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GENERALITY AND EXISTENCE: QUANTIFICATIONAL LOGIC IN HISTORICAL PERSPECTIVE

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Abstract. Frege explained the notion of generality by stating that each its instance is a fact, and added only later the crucial observation that a generality can be inferred from an arbitrary instance. The reception of Frege's quantifiers was a fifty-year struggle over a conceptual priority: truth or provability. With the former as the basic notion, generality had to be faced as an infinite collection of facts, whereas with the latter, generality was based on a uniformity with a finitary sense: the provability of an arbitrary instance.

§1. Frege's generality. The quantifiers, *all* for generality and *some* for existence, are as old as logic itself, but it was Frege in the *Begriffsschrift* who got a decisive hold of the principles of reasoning with them. Others, though, were slow in following him, partly because Frege explained generality in the first place by stating that each its instance is a "fact," and just later added the "illuminating" observation by which generality can be inferred from an *arbitrary* instance. It took over fifty years to arrive at a perfect understanding of the quantifiers, in the form of autonomous, purely formulated rules of inference for the universal and existential quantifiers in the work of Gentzen. In the meanwhile, a wide spectrum of attitudes towards the quantifiers and quantificational logic emerged, from a somewhat ambivalent refusal as in Skolem, to a whole-hearted platonistic acceptance as in Tarski.

Frege's introduction of quantificational logic in the *Begriffsschrift* contains a section 11 titled *Generality* (Die Allgemeinheit, p. 19) that explains the logic of the universal quantifier, modern notation $\forall x F(x)$: Its assertion means that whatever a is set in place of x, a "fact" $F(a)$ always results. One page later (p. 20), mere $\forall x F(x)$ without the assertion-component is declared to mean that $F(a)$ "is always a fact, whatever one should put in place of a." The quoted passage is, of course, the way a universal assumption is put to use and as such correct. How, then, to understand universal claims, against assumptions?

What I consider to be the crucial passage on Frege's universal quantifier is found on p. 21, again with the abuse of a modern notation:

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> *It is even illuminating that one can derive $A \supset \forall x F(x)$ from $A \supset F(a)$ if A is an expression in which a does not occur and if a stands in $F(a)$ only in the argument places.*[1]

In the former passage, Frege is giving what evolved into the standard semantical account of universality: $\forall x F(x)$ holds or is true in a domain whenever each instance $F(a)$ is true. Now we have, in all but trivial cases with a finite domain, an infinity of truths, and this went against the grain of finitists such as Skolem and Wittgenstein, as I shall explain. In the latter passage, the one Frege thought illuminating and even put in italics, he is not stating anything about truth, but just gives the essential principle for *reasoning about generality*, what became the rule of inference for the universal quantifier in Hilbert and Bernays' axiomatic logic, and the universal introduction rule of Gentzen's natural deduction.

After the quoted passage, Frege justifies the step of derivation in it: If $\forall x F(x)$ is denied, "one must be able to give a meaning to a such that $F(a)$ becomes denied." However, because $A \supset F(a)$ was admitted, "whatever a could be, the case in which A is admitted and $F(a)$ denied is excluded" (ibid., p. 22). Frege's justification gives his real interpretation of the universal quantifier, namely that there is *no counterexample*. The matter is seen clearly when Frege goes on to show by examples how his logic functions. In section 22, he gives the logical law $\forall x F(x) \supset F(a)$ and comments that if $\forall x F(x)$ is affirmed, $F(a)$ "cannot be denied. Our theorem expresses this." Instead of simply stating that any instance must hold, he is following the tradition of Aristotle who explained generality by: "A thing is predicated of all of another when there is nothing to be taken of which the other could not be said" (*Analytica Priora* 24b28).

Frege finished his *Grundgesetze der Arithmetik* I (Basic laws of arithmetic) fourteen years after the *Begriffsschrift*, in 1893. The propositional machinery is developed somewhat in relation to the earlier account, to make the construction and display of formal derivations easier. As for the quantifiers, the universal quantifier is taken into use with the motivation that one needs to make a distinction between the generality of a negation and the negation of a generality (p. 12). A generality is a truth if each its instance is a truth, and existence is defined in the usual way (ibid.). The inference to generality is presented inconspicuously twenty pages later, as a discourse about Latin and German (fraktur) letters, the former used as free and the latter as bound variables (p. 32). Frege's notation for the universal quantifier has a horizontal line in front of the formula, with a "notch" and a fraktur letter in the notch. The scope of a quantifier is the formula right of the horizontal line. The condition is that the letter, the quantified variable, must not remain free anywhere outside the scope, thus making the free variable an eigenvariable (pp. 32–33). On pp. 61–64, Frege summarizes his axioms and rules. Universal generalization is not presented as a rule of inference, but as modification (Verwandlung) of a proposition that is already at hand (p. 62).

[1] A reader of the Van Heijenoort edition will miss Frege's emphasis, for the translation of Frege's *Auch ist einleuchtend* is a bland *It is clear also.*

In sum, Frege presented the two principles, universal generalisation and instantiation, in what I take to be a reverse of the conceptual order of things, with a confusion as a result that took some five decades to clear, the final words set by the Göttingen logicians Hilbert, Bernays, and Gentzen. Aspects of this history are covered, in so many words, in Goldfarb (1979). Here I shall consider the reception of Frege's quantification theory through Russell to Skolem, the Göttingers, Wittgenstein's second coming as a philosopher, in particular his work on the foundations and philosophy of arithmetic from 1929 to about mid-1930s, and end up with the intuitionists Brouwer and Heyting.

§2. Enter Russell. Russell's book *The Principles of Mathematics* of 1903 contains his first published acknowledgement of Frege's achievement. The book is an old-style synthetic presentation of the foundations of mathematics as a whole, including even a lot of classical mechanics. No explicit logical notation is used, but the treatment is based on Peano's work. Russell thinks he can get along with a single primitive notion in logic, what he called the "formal implication" rendered as "ϕx implies ψx for all values of x" (p. 11). Peano had used the notation $\phi x \supset_x \psi x$ for such an implication with a free variable, typically an eigenvariable in an inductive step from x to its successor x'. That was the nature of Peano's arithmetic: the quantifiers were absent in his formalism.

Russell tells in the preface to his book that he had seen Frege's *Grundgesetze der Arithmetik* but added that he "failed to grasp its importance or to understand its contents," the reason being "the great difficulty of his symbolism" (p. xvi). Upon further study, he wrote a lengthy appendix with the title *The logical and arithmetical doctrines of Frege* (pp. 501–522), though with just a disappointing half a page dedicated to the formalism of logic. He notes the appearance of the universal quantifier in Frege (p. 519):

> He has a special symbol for assertion, and he is able to assert for all values of x a propositional function not stating an implication, which Peano's symbolism will not do. He also distinguishes, by the use of Latin and German letters, respectively, between *any* proposition of a certain propositional function and *all* such propositions.

Frege's Latin and German letters stand for free and bound variables.

In a less known paper of 1906, *The theory of implication*, Russell develops the calculus of logic formally, with negation and implication as primitives, but without quantifiers. The deductive formalism is taken over *verbatim* from Peano's work (for details, see von Plato 2013, section 14.3.C). The universal quantifier makes instead its next appearance in Russell's famous 1908 paper on the theory of types. Its section II is titled *All and any*. Mathematical reasoning proceeds through *any*: "In any chain of mathematical reasoning, the objects whose properties are being investigated are the arguments to *any* value of a propositional function" (p. 156). Still, reasoning with just free variables would not do, for bound variables are needed in definitions

(Russell's terminology for free and bound variables is "real" and "apparent"). Remarkably, his example is from mathematics proper (ibid.):

> We call $f(x)$ continuous for $x = a$ if, for every positive number $\sigma \ldots$ there exists a positive number $\varepsilon \ldots$ such that, for all values of δ which are numerically less than ε, the difference $f(a+\delta) - f(a)$ is numerically less than σ.

He goes on to explain that f appears in the definition in the *any*-mode, as an arbitrary function, and that σ, ε, and δ instead are just apparent variables without which the definition could not be made. Next Russell goes on to introduce a formal notation for the universal quantifier, $(x)\phi x$, presumably the first such notation in place of Frege's notch in the assertion sign, if we disregard the Π_x notation in Schröder's algebraic logic. The explanation, though, is a disappointment, for it is stated that $(x)\phi x$ denotes the proposition "ϕx is always true" (p. 156), a hopeless mixing of a proposition with an assertion that would never have occurred in Frege. Later, in the more formal section VI of the paper, this is corrected when the Fregean assertion sign $\vdash$ is put to use.

Russell's first example of a quantificational inference is: from $(x)\phi x$ and $(x)\phi x$ *implies* $(x)\psi x$ to infer $(x)\psi x$ (pp. 157–8):

> In order to make our inference, we must go from 'ϕx is always true' to ϕx, and from 'ϕx always implies ψx' to 'ϕx implies ψx,' where the x, while remaining any possible argument, is to be the same in both.

As can be seen, the rule is applied by which instances can be taken from a universal, after which the propositional rule of implication elimination can be applied. Then, since x is "any possible argument," ψx is always true, by which $(x)\psi x$ has been inferred (pp. 157–8). Here we have a clear case of the introduction of a universal quantifier. A further remarkable feature of Russell's example is its purely hypothetical character. He does read the universal propositions in the "is always true" mode, but the argument begins with: "Suppose that we know $(x)\phi x$," thus, we have here a universal assumption that is put into use by the rule of universal elimination.

One could add that bound variables are needed, not only in definitions, but also in theorems, say, in almost any standard result about continuous functions.

Russell ends his discussion of *all and any* in section II by praising Frege:

> The distinction between *all* and *any* is, therefore, necessary to deductive reasoning and occurs throughout in mathematics, though, so far as I know, its importance remained unnoticed until Frege pointed it out.

Russell's section VI of the type theory paper gives the formal machinery of his type theory. It includes an axiomatization of propositional logic with negation and disjunction as primitives, and implication elimination as the rule, formulated in the logicist manner as: "A proposition implied by a true premiss is true" (p. 170). The existential quantifier is defined through the universal one in the standard way, as $(\exists x).\phi x\ .=.\ \sim\{(x).\sim\phi x\}$, another notational novelty and presumably the first appearance of the existential quantifier not counting Schröder's Σ_x. The axioms and rules for the universal quantifier are:

(7) $\vdash: (x).\phi x. \supset .\phi y$
(8) If ϕy is true, where ϕy is any value of $\phi\hat{x}$, then $(x).\phi x$ is true.
(9) $\vdash: (x).\phi x. \supset .\phi a$, where a is any definite constant.

Axiom (7) gives the license to infer to a free-variable expression, used informally in examples we have quoted. In (8), the rule of universal generalization, the expression $\phi\hat{x}$ denotes the propositional function, as opposed to its particular value for an argument, as Russell explains in a footnote. Thus, it is a notation for functional abstraction. As to the last axiom, Russell sees no "infinity of facts" brought in by universal instantiation, contrary to many others who read Frege, but just writes (pp. 170–71): "It is the principle, that a general rule may be applied to particular cases."

Principles (7) and (9) give as contrapositions even the corresponding axioms for the existential quantifier, if it should be chosen as a primitive: a free-variable instance, resp. an instance with a constant, implies existence. As to the rule of inference, existence elimination would be the classical dual of universal introduction, but its formulation is tricky: Starting from Frege's generalization rule, its premiss and conclusion can be turned into the respective contrapositions by which, given $F(a) \supset A$ with a satisfying the same conditions as in Frege's rule, $(\exists x)F(x) \supset A$ can be derived. Existential elimination can be found in this form in Hilbert and Ackermann (1928). Can it be that late when the use of existential assumptions was put on a firm formal footing?

Russell's final word on logic is contained in the first volume of *Principia Mathematica, PM* for short, that appeared in 1910 and was co-authored with A. Whitehead. I take Russell to have been the driving force behind the enterprise and refer only to him even if details of *PM* may have originated with Whitehead.

Russell's years as an active logician were in fact not that many, less than ten anyway, and publications of note few in number. In 1927, in a preface to the second edition of *PM*, he makes a remark that is likely to startle a modern reader, namely that Sheffer's stroke, the single connective by which one can axiomatize classical propositional logic, is "the most definitive improvement resulting from work in mathematical logic during the past fourteen years" (p. *xiv*).

The presentation of logic in *PM* is somewhat different from Frege and the 1908 formulation that followed Frege. Part I, titled "Mathematical logic," begins with section A on "the theory of deduction" (pp. 90–126), followed by a "theory of apparent variables" (pp. 127–160). There are moreover things pertinent to propositional and predicate logic in the introductory part, such as on page 3 where we find a typical logicist slogan: "An inference is the dropping of a true premiss."

In first-order logic, as Russell calls it, *PM* has first both quantifiers as primitive, alongside negation and disjunction. The reason lies in Russell's reservation about applying the propositional connectives to quantified propositions. Motivated by ideas from type theory, Russell writes that, when applied to propositional and quantificational expressions, respectively, "negation and disjunction and their derivatives must have a different meaning" (p. 127). Russell's way out is to not to apply propositional connectives to quantified expressions at all, but to instead introduce negation and disjunction for quantified propositions as *defined* notions, here with Russell's numbering but the dot notation replaced by parentheses (p. 130):

$*9{\cdot}01\ \sim(x)\phi x \equiv (\exists x)\sim\phi x$
$*9{\cdot}02\ \sim(\exists x)\phi x \equiv (x)\sim\phi x$
$*9{\cdot}03\ (x)\phi x \vee p \equiv (x)(\phi x \vee p)$
$*9{\cdot}04\ p \vee (x)\phi x \equiv (x)(p \vee \phi x)$
$*9{\cdot}05\ (\exists x)\phi x \vee p \equiv (\exists x)(\phi x \vee p)$
$*9{\cdot}06\ p \vee (\exists x)\phi x \equiv (\exists x)(p \vee \phi x)$
$*9{\cdot}07\ (x)\phi x \vee (\exists y)\psi y \equiv (x)(\exists y)(\phi x \vee \psi y)$
$*9{\cdot}08\ (\exists y)\psi y \vee (x)\phi x \equiv (x)(\exists y)(\psi y \vee \phi x)$

The two straightforward cases with the same quantifier in both disjuncts are not listed. The effect of the rules is that all propositions with quantifiers become reduced to prenex normal form, a string of quantifiers followed by a propositional formula. Here we have also an explanation of the use of both quantifiers as primitives.

The axioms for the quantifiers are two:

$*9{\cdot}1\ \ \vdash \phi x \supset (\exists z)\phi z$
$*9{\cdot}11\ \vdash \phi x \vee \phi y \supset (\exists z)\phi z$

The latter axiom is needed only for proving the contractive implication:

$$(\exists x)\phi x \vee (\exists x)\phi x \supset (\exists x)\phi x$$

The rule of inference is universal generalization (p. 132): "When ϕy may be asserted, where y may be any possible argument, then $(x)\phi x$ may be asserted." The arbitrariness of y is further explained by: "if we can assert a wholly ambiguous value ϕy, that must be because all values are true." We see in the latter again, as in Frege, that the explanation goes from the truth of the universal proposition to any of its instances, not the other way around.

Formal derivations with the quantifiers in the system of *PM* become easily hopelessly tricky. The reason is the curious synthetic nature of reasoning in axiomatic logic, as compared to natural deduction, combined with the ban on propositional inferences with quantified formulas. To have a feasible system of derivation, Russell shows as a preparatory step that propositional inferences with the latter reduce to inferences within the propositional system.

The first example of quantificational inference is the derivation of:

$*9{\cdot}1\ \vdash (x)\phi x \supset \phi y$

The proof is a formal representation of the following steps of inference that synthesize the conclusion from an instance of excluded middle: First take the

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derivable propositional formula $\sim\phi y \vee \phi y$, then use existential introduction to conclude $(\exists x)(\sim\phi x \vee \phi y)$. Now the existential quantifier is moved in the left disjunct, to get $(\exists x)\sim\phi x \vee \phi y$, and finally definition $*9{\cdot}01$ is used in the reverse direction to get $(x)\phi x \supset \phi y$.

One would expect that in addition to $*9{\cdot}1$, also the rule of existence elimination would be shown a derivable rule, but that rule appears nowhere in *PM*. With hindsight, we know that the premiss of the rule has to be given by a formula such as $\phi y \supset p$, i.e., $\sim\phi y \vee p$, assumed to be derived. Now one can generalize to $(x)(\sim\phi x \vee p)$, then apply definitions $*9{\cdot}03$ and $*9{\cdot}01$ to get $\sim(\exists x)\phi x \vee p$, i.e., $(\exists x)\phi x \supset p$.

In paragraph 9 of section B, Russell gives an alternative account of quantification in which negation and disjunction apply to all propositions. Then existence can be defined in the usual way and its axioms replaced by ones for the universal quantifier, as in the 1908 paper.

Russell gives special emphasis to the notion of formal implication that he took over from Peano and defined in *PM* as (p. 139):

$$\phi x \supset_x \psi x \equiv (x)(\phi x \supset \psi x)$$

The importance of quantificational logic, against mere propositional logic, comes out in Russell's opinion as follows (pp. 20–21): In the latter, "material implications" $p \supset q$ between two propositions "can only be *known* when it is already known either that their hypothesis is false or that their conclusion is true." Such implications are useless, because "in the first case the conclusion need not be true, and in the second it is known already." Only formal implications serve the purpose of "making us know, by deduction, conclusions of which we were previously ignorant."

Russell's argument about the "uselessness" of implication in propositional logic is a logical fallacy, committed because Russell moves from $p \supset q$ to the classically equivalent $\sim p \vee q$, then appeals to the disjunction property for the latter, but that property need not hold in classical logic.

§3. Skolem: yes and no to the quantifiers. Thoralf Skolem was a solitary combinatorial genius who started, apparently on his own, to work with Ernst Schröder's algebra of logic. His long paper with the matchingly long title, *Untersuchungen über die Axiome des Klassenkalküls und über Produktations- und Summationsprobleme, welche gewisse Klassen von Aussagen betreffen*, was finished in 1917 and published in 1919. In it, he established many of the basic results of lattice theory, such as the independence of the axioms. He also defined what came later to be called Heyting algebras, i.e., lattices with an "arrow" operation that imitates implication and established many basic properties (see von Plato 2007 for details). These algebras relate to intuitionistic propositional logic in exactly the same way in which Boolean algebras relate to classical propositional logic. It is quite amazing that Skolem had been able to determine such a structure, considering that intuitionistic logic became formally well-defined only by Heyting's (1930) axiomatic presentation.

In 1919, Skolem became acquainted with the *Principia Mathematica*, but his reaction to Russell's universal and existential quantifiers was one of rejection. This is found in his paper on primitive recursive arithmetic, published in 1923 but written in the Fall of 1919. He states in the introduction that he wants to show that one can give a logical foundation to elementary arithmetic without the use of bound variables. The real motivation comes out in section 4 where the notion of divisibility is considered. Writing in Schröder's notation Σ for the existential quantifier, divisibility $D(a,b)$ can be defined as (p. 160):

$$D(a,b) \equiv \Sigma_x(a = bx)$$

Now comes the essential point:

> Such a definition refers to an infinite—and that is to say unexecutable—work, for the criterion of divisibility consists in *trying out, the whole series of numbers through*, whether one can find a number x such that $a = bx$ obtains.

The situation is saved here, because one can limit the search for x: a divisor of a cannot be greater than a. Therefore, universal and existential quantifiers are finitely bounded and can be used, the definition written, with $+$ Schröder's notation for disjunction, as (p. 161):

$$D(a,b) \equiv \sum_{1}^{a}{}_x(a = bx) \equiv ((a = b) + (a = 2b) + (a = 3b) + \cdots + (a = ba)).$$

The bound variable x has a finite upper bound, and Skolem concludes:

> Therefore this definition gives us a finite criterion of divisibility: one can determine in each case through a finite work—a finite number of operations—whether the proposition $D(a,b)$ holds or not.

In the next section, Skolem defines subtraction $b - c$ in which the problem is to check that $c \leqslant b$; otherwise subtraction is not defined. Thus we have, with $c \leqslant b$ defined as the negation $\overline{(c > b)}$ and a nice overloading of the symbol $+$ as either disjunction or sum (p. 165):

$$\overline{(c > b)} + \sum_{x}(x + b = c)$$

> Here the propositional summation [existential quantifier] in relation to x should be extended over "all" numbers from 1 to ∞. Though, even here it is not necessary to put into use such an actual infinity.

Towards the end of the paper, Skolem makes a remark that fixes his philosophy (p. 186):

> When a class of some objects is given, one would be tempted to say: These things come in a finite number n means that there *exists* a one-to-one correspondence between these objects and the first n numbers. However, there occurs a bound logical variable in this definition, and no limitation to the finite is given a priori for this variable, unless

> there is already at the start a result that states: *the number of possible correspondences is finite.* So, from the point of view of the strict finitism given here, such a result must be proved in advance, for the number of the objects concerned to be definable.

Here we have Skolem's approach that can be put concisely as: *Decidability is the only criterion of existence. All decision procedures have to terminate in a bounded number of steps.*

In elementary arithmetic, the atomic formulas are equations between numerical terms. If the latter don't contain variables, their values can be computed and thus equality becomes decidable, a situation that can be expressed so that the law of excluded middle, $a = b \vee \neg\, a = b$, holds for numerical terms. It of course holds in classical logic, but here the point is that it holds constructively. It is an easy exercise to show that the law of excluded middle for arbitrary propositional formulas becomes derivable. Thus, Skolem's primitive recursive arithmetic with its quantifier-free formulas obeys classical logic. In fact, one can interpret classical propositional logic as that special case of intuitionistic propositional logic in which the atomic formulas are decidable (as done in Negri and von Plato 2001, p. 207).

When the quantifiers are added to primitive recursive arithmetic, it becomes full intuitionistic arithmetic, usually called Heyting arithmetic. When the principle of indirect proof is further added, we get classical Peano arithmetic.

Skolem wrote another paper at about the time of the one on primitive recursive arithmetic, very famous for its first section that contains the Löwenheim–Skolem theorem. There one finds what to me seems a standard formulation of expressions of predicate logic, with conjunction, disjunction, negation, and universal and existential quantification. No logical rules of inference are given for these, but the reading is clear. Skolem gives a definition of what he calls *Zählaussage*, the clear idea being that these are expressions for things that can be counted, i.e., objects for which the language is the one of Russell's "first-order propositions." A counting expression is formed from relations through the five logical operations in which the Schröderian quantifiers Π and Σ range over individuals. The first two examples are (1920, p. 103):

> 1) $\Pi_x \Sigma_y R_{xy}$. This is in words: There is for every x a y such that the relation R_{xy} obtains between x and y.
>
> 2) $\Sigma_x \Pi_y \Sigma_z (R_{xy} + T_{xyz})$. This is in words: There is an x such that a z can be determined for every y so that either the binary relation R obtains between x and y or the ternary T between x, y, and z.

Had Skolem suddenly changed his mind about the Russellian quantifiers? Hardly so. He had found what is now the Löwenheim–Skolem theorem already in 1915–16, while in Göttingen, and dedicates the first section of the 1920 paper to it. The title is again monstrous, *Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen* (Logico-combinatorial investigations on the satisfiability or provability of mathematical propositions, together with a theorem on dense sets), and the contents a potpourri of three or four separate things. The last one is easiest: even Skolem himself writes that "the contents of this paragraph are completely independent of those of the preceding ones" (p. 130, footnote). As to the first section on the Löwenheim–Skolem theorem, that was something on which he wanted to make a statement, namely that nondenumerable infinity is a relative and even fictitious notion. The same is brought forth much more strongly in the 1922 version of the Löwenheim–Skolem theorem, with the explicit criticism of set theory at the end of the paper.

As to sections 2 and 3 of the 1920 paper, they are quite a different matter: Section 2 is titled, in translation, *Solution of the problem to decide if a given proposition of the calculus of groups is provable*. Section 3 has the title *A procedure for deciding whether a given proposition of descriptive elementary geometry follows from the axioms of incidence of the plane.* What to make of these? Everyone knows that Skolem formulated and proved in 1920 the result by which predicate logic is unable to distinguish between the denumerable and what is not denumerable: Any consistent collection of first-order formulas admits of a denumerable interpretation. Jean van Heijenoort, when delivering the English version of Skolem's argument, noted that the rest of Skolem's paper dealt with "decision procedures for the theory of lattices and elementary geometry, as well as with dense sets," but left these parts out of the translation. There is no explanation of the contents of these parts, and the same goes for Hao Wang's introduction to Skolem's *Selected Works in Logic*.

In the algebraic tradition of logic, no formal systems of proof were presented. Perhaps here is also the reason why Schröder's Π and Σ are not taken as having established quantification theory. Skolem's sections 2 and 3 are, however, completely different in this respect and present a "purely combinatorial conception of deduction," as Skolem himself formulates it, with explicit formal rules of proof.

It was around 1992 that Stanley Burris found out the following: The second section of Skolem's paper, on Schröder's "calculus of groups," contains a polynomial-time algorithm for the solution of the word problem for lattices. This result was otherwise believed to have stemmed from Cosmadakis (1988), but is now seen as one of the most important early results on lattice theory, with Skolem counted therefore among the founding fathers of the topic. The third chapter contains a solution to a similar derivability problem in plane projective geometry, analyzed in great detail in von Plato (2007) and extended to cover the axiom of noncollinearity in von Plato (2010). Both works are examples of what Skolem preached in his paper on primitive recursive arithmetic: only the decidable has meaning.

There are some final programmatic remarks in the paper on primitive recursive arithmetic in which Skolem writes that he is not really satisfied with the logical development of primitive recursive arithmetic, for it is too laborious with its logical notation:

> I shall soon publish another work on the foundations of mathematics that is free from this formal laboriousness. Even this work is through and through finitistic; it is built on the principle of Kronecker by which a mathematical determination is a real determination if and only if it leads to the goal in a *finite* number of attempts.

I believe that the following happened: Skolem was, after the paper on primitive recursive arithmetic, working towards the new paper on foundations, with chapters on a finitistic treatment of lattice theory and plane projective geometry as examples. At some point, he gave up and just packed together what he had: The Löwenheim–Skolem result, certainly unsatisfying against the light of the final remarks in the primitive recursive paper, the two finished chapters on how one should really be a finitist, and the "completely unrelated" result in infinitary combinatorics.

The second section of Skolem's paper deals with a specific derivability problem in lattice theory. We can see here in action the programmatic statement about the laboriousness of the logical formalism and the unimportance of notation. In the section on lattice theory, he considers what is known today as the *word problem for freely generated lattices*. We have a basic relation of weak partial order, $a \leqslant b$, assumed to be reflexive and transitive (axioms I and II). Next in lattice theory there are usually two operations, a lattice *meet* and *join*. Given two elements, $a \wedge b$ is the meet, the greatest element below a and b. Thus, the axioms are

$$a \wedge b \leqslant a \quad a \wedge b \leqslant b \quad c \leqslant a \,\&\, c \leqslant b \supset c \leqslant a \wedge b$$

The third axiom states that anything below both a and b is below $a \wedge b$. The axioms for join $a \vee b$ are duals of these. The word problem can now be expressed as: To find an algorithm for deciding if an atomic formula $a \leqslant b$ is derivable from given atomic formulas $a_1 \leqslant b_1, \dots, a_n \leqslant b_n$ in lattice theory or to show that there is none.

Skolem does not use the lattice operations, well known to him, but gives instead a *relational axiomatization* with two added three-place relations that we can write as $M(a, b, c)$ and $J(a, b, c)$ rendered as *the meet of a and b is c* and *the join of a and b is c*. He does not tell the reader why he does this. The axioms for meet, given above with lattice operations, become the following:

If $M(a, b, c)$, then $c \leqslant a$ and $c \leqslant b$.
If $M(a, b, c)$ and $d \leqslant a$ and $d \leqslant b$, then $d \leqslant c$.

The effect is that the axioms for meet and join use a quantifier pattern as in $\forall x \forall y \exists z M(x, y, z)$, existence axiom VI for meet. All of the axioms are without explicit notation in Skolem, in the style of: "There is for arbitrary x and y a z such that..."

Skolem's axioms I–V are rules for the production of new atomic formulas from given ones: New "pairs" and "triples," as he calls his atomic formulas for the order relation and the two lattice relations, are produced from the given atoms by these axioms. This means that the "arbitrary" $x, y, \dots$ in the axioms can be instantiated in any way by any $a, b, \dots$ known from the given

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atoms. When instead axiom VI is put to use, with the existential z instantiated, "there appear newly introduced letters." Skolem uses in fact $\alpha, \beta, \gamma \ \ldots$ as eigenvariables in the elimination of the existential quantifier. It is, on the whole, quite remarkable that Skolem was able to use the elimination rule of the existential quantifier in the proper way. In Göttingen, this insight was won towards the end of the 1920s, it seems. A very clear statement is found in Hilbert (1931): He gives first the rule of existential introduction and then adds (p. 121): "In the other direction, the expression $(Ex)A(x)$ can be replaced by $A(\eta)$, where η is a letter that has not occurred yet."

It would have been in the line of Skolem's "logic-free" approach to use the meet and join operations, for the axioms can be then written as free-variable formulas, as above. The relational axioms are easily proved from the ones with operations. The other direction is somewhat arduous, but it can be done (see von Plato 2013, p. 175). The advantage of Skolem's relational axiomatization is that there are no function terms, but just pure parameters $a, b, c, \ldots$ when the axiomatization is used. He proves in the paper that the existence axioms VI for meet and join are conservative over the rest of the axioms, I–V, for the derivability problem of atomic formulas from atomic assumptions.

It should be clear now that the two sections of the 1920 paper belong to the plan of a new logic-free approach to the foundations of mathematics envisaged in the final remarks of the paper on primitive recursive arithmetic. There is an essential tension, however: Take the meet and join existence axioms, and they are formulated in the typical existential form $\forall x \exists y A(x, y)$. This form is put into perfect even if informal use in the application of universal and existential instantiation, the former by the use of parameters as in $\exists y A(a, y)$, the latter through the use of eigenvariables as in $A(a, \alpha)$.

Perhaps the way out of the dilemma Skolem faced is to think of the axioms, not as anything that should be finitely verified by decision procedures, but as hypotheses. Very little in this direction can be found in Skolem, or in any of the early constructivistic literature. Following Kronecker's lead, the task was to take the natural numbers as concretely given objects, then to build up the basic structures of mathematics on strictly finitistic grounds. Abstract axiomatics, with no intended interpretation as in lattice theory, does not fit well into this picture. Whatever may have been Skolem's thought at this time, around 1920, the three papers (1920, 1922, 1923) were his last words on foundations for several years. By his paper of 1928 on applications of quantifier elimination, there is no criticism of quantificational logic.

§4. Logic in Göttingen. A recent essay by Reinhard Kahle, "David Hilbert and the *Principia Mathematica*," shows how enormously impressed Hilbert was by Russell's rendering of Frege's logic. At last, clarity was brought to inferences with the quantifiers, though the absorption of the novelties took its time.

The work of Hilbert, Bernays, and Wilhelm Ackermann of the 1920s on the logic of the *Principia Mathematica* is presented in some articles such as Hilbert (1922, 1923), Ackermann (1924), Bernays (1926), Hilbert (1927),

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and Hilbert (1931). The paper by Bernays was drawn from his *Habilitationsschrift* of 1918, and included the discovery that one of the propositional axioms of *PM* is in fact a theorem, as well as proofs of independence of the remaining propositional axioms through valuations that have more than two truth-values.

Around 1918, when Hilbert began his work on the basis of Russell, he used Russell's axioms with disjunction and negation as primitives. By the mid-1920s, he and others in Göttingen gave axiomatizations for all the standard connectives separately. The motivation for the connectives was the same as in axiomatic studies in geometry: To separate the role of the basic notions, especially negation. This move proved its worth when the axioms of intuitionistic logic were figured out, as in Heyting (1930). Bernays had in fact found the right axioms already in 1925, as he wrote in a letter to Heyting (found in Troelstra 1990).

The development of logic in Göttingen, excluding the work on separate axioms for each connective and what is known as Hilbert's ε-substitution method, is summarized in the Hilbert–Ackermann book of 1928, *Grundzüge der theoretischen Logik*, apparently largely written by Bernays. It contains many novelties such as explicit rules for the two quantifiers and a definition of derivability under assumptions in axiomatic logic, and has just one severe limitation, namely that its propositional part is limited to the negation-disjunction–fragment with implication a defined notion.

Hilbert–Ackermann has the two familiar quantifier axioms $(x)F(x) \rightarrow F(y)$ and $F(y) \rightarrow (Ex)F(x)$ and the following two rules, even these the work of Bernays as acknowledged by the authors (pp. 53–54):

> Let $B(x)$ be an arbitrary logical expression that depends on x, and A one that doesn't. If now $A \rightarrow B(x)$ is a correct formula, also $A \rightarrow (x)B(x)$ is. One obtains similarly from a correct formula $B(x) \rightarrow A$ the new $(Ex)B(x) \rightarrow A$.

Both rules are quite particular, reminiscent of the rules of sequent calculus. In the rule for existence, one has first that A follows under the assumption with an eigenvariable, then from existence. In the case of the first rule, if A is provable, the rule gives the simple rule of generalization, from $B(x)$ for x arbitrary to infer $(x)B(x)$. A similar dual rule for existence would be to instantiate $(Ex)A(x)$ by an eigenvariable, as in Skolem (1920), and even in Hilbert (1931). In this formulation, the rule does not guide the use of eigenvariables.

In Hilbert–Ackermann, both quantifiers are treated independently. Readers of the *Grundzüge* would not know the reasons that were somewhat particular. First of all, in Hilbert's first "post-Russell" paper of 1922 that aims explicitly at the justification of arithmetic, the logic is restricted to implication and universal quantification, with negation treated by taking in addition to equality $a = b$ also inequality $a \neq b$ as a primitive. The rules for universal quantification seem to be, even if the statement is anything but clear, universal generalization and instantiation, but no use is made of these (p. 167).

Hilbert's next paper, of 1923, is much clearer: At the suggestion of Bernays, there is just one "transfinite axiom," i.e., one that contains quantifiers, written $A(\tau A) \rightarrow A(a)$. The reading is (p. 156): "To each $A(a)$ is associated a determinate object τA such that if $A(\tau A)$, then $A(a)$ for all objects a." Hilbert now writes "definitional axioms" for the quantifiers:

1. $A(\tau A) \rightarrow (a)A(a)$
2. $(a)A(a) \rightarrow A(\tau A)$
3. $A(\tau \overline{A}) \rightarrow (Ea)A(a)$
4. $(Ea)A(a) \rightarrow A(\tau \overline{A})$

The first two axioms give some sort of license for inference to and from free-variable propositions. Thus, the former would be better rendered as a rule of inference, not an implication: Whenever $A(a)$ for an arbitrary a, $(a)A(a)$can be concluded. Hilbert does not elaborate on the third and fourth axioms; however, if in 1 we put $\overline{A}$ in place of A and take contrapositions, we get

$$\overline{(a)\overline{A}(a)} \rightarrow \overline{\overline{A}}(\tau \overline{A})$$

The antecedent is classically equivalent to existence, and the double negation in the consequent can be deleted, which brings us Hilbert's definitional axiom 4. Axiom 3 is obtained similarly from 2.

Hilbert's approach was used in Ackermann's 1924 paper, with the τ-operator for universality changed to its dual ε-operator for existence. The same is found in Hilbert (1927, p. 460), with the axiom:

$$A(a) \rightarrow A(\varepsilon(A))$$

Here, as Hilbert writes, "$\varepsilon(A)$ stands for an object for which $A(a)$ certainly holds if it holds for any object at all."

We have now the equivalences, dual to 1–4 above:

$$A(\varepsilon(A)) \leftrightarrow (Ea)A(a)$$

$$A(\varepsilon(\overline{A})) \leftrightarrow (a)A(a)$$

In Hilbert's last paper, the "Beweis des tertium non datur" of 1931, both quantifiers appear in a standard formulation, as in the Hilbert–Ackermann book.

Just five years after the Hilbert–Ackermann book put predicate logic on a firm and accessible basis, Gentzen had turned the quantifier axioms and rules into the natural deduction rules of predicate logic. The ones for the universal quantifier are:

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ A(y)\end{matrix}}{\forall x A(x)}\,\forall I \qquad \dfrac{\begin{matrix}\Gamma \\ \vdots \\ \forall x A(x)\end{matrix}}{A(a)}\,\forall E$$

As explained by Frege, the y in the premiss of generalization must be arbitrary, or an *eigenvariable* in rule $\forall I$ in the Gentzen formulation, i.e., one that is not free in any of the collection of open assumptions Γ the premiss $A(y)$ depends on.

By these rules, the role or meaning of a universal formula in a derivation depends on its place: if it is an assumption, it is put into use by instantiation. Otherwise, it is either a generality hidden in some other assumption, or it is an assertion that is proved by the introduction rule.

If the introduction of universality by Frege's or Gentzen's rule is followed by the taking of an instance, the two steps can be eliminated, as in Gentzen's normalization procedure for natural derivations (Gentzen 2008, a paper written in 1933):

$$\dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A(y)\end{matrix}}{\forall x A(x)}\,\forall I}{A(a)}\,\forall E \qquad \text{converts to} \qquad \begin{matrix}\Gamma\\ \vdots\\ A(a)\end{matrix}$$

Whenever a universality is instantiated, if it was derived in the canonical way through the universal introduction rule, the instance can be produced without the "transfinite" quantifier steps, to use a phrase of Hilbert's: Just take the derivation of the premiss $A(y)$ and since y is arbitrary, the substitution of y by a in the whole derivation gives a correct derivation of $A(a)$ from the assumptions Γ.

The above shows that Frege's "illuminating" addition to his explanation of universality, namely the principle of reasoning by which universality can be concluded, conveys meaning, and that instantiation is explained on its basis as in Gentzen's conversion. Specifically, if $A(a)$ is quantifier free, the conversion step should have pleased greatly Hilbert, for it is a proof transformation from the transfinite to the finitary that fits perfectly the Hilbert programme.

In Frege's time, logic was classical. Thus, he used implication and negation and universal quantification as primitive notions and defined the rest. Brouwer saw early on, in part already in his 1907 article, that such definitions are purely classical, but clear results in that direction were obtained much later. For example, the independence of the quantifiers in intuitionistic logic was proved first as late as in Prawitz (1965, pp. 59–62).

As mentioned, the idea to treat the connectives and quantifiers separately had already arisen in the tradition of classical logic, perhaps first in Bernays' (1918) attempt at isolating the axioms into groups for each logical operation. Each axiom and rule had to contain, however, at least an implication or a negation, as even seen in the two quantifier rules in Hilbert–Ackermann above. A perfect autonomy and purity of definition was achieved only with Gentzen's proof systems in 1933. By purity is meant that no other logical notions appear in the rules. The rules for the existential quantifier are:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A(a)\end{matrix}}{\exists x A(x)}\,\exists I \qquad\qquad \dfrac{\begin{matrix}\Gamma\\ \vdots\\ \exists x A(x)\end{matrix} \qquad \begin{matrix}\overset{1}{[A(y)]}, \Delta\\ \vdots\\ C\end{matrix}}{C}\,\exists E,1$$

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In rule $\exists E$, y is again an eigenvariable, not free in Δ, C, and the temporary assumption $A(y)$ is closed. Here, again, we see that the I-rule is used for arriving at an existential assertion, and the E-rule is instead used when an existential assumption is at hand or hidden in Γ. As with the universal quantifier, an introduction followed by the corresponding elimination gives rise to a reduction step:

$$\dfrac{\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A(a)\end{array}}{\exists x A(x)}\,{\scriptstyle \exists I} \qquad \begin{array}{c}\overset{1}{[A(y)]},\Delta\\ \vdots\\ C\end{array}}{C}\,{\scriptstyle \exists E,1} \qquad \text{converts to} \qquad \begin{array}{c}\Gamma\\ \vdots\\ A(a),\Delta\\ \vdots\\ C\end{array}$$

The lower part of the right derivation is obtained by substituting a for y in the derivation of the minor, right premiss of rule $\exists E$.

The famous *subformula property* is an easy corollary to the normalization theorem for natural deduction. It is even mentioned in an addition to Gentzen's unpublished 1933 proof of normalization. Gentzen calls it the "hillock theorem" (see von Plato 2008):

> Subformula theorem from the hillock theorem:
>
> There is, among the formulas that are not subformulas of the end-formula, one of a greatest grade. This is an inner hillock. Now one applies the hillock theorem.

In the printed thesis, Gentzen wrote that in the proof of a theorem, "no concepts beyond those that are contained in the final result, and that therefore have to be necessarily used in arriving at it, need be introduced" (1934–35, p. 177).

Gentzen formulated, as is well known, the normalization theorem within sequent calculus, as the *Hauptsatz* or cut elimination theorem. The motive was his failure to prove normalization for classical natural deduction. His detailed proof of normalization for intuitionistic natural deduction came known only in February 2005 when I found a handwritten version of his doctoral thesis. Having found a form of the cut elimination theorem that applied directly to both an intuitionistic and a classical sequent calculus, he had no special need for the normalization theorem and could afford never to mention it. That was, on the whole, quite unfortunate, for sequent calculus remained for a long time a rather esoteric discipline. The wider understanding of Gentzen's achievement, a logical calculus in which reasoning is analytic in the sense of the subformula property, started coming only with Prawitz' 1965 re-examination of natural deduction: Each of the logical connectives, as well as the two quantifiers, stand on their own feet with purely formulated rules that can be combined as needed.

§5. Intuitionistic predicate logic in the 1920s. Developments in the late 1920s led to a formal presentation of intuitionistic predicate logic, in the tradition of axiomatic logic, in Heyting's three-part article published in 1930. The axioms and rules for the quantifiers, found in the second part of

Heyting's article, were taken over from the book of Hilbert and Ackermann (1928). At that time, Brouwer had already presented many properties of intuitionistic logic in his papers, first of all of propositional logic, but even of predicate logic, especially in connection with his famous counterexamples to classical logical laws:

(A) *Brouwer's and Heyting's counterexamples:* When Brouwer developed intuitionistic mathematics, mainly in the 1920s, he did not create any separate formal intuitionistic logic, but had otherwise a perfect command of what was to count as intuitionistically valid in logic. Thus, Brouwer's 1928 paper *Intuitionistische Betrachtungen über den Formalismus* contains the, in fact quite astonishing, insight that the predicate-logical formula $\neg\neg\forall x(A(x) \vee \neg A(x))$ is not intuitionistically valid. First Brouwer makes note of the "intuitionistic consistency of the law of excluded middle," i.e., that the assumption of the inconsistency of excluded middle, $\neg(A \vee \neg A)$, leads to an "absurdity," by which $\neg\neg(A \vee \neg A)$ holds intuitionistically (p. 50). Then he shows by induction that the principle can be generalized to any finite "combination of mathematical properties," as in $\neg\neg((A_1 \vee \neg A_1) \,\&\, \ldots \,\&\, (A_n \vee \neg A_n))$. However, in Brouwer's terminology, "it turns out that the ... multiple law of excluded middle of the second kind [for an arbitrary instead of finite collection of mathematical properties] possesses no consistency." This is just Brouwer's way of expressing the intuitionistic failure of $\neg\neg\forall x(A(x) \vee \neg A(x))$. The counterexample consists of points of the intuitionistic unit continuum for which $\neg\neg\forall x(A(x) \vee \neg A(x))$ holds only if one of $\forall x A(x)$ and $\forall x \neg A(x)$ holds, which latter need not be the case (p. 52). The counterexample comes from the bar theorem by which it is not possible to divide the unit continuum in any nontrivial way. An accessible brief explanation of the bar theorem and its special case the fan theorem is found in Coquand (2004).

The intuitionistic failure of what is today usually called "the double negation shift" $\forall x \neg\neg A(x) \supset \neg\neg\forall x A(x)$, follows from Brouwer's counterexample to $\neg\neg\forall x(A(x) \vee \neg A(x))$. The law $\forall x \neg\neg(A(x) \vee \neg A(x))$ is easily proved, for we get by propositional logic that $\neg\neg(A(x) \vee \neg A(x))$ is provable with x arbitrary. Were the double negation shift an intuitionistic theorem, we could shift the double negation ahead of the universal quantifier in $\forall x \neg\neg(A(x) \vee \neg A(x))$, to conclude $\neg\neg\forall x(A(x) \vee \neg A(x))$, against the unprovability of the double negation of the "universal" excluded middle.

In Heyting's original paper on intuitionistic logic of 1930, *Die formalen Regeln der intuitionistichen Logik*, there are very few formal results. One such result is the independence of the propositional axioms, another the unprovability of $A \vee \neg A$. For the rest, Heyting just states things such as the "correctness" of the following (p. 44, here and later, the standard notation of today is used for uniformity):

$$\neg A \vee B \supset (A \supset B)$$
$$(A \supset B) \supset \neg(A \,\&\, \neg B)$$
$$A \vee B \supset \neg(\neg A \,\&\, \neg B)$$

The claim of correctness can be shown by formal derivations of these formulas in intuitionistic logic. However, when Heyting states that "the converses are all unprovable," there is not much at all to back up this claim. We could, say, consider for the last-mentioned the special case of the converse, with $B \equiv \neg A$:

$$\neg(\neg A \,\&\, \neg\neg A) \supset A \vee \neg A$$

The formula $\neg(\neg A \,\&\, \neg\neg A)$ is intuitionistically provable, so by the unprovability of excluded middle, even $\neg(\neg A \,\&\, \neg B) \supset A \vee B$ must be unprovable. The unprovability of the converse of the first formula is seen similarly, by setting $B \equiv A$ in it, with the result $(A \supset A) \supset \neg A \vee A$, and the second by setting $A \equiv \neg\neg B$. Heyting doesn't make notice of these possibilities.

On p. 50, we find the following formula, with an indication of a formal proof:

$$(A \vee \neg A) \supset (\neg\neg A \supset A)$$

"When the law of excluded middle holds for a definite mathematical proposition A, then even the reciprocity of the complementary species holds for A." The observation is followed by another: "This theorem cannot be inverted," left to Heyting's authority.

The relation between excluded middle and double negation is somewhat involved: Heyting like Brouwer sees that if $\neg\neg A \supset A$ is assumed as a general logical law, the law of excluded middle follows, simply by the intuitionistic derivability of $\neg\neg(A \vee \neg A)$ to which the double negation law is then applied, as in Brouwer (1928, somewhat hidden in the "fourth insight," p. 49). Still, the implication $(\neg\neg A \supset A) \supset A \vee \neg A$ is not a theorem of intuitionistic logic.

By the above, Brouwer and Heyting had obviously a deep grasp of the properties of intuitionistic logic, but they had little in their hands by the way of means for proving their points.

In the second installment of Heyting's 1930 essay, with the slightly modified title *Die formalen Regeln der intuitionistichen Mathematik*, the following are proved in intuitionistic predicate logic (p. 65):

6.77. $\exists x \neg\neg A(x) \supset \neg\neg\exists x A(x)$

6.78. $\neg\neg\forall x A(x) \supset \forall x \neg\neg A(x)$

The proof sketches for these are followed by a passage that we quote in full:

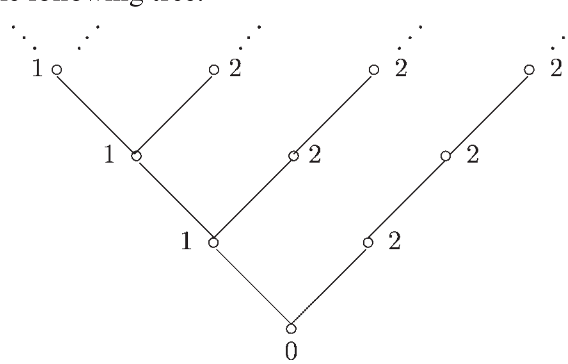
> *Remark*. The inverses of both of the above formulas are not provable. For 6. 77, this follows easily from the meaning of $\exists x$. For 6. 78, we show it by an example already used by Brouwer (Math. Ann. 93, p. 256), of a set A of all choice sequences that consist of only the signs 1 and 2, with 2 always followed by 2. We associate to each natural number an element of A in the following way: To 1 belongs the sequence of ones throughout; to 2 belongs 2 2 2 2 ..., to 3 belongs 1 2 2 2 ..., to 4 belongs 1 1 2 2 ..., etc.

> Let a mean the sentence: "x is associated to a natural number"; let $\forall x\, a$ mean: "a holds for each element of A." Then $\forall x \neg\neg a$ holds, for were an element of A not associated to any number, there would have to occur in it neither ever, nor once, a 2. However, even $\neg \forall x\, a$ holds, for if there were a natural number associated to each element of A, there would be a natural number z such that, for each two elements that are equal in the z first signs, the same number would be associated (Brouwer, Math. Ann. 97 p. 66); so the same number would be associated to all elements that begin with z ones.
>
> However, $\forall x \neg\neg a\ \&\ \neg \forall x\, a$ is by 4. 5 2 1 [a previous derived formula] incompatible with the inverse of 6. 78.[2]

What kind of principles about choice sequences are hidden in this argument?

(B) *Heyting's counterexample in detail:* We consider now Heyting's direct counterexample to the double negation shift in detail. The set of choice sequences $\mathcal{S}$ consists of infinite sequences of 1's and 2's with the condition that whenever there is a first occurrence of 2, all successive members of the sequence are 2's. The variables $x, y, z, \dots$ range over $\mathcal{S}$, and $n, m, k, \dots$ over $\mathcal{N}$. The notation $\overline{x}_k$ stands for the initial segment of x of length k.

If we add 0 as a root element, the choice sequences can be depicted as branches in the following tree:



[2]The German original reads: *Bemerkung.* Die Umkehrungen der beiden vorstehenden Formeln sind nicht beweisbar. Für 6. 77 folgt dies leicht aus der Bedeutung von (Ex). Für 6. 78 zeigen wir es an dem schon von Brouwer benuzten (Math. Ann. 93, S. 256) Beispiel der Menge A aller Wahlfolgen, welche nur aus den Ziffern 1 und 2 bestehen, während nach einer 2 immer wieder eine 2 folgt. Wir ordnen jeder natürlichen Zahl ein Element von A zu in folgender Weise: Zu 1 gehört die Folge von lauter Einsen; zu 2 gehört 2 2 2 2 . . . , zu 3 gehört 1 2 2 2 . . . , zu 4 gehört 1 1 2 2 . . . , usw.

a bedeute den Satz: "x ist einer natürlichen Zahl zugeordnet"; $(x)\, a$ bedeute: "a gilt für jedes element von A." Dann gilt $(x)\neg\neg a$, denn wenn ein Element von A keiner natürlichen Zahl zugeordnet wäre, so müsste in ihm weder niemals, noch einmal eine 2 auftreten. Es gilt aber auch $\neg(x)\, a$, denn wenn jedem Element von A eine natürliche Zahl zugeordnet wäre, so gäbe es eine solche natürliche Zahl z, dass je zwei Elementen, die in den ersten z Ziffern übereinstimmen, dieselbe Zahl zugeordnet wäre (Brouwer, Math. Ann. 97 S. 66); also wäre allen Elementen, die mit z Einsen anfangen, dieselbe Zahl zugeordnet.

$(x)\neg\neg a \wedge \neg(x)\, a$ ist aber nach 4. 5 2 1 unverträglich mit der Umkehrung von 6. 78.

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Next we define a function f over $\mathcal{N}^+$ as follows:

$$\begin{aligned} f(1) &= \langle 1\,1\,1\ldots\rangle \\ f(2) &= \langle 2\,2\,2\ldots\rangle \\ f(3) &= \langle 1\,2\,2\ldots\rangle \\ f(4) &= \langle 1\,1\,2\,2\ldots\rangle \\ &\vdots \\ f(n) &= \langle \underbrace{1\ldots1}_{n-2}\,2\,2\ldots\rangle \end{aligned}$$

Property $A(x)$ (Heyting's a) is: *To x is associated a natural number.* In terms of f, we can write $A(x) \equiv \exists n. f(n) = x$

LEMMA 5.1. $\forall x \neg\neg\exists n. f(n) = x$.

PROOF. Assume $\exists x \neg\exists n. f(n) = x$. With y the eigenvariable in existence elimination, we have the assumption $\neg\exists n. f(n) = y$.

Assume 2 occurs in y. *Then it can be decided* when a first occurrence of 2 takes place: $y = \langle \underbrace{1, \ldots, 1}_{k}, 2, 2, \ldots\rangle$. Now $f(k+2) = y$ against assumption.

Therefore 2 *does not occur in* y, and $y = \langle 1, 1, \ldots\rangle$ and $f(1) = y$ against assumption.

Conclusion: $\neg\exists x \neg\exists n. f(n) = x$, or equivalently, $\forall x \neg\neg\exists n. f(n) = x$. ⊣

There are two specific points, rendered in *italics* in the above: First, if the digit 2 occurs in a choice sequence from the collection $\mathcal{S}$, it is assumed decidable when it occurs for a first time. Secondly, if it doesn't occur, it is concluded that all digits must be 1's. What sort of principles about choice sequences warrant these steps? First, if there is 2 in the sequence y, it can be found. Nothing, however, gives a bound on how far one has to inspect the initial segments of y. The second principle is that if 2 does not occur in y, then $y = \langle 1, 1, \ldots\rangle$. As there are just two alternatives, this seems a justified step.

Heyting's argument seems somehow to go in two directions: First there is a sequence associated to each natural number. Then he considers sequences and asks what numbers are associated to them. Easy "Brouwerian counterexamples" can be used to show that not every sequence constructed by the choice conditions stipulated need have an associated natural number. Example: Let the first member of g be 1 and the n-th member 1 if $2n$ is the sum of two primes, and 2 otherwise. As long as Goldbach's conjecture remains unsettled, no number is associated to g.

LEMMA 5.2. $\neg\forall x \exists n. f(n) = x$.

PROOF. Assume $\forall x \exists n. f(n) = x$. By the bar theorem,

$$\exists k \forall y \forall z (\overline{y}_k = \overline{z}_k \supset f(k) = y \,\&\, f(k) = z)$$

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In particular, with $\overline{y}_k = \overline{z}_k = \langle \underbrace{1, \ldots, 1}_{k-2} \rangle$, $y = z$, but this need not be so.

Therefore $\neg\forall x \exists n. f(n) = x$. ⊣

The fan theorem is actually sufficient for the result, because the tree of choice sequences pictured above has a binary branching.

THEOREM 5.3. $\forall x \neg\neg A(x) \supset \neg\neg\forall x A(x)$ *is not intuitionistically valid.*

PROOF. Let $A(x)$ be $\exists n. f(n) = x$, then by lemma 1, $\forall x \neg\neg\exists n. f(n) = x$, and by lemma 2, $\neg\forall x \exists n. f(n) = x$. ⊣

Heyting's article established intuitionistic logic as a separate discipline in 1930. Choice sequences were taken into use in studying its properties right away, but this aspect seems not to have been discussed in previous literature, unless I have missed something. It is not easy to judge what Heytings' arguments with choice sequences amount to in detail; At least we have that the space of choice sequences is not discrete, for the equality of two choice sequences cannot be decidable, as the Goldbach example shows.

(C) *Failed recognition of a crucial fact:* There is a letter from Gentzen to Heyting, written 23 January 1934, in which Gentzen writes as follows (the letter is included in the collection von Plato 2015):

> My dissertation will appear in the Mathem. Zeitschrift under the title "Untersuchungen über das logische Schliessen." I prove therein a quite general theorem about intuitionistic and classical propositional and predicate logic [the cut elimination theorem]. The decision procedure for intuitionistic propositional logic results as a simple application of this theorem. One can also show with it the intuitionistic unprovability of simple formulas of predicate logic, such as $(x)\neg\neg A\,x\,.\supset\neg\neg(x)A\,x$. I have not studied how far, in the end, one could go. I am now working with the proof of the consistency of analysis that has been since 2 years my real aim.

Gentzen must have studied in detail Brouwer's 1928 paper as well as Heyting's 1930 work in which the double negation shift and the intuitionistic unprovability of $\neg\neg(x)(A\,x \vee \neg A\,x)$ appear. Gentzen obtained unprovability results syntactically from an analysis of cut-free derivations in sequent calculus, thereby showing that these results belong to pure intuitionistic logic, rather than depending on the intuitionistic theory of real numbers. Heyting was initially enthusiastic about natural deduction, as is shown by his series of Dutch papers from 1935 on (see von Plato 2012 for details). It is difficult to understand why he later completely ignored these developments. Imagine what it would mean if things were otherwise than envisaged by Gentzen: The use of intuitionistic reals would be essential for showing the failure of classically provable formulas of predicate logic. Would not intuitionistic predicate logic be incomplete if that were the case? So, why was he not positively alarmed by Gentzen's sensational discovery of a method of syntactic unprovability in intuitionistic predicate logic?

§6. Stricter than Skolem: Wittgenstein and Goodstein. Ludwig Wittgenstein thought he had solved the problems of philosophy in his little book on logical investigations that got baptized into the *Tractatus Logico-Philosophicus* with publication in 1922 under the patronage of Russell.

A careful reader of the *Tractatus* will notice the total absence of the notion of inference or deduction in it. There is instead the semantical method of truth tables by which it can be determined whether a propositional formula is a tautology. How the method is to be extended to the quantifiers is nowhere explained: At 6.1201, the principle of universal instantiation $(x)fx \supset fa$ is simply called a "tautology."

The *Principia* had made it clear that there is no quantificational logic without a *rule* of generalization. Wittgenstein does not see that this rule is crucial, as is shown by the passage 6.1271 in the *Tractatus* where he states that all of logic follows from one basic law, the "conjunction of Frege's *Grundgesetze*." Rules can be no parts of such conjunctions. Moreover, the *Principia* made it clear that the notion of tautology does not extend to the quantifiers. Therefore even the rule of detachment is essentially needed. Wittgenstein missed both of these points and my conclusion is, unfortunately: The impatient philosopher had never made it to page 130+ of the *Principia*!

Realizing later that there are still things to do beyond the *Tractatus*, Wittgenstein turned to philosophy around 1928, and was greatly interested in the philosophy of mathematics. Some of his discussions of the time can be found recorded *verbatim*, through shorthand notes by Friedrich Waismann, in the book manuscript *Philosophische Bemerkungen*, dated November 1930 and published in 1964. The notes are on pages 317–346.

Wittgenstein went to Cambridge in 1929 and became a professor. He prepared long manuscripts on the basis of his lectures that have been published many years after his death in 1951. He also dictated shorter pieces to his students and friends, such as one known as *The Blue and Brown Books*, with several more of these still to be published today.

(A) *Problems with generality and existence:* The book manuscripts, such as the *Philosophische Grammatik* that was written around 1933, contain lengthy discussions of themes related to logic. Regarding the quantifiers, it emerges from these discussions that Wittgenstein was at great pains at understanding them: As in the *Tractatus*, there is no trace of the rule of universal introduction, but quantifiers are instead simply logical expressions of a certain form. Generality is first taken as a "logical product" and existence as a "logical sum," the latter written, with f a predicate, as (p. 269):

$$fa \vee fb \vee fc \vee \dots$$

Generality covers all cases, but its explanation as a "product" of instances becomes infinitistic, and that was not acceptable for Wittgenstein (p. 268). In the absence of a rule of generalization, one gets at most that a universality implies any of its instances. Likewise, existence cannot be a summing up of all the disjunctive possibilities for its introduction, because there is an infinity of such. The dual to universal generalization is existential elimination and in its absence, one gets only that an instance implies existence.

In the *Philosophische Grammatik*, Wittgenstein discusses at length an example, in translation the phrase *The circle is in the square*, illustrated by a drawing of a rectangle and a circle inside (p. 260). It is clearly correct to say that there is a circle in the square, but the statement does not fix any specific circle. Wittgenstein sees that there is a generality behind existence and ponders on the matter page after page; all this because he does not know that there should be a rule of existential elimination, the one Skolem used in an informal way, Hilbert and Bernays wrote in an axiomatic form, and finally Gentzen as a pure rule of natural deduction. Wittgenstein's "generic circle" is correctly presented through the eigenvariable of an existential elimination.

Wittgenstein's first works in his "second period" as a philosopher of logic and mathematics include two specific achievements, both of them somewhat cryptic and clarified only decades later. The first is a constructivization of Euler's proof of the infinity of primes, reconstructed in detail in Mancosu and Marion (2003). The second discovery derives from Wittgenstein's careful reading of Skolem's paper on primitive recursive arithmetic. Both are directly relevant to Wittgenstein's modest understanding of the quantifiers, and to his philosophy of mathematics:

A direct statement of the infinity of primes could be: For any n, there is an m such that $m > n$ and m is prime. The logical form of Euler's argument is: Assume that there is a number n such that for any $m > n$, m is divisible. A contradiction follows. From Euler's argument, we could at most infer that for any n, it is impossible that there should not be a number m such that $m > n$ and m is prime; Still, no way of actually producing a prime greater than n need have been given by the proof. Wittgenstein turned the indirect inference into a direct one. The context was a manuscript of Heinrich Behmann's in which the latter claimed to be able to convert any classical proof into a constructive one. After criticism by, inter alia, Gödel, Behmann withdrew publication. The full story of the Behmann affair is found in Mancosu (2002).

The nature of indirect existence proofs was debated a lot in the 1920s, because of the intuitionistic criticisms of such classical proofs by Brouwer. Wittgenstein's interpretation was that two notions of existence are in fact involved, and that there is no content in denying the law of excluded middle: One just adapts different rules of proof and the sense of the theorems is different. One of these could be called classical existence, the other constructive existence.

So far, so good. However, considering Wittgenstein's wanting understanding of the quantifier rules, it is not surprising that he got some of the properties of universal and existential quantification wrong. He certainly understood the law of excluded middle and the related law of double negation. In the case of indirect existence proofs, the latter can be put in the form of $\neg\neg\exists x A(x) \supset \exists x A(x)$, a law that fails intuitionistically. The properties of intuitionistic logic were not perfectly understood in the early 1930s in general, and here Wittgenstein seems to have committed a specific mistake even though I have so far not found it directly in any text of his: Instead, his pupil Reuben Louis Goodstein followed his lectures in Cambridge in

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1931–34 and started work on a topic to which I shall soon turn. In the meanwhile he published an article titled *Mathematical systems*, in the well-known philosophical journal *Mind* in 1939. It was a statement of what he took to be Wittgenstein's philosophy of mathematics. The article contains many exclamations and positions that should perhaps best be described as silly, but there are even indications that Wittgenstein was not displeased with it, contrary to some writings of other pupils of his.

In the paper, Goodstein maintains that the inference from $\neg\exists x \neg A(x)$ to $\forall x A(x)$ is intuitionistically legitimate. The converse implication is intuitionistically provable, so with the claimed inference, the universal quantifier could be defined by the existential one. Instead, this particular argument against intuitionism and for the "strict finitism" of Wittgenstein and Goodstein is just fallacious: In Goodstein (1951, p. 49), written under Wittgenstein's influence around 1940, it is stated that "some constructivist writers maintain that. . . a 'reduction' proof of universality is acceptable." In Goodstein (1958, p. 300), we find again that Brouwer rejects indirect existence proofs, here $\neg(\forall x)\neg P(x) \rightarrow (\exists x)P(x)$, "whilst retaining the converse implication $\neg(\exists x)\neg P(x) \rightarrow (\forall x)P(x)$." In other words, if $(\exists x)\neg P(x)$ turns out impossible, a reduction gives $(\forall x)P(x)$; certainly not anything Brouwer or any other constructivist thinker would have ever proposed. Goodstein's early paper of 1939 has very likely the same claim, but there is in the formula a dot instead of a negation, on p. 66 of his article. Such a dot has no place there and I take it to be a misprint.

The reason for the above misunderstanding is somewhat subtle. The intuitionistically invalid implication $\neg\exists x \neg A(x) \supset \forall x A(x)$ is perhaps at a first sight rather close to $\neg\exists x A(x) \supset \forall x \neg A(x)$. The latter is intuitionistically provable, in fact one of the first examples of intuitionistically correct inference that Gentzen gave when he presented the calculus of natural deduction in his thesis (1934–35). The argument is very easy: To prove the implication assume $\neg\exists x A(x)$. To prove $\forall x \neg A(x)$, try to prove $\neg A(x)$ for an arbitrary x. For this, in turn, assume $A(x)$ and try to derive a contradiction. It comes in one step: by rule $\exists I$, $\exists x A(x)$ follows, a contradiction. Therefore $\neg A(x)$ follows from the assumption $\neg\exists x A(x)$. The variable x is not free in the assumption, so rule $\forall I$ gives $\forall x \neg A(x)$, and $\neg\exists x A(x) \supset \forall x \neg A(x)$ can be concluded.

One could think that if $\neg\exists x A(x) \supset \forall x \neg A(x)$ is intuitionistically provable, it makes no difference to have $\neg A(x)$ under the negated existence, and $A(x)$ under the universal, instead of the other way around as in the above proof, but this is not in the least so: With $\neg A(x)$ in place of $A(x)$, we do get from what was proved above $\neg\exists x \neg A(x) \supset \forall x \neg\neg A(x)$ as an instance, but the double negation cannot be deleted.

Wittgenstein was not alone with his problems: The correspondence between Heyting and Oskar Becker gives ample illustration of how difficult it was to get intuitionistic logic right, even for people who tried hard (see Van Atten 2005).

A tentative conclusion can be drawn from this little story: Part of the motivation of Wittgenstein's refusal of the quantifiers, even the intuitionistic ones,

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in favour of a strict finitism as in Skolem, was based on misunderstanding the nature of the intuitionistic quantifiers.

We come now to Wittgenstein's second specific discovery:

(B) *From induction to recursion:* In 1945, there appeared in the *Proceedings of the London Mathematical Society* a long article titled "Function theory in an axiom-free equation calculus." The bearing idea of the work was to recast primitive recursive arithmetic in an even stricter mould than the quantifier-free calculus of Skolem. Even logic had to go, and the venerated principle of arithmetic induction as well. The latter is replaced by a principle by which two recursive functions defined by the same equations are the same (p. 407): "If two functions signs 'a', 'b' satisfy the same introductory equations, then '$a = b$' is a proved equation." A footnote added to this principle tells the following: "This connection of induction and recursion has been previously observed by both Wittgenstein and Bernays." The author of the paper, this time not in the least silly, was Wittgenstein's student Goodstein. The full story of his paper can be recovered through the correspondence he had with Paul Bernays. In the opening letter of 29 July 1940, he writes:

> The manuscript which accompanies this letter gives some account of a new formal calculus for the Foundations of Mathematics on which I have been working for the past six years.

Unfortunately, the original version of the paper is not to be found. The most we know are some comments by Bernays such as the following from his first letter to Goodstein, of 28 November 1940:

> Generally my meaning is that your attempt could be quite as well, and perhaps even better appreciated, if you could deliver it from the polemics against the usual mathematical logics which seem to me somewhat attackable, in particular as regards your arguments on the avoidability of quantifiers. Of course in your calculus, like in the recursive number theory, quantifiers are not needed. But with respect to the "current works on mathematical philosophy" the thesis that "the apparent need for the sign '(x)' arose from a confusion of the two different uses ... of variable signs" can hardly be maintained. In fact the possibility of taking $f(x) = 0$ instead of $(x)(f(x) = 0)$ consists [*sc*. exists] only, if the formula in question stands separately and not as a part of a complex logical structure.
>
> So for instance the negation of $(x)(f(x)=0)$, that is $\sim(x)(f(x)=0)$, has of course to be distinguished from the proposition $(x).\sim(f(x) = 0)$; if here the sign"(x)" is left out, then really a confusion is arising. Thus there neither is the possibility of taking simply the proposition $\sim(f(x)=0)$ instead of $(Ex).\sim(f(x)=0)$ (or else one would have to add artificial conventions).

Bernays mentions also that he had presented in 1928 at the Göttingen Mathematical Society "the possibility of taking instead of the complete induction the rule of equalizing recursive terms satisfying the same recursive equations," a discovery he left unpublished. Bernays' first letter to Goodstein is ten pages long, typewritten single-spaced, and it displays his full command of Goodstein's calculus. Goodstein was enormously impressed as can be seen from his letters and thankfully revised his paper and cleared it of polemics, adding all the references to a literature that had been unknown to him; quite embarrassingly, even the extensive treatment of primitive recursive arithmetic in the first volume of the *Grundlagen der Mathematik*, Section 7, pp. 287–343 belonged there.

Before going to the replacement of induction by recursion, a brief word about the disposal of logic: Goodstein, as well as Bernays before him (*Grundlagen*, pp. 310–12), noticed that propositional logic can be reduced to equational reasoning in primitive recursive arithmetic. Equations $a = b$ can be turned into equivalent equations of the form $t = 0$, and now conjunction and negation can be defined: $a = b \,\&\, c = d$ turns into $t + s = 0$, and $\neg\, a = b$ into $1 - t = 0$, with subtraction truncated so that whenever $t > 0$, we have $1 - t = 0$.[3]

The Wittgensteinian background of Goodstein's "logic-free" and "induction-free" arithmetic calculus is not mentioned in the book *Recursive Number Theory* that Goodstein published in the prestigious yellow logic series of the North-Holland Publishing Company in 1957. Instead, when Wittgenstein's book manuscript *Philosophische Grammatik* came out in 1969, one could find his discovery of the way from proof by induction to proof by recursion equations clearly stated, and developed to some extent mainly through a few examples (*Philosophische Grammatik, PG* below, pp. 397–450). The text was written between 1932–34, the years during which Goodstein attended Wittgenstein's lectures. The crucial discovery comes out on the very first page devoted to the topic (*PG*, p. 397), where Wittgenstein considers the associative law for sum in elementary arithmetic, denoted by A:

$$(a + b) + c = a + (b + c) \qquad A$$

Skolem's 1923 paper on primitive recursive arithmetic, Wittgenstein's source for the topic of elementary arithmetic, gives the standard inductive proof for A, based on the recursive definition of sum by the recursion equations:

$$a + 0 = a$$
$$a + (b + 1) = (a + b) + 1$$

If one counts the natural numbers from 1 on, the second equation gives the base case of the inductive proof. For the step case, one assumes A for c and proves it for $c+1$, i.e., $(a+b)+(c+1) = a+(b+(c+1))$. The left side is by the recursion equation equal to $((a+b)+c)+1$, then applying the inductive hypothesis to $(a + b) + c$ one gets $((a + b) + c) + 1 = ((a + (b + c)) + 1$, and finally by two applications of the recursion equation in the opposite direction $((a + (b + c)) + 1 = a + ((b + c) + 1) = a + (b + (c + 1))$.

[3] Some details: Define the predecessor function p by $p(0) = 0$ and $p(a + 1) = a$, then the "truncated" subtraction $a \dot{-} b$ by $a \dot{-} 0 = a$ and $a \dot{-} (b + 1) = p(a \dot{-} b)$. Finally, define the absolute distance between a and b by $d(a, b) = (a \dot{-} b) + (b \dot{-} a)$. Now $d(a, b) = 0$ whenever $a = b$, and negation can be defined by $\neg\, a = b \equiv 1 \dot{-} d(a, b) = 0$, and conjunction further by $a = b \,\&\, c = d \equiv d(a, b) + d(c, d) = 0$. A similar development is found in Curry (1941).

In *PG*, p. 397, Wittgenstein gives the proof as follows:

> What Skolem calls the recursive proof of A can be written as follows:
>
> $$\left.\begin{array}{r} a+(b+1)=(a+b)+1 \\ a+(b+(c+1))=a+((b+c)+1)=(a+(b+c))+1 \\ (a+b)+(c+1)=((a+b)+c)+1 \end{array}\right\} B$$

We have to put emphasis on Wittgenstein's words "can be written," for this is not Skolem's proof by induction, but another proof that Wittgenstein goes on to explain in the following words:

> In the proof [B], the proposition proved clearly does not occur at all.– One should find a general stipulation that licenses the passage to it. This stipulation could be expressed as follows:
>
> $$\left.\begin{array}{ll} \alpha & \varphi(1)=\psi(1) \\ \beta & \varphi(c+1)=F(\varphi(c)) \\ \gamma & \psi(c+1)=F(\psi(c)) \end{array}\right\} \qquad \begin{array}{c} \Delta \\ \varphi(c)=\psi(c) \end{array}$$
>
> When three equations of the forms α, β, γ have been proved, we shall say: "the equation Δ has been proved for all cardinal numbers."

Here we see the essence of the argument: Two functions φ and ψ that obey the same recursion equations, are the same function. Wittgenstein himself writes (*PG*, p. 398):

> I can now state: The question whether A holds for all cardinal numbers shall mean: Do equations α, β, and γ hold for the functions
>
> $$\varphi(\xi)=a+(b+\xi), \qquad \psi(\xi)=(a+b)+\xi$$

Wittgenstein's principle can be considered, as in the letter of Bernays quoted above, a "rule of equalizing recursive terms." Taken as a rule, it is readily seen to be a derivable rule in *PRA:* Given its premisses for two functions φ and ψ, the conclusion follows by the principle of induction. These premisses are:

$$\begin{array}{ll} \alpha & \varphi(1)=\psi(1) \\ \beta & \varphi(c+1)=F(\varphi(c)) \\ \gamma & \psi(c+1)=F(\psi(c)) \end{array}$$

We want to derive $\varphi(x)=\psi(x)$ for an arbitrary x. Equation α gives the base case of induction. For the inductive case, assume $\varphi(y)=\psi(y)$. By β, we get $\varphi(y+1)=F(\varphi(y))$, by the inductive hypothesis $F(\varphi(y))=F(\psi(y))$, and by equation γ next $F(\psi(y))=\psi(y+1)$, so the equation $\varphi(y+1)=\psi(y+1)$ follows. By the principle of induction, $\varphi(x)=\psi(x)$ follows for arbitrary x.

By the above, we see that Wittgenstein's rule contains the essential steps that lead from y to the successor $y+1$, i.e., the inductive step, in a somewhat disguised form.

The way from Wittgenstein's uniqueness principle for recursion equations to induction is similar: Assume given the premisses of induction, $\varphi(1)=\psi(1)$ and $\varphi(y)=\psi(y)\supset\varphi(y+1)=\psi(y+1)$ for an arbitrary y, and the

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task is to prove $\varphi(x) = \psi(x)$ for arbitrary x. The recursive functions φ and ψ are defined by some recursions equations that for the successor case have the forms:

$$\begin{array}{ll} \beta & \varphi(c+1) = F(\varphi(c)) \\ \gamma & \psi(c+1) = G(\psi(c)) \end{array}$$

If $\varphi(y) = \psi(y)$, then $\varphi(y+1) = \psi(y+1)$ by the assumption that the inductive step is given. By the recursion equations, this latter equation gives at once $F(\varphi(y)) = G(\psi(y))$. Therefore, when the arguments $\varphi(y)$ and $\psi(y)$ of F and G are equal, the values are equal. We have now altogether:

$$\begin{array}{ll} \alpha & \varphi(1) = \psi(1) \\ \beta & \varphi(c+1) = F(\varphi(c)) \\ \gamma & \psi(c+1) = F(\psi(c)) \end{array}$$

By Wittgenstein's uniqueness rule, $\varphi(x) = \psi(x)$ follows for any x, i.e., we have reached the conclusion of induction.

Wittgenstein's book does not reveal the motive for preferring proofs by recursion equations to proofs by induction, but in 1972, Goodstein published a paper "Wittgenstein's philosophy of mathematics" in which the matter is explained. In reference to the *Philosophische Grammatik* that had come out three years earlier, Goodstein recalls Skolem's inductive proof and then adds (p. 280):

> In his lectures Wittgenstein analysed the proof in the following way. He started by criticizing the argument as it stands by asking what it means to *suppose* that (1) [associativity] holds for some value C of c. If we are going to deal in suppositions, why not simply suppose that (1) holds for any c.

Goodstein now gives a very clear, intuitive explanation of why Wittgenstein's method works: With $c = 0$, $(a+b)+0 = a+b = a+(b+0)$. Thus, the ground values of Wittgenstein's φ- and ψ-functions are the same, here $\varphi(0) = \psi(0)$ with the natural numbers starting from 0 instead of 1 as in the 1920s. For the rest, when c grows by one, $\varphi(c)$ and $\psi(c)$ obtain their values in the same way, here, both growing by 1, by which $(a+b)+c$ and $a+(b+c)$ are always equal.

As the above-quoted clear recollection on the part of Goodstein shows, Wittgenstein was led to propose a finitism that was even stricter than that of Skolem, in that *assumptions with free variables* were to be banned. These assumptions are a crucial component in inductive inference, where one assumes a property $A(n)$ for an arbitrary natural number n then shows that the successor of n has the property, expressed as $A(n+1)$. However, the assumption $A(n)$ is a far cry from assuming, say in the case of associativity, that the inductive predicate "holds for any c" as Goodstein suggests at the end of the quote. It is the simplest error in inference with the quantifiers to assume $A(x)$, then to conclude $\forall x A(x)$: The eigenvariable condition in universal generalization is that x must not occur free in any assumption on which its premiss $A(x)$ depends, but here one must keep in mind that if $A(x)$

itself is an assumption, it depends on itself so to say, thus, x is free in an assumption. More generally: To assume $A(x)$ is not the same as to assume $A(x)$ provable and only the latter gives $\forall x A(x)$. No amount of philosophical reflection in Wittgenstein can replace the command over quantificational inferences that results from Gentzen's formulation of the quantifier rules in terms of natural deduction.

* * *

I began by stating that the quantifiers are as old as logic itself. That was an implicit reference to Aristotle's syllogistic, a theory of the four quantifiers *every, no, some*, and *not some*, what they mean when prefixed to the indefinite form of predication *A is a B*, and what the correct forms of inference are. Even if Frege was proud to present a formalization of the syllogistic inferences in terms of predicate logic, as the final example of his new notation in the *Begriffsschrift*, no formal quantifiers in the modern sense are needed for their theory, ones that would bind variables. The four kinds of quantified propositions *Every A is a B, Some A is a B, No A is a B*, and *Not some A is a B* can be treated simply as atomic formulas (as in section 14.1 of von Plato 2013). In the light of this fact, it should be no surprise that Aristotle's quantifiers and syllogistic played no role in ancient Greek mathematics.

In Greek mathematical texts, one finds generality and existence treated informally, but according to our best experts on the topic, explicit quantifiers are practically absent (see, e.g., Acerbi 2010, p. 33). A typical pattern would be to begin a theorem by an assumption like *Given two points A and B such that...* with a claim like *to construct a triangle such that....* These situations are clear free-variable inferences. Much confusion has been caused by a similar situation in which the given is taken as an assumption about existence. In that case, the free variables act as the eigenvariables of existential elimination, a tricky move as witnessed by its formal representation as late as in 1928. Moreover, such an existential assumption, rendered as $\exists x A(x)$ in modern notation, can have a consequence B in which the eigenvariable does not occur. Then the result can be $\forall x(A(x) \supset B)$ when instead one would expect it to be $\exists x A(x) \supset B$. The same is seen informally in ancient mathematics, as discussed in my essay review of Acerbi's book (von Plato 2013a). Thus, sometimes the givens act as free variables that lead to generality of a conditional, at other times as eigenvariables in the elimination of an existential assumption. An explanation of what happens here is hidden in Russell's prenex rules for the quantifiers: The application of definitions $*9\cdot03$ and $*9\cdot01$ to $\exists x A(x) \supset B$ gives at once $\forall x(A(x) \supset B)$ with the, certainly less intuitive, consequence that these two expressions are *logically equivalent*.

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