to higher demand for insurance coverage. This point will be echoed in the numerical examples in Section 5.

This result echoes Equations (8) and (9) in Gollier (2014) *in an EU-maximization framework.*

Remark 3.2. *The solution to the budget-free version of Problem 2, that is, solving (3.2) without the constraint (3.3), could be derived straightforwardly with Proposition 3.1. This optimal solution, denoted by* $\eta^*(x;0)$*, can be derived implicitly from the representation (3.4) by setting* $\lambda = 0$*. Without the budget constraint, we can derive the global optimal indemnity function* $\eta^*(x;0)$ *and the corresponding budget level* $\pi_0^* = (1+\xi)\int_0^M S_Q(x)\eta^*(x;0)dx$*.*

Remark 3.3. *The KKT conditions* (3.8) *and* (3.9) *in fact indicate the following procedure to deal with the problem with budget constraint* π_0*. We first solve the budget-free problem as in Remark 3.2 by setting* $\lambda = 0$*. If the resultant premium* $(1+\xi)\int_0^M S_Q(x)\eta^*(x;0)dx \le \pi_0$*, then* $\eta^*(x;0)$ *is the required solution. If* $(1+\xi)\int_0^M S_Q(x)\eta^*(x;0)dx > \pi_0$*, then as indicated by* (3.9)*, we seek a* $\lambda > 0$ *such that* $(1+\xi)\int_0^M S_Q(x)\eta^*(x;\lambda)dx = \pi_0$*. This procedure is followed in all numerical examples.*

Remark 3.4. *For a DM who nonambiguously assumes that the loss distribution is, say,* F_1*, the optimal indemnity function can be easily obtained by setting* $\alpha_1 = 1$ *in Corollary 3.1. Note that* F_1 *may be different from* F_Q*, which is assumed by the insurer. Thus, the problem becomes a special case of a more general optimal insurance problem with heterogeneous beliefs, which was studied, for example, by Boonen* (2016) *and Boonen and Ghossoub* (2019)*. In particular, Boonen and Ghossoub* (2019) *considered one insurer and multiple reinsurers who have different distributional assumptions. For comparison, we note that this paper assumes that there is only one insured and one insurer. However, in our model, the insured has several candidate loss distributions and is uncertain about which one is true.*

3.2. The worst case

An extreme ambiguity-averse DM would only consider the worst-case scenario (e.g., $a \to \infty$ when $\phi(x) = e^{ax}$). This situation is related to robust insurance,

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which has been studied in the literature by, for example, Asimit *et al.* (2017). Similar to Equation (2.4) of Asimit *et al.* (2017), if only the worst case is considered, we need to solve the following:

Problem 4 (Worst-case scenario).

$$\begin{aligned} &\min_{I\in\mathcal{C}} \max_{i\in\{1,2,\ldots,n\}} \mathbf{H}_{g,F_i}(X-I(X)+\pi(I(X))), \\ &s.t.\ \pi(I(X))\le\pi_0. \end{aligned} \tag{3.17}$$

The solution to Problem 4 can be obtained via Proposition 3.2 by setting $\phi(x)=e^{ax}$ and letting a approach infinity. However, it is not possible to provide explicit solution this way. Thus, we next obtain it through establishing a connection between the solutions to Problems 4 and 3 directly.

First note that for arbitrary $I\in\mathcal{C}$ and $(\alpha_1,\ldots,\alpha_n)\in\mathcal{A}$, we have:

$$\max_{i\in\{1,2,\ldots,n\}} \mathbf{H}_{g,F_i}(X-I(X)+\pi)\ge\sum_{i=1}^{n}\alpha_i\mathbf{H}_{g,F_i}(X-I(X)+\pi),$$

which naturally leads to

$$\max_{i\in\{1,2,\ldots,n\}} \mathbf{H}_{g,F_i}(X-I(X)+\pi)\ge\max_{(\alpha_1,\ldots,\alpha_n)\in\mathcal{A}}\sum_{i=1}^{n}\alpha_i\mathbf{H}_{g,F_i}(X-I(X)+\pi). \tag{3.18}$$

In addition, since for each i,

$$\mathbf{H}_{g,F_i}(X-I(X)+\pi)\le\max_{(\alpha_1,\ldots,\alpha_n)\in\mathcal{A}}\sum_{i=1}^{n}\alpha_i\mathbf{H}_{g,F_i}(X-I(X)+\pi),$$

we also have

$$\max_{i\in\{1,2,\ldots,n\}} \mathbf{H}_{g,F_i}(X-I(X)+\pi)\le\max_{(\alpha_1,\ldots,\alpha_n)\in\mathcal{A}}\sum_{i=1}^{n}\alpha_i\mathbf{H}_{g,F_i}(X-I(X)+\pi). \tag{3.19}$$

Combining inequalities (3.18) and (3.19), we can reformulate Problem 4 as follows.

Problem 4a (Second form of Problem 4).

$$\begin{aligned} &\min_{I\in\mathcal{C}} \max_{(\alpha_1,\ldots,\alpha_n)\in\mathcal{A}} \sum_{i=1}^{n}\alpha_i\mathbf{H}_{g,F_i}(X-I(X)+\pi(I(X))), \\ &s.t.\ \pi(I(X))\le\pi_0. \end{aligned}$$

To proceed, we need to present here the well-known minimax Theorem proposed by Fan (1953).

Lemma 3.1 (Minimax Theorem). *Let Ξ_1 be a compact convex Hausdorff topological vector space and Ξ_2 be a convex set. If $\mathcal{H}$ is a real-valued function defined on $\Xi_1 \times \Xi_2$ such that*

- *$\xi_1 \to \mathcal{H}(\xi_1, \xi_2)$ is convex and lower-semicontinuous on Ξ_1 for each $\xi_2 \in \Xi_2$;*
- *$\xi_2 \to \mathcal{H}(\xi_1, \xi_2)$ is concave on Ξ_2 for each $\xi_1 \in \Xi_1$,*

then

$$\min_{\xi_1\in\Xi_1}\max_{\xi_2\in\Xi_2}\mathcal{H}(\xi_1,\xi_2)=\max_{\xi_2\in\Xi_2}\min_{\xi_1\in\Xi_1}\mathcal{H}(\xi_1,\xi_2).$$

It is easy to check that both $\mathcal{C}$ and $\mathcal{A}$ are convex sets. With Arzelà–Ascoli theorem, the set $\mathcal{C}'$ is also compact (see also Liu *et al.* (2020)). With the integral representation of the objective function, one can see $\sum_{i=1}^{n}\alpha_i\mathbf{H}_{g,F_i}(X-I(X)+\pi)$ is linear in both $(\alpha_1 .\dots,\alpha_n)$ and $I(x)$. Thereby all the requirements for applying the Lemma 3.1 are met. Then Problem 4a can be written as follows:

Problem 4b (Third form of Problem 4).

$$\max_{(\alpha_1,\dots,\alpha_n)\in\mathcal{A}}\ \min_{I\in\mathcal{C}}\ \sum_{i=1}^{n}\alpha_i\mathbf{H}_{g,F_i}(X-I(X)+\pi(I(X))),$$
$$s.t.\ \pi(I(X))\le\pi_0.$$

Since the minimization part of Problem 4b has the same format as Problem 3, its solution is obtained in Corollary 3.1. Thus, Problem 4b becomes as follows:

Problem 4c (Fourth form of Problem 4).

$$\max_{(\alpha_1,\dots,\alpha_n)\in\mathcal{A}}\ V(\alpha_1,\dots,\alpha_n),$$

where

$$V(\alpha_1,\dots,\alpha_n)=\sum_{i=1}^{n}\alpha_i\left(\mathbf{H}_{g,F_i}(X)+\int_0^M h_i(x)\eta(x;\lambda,\alpha_1,\dots,\alpha_n)dx\right).$$

We then have the following corollary.

Corollary 3.2. *The optimal marginal indemnity function that solves Problem 4 has the same form as the solution to Problem 3, but with a different set of subjective weights $(\alpha_1^*,\dots,\alpha_n^*)$, which are obtained by solving Problem 4c.*

Remark 3.5. *The worst-case analysis here is very similar to that of Birghila and Pflug (2019), in which the authors discussed a model with infinite candidate distributions and applied the saddle point in Sion's minimax theory to derive the characterization of optimal solution. However, in that paper, the methodology to*

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determine the saddle point, or more precisely F^* in their Proposition 3.1, is not provided. Our paper focuses on a model with finite candidate distributions, and the "saddle point" is obtained through optimizing $\alpha_1, \ldots, \alpha_n$ in Problem 4c.

4. The optimal insurance contract with VaR

In this section, we examine a special albeit important case when the candidate distributions of the loss X belong to the set $\mathcal{M} = \{F_1, F_2\}$, which were assigned probability weight α and $1-\alpha$ by the DM. The DM applies VaR as the risk measure. Specifically, VaR is defined as

$$VaR_\beta(X) = \inf\{x \in [0, M] \mid F_X(x) \geq \beta\},$$

with the corresponding distortion function given by $g_\beta(t) = \mathbb{1}_{[1-\beta,1]}(t)$. We denote $b_1 = F_1^{-1}(\beta), b_2 = F_2^{-1}(\beta)$ and assume that $b_1 < b_2$. As shown in the following derivation, for this case, we can directly apply Proposition 3.2 to obtain the optimal indemnity function for an ambiguity-averse DM. Thus we will present the general solution first and then discuss the special cases of decisions for nonambiguous, ambiguous-neutral, and extreme ambiguous DM.

4.1. The general solution for the VaR case

We begin with the following general result for an ambiguity-averse DM.

Proposition 4.1. *If there are two candidate distributions and VaR is the risk measure, the solution to Problem 1 is given by*

$$I^*(x) = (x \wedge b_1 - x_1^*)_+ + (x \wedge b_2 - x_2^*)_+,$$

for some values $x_1^* \in [0, b_1]$ and $x_2^* \in [b_1, b_2]$.

The proof of the proposition is given in Appendix C. We next determined the values of x_1^* and x_2^*. We first present the results when the budget constraint is not binding and then those for the binding case.

Scenario A: The budget constraint is not binding

In this case, the KKT multiplier is zero. Then the two roots of Equation (D1) are given by

$$x_1 = F_Q^{-1}\left(\frac{\xi}{1+\xi}\right), \quad x_2 = F_Q^{-1}\left(\frac{\xi+\tilde{\alpha}}{1+\xi}\right).$$

Since x_1 is fixed and x_2 has a one-to-one relation with $\tilde{\alpha}$, we next determine optimal value of x_2. In terms of x_2, the objective function V in Proposition 3.2 can be rewritten as:

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$$\tilde{V}(x_2) = \alpha\phi\left(x_1 + C + (1+\xi)\int_{x_2}^{b_2} S_Q(x)dx\right) + (1-\alpha)\phi\left(x_2 + x_1 - b_1 + C + (1+\xi)\int_{x_2}^{b_2} S_Q(x)dx\right),$$

where $C = (1+\xi)\int_{x_1}^{b_1} S_Q(x)dx$ is a constant. Since $\phi(\cdot)$ is a strictly increasing convex function, it is easy to check that $\tilde{V}''(x_2) > 0$ and we have the following results:

- Case 1: If $S_Q(b_1) \le \frac{1-\alpha}{1+\xi}$, then

$$\tilde{V}'(b_1) = \phi'\left(x_1 + C + (1+\xi)\int_{b_1}^{b_2} S_Q(x)dx\right)\left\{1 - \alpha - (1+\xi)S_Q(b_1)\right\} \ge 0,$$

  and the function $\tilde{V}(x_2)$ reaches its minimum at $x_2 = b_1$. In this case, $x_1^* = x_1 \wedge b_1$, where $x_1 \wedge b_1 = min(x_1, b_1)$ and $x_2^* = b_1$. Therefore, the optimal marginal indemnity function is given by

$$\eta^*(x) = \mathbb{1}_{(x_1^*, b_1)}(x) + \mathbb{1}_{(b_1, b_2)}(x) = \mathbb{1}_{(x_1^*, b_2)}(x).$$

  The corresponding optimal indemnity function is

$$I^*(x) = \left(x \wedge b_2 - x_1^*\right)_+ .$$

- Case 2: If $S_Q(b_2) \ge \frac{1-\alpha}{1+\xi} \cdot \frac{\phi'(x_1+C+b_2-b_1)}{\alpha\phi'(x_1+C)+(1-\alpha)\phi'(x_1+C+b_2-b_1)}$, then

$$\tilde{V}'(b_2) = -\alpha(1+\xi)\phi'(x_1 + C)S_Q(b_2) + (1-\alpha)\phi'(x_1 + C + b_2 - b_1)\left\{1 - (1+\xi)S_Q(b_2)\right\} \le 0.$$

  Thereby the function $\tilde{V}(x_2)$ reaches its minimum at $x_2^* = b_2$. Thus the optimal indemnity function is

$$I^*(x) = (x \wedge b_1 - x_1^*)_+.$$

- Case 3: If $S_Q(b_1) > \frac{1-\alpha}{1+\xi}$ and $S_Q(b_2) < \frac{1-\alpha}{1+\xi} \cdot \frac{\phi'(x_1+C+b_2-b_1)}{\alpha\phi'(x_1+C)+(1-\alpha)\phi'(x_1+C+b_2-b_1)}$, then there exists a $x_2^* \in (b_1, b_2)$ such that $\tilde{V}'(x_2^*) = 0$. The obtained optimal indemnity function is

$$I^*(x) = (x \wedge b_1 - x_1^*)_+ + (x \wedge b_2 - x_2^*)_+.$$

Scenario B: The budget constraint is binding

If the indemnity function found above results in premium exceeding π_0, one needs to identify a KKT multiplier such that $\eta^*(x; \tilde{\lambda}, \tilde{\alpha})$ satisfies

$$(1+\xi)\int_0^M S_Q(x)\eta^*(x; \tilde{\lambda}, \tilde{\alpha})dx = \pi_0,$$

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which due to Equation (D2) becomes

$$\int_{x_1}^{b_1} S_Q(x)dx + \int_{x_2}^{b_2} S_Q(x)dx = \frac{\pi_0}{1+\xi}. \tag{4.1}$$

Hence we may treat x_1 as an implicit function of x_2 and write

$$x_1 = \tau(x_2).$$

In addition, differentiating both sides of (4.1) yields

$$\tau'(x_2) = -S_Q(x_2)/S_Q(x_1).$$

In terms of x_2, the objective function V in Proposition 3.2 can be rewritten as:

$$\tilde{V}(x_2) = \alpha\phi\,(\tau(x_2)+\pi_0) + (1-\alpha)\phi\,(\tau(x_2)+\pi_0+x_2-b_1).$$

It is easy to check that $\tilde{V}''(x_2) > 0$, thus $\tilde{V}(x_2)$ is strictly convex and admits one unique minimum point on $[b_1, b_2]$. The minimum point is determined as follows:

- Case 1: $\tilde{V}'(b_1) \geq 0$
  - if $(1+\xi)\int_{b_1}^{b_2} S_Q(x)dx \leq \pi_0$, $x_1^* = \tau(b_1)$ and $x_2^* = b_1$;
  - if $(1+\xi)\int_{b_1}^{b_2} S_Q(x)dx > \pi_0$, $x_1^* = b_1$ and x_2^* is the solution to $(1+\xi)\int_{x_2}^{b_2} S_Q(x)dx = \pi_0$.
- Case 2: $\tilde{V}'(b_2) \leq 0$
  - if $(1+\xi)\int_0^{b_1} S_Q(x)dx \geq \pi_0$, $x_1^* = \tau(b_2)$ and $x_2^* = b_2$;
  - if $(1+\xi)\int_0^{b_1} S_Q(x)dx < \pi_0$, $x_1^* = 0$ and x_2^* solves $\int_0^{b_1} S_Q(x)dx + \int_{x_2}^{b_2} S_Q(x)dx = \frac{\pi_0}{1+\xi}$.
- Case 3: $\tilde{V}'(b_1) < 0$ and $\tilde{V}'(b_2) > 0$
  There exists a $\hat{x}_2 \in (b_1, b_2)$ such that $V'(x_2)\big|_{x_2=\hat{x}_2} = 0$.
  - if $\int_0^{b_1} S_Q(x)dx + \int_{\hat{x}_2}^{b_2} S_Q(x)dx \leq \frac{\pi_0}{1+\xi}$, $x_1^* = 0$ and x_2^* solves $\int_0^{b_1} S_Q(x)dx + \int_{x_2}^{b_2} S_Q(x)dx = \frac{\pi_0}{1+\xi}$.
  - if $\int_{\hat{x}_2}^{b_2} S_Q(x)dx \geq \frac{\pi_0}{1+\xi}$, $x_1^* = b_1$ and x_2^* solves $(1+\xi)\int_{x_2}^{b_2} S_Q(x)dx = \pi_0$.
  - if $\int_0^{b_1} S_Q(x)dx + \int_{\hat{x}_2}^{b_2} S_Q(x)dx > \frac{\pi_0}{1+\xi}$ and $\int_{\hat{x}_2}^{b_2} S_Q(x)dx < \frac{\pi_0}{1+\xi}$, $x_1^* = \tau(\hat{x}_2)$ and $x_2^* = \hat{x}_2$.

The rest of this section applies these results to the ambiguity-neutral, nonambiguous, and worst-case problems.

4.2. Nonambiguous case

When the DM is certain that the distribution of the underlying loss is, say, F_1, then $\alpha = 1$. By (D1), we have

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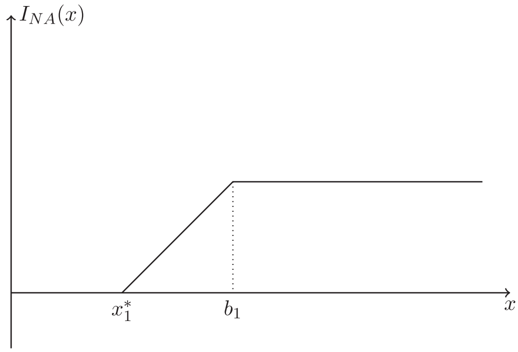

FIGURE 1: The optimal indemnity function $I^*_{NA}(x)$ when F_1 is the distribution of X.

$$x_1 = F_Q^{-1}\left(1 - \frac{1}{(1+\lambda)(1+\xi)}\right), \tag{4.2}$$

and $x_2 \to M$. The parameter λ satisfies $\lambda((1+\xi)\int_0^M S_Q(x)\eta^*(x;\lambda)dx - \pi_0) = 0$.

Applying Proposition 4.1, we obtain the optimal indemnity function:

$$I^*_{NA}(x) = \left(x \wedge b_1 - x_1^*\right)_+, \tag{4.3}$$

where $x_1^* = x_1 \wedge b_1$. Figure 1 illustrates the function $I^*_{NA}(x)$.

The value of λ depends on whether the budget constraint is binding. The derivation is a special case of Scenario A and B in Section 4.1, so the details are omitted here. We note that this result is well known in the literature. See, for example, Cai *et al*. (2008).

4.3. The ambiguity-neutral case

When the DM is ambiguity-neutral, $\phi(x) = x$, we have

$$x_1 = F_Q^{-1}\left(1 - \frac{1}{(1+\lambda)(1+\xi)}\right), \tag{4.4}$$

and

$$x_2 = F_Q^{-1}\left(1 - \frac{1-\alpha}{(1+\lambda)(1+\xi)}\right), \tag{4.5}$$

where λ is determined by the KKT conditions.

Applying Proposition 4.1, the optimal indemnity function is given by

$$I^*_{AN}(x) = \left(x \wedge b_1 - x_1^*\right)_+ + \left(x \wedge b_2 - x_2^*\right)_+, \tag{4.6}$$

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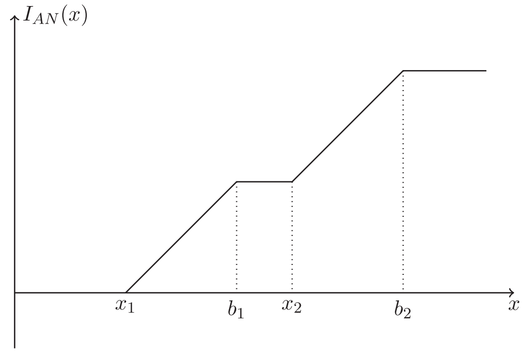

FIGURE 2: The optimal indemnity function $I^*_{AN}(x)$ when $b_1 \le x_2 \le b_2$.

where $x_1^* = x_1 \wedge b_1$ and $x_2^* = \max(x_2, b_1) \wedge b_2$. Figure 2 illustrates the function $I^*_{AN}(x)$. Again, the determination of the KKT multiplier λ is a special case of the discussions in Section 4.1 and therefore omitted here.

Remark 4.1. *In all cases, the optimal indemnity function provides coverage for the losses in the layer* $[x_1^*, b_1]$ *and* $[x_2^*, b_2]$ *for some values* $x_1^* \in [0, b_1]$ *and* $x_2^* \in [b_1, b_2]$. *The two attachment points* x_1^* *and* x_2^* *depend on the safety loading factor, the budget, as well as the ambiguity aversion level. Particularly, when the DM is ambiguity-neutral and there is no budget constraint, some observations for the optimal indemnity function are presented below:*

- *By* (4.4), *if* $\frac{1}{1+\xi} < S_Q(b_1)$, *then* $x_1^* = b_1$ *and no coverage in the layer* $[0, b_1]$ *is purchased;*
- *By* (4.5), *when* $\frac{1-\alpha}{1+\xi} < S_Q(b_2)$, *then* $x_2^* = b_2$ *and no coverage in the layer* $[b_1, b_2]$ *is purchased;*
- *If* $\frac{1}{1+\xi} < S_Q(b_1)$ *and* $S_Q(b_2) < \frac{1-\alpha}{1+\xi}$, *no coverage in the layer* $[0, b_1]$ *is purchased, but some coverage in the layer* $[b_1, b_2]$ *is purchased.*

Intuitively, losses in the layer $[0, b_1]$ *contribute to the VaR under both distribution* F_1 *and* F_2, *while losses in the layer* $[b_1, b_2]$ *contribute to the VaR under* F_2 *only, which was assigned a subjective probability of* $1-\alpha$ *to be true. Therefore, as shown in the first two bullet points above, in making the decision on insurance policy, the DM compares the inverse of the risk loading in the premium* $\frac{1}{1+\xi}$ *(inverse price) weighted by the subjective probability assigned (*$1 = \alpha + 1 - \alpha$ *for layer* $[0, b_1]$, *and* $1-\alpha$ *for layer* $[b_1, b_2]$*) with the corresponding survival probability. If the weighted inverse price is too low (price too high) relative to the survival probability, then no coverage is purchased.*

Notice that in no cases coverage will be purchased for losses above b_2, *because they do not contribute to the VaR under either distributional assumptions of the underlying loss* X.

4.4. The worst case

Now we seek the solution for the most ambiguity-averse DM, who only considers the worst-case scenario. As stated in Corollary 3.2, the form of the optimal marginal indemnity function is the same as that in Section 4.3. However, we need to determine the value of α^* as in Problem 4c.

Referring to (4.5), there is a one-to-one relationship between α and x_2. Therefore, to solve Problem 4c, we next determine the optimal value x_2^* over $[b_1, b_2]$ instead of α^* over [0, 1].

We first consider the budget-free case so that $\lambda = 0$. Define $C = (1+\xi)\int_{x_1}^{b_1} S_Q(x)dx$ and considering the distortion function of VaR and the form of I^*_{AN}, Problem 4c can be written as

$$\max_{\alpha\in[0,1]} \alpha\left(x_1 + C + (1+\xi)\int_{x_2}^{b_2} S_Q(x)dx\right) + (1-\alpha)\left(x_2 + x_1 - b_1 + C + (1+\xi)\int_{x_2}^{b_2} S_Q(x)dx\right). \tag{4.7}$$

Applying the relationship $x_2 = F_Q^{-1}\left(\frac{\xi+\alpha}{1+\xi}\right)$, this is equivalent to

$$\max_{x_2\in[b_1,b_2]} V(x_2), \tag{4.8}$$

where

$$V(x_2) = (1+\xi)S_Q(x_2)(x_2 - b_1) + x_1 + C + (1+\xi)\int_{x_2}^{b_2} S_Q(x)dx.$$

Since

$$V'(x_2) = (1+\xi)S'_Q(x_2)(x_2 - b_1) \le 0,$$

the function $V(x_2)$ reaches its maximum at $x_2^* = b_1$. Thereby the optimal indemnity function is

$$I^*_{WC}(x) = (x\wedge b_1 - x_1)_+ + (x\wedge b_2 - b_1)_+ = (x\wedge b_2 - x_1)_+.$$

This result is intuitive. The DM purchases full coverage for the losses in layer $[b_1, b_2]$ because the worst case is considered (see Figure 3(a)).

The premium corresponding to the optimal marginal indemnity function $I^*_{WC}(x)$ is $(1+\xi)\int_{x_1}^{b_2} S_Q(x)dx$. If this premium level exceeds the budget level π_0, one needs a KKT coefficient λ such that $x_2 = F_Q^{-1}\left(1 - \frac{1-\alpha}{(1+\lambda)(1+\xi)}\right)$ and the budget constraints is binding. That is,

$$\int_{x_1}^{b_1} S_Q(x)dx + \int_{x_2}^{b_2} S_Q(x)dx = \frac{\pi_0}{1+\xi}, \tag{4.9}$$

based on which we write x_1 as an implicit function of x_2, that is, $x_1 = \tau(x_2)$. In this case, Problem 4c can be written as

$$\begin{aligned} &\max_{\alpha\in[0,1]}\ \alpha\left(b_1 - I^*_{AN}(b_1) + \pi_0\right) + (1-\alpha)\left(b_2 - I^*_{AN}(b_2) + \pi_0\right), \\ \Longrightarrow\ &\max_{\alpha\in[0,1]}\ \alpha\,(b_1 - (b_1 - x_1) + \pi_0) + (1-\alpha)\,(b_2 - (b_2 - x_2 + b_1 - x_1) + \pi_0), \\ \Longrightarrow\ &\max_{\alpha\in[0,1]}\ (1-\alpha)(x_2 - b_1) + x_1 + \pi_0. \end{aligned}$$

Since $x_2 = F_Q^{-1}\left(1 - \frac{1-\alpha}{(1+\lambda)(1+\xi)}\right)$, we have $1-\alpha = (1+\lambda)(1+\xi)S_Q(x_2)$. Then the above problem is further equivalent to

$$\max_{x_2\in[b_1,b_2]} V(x_2),$$

where

$$V(x_2) = (1+\lambda)(1+\xi)S_Q(x_2)(x_2 - b_1) + \tau(x_2),$$

where x_1 is expressed as a function of x_2 through (4.9).

It can be verified that

$$\tau'(x_2) = -S_Q(x_2)/S_Q(x_1) = -(1+\lambda)(1+\xi)S_Q(x_2),$$

and

$$V'(x_2) = (1+\xi)S'_Q(x_2)(x_2 - b_1) \le 0.$$

As such, we have the following.

- If $(1+\xi)\int_{b_1}^{b_2} S_Q(x)dx \ge \pi_0$, then
$$I^*_{WC}(x) = (x \wedge b_2 - x_2^*)_+,$$
where x_2^* solves $(1+\xi)\int_{x_2}^{b_2} S_Q(x)dx = \pi_0$.
- If $(1+\xi)\int_{b_1}^{b_2} S_Q(x)dx < \pi_0$, then
$$I^*_{WC}(x) = (x \wedge b_2 - \tau(b_1))_+.$$

The solution shows that the DM always choose to purchase coverage for the loss in the layer $[b_1, b_2]$ as much as budget allows (see Figure 3(b)).

5. Numerical illustrations

This section provides numerical examples based on the following setting.

The DM has the following two exponential distributions as candidates for the underlying losses:

$$F_1(x) = 1 - e^{-\frac{x}{1000}}, \quad x \ge 0,$$

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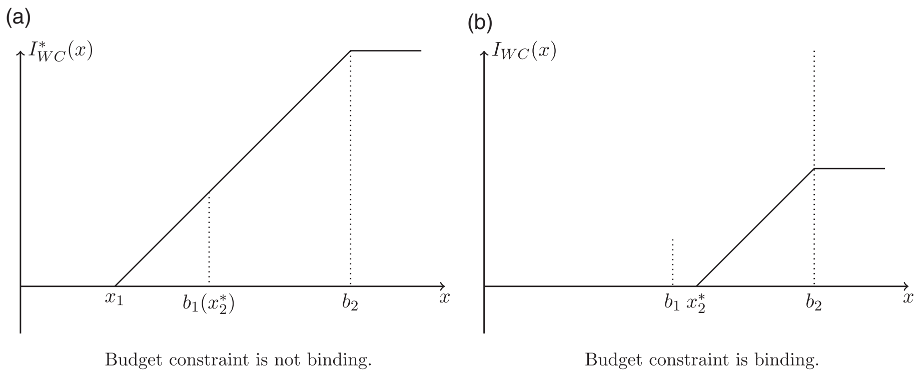

FIGURE 3: The optimal indemnity function $I^*_{WC}(x)$.

and

$$F_2(x) = 1 - e^{-\frac{x}{2500}}, \quad x \geq 0,$$

which were assigned weights $\alpha = 0.995$ and $1 - \alpha = 0.005$ respectively. Assume that the DM is ambiguity averse in the sense of KMM model with the function

$$\phi(t) = e^{at},$$

where a reflects its ambiguity aversion level. Moreover, the DM uses $\mathrm{VaR}_{0.9}(\cdot)$ as its risk measure.

The insurer assumes that the underlying losses follow an exponential distribution with the survival function

$$S_Q(x) = e^{-\frac{x}{2000}},$$

and charge premium according to the expectation principle

$$\pi(I(X)) = 1.5 \cdot \mathbf{E}_Q[I(X)].$$

In the following two subsections, we first derive the exact form of the optimal indemnity function, then we illustrate the impacts of model uncertainty to an ambiguity-averse DM.

5.1. The optimal indemnity function

This section illustrates how the DM's ambiguity aversion level and the budget-constraints affect the optimal indemnity function.

5.1.1. *Budget-free case*

When there is no budget constraint, as per Section 4 the optimal indemnity function takes the form

$$I^*(x) = (x \wedge b_1 - x_1^*)_+ + (x \wedge b_2 - x_2^*)_+,$$

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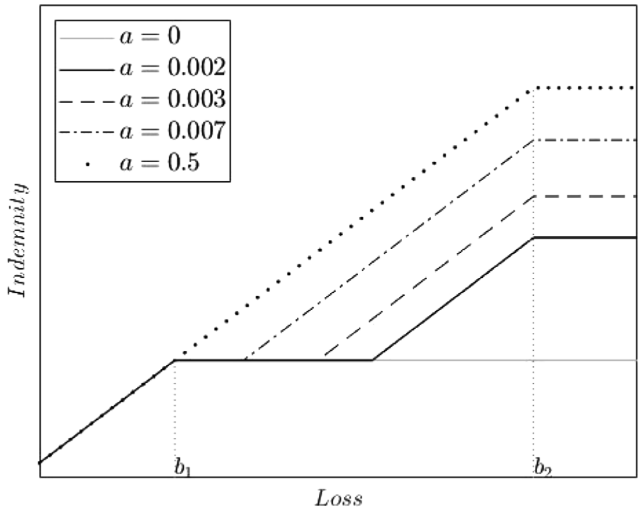

FIGURE 4: The optimal indemnity function for the budget-free case.

where $b_1 = 2303$ and $b_2 = 5757$, and the values of x_1^* and x_2^* are determined by the ambiguity aversion parameter a.

Figure 4 displays these indemnity functions. One can see that when the DM gets more ambiguity averse, more insurance coverage is purchased. For an extreme ambiguous-averse DM ($a = 0.5$), all losses in the layer (b_1, b_2) are covered. This result coincides with the worst-case scenario. Notice that no coverage is purchased for the loss $x > b_2$, since losses beyond b_2 do not contribute to the DM's VaR.

5.1.2. *Budget-constrained case*

In this section, we assume that the DM has budget $\pi_0 = 700$, which is about 1/4 of the premium if the largest coverage is purchased. Based on the analysis done in Section 4, one needs to figure out the optimal x_1 and x_2 subject to

$$\int_{x_1}^{b_1} S_Q(x)dx + \int_{x_2}^{b_2} S_Q(x)dx = \frac{\pi_0}{1+\xi}.$$

The optimal indemnity functions are exhibited in Figure 5(a). It shows that as the DM becomes more ambiguity averse, it would like to sacrifice the coverage of small losses for more coverage of large losses. Figure 5(b) shows the optimal indemnity functions for an ambiguity-averse DM with $a = 0.0002$ but with different budget level. It indicates that if a larger budget is available, more coverage for both small and large losses are purchased.

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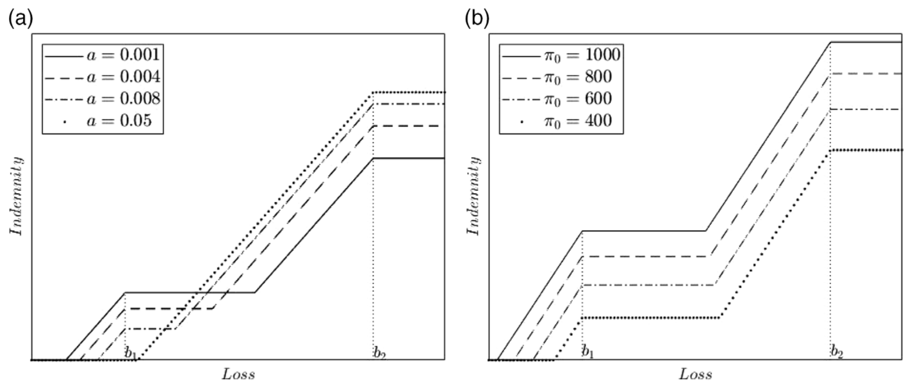

FIGURE 5: (a) The optimal indemnity functions for different ambiguity aversion levels in the presence of budget constraint; (b) The optimal indemnity functions for different budget levels.

5.2. Analyzing suboptimality of the insurance contracts

In the previous sections, we have derived the optimal insurance contract for an ambiguity-averse DM when exact information of the distribution of underlying losses is unknown (ambiguous). By construction, these insurance contracts are optimal in the sense of minimizing the DM's risk based on Problem 1. However, these contracts are not optimal in the general sense as they are not derived based on the true distribution of losses. In real life, our model could be interpreted as the following: the true loss distribution (θ_0) is an element in $\mathcal{M}$ which can be learnt (with extra cost) but the DM decides to act ambiguously (suboptimally). One natural question is what is the cost to the DM (either in terms of extra budget or extra risk) of acting ambiguously. From a practical perspective, DMs may wonder whether they should spend extra resources to get a better sense (estimate) of θ_0. If the cost of acting ambiguously is not high, then it may not be worth learning more about θ_0. Such analysis sheds light on the benefits of learning more about the distribution of losses.

The suboptimal analysis originates from the financial literature, see Liu and Pan (2003) for a pioneer use in portfolio optimization, and more recently Branger *et al.* (2013) and Escobar *et al.* (2015). The type of analysis and framework proposed here shed light on the financial implications of using suboptimal contracts, therefore helping the DM to better understand and measure the influence of its prior belief.

5.2.1. *Measuring the cost of suboptimality*

In this section we provide a mathematical framework to measure the impact of applying a suboptimal contract. For simplicity of presentation , we assume there are two DMs where DM 1 knows exactly that the true loss distribution is F_j where j could be 1 or 2, while DM 2 only knows that the true loss distribution belongs to the set $\mathcal{M}$.

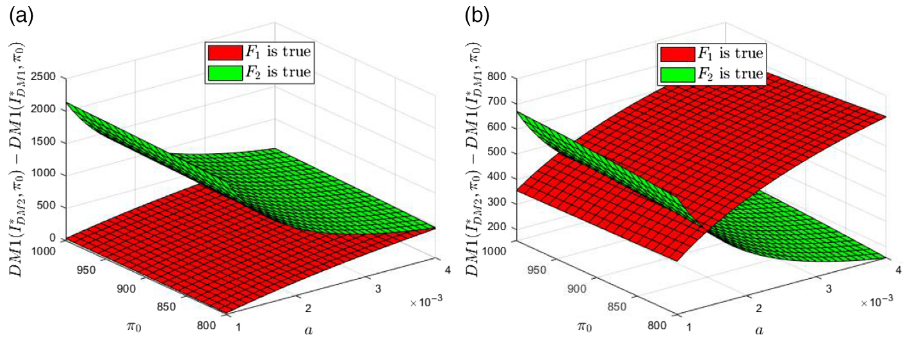

FIGURE 6: The surface plots of $DM1(I^*_{DM2}, \pi_0) - DM1(I^*_{DM1}, \pi_0)$ with different α: (a) $\alpha = 0.995$; (b) $\alpha = 0.9$.

With a fixed budget π_0, DM 1 selects I^*_{DM1} which solves

$$\min_{I\in\mathcal{C}} DM1(I, \pi_0) = \mathbf{H}_{g,F_j}(X - I(X) + \pi(I)) \quad \text{s.t. } \pi(I) \le \pi_0, \tag{5.1}$$

and the DM 2 selects I^*_{DM2} which solves

$$\min_{I\in\mathcal{C}} DM2(I, \pi_0) = \sum_{i=1}^{n} \alpha_i \phi\left(\mathbf{H}_{g,F_i}(X - I(X) + \pi(I))\right) \quad \text{s.t. } \pi(I) \le \pi_0. \tag{5.2}$$

As F_j is indeed the true distribution, the lowest risk level is $DM1(I^*_{DM1}, \pi_0)$. Herein, one can compare the risk level $DM1(I^*_{DM2}, \pi_0)$ and $DM1(I^*_{DM1}, \pi_0)$ to see how unfavorable it is by choosing the suboptimal indemnity function I^*_{DM2}. It is worth mentioning that the risks compared here could be of practical sense. For example, the obtained risks by using the VaR and TVaR are usually referred to as standards of the regulatory capital. In that case, the difference $DM1(I^*_{DM2}, \pi_0) - DM1(I^*_{DM1}, \pi_0)$ could be read as the extra/unnecessary regulatory capital.

Specifically, we consider $\alpha = 0.995$ and 0.9 to examine the influence of prior belief. As either F_1 or F_2 could be true, both cases are investigated.

Figure 6 shows the surface plots of the difference in risk, $DM1(I^*_{DM2}, \pi_0) - DM1(I^*_{DM1}, \pi_0)$, that the DM 2 has to bear under the different budgets and ambiguity aversion levels. Several observations could be concluded below.

- In all cases, with the same budget level, DM 2, who does not have the exact loss distribution, will bear more risk.
- If F_1 (the less severe distribution) is the true distribution, then the difference is increasing w.r.t a. This is as expected because a higher ambiguity aversion level would distort the DM's belief more. This results in giving more weights to F_2, which is the wrong distribution.
- If F_2 is true, the difference is decreasing w.r.t a. This is because a more ambiguity-averse DM will give more weights to the more severe distribution F_2, which happens to be true in this case.

- When α changes from 0.995 to 0.9, the red surface apparently moves upward while the green surface moves downward. This change is also anticipated. If the DM 2 initially puts more weights to F_2, the extra risk it needs to bear is lower when the true loss distribution is F_2. In contrast, DM 2 has to bear more extra risk if F_1 is the true loss distribution.

All these findings show that an ambiguity-averse DM would more or less bear some additional risk in an uncertain environment.

6. Concluding remarks and future research

As one of the most recent developments in modeling the preference under ambiguity, the KMM model has drawn significant interest in both theoretical and practical fields. A study of the distortion risk measures within the framework of KMM model has been conducted by Robert and Therond (2014). This paper builds on the work of Robert and Therond (2014) and studies the design of optimal insurance contract under model uncertainty for an ambiguity-averse DM.

We show that the optimal indemnity functions could be characterized through a simple application of the calculus of variation. However, the explicit solution does not exist for the general case. Nevertheless, the form of the optimal indemnity functions indicates that the solution is indeed that of the ambiguity-neutral case but with distorted preassigned probabilities for the candidate distributions. This finding enables us to translate the original infinite-dimensional optimization problem to a tractable finite-dimensional one. As an important specific example, we provide detailed analysis for a two-state VaR problem. In addition, our numerical examples illustrate the potential impacts of model uncertainty and ambiguity aversion on the optimal insurance contract.

This paper is limited to the condition of finite candidate distributions. It would be meaningful to extend our results to the case with infinite candidate distributions in the future.

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APPENDIX A. PROOF OF PROPOSITION 3.1

We apply the calculus of variation. Suppose that the solution to Problem 2 is η^* and consider its perturbation

$$\eta_\epsilon(x) = (1-\epsilon)\eta^*(x) + \epsilon\eta(x),$$

where $\epsilon \in (0, 1)$, $\eta(x) \in \mathcal{C}'$, and $\int_0^M S_Q(x)\eta(x)dx \leq \pi_0$.

Let

$$V(\epsilon) = \sum_{i=1}^{n} \alpha_i \phi\left(\mathbf{H}_{g,F_i}(X) + \int_0^M h_i(x)\eta_\epsilon(x)dx\right).$$

Due to the assumption $\phi''(\cdot) > 0$, the necessary and sufficient condition for η^* to be optimal is:

$$V'(\epsilon)\big|_{\epsilon=0} \geq 0, \tag{A1}$$

which can be further simplified to be

$$\int_0^M \left\{ \sum_{i=1}^n \alpha_i h_i(x) \phi' \left(\mathbf{H}_{g,F_i}(X) + \int_0^M h_i(x)\eta^*(x)dx \right) \right\} \eta(x)dx,$$
$$\geq \int_0^M \left\{ \sum_{i=1}^n \alpha_i h_i(x) \phi' \left(\mathbf{H}_{g,F_i}(X) + \int_0^M h_i(x)\eta^*(x)dx \right) \right\} \eta^*(x)dx.$$

The inequality above indicates that η^* can be obtained through solving the following problem:

$$\begin{aligned} &\min_{\eta \in \mathcal{C}'} \int_0^M \left\{ \sum_{i=1}^n \alpha_i h_i(x) \phi' \left(\mathbf{H}_{g,F_i}(X) + \int_0^M h_i(x)\eta^*(x)dx \right) \right\} \eta(x)dx, \\ &\text{s.t. } (1+\xi) \int_0^M S_Q(x)\eta(x)dx \leq \pi_0, \quad \xi \geq 0. \end{aligned} \tag{A2}$$

Therefore, the solution to Problem 2 can be obtained by applying the KKT conditions to (A2), which yields:

$$\min_{\eta \in \mathcal{C}'} \int_0^M K(x)\eta(x)dx, \tag{A3}$$

where

$$K(x) = \lambda(1+\xi)S_Q(x) + \sum_{i=1}^n \alpha_i h_i(x) \phi' \left(\mathbf{H}_{g,F_i}(X) + \int_0^M h_i(x)\eta^*(x)dx \right).$$

The integral in (A3) could be minimized through minimizing the integrand $K(x)\eta(x)$ pointwisely (see, e.g., Ludkovski and Young (2009) for similar discussions). In particular, since $\eta(x)$ is bounded by [0, 1], the solution to (A3) is given by

- $\eta^*(x) = 1$ if $K(x) < 0$;
- $\eta^*(x) = \gamma(x)$ is an arbitrary function in $\mathcal{C}'$, if $K(x) = 0$;
- $\eta^*(x) = 0$ if $K(x) > 0$;. ■

APPENDIX B. PROOF OF COROLLARY 3.1

The form of the optimal indemnity function is obvious from Proposition 3.1. We next show the existence of $\lambda \geq 0$ such that the budget constraint is satisfied. First, as shown in Remark 3.2, let $\eta^*(x; 0, \alpha_1, \cdots, \alpha_n)$ solve (3.2) without the constraint (3.3) and let the corresponding budget level be $\pi_0^* = (1+\xi)\int_0^M S_Q(x)\eta^*(x; 0, \alpha_1, \cdots, \alpha_n)dx$. Then if $\pi_0^* \leq \pi_0$, we take $\lambda = 0$. If $\pi_0^* > \pi_0$, we next show that there exists a $\lambda > 0$ such that the premium of the optimal policy $\pi_\lambda^* = (1+\xi)\int_0^M S_Q(x)\eta^*(x; \lambda, \alpha_1, \cdots, \alpha_n)dx = \pi_0$.

First, from (3.12) and (3.13), we can see that for $\lambda_1 \geq \lambda_2$, $\tilde{D}_{\lambda_1} \subseteq \tilde{D}_{\lambda_2}$, $\tilde{D}_{\lambda_1} \cup \tilde{E}_{\lambda_1} \subseteq \tilde{D}_{\lambda_2} \cup \tilde{E}_{\lambda_2}$. Therefore, $\eta^*(x, \lambda, \alpha_1, \ldots, \alpha_n)$ is decreasing w.r.t λ.

Second, when $\lambda \downarrow 0$, $\eta^*(x, \lambda, \alpha_1, \ldots, \alpha_n) \uparrow \eta^*(x, 0, \alpha_1, \ldots, \alpha_n)$ for each $x \in [0, M]$; similarly if $\lambda \to +\infty$, $D_\lambda \cup E_\lambda \downarrow \emptyset$, then $\eta^*(x, \lambda, \alpha_1, \ldots, \alpha_n) \downarrow 0$ for each $x \in [0, M]$.

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Third, if $\{\lambda_m\}_{m=1,2,...} \to \lambda_0$, then $\{\eta^*(x, \lambda_m, \alpha_1, \ldots, \alpha_n)\}_{m=1,2,...}$ converges to $\eta^*(x, \lambda_0, \alpha_1, \ldots, \alpha_n)$ pointwisely. Thus we can apply Lebesgue's Dominated Convergence Theorem to conclude that,

$$\lim_{m\to\infty} \int_0^M S_Q(x)\eta^*(x, \lambda_m, \alpha_1, \ldots, \alpha_n)dx = \int_0^M S_Q(x)\eta^*(x, \lambda_0, \alpha_1, \ldots, \alpha_n)dx.$$

In other words, the integral $\int_0^M S_Q(x)\eta^*(x, \lambda, \alpha_1, \ldots, \alpha_n)dx$ is continuous w.r.t λ.

Based on the above three points, we conclude that there exists a $\lambda \geq 0$ such that the budget constraint, $\int_0^M S_Q(x)\eta^*(x, \lambda, \alpha_1, \ldots, \alpha_n)dx \leq \pi_0$ is satisfied. ■

APPENDIX C. PROOF OF PROPOSITION 3.2:

First note that the sets D_λ and E_λ in Equations (3.5) and (3.6) can be rewritten as

$$D_{\tilde{\lambda}} = \left\{x \mid x \in [0, M],\ (1+\tilde{\lambda})(1+\xi)S_Q(x) < \sum_{i=1}^{n} \tilde{\alpha}_i g(S_i(x))\right\}, \tag{C1}$$

$$E_{\tilde{\lambda}} = \left\{x \mid x \in [0, M],\ (1+\tilde{\lambda})(1+\xi)S_Q(x) = \sum_{i=1}^{n} \tilde{\alpha}_i g(S_i(x))\right\}, \tag{C2}$$

where

$$\tilde{\lambda} = \frac{\lambda}{\sum_{i=1}^n \alpha_i \phi'\left(\mathbf{H}_{g,F_i}(X) + \int_0^M h_i(x)\eta^*(x; \lambda)dx\right)}, \tag{C3}$$

$$\tilde{\alpha}_i = \frac{\alpha_i \phi'\left(\mathbf{H}_{g,F_i}(X) + \int_0^M h_i(x)\eta^*(x; \lambda)dx\right)}{\sum_{i=1}^n \alpha_i \phi'\left(\mathbf{H}_{g,F_i}(X) + \int_0^M h_i(x)\eta^*(x; \lambda)dx\right)}, \quad i = 1, 2, \ldots, n. \tag{C4}$$

Therefore, for any function $\eta(x, \lambda)$ that has the form (3.4) to (3.7), there corresponding to it exists a function $\eta(x; \tilde{\lambda}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ that has the form (3.11) to (3.14). Therefore, finding the solution to problem 2, $\eta^*(x, \lambda)$, is equivalent to finding the solution to problem (3.15). ■

APPENDIX D. PROOF OF PROPOSITION 4.1

In this case, the function $\tilde{K}(x)$ in Corollary 3.1 becomes

$$\begin{aligned}&(1+\tilde{\lambda})(1+\xi)S_Q(x) - \tilde{\alpha} g_\beta\,(S_1(x)) - (1-\tilde{\alpha}) g_\beta\,(S_2(x)) \\ &= \begin{cases} (1+\tilde{\lambda})(1+\xi)S_Q(x) - 1, & x \in [0, b_1], \\ (1+\tilde{\lambda})(1+\xi)S_Q(x) - (1-\tilde{\alpha}), & x \in (b_1, b_2], \\ (1+\tilde{\lambda})(1+\xi)S_Q(x), & x \in (b_2, M], \end{cases}\end{aligned} \tag{D1}$$

which is shown in the illustrative Figure D.1.

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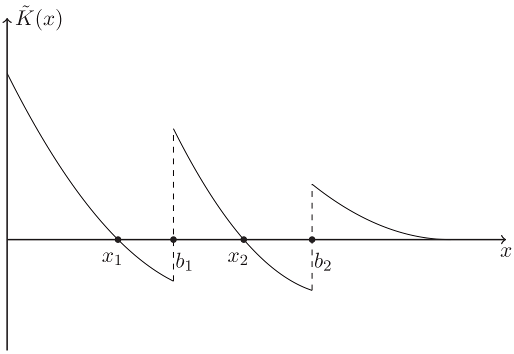

FIGURE D.1: The shape of function $\tilde{K}(x)$.

As indicated in Figure D.1, we let x_1 and x_2 denote the roots of the first two equations in (D1), disregarding the bounds for x. Then the optimal contract takes the form

$$\eta(x) = \mathbb{1}_{(x_1,b_1)}(x) + \mathbb{1}_{(x_2,b_2)}(x), \tag{D2}$$

with the understanding that $(x, b) = \emptyset$ for $x > b$. Note that the values of roots x_1 and x_2 depend on the value of $\tilde{\alpha}$. Therefore, determining the optimal value of $\tilde{\alpha}$ in Proposition 3.2 is equivalent to determining the optimal values of x_1 and x_2, which we denote by x_1^* and x_2^*. ■

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