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Potential kernel, hitting probabilities and distributional asymptotics

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Abstract. $\mathbb{Z}^d$-extensions of probability-preserving dynamical systems are themselves dynamical systems preserving an infinite measure, and generalize random walks. Using the method of moments, we prove a generalized central limit theorem for additive functionals of the extension of integral zero, under spectral assumptions. As a corollary, we get the fact that Green–Kubo's formula is invariant under induction. This allows us to relate the hitting probability of sites with the symmetrized potential kernel, giving an alternative proof and generalizing a theorem of Spitzer. Finally, this relation is used to improve, in turn, the assumptions of the generalized central limit theorem. Applications to Lorentz gases in finite horizon and to the geodesic flow on Abelian covers of compact manifolds of negative curvature are discussed.

Key words: central limit theorem, probabilistic potential theory, classical ergodic theory, infinite measure, null recurrent process
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1. *Introduction*

Given a recurrent random walk (S_n) on $\mathbb{Z}^d$, with $d \in \{1, 2\}$, a natural question is how much time the walker spends in any region of the space—the so-called occupation times. More generally, one may choose an observable $f : \mathbb{Z}^d \to \mathbb{R}$, and consider the Birkhoff averages $n^{-1} \sum_{k=0}^{n-1} f(S_k)$. When f is summable and the walk is well-behaved, it is known that $a_n^{-1} \sum_{k=0}^{n-1} f(S_k)$ converges in distribution to a Mittag–Leffler random variable, for well-chosen coefficients $(a_n)_{n\geq 0}$ [**45**]. This behaviour generalizes to null-recurrent Markov processes [**1**, **21**].

When f has integral zero, this family of results is not sharp enough, and we must look at a higher order. In the same way that a central limit theorem replaces the weak law of large numbers, one can get a generalized central limit theorem for observables of null-recurrent Markov processes. Typically, $a_n^{-1/2} \sum_{k=0}^{n-1} f(S_k)$ converges in distribution, with an explicit limit. The story of these central limit theorems starts from Dobrushin [**23**] where (S_n) is the simple random walk on $\mathbb{Z}$. Then these results were generalized to Markov processes [**36**, **38**, **41**], and later included invariance principles [**10**, **11**, **37**].

In this article, we are interested not in Markov processes, but in a family of dynamical systems preserving an infinite measure: $\mathbb{Z}^d$-extensions, which are a generalization of random walks. Starting from a dynamical system preserving a probability measure (A, μ, T) and a function $F : A \to \mathbb{Z}^d$, we work with the transformation $\widetilde{T} : (x, p) \mapsto (T(x), p + F(x))$ on $A \times \mathbb{Z}^d$. This class of systems includes random walks on $\mathbb{Z}^d$, as well as, for instance, Lorentz gases [**14**, **15**] and the geodesic flow on Abelian covers of complete manifolds [**39**, **53**, **59**]. Given an observable $f : A \times \mathbb{Z}^d \to \mathbb{R}$, we want to understand the limit in the distribution of $\sum_{k=0}^{n-1} f \circ \widetilde{T}^k$.

In two previous works by the second-named author [**67**, **68**], adapting previous methods [**18**–**20**], the case where (A, μ, T) is a Gibbs–Markov map was investigated. In the current article, we are able to get a generalized central limit theorem under spectral hypotheses on the transfer operator of the system (A, μ, T), which has much wider applications. The downside is that we need the observable $f(x, p)$ to depend only on p and decay fast enough at infinity. This is Theorem 2.4, which we prove using the method of moments. The computation of the asymptotics of the moments for this specific problem is, to our knowledge, new; but this method has proven to be very fruitful for closely related questions, such as the distributional limit of occupation times [**1**, **5**, **54**, **64**, **65**]. We then apply Theorem 2.4 to Lorentz gases with finite horizon.

An interesting corollary of Theorem 2.4 and [**68**, Theorem 6.8] is that, for $\mathbb{Z}^d$-extensions of Gibbs–Markov maps, Green–Kubo's formula—which appears as the asymptotic variance in the central limit theorem—is invariant under induction. This is the content of Corollary 2.13. By choosing the observable f carefully, in Theorem 2.7 we are able to relate the probability that an excursion from $A \times \{0\}$ hits a site $A \times \{p\}$, and the symmetrized potential kernel associated to the $\mathbb{Z}^d$-extension. Our proof relies on the first hitting time of small target statistics. This method provides a new proof of an earlier proposition by Spitzer [**61**, Ch. III.11, P5], and generalizes it to $\mathbb{Z}^d$-extensions (for which harmonic analysis as used in [**61**] does not make sense). Finally, the estimates from Theorem 2.7 are used to relax the assumptions from [**68**]: in Theorem 2.11, the observables need only to decay polynomially at infinity, instead of having bounded support. We apply it to the geodesic flow on Abelian covers of compact manifolds with negative curvature.

This article is organized as follow. We present our setting and our results in §2, as well as our applications to Lorentz gases (§2.4.1) and to geodesic flows (§2.4.2). In §3 we present our spectral assumptions, and prove Theorem 2.4 using the method of moments. In §4 we prove Theorems 2.7 and 2.11, and in §5 the two applications mentioned above. We discuss Green–Kubo's formula in the Appendix.

2. *Main results*

2.1. *Setting and goals.* We consider conservative ergodic dynamical systems given by $\mathbb{Z}^d$-extensions of probability-preserving dynamical systems, where the underlying dynamical system is sufficiently hyperbolic and $d \in \{1, 2\}$. We shall deem a system hyperbolic enough if its transfer operator satisfies good properties. For some applications, we use the stronger assumption that the underlying dynamical system is Gibbs–Markov.

Let (A, μ, T) be a probability-preserving dynamical system. Let $F : A \to \mathbb{Z}^d$ with $d \in \{1, 2\}$ be a μ-integrable function such that $\int_A F\, d\mu = 0$. The $\mathbb{Z}^d$-*extension* $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ of (A, μ, T) with step function F is the dynamical system given by the following.

- $\widetilde{A} := A \times \mathbb{Z}^d$;
- $\tilde{\mu} := \sum_{p\in\mathbb{Z}^d} \mu \otimes \delta_p$;
- $\widetilde{T}(x, p) = (T(x), p + F(x))$.

Note that $\widetilde{T}$ preserves the infinite measure $\tilde{\mu}$. We shall always assume that $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ is ergodic. If (A, μ, T) has a Markov partition π, we may also assume that the step function F is $\sigma(\pi)$-measurable—that is, constant almost everywhere on elements of the partition. We then say that $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ is a *Markovian* $\mathbb{Z}^d$*-extension* of (A, μ, T).

Let $S_n := S_n^T F := \sum_{k=0}^{n-1} F \circ T^k$ be the second coordinate of $\widetilde{T}^n(x, 0)$. Heuristically, the sequence $(S_n)_{n\geq 0}$, under the distribution μ, behaves much like a random walk, the randomness being generated by the dynamical system (A, μ, T). Indeed, this family of extensions includes every random walk on $\mathbb{Z}^d$, as well as some physically or geometrically interesting systems such as Lorentz gases (§2.4.1) or the geodesic flow on $\mathbb{Z}^d$-periodic manifolds of negative curvature (§2.4.2)†.

In the present paper, we will make assumptions ensuring the convergence in distribution of $(S_n/\mathfrak{a}_n)_n$ to a Lévy stable distribution, for some normalizing sequence $(\mathfrak{a}_n)_n$. Our main goals are the following:

(A) Given $f : \widetilde{A} \to \mathbb{R}$ integrable and such that $\int_{\widetilde{A}} f\, d\tilde{\mu} = 0$, we are interested in the asymptotic behaviour of the ergodic sum

$$S_n^{\widetilde{T}} f = \sum_{k=0}^{n-1} f \circ \widetilde{T}^k,$$

as $n \to +\infty$. More precisely, we are looking for a non-trivial strong convergence in distribution:

$$\frac{S_n^{\widetilde{T}} f}{\mathfrak{A}_n} \overset{\text{dist.}}{\Rightarrow} \sigma(f)\mathcal{Y} \quad \text{with } \mathfrak{A}_n := \sqrt{\sum_{k=1}^{n} \mathfrak{a}_k^{-d}}, \tag{2.1}$$

where $\sigma(f)$ is some constant, which depends on the pushforward of the measure $\tilde{\mu}$ by $(f \circ \widetilde{T}^n)_{n\geq 0}$, whereas the random variable $\mathcal{Y}$ depends only on‡ the distribution of $(F \circ T^k)_k$ (with respect to μ).

(B) In the context of Gibbs–Markov maps, we consider the probability $\alpha(p)^{-1}$, starting from $A \times \{0\}$ endowed with the measure μ, to visit $A \times \{p\}$ before coming back

† Up to some lengthy, but in our case not particularly challenging, legwork to go from discrete time to continuous time.

‡ Up to a constant, $\mathcal{Y}$ actually depends only on the index α of the Lévy stable distribution that is the limit of $(S_n/\mathfrak{a}_n)_n$.

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to $A \times \{0\}$. By applying the limit theorems we have proved before to $f_p(x, q) := (\mathbf{1}_{\{p\}} - \mathbf{1}_{\{0\}})(q)$, we are able to prove that

$$\alpha(p) \sim \frac{\sigma(f_p)}{2} \quad \text{as } p \to \infty,$$

which provides a new proof of [**61**, Ch. III.11, P5], and generalizes it to systems that are not random walks.

The next sub-sections present in more details these two goals, and the precise statements we get.

2.2. *Distributional limit theorems.*

2.2.1. *Convergence and limit distributions.*

When working with spaces endowed with an infinite measure, there is no natural notion of convergence in distribution. We shall instead use the notion of *strong convergence in distribution*. The reader may consult e.g. [**1**, Ch. 3.6] for an introduction to this notion and applications to ergodic dynamical systems whose invariant measure is infinite.

Definition 2.1. (Strong convergence in distribution) Let $(\widetilde{A}, \tilde{\mu})$ be a measured space. Let $(X_n)_{n\geq 0}$ be a sequence of measurable functions from $\widetilde{A}$ to $\mathbb{R}$. Let X be a real-valued random variable. We say that (X_n) converges strongly in distribution to X if, for every probability measure $\nu \ll \tilde{\mu}$,

$$X_n \to_{n\to+\infty} X \quad \text{in distribution on } (\widetilde{A}, \nu).$$

Now that we have defined our mode of convergence, we introduce our limit objects: Mittag–Leffler random variables, and Mittag–Leffler–Gaussian mixtures.

Definition 2.2. (*ML* and *MLGM* random variables) Let $\gamma \in [0, 1]$. Let X be a non-negative real-valued random variable. We say that X follows a standard Mittag–Leffler distribution of index γ if, for all $z \in \mathbb{C}$ (or all $z \in B(0, 1)$ if $\alpha = 0$),

$$\mathbb{E}[e^{zX}] = \sum_{n=0}^{+\infty} \frac{\Gamma(1+\gamma)^n z^n}{\Gamma(1+n\gamma)}.$$

If this is the case, we shall write that X has a $ML(\gamma)$ distribution.

Let X be a real-valued random variable. We say that X follows a standard Mittag–Leffler–Gaussian mixture distribution of index γ if X has the same distribution as $\sqrt{Y} \cdot Z$, where Y and Z are two independent random variables with respective distribution $ML(\gamma)$ and standard normal $\mathcal{N}(0, 1)$. If this is the case, we shall write that X has a $MLGM(\gamma)$ distribution. See [**66**, Ch. 1.4] for a partial description of the MLGM distributions.

For $\gamma = 0$, these distributions take more common forms: a $ML(0)$ distribution is an exponential distribution of parameter 1, while a $MLGM(0)$ distribution is a Laplace distribution of parameter $1/\sqrt{2}$, with density $2^{-(1/2)} e^{-\sqrt{2}|x|}$ with respect to the Lebesgue measure. A $ML(1/2)$ random variable is the absolute value of a centered Gaussian of variance $\pi/2$.

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2.2.2. *Main distributional theorem.* Mittag–Leffler distributions appear when one studies the distributional convergence of the local time of null recurrent Markov processes, or chaotic enough σ-finite ergodic dynamical systems. For the Brownian motion, the result goes back to Lévy [**45**], and to Darling–Kac's theorem for Markov chains [**21**]. We refer the reader to [**47**] for α-stable Lévy processes, and to [**1**, Corollary 3.7.3] for dynamical systems in infinite ergodic theory. For instance [**1**, Corollary 3.7.3] and Hopf's ergodic theorem [**32**, §14, Individueller Ergodensatz für Abbildungen] yield the following.

PROPOSITION 2.3. *Let $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ be a measure-preserving transformation of a σ-finite measure space. Assume that $\widetilde{T}$ is pointwise dual ergodic with return sequence $(a_n)_n$ (see* [**1**, Ch. 3.5] *for definitions). Assume that $(a_n)_n$ has regular variation of index $\alpha \in [0, 1]$, i.e. $a_n = n^{1/\alpha} L(n)$ for some sequence L, which varies slowly at infinity. Then, for all $f \in \mathbb{L}^1(\widetilde{A}, \tilde{\mu})$,*

$$\frac{S_n^{\widetilde{T}} f}{a_n} \Rightarrow \int_{\widetilde{A}} f \, d\tilde{\mu} \cdot \mathcal{Y},$$

where $\mathcal{Y}$ is a standard ML(α) random variable and the convergence is strong in distribution.

However, this kind of result is not sharp enough when the integral of the observable f is zero. We want to get more precise asymptotics, that is to say, some kind of central limit theorem for observables of σ-finite ergodic dynamical systems whose integral is 0. We need to add some regularity condition on the observable f, as well as stronger integrability conditions—as is usual in ergodic theory, for instance to get a central limit theorem [**48**, **49**]. In this article, we shall prove the following result.

THEOREM 2.4. *Let $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ be an ergodic and aperiodic $\mathbb{Z}^d$-extension of (A, μ, T) with step function F and $\alpha \in [d, 2]$. Assume Hypothesis 3.1. Let $(\mathfrak{a}_n)_n$ be an α^{-1}-regularly varying sequence of positive numbers and Y be an α-stable random variable Y such that*

$$S_n/\mathfrak{a}_n \overset{\text{distrib.}}{\Rightarrow} Y.$$

Let $\mathfrak{A}_n := \sqrt{\sum_{k=1}^n \mathfrak{a}_k^{-d}}$. Let $\beta : \mathbb{Z}^d \to \mathbb{R}$ be such that:

- $\sum_{p\in\mathbb{Z}^d} |p|^{(\alpha-d)/2+\varepsilon} |\beta(p)| < +\infty$ *for some $\varepsilon > 0$;*
- $\sum_{p\in\mathbb{Z}^d} \beta(p) = 0$.

Let $f(x, p) := \beta(p)$. Then the following sum over k is absolutely convergent:

$$\sigma_{GK}^2(f, \widetilde{A}, \tilde{\mu}, \widetilde{T}) = \int_{\widetilde{A}} f^2 \, d\tilde{\mu} + 2 \sum_{k\geq 1} \int_{\widetilde{A}} f \cdot f \circ \widetilde{T}^k \, d\tilde{\mu}. \tag{2.2}$$

Moreover,

$$\frac{S_n^{\widetilde{T}} f}{\sqrt{\Phi(0)}\mathfrak{A}_n} \Rightarrow \sigma_{GK}(f, \widetilde{A}, \tilde{\mu}, \widetilde{T}) \mathcal{Y}, \tag{2.3}$$

where $\mathcal{Y}$ is a standard MLGM$(1 - d/\alpha)$ random variable and the convergence is strong in distribution, and where Φ is the continuous version of the density function of Y.

Under the hypotheses of Theorem 2.4, we have, in addition,

$$\sigma_{GK}^2(f, \widetilde{A}, \tilde{\mu}, \widetilde{T}) = \sum_{a\in\mathbb{Z}^d} \beta(a)^2 + 2 \sum_{k\geq 1} \sum_{a,b\in\mathbb{Z}^d} \beta(a)\beta(b)\mu(S_k = a - b). \tag{2.4}$$

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Remark 2.5. For a definition of aperiodicity in the setting of Gibbs–Markov maps, see Definition 4.9. An assumption of aperiodicity is not necessary in the statement in the theorem, but appears as a result of Hypothesis 3.1, and we prefer to make this assumption explicit.

We do not expect aperiodicity to be necessary in the statement of Theorem 2.4, up to the necessary modification in Hypothesis 3.1. Proving this generalization would be straightforward if f were allowed to depend on x; however, allowing such a dependence would make the proof of Theorem 2.4 much more difficult. We choose to leave the non-aperiodic case aside, except for a couple of later results, Theorems 2.7 and 2.11.

Theorem 2.4 shall be proved in §3 with the method of moments and is based on refinements of the local limit theorem for S_n, which says that $\mathbb{P}(S_n = 0) \sim \Phi(0)\mathfrak{a}_n^{-d}$. Under our hypotheses, the normalization $\sqrt{\Phi(0)}\mathfrak{A}_n$ is equivalent to $\sqrt{\sum_{k=0}^{n-1} \mu(S_k = 0)}$. See e.g. [**4**] for a spectral proof of the local limit theorem, which holds under Hypothesis 3.1, and implies the equivalence of the normalizations.

In special cases, the normalization $\mathfrak{A}_n$ can be made explicit:

$$\mathfrak{A}_n \sim \begin{cases} \sqrt{\dfrac{\alpha}{\alpha - 1}\dfrac{n}{\mathfrak{a}_n}} & \text{if } d = 1 \text{ and } \alpha > 1, \\ \sqrt{\log n} & \text{if } d = \alpha \text{ and } a_n \sim n^{1/\alpha}, \\ \sqrt{\log \log n} & \text{if } d = \alpha \text{ and } a_n \sim (n \log n)^{1/\alpha}. \end{cases}$$

2.2.3. *Symmetrized potential kernel.* The case when $f = f_p$ of Theorem 2.4 is especially interesting. Then the computation of $\sigma_{GK}(f_p, \widetilde{A}, \tilde{\mu}, \widetilde{T})$ boils down to an estimation of the symmetrized potential kernel g of the $\mathbb{Z}^d$-extension:

$$\sigma_{GK}^2(f_p, \widetilde{A}, \tilde{\mu}, \widetilde{T}) = 2g(p) - 2,$$

with

$$g(p) := \sum_{n \geq 0} (2\mu(S_n = 0) - \mu(S_n = p) - \mu(S_n = -p)),$$

which is well-defined under the assumptions of Theorem 2.4. We estimate the asymptotic growth of $g(p)$ in §3.5, adapting the methods of [**61**] to dynamical systems. We get the following.

PROPOSITION 2.6. *Let $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ be an ergodic, aperiodic $\mathbb{Z}^d$-extension of (A, μ, T) with step function F. Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a complex Banach space of functions defined on A. Assume Hypothesis 3.1 holds with $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $\alpha \in [d, 2]$. If $\alpha = d$, let I be the functions defined by equation (3.55).*

If $d = 1$ and $\alpha \in (1, 2]$,

$$g(p) \sim_{p \to \infty} \frac{1}{\vartheta(1 + \zeta^2)\Gamma(\alpha)\sin((\alpha - 1)\pi/2)} \frac{|p|^{\alpha - 1}}{L(|p|)}.$$

If $d = \alpha = 1$,

$$g(p) \sim_{p \to \infty} \frac{2}{\pi\vartheta(1 + \zeta^2)} I(|p|^{-1}).$$

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If $d = \alpha = 2$,

$$g(p) \sim_{p\to\infty} \frac{2}{\pi\sqrt{\det(\Sigma)}} I(|p|^{-1}).$$

2.3. *Hitting probabilities of excursions.* We leave aside for a moment the distributional asymptotics of the Birkhoff sums, and focus on the probability that an excursion hits a given site (§4). We now assume that (A, μ, T) is a Gibbs–Markov map. The leading theme of this section is the study of the probability that an excursion from $A \times \{0\}$ hits $A \times \{p\}$, and its asymptotics as p goes to infinity.

2.3.1. *Induced transformations.* Let us describe the terminology. We define $\varphi_{\{0\}} : A \to \mathbb{N}_+ \cup \{\infty\}$, where $\varphi_{\{0\}}(x)$ is the length of an excursion starting from $(x, 0)$:

$$\varphi_{\{0\}}(x) := \inf\{k > 0 : S_k(x) = 0\}.$$

Then, define the corresponding induced map by $\widetilde{T}_{\{0\}}(x) := T^{\varphi_{\{0\}}(x)}(x)$, which is well-defined for μ-almost every $x \in A$. Note that $(A, \mu, \widetilde{T}_{\{0\}})$ is a measure-preserving ergodic dynamical system [**34**]. For any observable $f : \widetilde{A} \to \mathbb{R}$ and any $x \in A$, let

$$f_{\{0\}}(x) := \sum_{k=0}^{\varphi_{\{0\}}(x)-1} f \circ \widetilde{T}^k(x, 0).$$

Let us introduce a few more objects: the time N_p that an excursion from $A \times \{0\}$ spends at $A \times \{p\}$, and the inverse probability $\alpha(p)$ that an excursion from $A \times \{0\}$ hits $A \times \{p\}$, and the number of times $N_{0,p}$ that the system goes back to $A \times \{0\}$ before hitting $A \times \{p\}$. Formally,

$$N_p(x) := \#\{k = 0, \ldots, \varphi_{\{0\}}(x) - 1 : S_k(x) = p\} = 1 + f_{p,\{0\}}(x),$$
$$\alpha(p) := \mu(N_p > 0)^{-1} = \mu(\exists 0 \le k < \varphi(x) : S_k(x) = p)^{-1},$$

and

$$N_{0,p}(x) := \inf\{n \ge 0 : T^n_{\{0\}}(x) \in \{N_p > 0\}\}.$$

The following theorem explains how these quantities are related in the limit $p \to \infty$.

THEOREM 2.7. *Let $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ be a conservative and ergodic† Markovian $\mathbb{Z}^d$-extension of a Gibbs–Markov map (A, μ, T). Then the following are true.*

- *As $p \to +\infty$,*

$$\alpha(p) = \alpha(-p) \sim \mathbb{E}_\mu[N_p | N_p > 0] \sim \mathbb{E}_\mu[N_{0,p}] \sim \frac{\sigma^2_{GK}(f_{p,\{0\}}, A, \mu, \widetilde{T}_{\{0\}})}{2} \sim \frac{\mathbb{E}_\mu[f^2_{p,\{0\}}]}{2}.$$

- *The conditional distributions of $\alpha(p)^{-1} N_p$ given $\{N_p > 0\}$ have exponential tails, uniformly in p: there exist $C \ge 0$ and $\kappa > 0$ such that, for every $t > 0$,*

$$\sup_{p\in\mathbb{Z}^d} \mu((\alpha(p))^{-1} N_p > t | N_p > 0) \le C e^{-\kappa t}.$$

† The extension needs not be aperiodic for this theorem.

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- *The random variables $\alpha(p)^{-1}N_p$ conditioned on $\{N_p > 0\}$ converge in distribution and in moments to an exponential random variable of parameter* 1 *as p goes to infinity. In particular, for all $q > 1$,*

$$\mathbb{E}_\mu[|f_{p,\{0\}}|^q] \sim \Gamma(1+q)\alpha(p)^{q-1}.$$

The equality $\alpha(p) = \alpha(-p)$ holds for any recurrent group extension of a probability-preserving dynamical system. The remaining points rely much more on the Gibbs–Markov structure.

The proof of Theorem 2.7 rests on two main points: the exponential tightness of $\alpha(p)^{-1}N_p$ given $\{N_p > 0\}$ (§4.3), and its convergence to an exponential random variable (§4.4). The later point is an interesting application of the general fact that, for many hyperbolic dynamical systems, the hitting time of small balls, once renormalized, converges in distribution to an exponential random variable (see e.g. the reviews [**17**, **30**, **58**]). Once we have tightness and convergence in distribution, we can evaluate the moments of N_p.

For random walks, many estimates are more explicit. For instance, the conditional distribution of N_p given $\{N_p > 0\}$ is geometric, so its moments are exactly known (as functions of $\alpha(p)$). With this improvement, one can recover part of [**61**, Ch. III.11, P5]—that is, the equivalents in Theorem 2.7 and Corollary 2.9 can be made into equalities:

$$\begin{aligned}\alpha(p) &= \alpha(-p) = \mathbb{E}_\mu[N_p | N_p > 0] = 1 + \mathbb{E}_\mu[N_{0,p}] \\ &= 1 + \frac{\sigma_{GK}^2(f_{p,\{0\}}, A, \mu, \widetilde{T}_{\{0\}})}{2} = 1 + \frac{\mathbb{E}_\mu[f_{p,\{0\}}^2]}{2} = g(p).\end{aligned}$$

2.3.2. *Induction invariance of the Green–Kubo formula.* While Theorem 2.7 gives asymptotic relationships between many quantities, it does not provide any way to effectively compute them. For random walks, $\alpha(p)$ and $g(p)$ are related through a probabilistic interpretation of the symmetrized potential kernel.

PROPOSITION 2.8. [**61**, Ch. III.11, P5] *Consider an ergodic aperiodic random walk on $\mathbb{Z}^2$. For all $p \in \mathbb{Z}^2$,*

$$\alpha(p) = g(p).$$

We are able to generalize this proposition to a larger class of dynamical systems. To our knowledge, our proof of Proposition 2.8 is new even for random walks. We leverage Theorem 2.4 and [**68**, Theorem 6.8]. Whenever the hypotheses of these theorems coincide, their conclusions must be the same. Hence, the scaling factors before the *MLGM* distribution must be the same, that is,

$$\sigma_{GK}^2(f, \widetilde{A}, \tilde{\mu}, \widetilde{T}) = \sigma_{GK}^2(f_{\{0\}}, A, \mu, \widetilde{T}_{\{0\}}). \tag{2.5}$$

If we apply this observation to $f = f_p$, we get the following.

COROLLARY 2.9. *Let $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ be an aperiodic Markovian $\mathbb{Z}^d$-extension of a Gibbs–Markov map $(A, \pi, \lambda, \mu, T)$ with step function F. Assume that the extension is ergodic, conservative, and either of the following hypotheses.*

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- *$d = 1$ and F is in the domain of attraction of an α-stable distribution, with $\alpha \in (1, 2]$.*
- *$d = 1$ and $\int_A e^{iuF}\, d\mu = e^{-\vartheta|u|[1-i\zeta \operatorname{sgn}(u)]L(|u|^{-1})} + o(|u|L(|u|^{-1}))$ at 0, for some real numbers $\vartheta > 0$ and $\zeta \in \mathbb{R}$ and some function L with slow variation.*
- *$d = 2$ and F is in the domain of attraction of a non-degenerate Gaussian random variable.*

Then, as $p \to +\infty$,

$$\alpha(p) \sim g(p).$$

Remark 2.10. (1-stable laws) The description of the distributions in the basin of attraction of a 1-stable law is notoriously difficult [**2**]. As in Hypothesis 3.1, we choose to make a spectral assumption. It does not capture all such distributions, but includes e.g. symmetric distributions. We believe that this assumption can be significantly weakened if needed.

2.3.3. *An improved distributional limit theorem.* Proposition 2.6 provides a first-order estimate of $\alpha(p)$, depending on the tails of F. We can use this estimate to run an (improved version of an) argument by Csáki, Csörgő, Földes and Révész [**18**, Lemma 3.1]. we get more explicit integrability conditions than in [**68**, Theorem 6.8] for observables of $\mathbb{Z}^d$-extensions, which yields a new distributional limit theorem. Note that aperiodicity is not required for this result.

THEOREM 2.11. *Let $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ be a Markovian $\mathbb{Z}^d$-extension of a Gibbs–Markov map $(A, \pi, \lambda, \mu, T)$ with step function F. Assume that the extension is ergodic, conservative, and either of the following hypotheses.*

- *$d = 1$ and F is in the domain of attraction of an α-stable distribution, with $\alpha \in (1, 2]$.*
- *$d = 1$ and $\int_A e^{iuF}\, d\mu = e^{-\vartheta|u|[1-i\zeta \operatorname{sgn}(u)]L(|u|^{-1})} + o(|u|L(|u|^{-1}))$ at 0, for some real numbers $\vartheta > 0$ and $\zeta \in \mathbb{R}$ and some function L with slow variation.*
- *$d = 2$ and F is in the domain of attraction of a non-degenerate Gaussian random variable.*

Let $f : \widetilde{A} \to \mathbb{R}$ be such that:

- *the family of function $(f(\cdot, p))_{p\in\mathbb{Z}^d}$ is uniformly locally η-Hölder for some $\eta > 0$;*
- *$\int_{\widetilde{A}} (1 + |p|)^{(\alpha-d)/2+\varepsilon} \|f(\cdot, p)\|_{\mathbb{L}^q(A,\mu)}\, d\tilde{\mu}(x, p) < +\infty$ for some $\varepsilon > 0$ and $q > 2$;*
- *$\int_{\widetilde{A}} f\, d\tilde{\mu} = 0$.*

Then,

$$\frac{S_n^{\widetilde{T}} f}{\sqrt{\Phi(0)\mathfrak{A}_n}} \Rightarrow \sigma_{GK}(f_{\{0\}}, A, \mu, \widetilde{T}_{\{0\}})\mathcal{Y}, \tag{2.6}$$

where $\mathcal{Y}$ is a standard MLGM$(1 - d/\alpha)$ random variable, the convergence is strong in distribution, and

$$\sigma_{GK}^2(f_{\{0\}}, A, \mu, \widetilde{T}_{\{0\}}) := \lim_{n\to+\infty} \int_A f_{\{0\}}^2 d\mu + 2\sum_{k=1}^{n} \int_A f_{\{0\}} \cdot f_{\{0\}} \circ \widetilde{T}_{\{0\}}^k\, d\mu,$$

where the limit is taken in the Cesàro sense.

Remark 2.12. (Optimal exponent in the summability assumption) We consider the case when $d = 1$ and $\alpha = 2$. In [**18**] and some subsequent works by the same authors, the

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condition required for f is

$$\sum_{p\in G} |p|^{1+\varepsilon} |\beta(p)| < +\infty. \tag{2.7}$$

The reason is that the authors used Jensen's inequality in their proof [**18**, Lemma 2.1], which is in this context less efficient than Minkowski's inequality, which we used in the proof of Lemma 4.19. This small modification can be implemented in their proof, which improves by a factor 2 some requirements in their works, e.g. [**18**, Theorem 1] and [**19**, Example 3.3].

Finally, the hypotheses of Theorem 2.4 and of Theorem 2.11 have a greater overlap than those of Theorem 2.4 and [**68**, Theorem 6.8], so we can improve the observation in equation (2.5).

COROLLARY 2.13. (Induction invariance of the Green–Kubo formula) *Let $(A, \pi, \lambda, \mu, T)$ be an ergodic Gibbs–Markov map. Assume that the step function $F : A \to \mathbb{Z}^d$ is $\sigma(\pi)$-measurable, integrable, aperiodic, and that $\int_A F\, d\mu = 0$. We also assume that the distribution of F with respect to μ is in the domain of attraction of an α-stable distribution, and that the Markovian $\mathbb{Z}^d$-extension $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ is conservative and ergodic.*

Let $\beta : \mathbb{Z}^d \to \mathbb{R}$ be such that:
- $\sum_{p\in\mathbb{Z}^d} |p|^{(\alpha-d)/2+\varepsilon} |\beta(p)| < +\infty$ *for some* $\varepsilon > 0$;
- $\sum_{p\in\mathbb{Z}^d} \beta(p) = 0$.

Let $f(x, p) := \beta(p)$. Then,

$$\sigma_{GK}(f, \widetilde{A}, \tilde{\mu}, \widetilde{T}) = \sigma_{GK}(f_{\{0\}}, A, \mu, \widetilde{T}_{\{0\}}). \tag{2.8}$$

See Appendix A for a discussion of this corollary.

2.4. *Applications.* To finish this introduction, we present some applications of our results to more concrete dynamical systems: the geodesic flow on Abelian covers in negative curvature, and Lorentz gases (i.e. periodic planar billiards). The proofs can be found in §5.

2.4.1. *Periodic planar billiard systems.* Lorentz gases—that is, periodic or quasi-periodic convex billiards—are classical dynamical systems, whose initial motivation comes from the modelization of a gas of electrons in a metal. The electron is then seen as bouncing on the atoms of the metal, which act as scatterers.

In the plane and with a finite horizon, Lorentz gases exhibit classical diffusion, and the trajectory of a particle behaves much like a random walk in the Euclidean space. For instance, the trajectories are chaotic [**60**], satisfy a central limit theorem [**14**, **15**], a local limit theorem [**62**], an almost sure invariance principle [**27**] (i.e. the renormalized trajectories converge in a strong sense to the trajectories of a Brownian motion), etc. We refer the reader to [**16**] for more information of billiards. While the infinite horizon case is also well-known [**22**, **63**], it presents many non-trivial difficulties, so we shall restrict ourselves to finite horizon planar billiards.

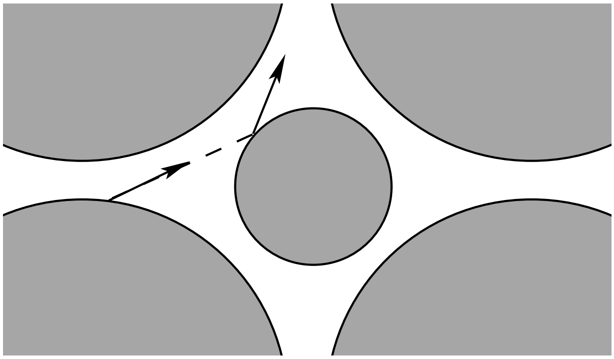

FIGURE 1. A Sinai billiard with finite horizon.

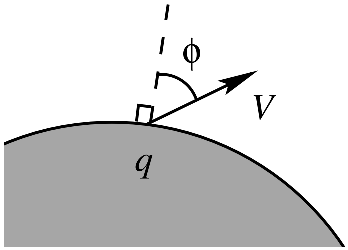

FIGURE 2. A single collision.

Choose a $\mathbb{Z}^2$-periodic locally finite configuration of obstacles $(p + O_i : i \in \mathcal{I},\ p \in \mathbb{Z}^2)$, where $\mathcal{I}$ is a finite set. We assume that the obstacles $O_i + p$ are convex open sets, with pairwise disjoint closures (so that there is no cusp), that their boundaries are $\mathcal{C}^3$ and have non-vanishing curvature. We assume moreover that the horizon is finite: every line in $\mathbb{R}^2$ meets at least one obstacle (Figure 1). The billiard domain is the complement in $\mathbb{R}^2$ of the union of the obstacles $Q := \mathbb{R}^2 \setminus \bigcup_{i\in\mathcal{I}, p\in\mathbb{Z}^2}(p + O_i)$.

We consider a point particle moving at unit speed in the billiard domain Q, bouncing on obstacles with the classical reflection law: the incident angle equals the reflected angle and going straight on between two collisions (Figure 2). This is the billiard flow, whose configuration space is (up to a set of zero measure) $Q \times \mathbb{S}_1$. Now, consider this model at collision times; the configuration space is then given by $\Omega := \partial Q \times [-\pi/2, \pi/2]$. The space Ω is endowed with the Liouville measure $\tilde{\nu}$, which has density $\cos(\phi)$ in (x, ϕ) with respect to the Lebesgue measure (see the picture), and is invariant under the collision map.

For every $p \in \mathbb{Z}^2$, we call *cell* any set $\mathcal{C}_p := \bigcup_{i\in\mathcal{I}}(p + \partial O_i)$ and attribute to this cell a value $\beta(p)$ given by a function $\beta : \mathbb{Z}^2 \to \mathbb{R}$ (Figure 3). We assume that the particle wins the value $\beta(p)$ associated to $\mathcal{C}_p$ each time it touches it. We are interested in the behaviour, as $n \to +\infty$, of the total amount $\mathcal{Y}_n$ won by the particle after the nth reflection.

We write $S_n(x)$ for the index in $\mathbb{Z}^2$ of the cell touched at the nth reflection time by a particle starting from $x \in \Omega$. Recall that $(S_n/\sqrt{n})_n$ converges strongly in distribution (with respect to the Lebesgue measure on Ω) to a centered Gaussian random variable with positive definite covariance matrix Σ [**14**, **15**, **70**].

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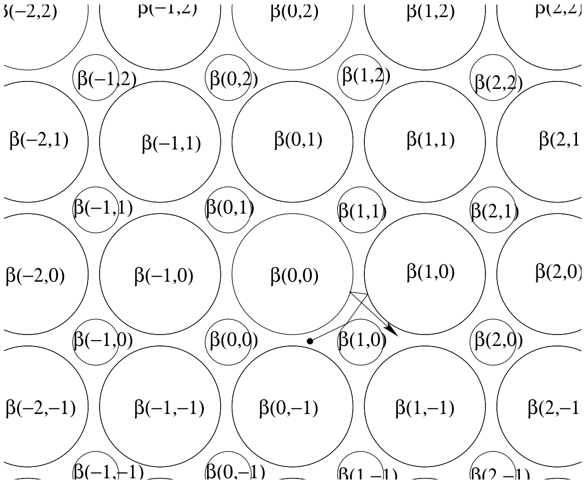

FIGURE 3. A periodic billiard table, and the observable β.

If β is summable and $\sum_{p\in\mathbb{Z}^2}\beta(p)\neq 0$, then $\mathcal{Y}_n/\log(n)$ converges strongly in distribution to $(\sum_{p\in\mathbb{Z}^2}\beta(p)\neq 0)\mathcal{E}$, where $\mathcal{E}$ has a non-degenerate exponential distribution. This follows e.g. from [**1**, Corollary 3.7.3] and Young's construction [**70**], and is also done in [**22**]. In another direction, if $(\beta(p))_{p\in\mathbb{Z}^2}$ is a sequence of independent identically distributed random variables independent of the billiard, the asymptotic behaviour of $(\mathcal{Y}_n)$ is markedly different [**51**].

We present two applications of Theorem 2.4, the first for (hidden) $\mathbb{Z}$-extensions, and the second for $\mathbb{Z}^2$-extensions.

COROLLARY 2.14. *With the above notation, assume that:*

- $\beta(a,b)=\tilde{\beta}(a)$ *for some function* $\tilde{\beta}$*;*
- *there exists* $\varepsilon>0$ *such that* $\sum_{p\in\mathbb{Z}}|p|^{1/2+\varepsilon}|\tilde{\beta}|(p)<+\infty$*;*
- $\sum_{p\in\mathbb{Z}}\tilde{\beta}(p)=0$.

Then,

$$\lim_{n\to+\infty}\frac{1}{n^{1/4}}\mathcal{Y}_n=\sigma(f)\mathcal{Y},$$

where the convergence is strong in distribution on (Ω,Leb)*, the random variable* $\mathcal{Y}$ *follows standard MLGM*$(1/2)$ *distribution, and*

$$\sigma(f)^2=\sqrt{\frac{2}{\pi\,\Sigma_{1,1}}}\sum_{k\in\mathbb{Z}}\sum_{a,b\in\mathbb{Z}^2}\beta(a)\beta(b)\tilde{\nu}(S_k=a-b|\mathcal{C}_0).$$

In addition, $\sigma(f)=0$ *if and only if* f *is a coboundary.*

COROLLARY 2.15. *With the above notation, assume that:*

- *there exists* $\varepsilon>0$ *such that* $\sum_{p\in\mathbb{Z}^2}|p|^{\varepsilon}|\beta|(p)<+\infty$*;*
- $\sum_{p\in\mathbb{Z}^2}\beta(p)=0$.

Then,

$$\lim_{n\to+\infty}\frac{1}{\sqrt{\log(n)}}\mathcal{Y}_n=\sigma(f)\mathcal{Y},$$

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where the convergence is strong in distribution on (Ω, Leb)*, the random variable* $\mathcal{Y}$ *follows a Laplace distribution of parameter* $1/\sqrt{2}$*, and*

$$\sigma(f)^2 = \frac{1}{2\pi\sqrt{\det(\Sigma)}} \sum_{k\in\mathbb{Z}} \sum_{a,b\in\mathbb{Z}^2} \beta(a)\beta(b)\tilde{\nu}(S_k = a - b|\mathcal{C}_0).$$

In addition, $\sigma(f) = 0$ *if and only if* f *is a coboundary.*

2.4.2. *Geodesic flow on Abelian covers.* The geodesic flow on a connected, compact manifold with negative sectional curvature is a well-known example of a hyperbolic dynamical system. The geodesic flow on Abelian covers of such manifolds provides a class of dynamical systems, which preserve a σ-finite measure, for instance the Liouville measure. They are also more tractable than billiards, as they do not have singularities. These geodesic flows have been studied extensively, for instance to count periodic orbits on the basis manifold of given length in a given homology class [**39**, **53**, **59**]. There are extensions to Anosov flows [**8**] as well as to surfaces with cusps [**3**]. Finally, let us mention that the geodesic flow on periodic manifolds is also used to study the horocycle flow on the same manifolds [**7**, **42–44**].

Limit theorems for observables with integral zero have already been obtained in this context [**68**], but the improvement we get with Theorem 2.11 translates into a limit theorem, which is valid for a wider class of observables. Instead of having compact support, the observables need only to decay polynomially fast at infinity.

Let M be a compact, connected manifold with a Riemannian metric of negative sectional curvature. Let $\varpi : N \to M$ be a connected $\mathbb{Z}^d$-cover of M. Given a Gibbs measure μ_M on T^1M, we endow T^1N with a σ-finite measure μ_N by lifting μ_M locally. We refer the reader to [**50**, Ch. 11.6] for more details about Gibbs measures in this context.

Let $(g_t)_{t\in\mathbb{R}}$ be the geodesic flow on T^1N. In §5.2, we shall prove the following proposition, which is a generalization of [**68**, Proposition 6.12].

PROPOSITION 2.16. *Let* μ_N *be the lift of a Gibbs measure* μ_M *corresponding to a Hölder potential. Assume that the extension* $(N, (g_t), \mu_N)$ *is ergodic and recurrent. Fix* $x_0 \in T^1N$*. Let* f *be a real-valued Hölder function on* T^1N*. Assume that:*

- *there exists* $\varepsilon > 0$ *such that* $\int_{T^1N} d(x_0, x)^{1-d/2+\varepsilon}|f|(x)\, d\mu_N(x) < +\infty$*;*
- $\int_{T^1N} f\, d\mu_N = 0$.

If $d = 1$*, there exists* $\sigma(f) \geq 0$ *such that*

$$\lim_{t\to+\infty} \frac{1}{t^{1/4}} \int_0^t f \circ g_s(x, v)\, ds = \sigma(f)\mathcal{Y}_{1/2},$$

where the convergence is strong in distribution on (T^1N, μ_N)*, and* $\mathcal{Y}_{1/2}$ *follows a standard MLGM*$(1/2)$ *distribution.*

If $d = 2$*, there exists* $\sigma(f) \geq 0$ *such that*

$$\lim_{t\to+\infty} \frac{1}{\sqrt{\log(t)}} \int_0^t f \circ g_s(x, v)\, ds = \sigma(f)\mathcal{Y}_0,$$

where the convergence is strong in distribution on (T^1N, μ_N)*, and* $\mathcal{Y}_0$ *follows a standard MLGM*(0) *distribution.*

In both cases, $\sigma(f) = 0$ *if and only if* f *is a measurable coboundary.*

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Remark 2.17. (Recurrent extensions of Gibbs measures) Given a Hölder potential $F : T^1M \to \mathbb{C}$, let $\hat{F}(x, v) := F(x, -v)$. We say that the potential is reversible if F and $\hat{F}$ are cohomologous, that is, if there exists a Hölder function u such that $\int_0^t f \circ g_s \, ds = u \circ g_t - u$ for all t. In this case, we also say that $F - \hat{F}$ is a Hölder coboundary. We say that a Gibbs measure is reversible if it is associated to a reversible potential.

For instance, both the Liouville measure and the maximal entropy measure are reversible, because the associated potentials (constants for the maximal entropy measure, and the log-Jacobian of the flow restricted to the unstable direction for the Liouville measure) are reversible.

If μ_M is a reversible Gibbs measure and $d \in \{1, 2\}$, then the geodesic flow on (T^1N, μ_N) is both ergodic and recurrent (see [**55**] for the constant curvature case, although the proof works as well in variable curvature).

The only difference between Proposition 2.16 and [**68**, Proposition 6.12] is that the assumption that f has compact support is relaxed to $\int_{T^1N} d(x_0, x)^{1-d/2+\varepsilon} |f|(x) \, d\mu_N(x) < +\infty$ for some $\varepsilon > 0$.

Note that our work gives us more information on this system; for instance, Theorem 2.7 can be adapted to yield an asymptotic equivalence of the probability that, starting from some nice Poincaré section A_1, the geodesic flow reaches a faraway Poincaré section A_2 before returning to A_1. However, the geometric interpretation of these sections is less evident than for others systems, such as billiards.

3. *Theorem 2.4: assumptions and proof*

This section is mostly devoted to the proof of Theorem 2.4. It is organized as follows. The spectral hypotheses are presented in §3.1. The following three subsections contain, respectively, a sketch of the proof of Theorem 2.4, the full proof of the theorem, and a proof of the more technical estimates we use. Finally, in §3.5 we prove Proposition 2.6.

3.1. *General spectral assumptions.*

Let P be the transfer operator associated to $h \mapsto h \circ T$, that is,

$$\int_A Pf \cdot g \, d\mu = \int_A f \cdot g \circ T \, d\mu \quad \text{for all } f \in \mathbb{L}^1(A, \mu), \text{ for all } g \in \mathbb{L}^\infty(A, \mu).$$

We consider the family $(P_u)_{u \in \mathbb{T}^d}$ of operators defined by $P_u : h \mapsto P(e^{i\langle u, F \rangle} h)$, where $\langle \cdot, \cdot \rangle$ stands for the usual scalar product in $\mathbb{R}^d$. Note that

$$P_u^k(h) = P^k(e^{i\langle u, S_k \rangle} h). \tag{3.1}$$

We make the following assumptions. Thanks to perturbation theorems (see namely [**28**, **31**, **40**, **48**, **49**] for the general method, and [**4**] for an application to Gibbs–Markov maps), they hold for a wide variety of hyperbolic dynamical systems.

Hypothesis 3.1. (Spectral hypotheses) The stochastic process $(S_n)_n$ is recurrent. There exists an integer $M \geq 1$ and a μ-essential partition of A in M measurable sub-sets $(A_j)_{j \in \mathbb{Z}/M\mathbb{Z}}$ such that $T(A_j) = A_{j+1}$ for all $j \in \mathbb{Z}/M\mathbb{Z}$ ($M = 1$ if T is mixing).

There exists a complex Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ of functions defined on A, on which P acts continuously, and such that:

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- $\mathbb{E}_\mu[\cdot]$ defines a linear continuous form on $\mathcal{B}$;
- $\mathbf{1} \in \mathcal{B}$ and for every j, the multiplication by $\mathbf{1}_{A_j}$ belongs to $\mathcal{L}(\mathcal{B}, \mathcal{B})$, where $(\mathcal{L}(\mathcal{B}, \mathcal{B}), \|\cdot\|_{\mathcal{L}(\mathcal{B},\mathcal{B})})$ stands for the Banach space of linear continuous endomorphisms of $\mathcal{B}$;
- there exist a neighbourhood U of 0 in $\mathbb{T}^d$, two constants $C > 0$ and $r \in (0, 1)$, two continuous functions $\lambda_\cdot : U \to \mathbb{C}$ and $\Pi_\cdot : U \to \mathcal{L}(\mathcal{B}, \mathcal{B})$ such that, for all $u \in U$,

$$P_u = \lambda_u \Pi_u + R_u, \tag{3.2}$$

with

$$\Pi_u R_u = R_u \Pi_u = 0, \tag{3.3}$$
$$\Pi_u^{M+1} = \Pi_u, \tag{3.4}$$
$$\lambda_0 = 1, \tag{3.5}$$
$$\Pi_0 = M \sum_{j\in\mathbb{Z}/M\mathbb{Z}} \mathbb{E}_\mu[\mathbf{1}_{A_j} \cdot]\mathbf{1}_{A_{j+1}}, \tag{3.6}$$
$$\sup_{v\in U} \|R_v^k\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \le Cr^k, \tag{3.7}$$
$$\sup_{v\in[-\pi,\pi]^d\setminus U} \|P_v^k\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \le Cr^k; \tag{3.8}$$

- if $d = 1$, there exists $\alpha \in [1, 2]$ such that, for all $u \in U$,

$$\lambda_u = e^{-\psi(u)L(|u|^{-1})} + o(|u|^\alpha L(|u|^{-1})),$$

as u goes to 0, where $\psi(u) = \vartheta|u|^\alpha[1 - i\zeta \operatorname{sgn}(u)]$ for some real numbers $\vartheta > 0$ and $\zeta \in \mathbb{R}$ such that $|\zeta| \le \tan(\pi\alpha)/2$ if $\alpha > 1$. We set $\Sigma := 1$;
- if $d = 2$, there exists an invertible positive symmetric matrix Σ such that, for all $u \in U$,

$$\lambda_u = e^{-\psi(\sqrt{\Sigma}u)L(|\sqrt{\Sigma}u|^{-1})} + o(|u|^2 L(|u|^{-1})),$$

as u goes to 0, where $\psi(u) := |u|^2/2$ and L is slowly varying at infinity. We set $\vartheta := 1/2$.

Hypothesis 3.1 implies the ergodicity of T and the mixing of $(T^M)_{|A_j}$ for all $j \in \mathbb{Z}/M\mathbb{Z}$ as soon as $\mathcal{B}$ is dense in $\mathbb{L}^1(A, \mu)$. If the system is not mixing, then it is expected that the transfer operators has multiple eigenvalues of modulus 1. The following proposition asserts that, in this case, the standard spectral techniques yield a decomposition as in equation (3.2).

PROPOSITION 3.2. *Assume the beginning of Hypothesis 3.1 and its first two items, and that $(A_0, \mu(\cdot|A_0), (T^M)_{|A_0})$ is mixing. Assume in addition that there exist a neighbourhood U of 0 in $\mathbb{T}^d$, two constants $C > 0$ and $r \in (0, 1)$ and continuous functions $\tilde{\lambda}_\cdot, \lambda_{0,\cdot}, \ldots, \lambda_{K-1,\cdot} : U \to \mathbb{C}$ and $\tilde{\Pi}_\cdot, \Pi_{0,\cdot}, \ldots, \Pi_{K-1,\cdot}, \tilde{R}_\cdot, R_\cdot : U \to \mathcal{L}(\mathcal{B}, \mathcal{B})$ such*

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that, for all $u \in U$

$$\begin{aligned} P_u &= \sum_{j \in \mathbb{Z}/K\mathbb{Z}} \lambda_{j,u} \Pi_{j,u} + R_u, \\ \Pi_{j,u} R_u &= R_u P_{j,u} = 0, \\ \Pi_{j,u} \Pi_{j',u} &= \delta_{j,j'} \Pi_{j,u}, \\ |\lambda_{j,0}| &= 1 \\ \sup_{v \in U} \|R_v^k\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} &\leq C r^k, \end{aligned}$$

and $\mathbf{1}_{A_0} P_u^M(\mathbf{1}_{A_0}) = \tilde{\lambda}_u \tilde{\Pi}_u + \tilde{R}_u$, *with*

$$\begin{aligned} \tilde{\lambda}_0 &= 1, \\ \tilde{\Pi}_u^2 &= \tilde{\Pi}_u, \\ \tilde{\Pi}_0 &= \mu(\cdot|A_0)\mathbf{1}_{A_0}, \\ \tilde{\Pi}_u \tilde{R}_u &= \tilde{R}_u \tilde{\Pi}_u = 0, \\ \|\tilde{R}_u^k\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} &\leq C r^k. \end{aligned}$$

Then $P_u = \lambda_u \Pi_u + R_u$ for all $u \in U$, and the equations (3.3)–(3.7) are satisfied. If moreover $u \mapsto P_u$ is continuous on $\mathbb{T}^d$ and P_u admits no eigenvalue of modulus 1 for $u \neq 0$, then equation (3.8) is also satisfied, up to an increase of $C > 0$ and $r \in (0, 1)$.

Proof. Up to taking a smaller U, we assume that $|\lambda_{j,\cdot}| > C^{1/M} r$ and $|\tilde{\lambda}_\cdot| > C r^M$. Then $\tilde{\lambda}_u = \lambda_{j,u}^M$ for every $j \in \mathbb{Z}/M\mathbb{Z}$, and $\tilde{\Pi}_u = \sum_{j \in \mathbb{Z}/M\mathbb{Z}} \mathbf{1}_{A_0} \Pi_{j,u}(\mathbf{1}_{A_0} \cdot)$. Hence we can take $K = M$ and, up to a permutation of indices, we assume that $\lambda_{j,u} = \lambda_u \xi^j$ with $\xi := e^{2i\pi/M}$ and $\lambda_0 = 1$ ($P\mathbf{1} = \mathbf{1}$ ensures that 1 is an eigenvalue of P_0, and this convention yields equation (3.5)). Hence $P_u = \lambda_u \Pi_u + R_u$, with

$$\begin{aligned} \lambda_u &:= \lambda_{0,u}, \\ \Pi_u &:= \sum_{j \in \mathbb{Z}/M\mathbb{Z}} \xi^j \Pi_{j,u}. \end{aligned}$$

Note that $\Pi_u R_u = R_u \Pi_u = 0$ and that $\Pi_u^{M+1} = \sum_{j \in \mathbb{Z}/M\mathbb{Z}} \xi^{j(M+1)} \Pi_{j,u} = \Pi_u$, which proves equations (3.3) and (3.4). In the general case, it remains to prove (3.6).

Let f be an eigenvector for the eigenvalue ξ^j of P. For all $k \in \mathbb{Z}/M\mathbb{Z}$,

$$P(\mathbf{1}_{A_k} f) = \xi^j \mathbf{1}_{A_{k+1}} f, \tag{3.9}$$

so that $P^M(\mathbf{1}_{A_k} f) = \mathbf{1}_{A_k} f$. Since T^M is mixing, f must be constant on each A_k; using equation (3.9), we get that f is proportional to $\sum_{k \in \mathbb{Z}/M\mathbb{Z}} \xi^{-jk} \mathbf{1}_{A_k}$. We conclude that

$$\Pi_{j,0} = \sum_{k \in \mathbb{Z}/M\mathbb{Z}} \xi^{-jk} \mathbf{1}_{A_k} \mathbb{E}_\mu \bigg[\sum_{\ell \in \mathbb{Z}/M\mathbb{Z}} \xi^{j\ell} \mathbf{1}_{A_\ell} \cdot \bigg],$$

and from there that $\Pi_0 = M \sum_{j \in \mathbb{Z}/M\mathbb{Z}} \xi^j \Pi_{j,0} = \sum_{k \in \mathbb{Z}/M\mathbb{Z}} \mathbf{1}_{A_{k+1}} \mathbb{E}_\mu[\mathbf{1}_{A_k} \cdot]$.

Finally, equation (3.8) comes from [**4**]. □

For every n, we set

$$\mathfrak{a}_n := \inf\{x > 0 : n|x|^{-\alpha} L(x) \geq 1\}, \tag{3.10}$$

so that $nL(\mathfrak{a}_n) \sim \mathfrak{a}_n^\alpha$. The sequence $(\mathfrak{a}_n)$ is then regularly varying of index $1/\alpha$. Under Hypothesis 3.1, $\mathbb{E}_\mu[e^{i\langle t, S_n\rangle/\mathfrak{a}_n}] \sim (\lambda_{t/\mathfrak{a}_n})^n \sim e^{-\psi(\sqrt{\Sigma}t)}$ for every $t \in \mathbb{R}^d$. Thus, the sequence $(S_n/\mathfrak{a}_n)_n$ converges in distribution to an α-stable random variable with characteristic function $e^{-\psi(\sqrt{\Sigma}\cdot)}$.

3.2. *Strategy of the proof.* Given the length of the proof and the technicality of some of its parts, we give here a brief outline of how the method of moments can be applied to our problem.

Let $\mathcal{Z}_n(\beta)(x) := \sum_{k=1}^n \beta(S_n^T F(x))$ be the Birkhoff sum of β starting from $(x, 0)$. The proof consists in showing the convergence for every m of the mth moment of $\mathcal{Z}_n(\beta)$:

$$\mathbb{E}_\mu\left[\left(\frac{\mathcal{Z}_n(\beta)}{\mathfrak{A}_n}\right)^m\right] = \mathbb{E}_\mu\left[\left(\frac{\sum_{k=1}^n \sum_{a\in\mathbb{Z}^d} \beta(a)\mathbf{1}_{\{S_k=a\}}}{\mathfrak{A}_n}\right)^m\right]$$
$$= \mathfrak{A}_n^{-m} \sum_{k_1,\ldots,k_m=1}^{n} \sum_{a_1,\ldots,a_m\in\mathbb{Z}^d} \beta(a_1)\cdots\beta(a_m)\mu(S_{k_1} = a_1, \ldots, S_{k_m} = a_m).$$

Hence we have to deal with quantities of the following form:

$$\sum_{1\leq k_1<\cdots<k_q\leq n} \sum_{a_1,\ldots,a_q\in\mathbb{Z}^d} \beta(a_1)^{N_1}\cdots\beta(a_q)^{N_q}\mu(S_{k_1} = a_1, \ldots, S_{k_q} = a_q),$$

where $N_1 + \cdots + N_q = m$. Let us write $A_{n;q;N_1,\ldots,N_q}$ for this quantity, which behaves roughly as

$$\sum_{1\leq k_1<\cdots<k_q\leq n} \sum_{a_1,\ldots,a_q\in\mathbb{Z}^d} \beta(a_1)^{N_1}\cdots\beta(a_q)^{N_q}$$
$$\times \mu(S_{k_1} = a_1)\mu(S_{k_2-k_1} = a_2 - a_1)\cdots\mu(S_{k_q-k_{q-1}} = a_q - a_{q-1}).$$

This equation would actually be exact if $(S_n)_n$ were a random walk. Then, put $k_0 := 0$ and $\ell_i := k_i - k_i - 1$, so that

$$A_{n;q;N_1,\ldots,N_q} \sim \sum_{\ell_1+\cdots+\ell_q\leq n} \sum_{a_1,\ldots,a_q\in\mathbb{Z}^d} \prod_{i=1}^q (\beta(a_i)^{N_i}\mu(S_{\ell_i} = a_i - a_{i-1})).$$

We prove that

$$A_{n;q;N_1,\ldots,N_q} = O(\mathfrak{A}_n^m)$$

and even that

$$A_{n;q;N_1,\ldots,N_q} = o(\mathfrak{A}_n^m)$$

except if $(N_1, \ldots, N_q)$ is made of 2s and of pairs of consecutive 1s and of nothing else, which implies that m is even. In particular, for all odd m,

$$\mathbb{E}_\mu[\mathcal{Z}_n^m] = o(\mathfrak{A}_n^m).$$

This is the content of Lemma 3.4, which is by far the most technical part of our proof. This is also the point where we use the fact that β has zero sum; otherwise, we would get $A_{n;q;N_1,\ldots,N_q} = \Theta(\mathfrak{A}_n^{2m})$ for $(N_1, \ldots, N_q) = (1, \ldots, 1)$.

If $(N_1, \ldots, N_q)$ is made of 2s and disjoint pairs of consecutive 1, then it contains $(m-q)$ times the value 2 and $(q-m/2)$ pairs $(N_i, N_{i+1}) = (1, 1)$. Then, we shall prove that

$$
\begin{aligned}
A_{n;q;N_1,\ldots,N_q} &\sim \sum_{\ell_1+\cdots+\ell_{m/2}\le n} \prod_{i=1}^{m-q} \bigg(\sum_{a\in\mathbb{Z}^d} \beta(a)^2 \mu(S_{\ell_i} = a) \bigg) \\
&\quad \times \prod_{i=m-q+1}^{m/2} \bigg(\sum_{\ell\ge 1} \sum_{a,b\in\mathbb{Z}^d} \beta(a)\beta(b)\mu(S_\ell = b-a)\mu(S_{\ell_i} = a) \bigg) \\
&\sim \sum_{\ell_1+\cdots+\ell_{m/2}\le n} \prod_{i=1}^{m-q} \bigg(\sum_{a\in\mathbb{Z}^d} \beta(a)^2 c\mathfrak{a}_{\ell_i}^{-d} \bigg) \\
&\quad \times \prod_{i=m-q+1}^{m/2} \bigg(\sum_{\ell\ge 1} \sum_{a,b\in\mathbb{Z}^d} \beta(a)\beta(b)\mu(S_\ell = b-a) c\mathfrak{a}_{\ell_i}^{-d} \bigg) \\
&\sim c^{m/2} \bigg(\sum_{a\in\mathbb{Z}^d} \beta(a)^2 \bigg)^{m-q} \bigg(\sum_{\ell\ge 1} \sum_{a,b\in\mathbb{Z}^d} \beta(a)\beta(b)\mu(S_\ell = b-a) \bigg)^{q-m/2} \\
&\quad \times \sum_{\ell_1+\cdots+\ell_{m/2}\le n} \prod_{i=1}^{m/2} \mathfrak{a}_{\ell_i}^{-d} \\
&\sim K_{m,q} \bigg(\sum_{\ell=1}^{n} \mathfrak{a}_\ell^{-d} \bigg)^{m/2} = K_{m,q} \mathfrak{A}_n^m,
\end{aligned}
$$

where the constants c and $K_{m,q}$ are explicit and yield the *MLGM* random variables.

3.3. *Proof of Theorem 2.4.* In this section we prove Theorem 2.4. To prove the strong convergence in distribution, it is actually sufficient to prove the convergence in distribution with respect to some absolutely continuous probability measure [**72**, Theorem 1]. At first, we prove the convergence of $(S_n^{\widetilde{T}} f/\mathfrak{A}_n)_n$ under the probability measure $\mu_0 := \mu \otimes \delta_0$, i.e. the convergence of $(\mathcal{Z}_n(\beta)/\mathfrak{A}_n)_n$ under the probability measure μ, where

$$
\mathcal{Z}_n(\beta) = \sum_{k=1}^{n} \beta(S_k).
$$

We use the method of moments. Let $m \ge 0$ be an integer, which is fixed for the remainder of this proof. Then, for all n

$$
\begin{aligned}
\mathbb{E}_\mu[\mathcal{Z}_n(\beta)^m] &= \mathbb{E}_\mu\bigg[\bigg(\sum_{k=1}^{n} \beta(S_k)\bigg)^m\bigg] \\
&= \sum_{k_1,\ldots,k_m=1}^{n} \sum_{d_1,\ldots,d_m\in\mathbb{Z}^d} \mathbb{E}_\mu\bigg[\prod_{s=1}^{m} \beta(d_s)\mathbf{1}_{\{S_{k_s}=d_s\}}\bigg].
\end{aligned}
$$

We delete the terms that are null, and regroup those that are equal. Let us consider one of the terms $\prod_{s=1}^{m} \beta(d_s)\mathbf{1}_{\{S_{k_s}=d_s\}}$. We may assume that $d_s = d_{s'}$ as soon as $k_s = k_{s'}$;

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otherwise, $\mathbf{1}_{\{S_{k_s}=d_s\}}\mathbf{1}_{\{S_{k_{s'}}=d_{s'}\}}=0$ and the whole product is zero. Let $q:=\#\{k_1,\ldots,k_m\}$. Then $\{k_1,\ldots,k_m\}=\{n_1,\ldots,n_q\}$ with $1\le n_1<\cdots<n_q\le n$. We set $N_j:=\#\{i=1,\ldots,m:k_i=n_j\}$ for the multiplicity of n_j in $(k_1,\ldots,k_m)$, and $a_j:=d_i$ if $k_i=n_j$. We write $\mathbf{a}:=(a_1,\ldots,a_q)$, $\mathbf{N}:=(N_1,\ldots,N_q)$ and $\mathbf{n}:=(n_1,\ldots,n_q)$, and set, by convention, $n_0:=0$ and $a_0:=0$. Observe that

$$\prod_{s=1}^{m}\beta(d_s)\mathbf{1}_{\{S_{k_s}=d_s\}}=\prod_{j=1}^{q}\beta(a_j)^{N_j}\mathbf{1}_{\{S_{n_j}=a_j\}}$$

and that the number of m-uplets $(k_1,\ldots,k_m)$ giving the same pair $(\mathbf{n},\mathbf{N})$ is equal to the number $c_{\mathbf{N}}$ of maps $\phi:\{1,\ldots,m\}\to\{1,\ldots,q\}$ such that $|\phi^{-1}(\{j\})|=N_j$ for all $j\in\{1,\ldots,q\}$. Hence

$$\mathbb{E}_\mu[\mathcal{Z}_n(\beta)^m]=\sum_{q=1}^{m}\sum_{\substack{N_j\ge1\\N_1+\cdots+N_q=m}}c_{\mathbf{N}}\sum_{1\le n_1<\cdots<n_q\le n}\sum_{\mathbf{a}\in(\mathbb{Z}^d)^q}\mathbb{E}_\mu\bigg[\prod_{j=1}^{q}(\beta(a_j)^{N_j}\mathbf{1}_{\{S_{n_j}=a_j\}})\bigg].$$

For all $n\ge1$, for all $1\le q\le m$ and for all $\mathbf{N}=(N_j)_{1\le j\le q}$ such that $N_j\ge1$ and $\sum_{j=1}^{q}N_j=m$, we define

$$\begin{aligned}A_{n;q;\mathbf{N}}&:=\sum_{1\le n_1<\cdots<n_q\le n}\sum_{\mathbf{a}\in(\mathbb{Z}^d)^q}\mathbb{E}_\mu\bigg[\prod_{j=1}^{q}(\beta(a_j)^{N_j}\mathbf{1}_{\{S_{n_j}=a_j\}})\bigg]\\&=\sum_{1\le n_1<\cdots<n_q\le n}\sum_{\mathbf{a}\in(\mathbb{Z}^d)^q}\mathbb{E}_\mu\bigg[\prod_{j=1}^{q}(\beta(a_j)^{N_j}\mathbf{1}_{\{S_{n_j}-S_{n_{j-1}}=a_j-a_{j-1}\}})\bigg],\end{aligned}$$

so that

$$\mathbb{E}_\mu[\mathcal{Z}_n(\beta)^m]=\sum_{q=1}^{m}\sum_{\substack{N_j\ge1\\N_1+\cdots+N_q=m}}c_{\mathbf{N}}A_{n;q,\mathbf{N}}.\tag{3.11}$$

Instead of working with a sequence of times (n_j) and positions (a_j), it shall be more convenient to work with time increments and position increments. Let $1\le n_1<\cdots<n_q\le n$. We can describe this sequence with integers $(\ell_1,\ldots,\ell_q)$ by taking $\ell_1:=n_1$ and $\ell_j:=n_j-n_{j-1}$ for all $2\le j\le q$. In what follows, sequences $(\ell_1,\ldots,\ell_q)\in\{1,\ldots,n\}^q$ and sequences $(n_1,\ldots,n_q)$ such that $1\le n_1<\cdots<n_q\le n$ shall be related in this way. Let $E_{q,n}$ be the set defined by

$$E_{q,n}=\bigg\{\boldsymbol{\ell}=(\ell_1,\ldots,\ell_q)\in\{1,\ldots,n\}^q:\sum_{j=1}^{q}\ell_j\le n\bigg\}.$$

Then summing over all $\mathbf{n}=(n_1,\ldots,n_q)$ such that $1\le n_1<\cdots<n_q\le n$ is the same as summing over all $\boldsymbol{\ell}$ in $E_{q,n}$, whence

$$A_{n;q;\mathbf{N}}=\sum_{\mathbf{a}\in(\mathbb{Z}^d)^q}\bigg[\bigg(\prod_{j=1}^{q}\beta(a_j)^{N_j}\bigg)\sum_{\boldsymbol{\ell}\in E_{q,n}}\mathbb{E}_\mu\bigg[\prod_{j=1}^{q}\mathbf{1}_{\{S_{\ell_j}=a_j-a_{j-1}\}}\circ T^{n_{j-1}}\bigg]\bigg].\tag{3.12}$$

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A single coefficient $A_{n;q;\mathbf{N}}$ is the contribution to the mth moment of $\mathcal{Z}_n(\beta)$ by paths of length q and with weights (N_j). Our goal is to find a sub-family of such weighted paths, which is manageable enough so that we can estimate the behaviour of the $A_{n;q;\mathbf{N}}$, and large enough so that it makes for almost all $\mathbb{E}_\mu[\mathcal{Z}_n(\beta)^m]$ as n goes to infinity. However, in order to benefit from the fact that $\sum_{a\in\mathbb{Z}^d}\beta(a)=0$, we use transfer operators, and a decomposition, which leverages this equality to make some further simplifications†.

For all $\ell\in\mathbb{N}$ and $a\in\mathbb{Z}^d$, we define an operator $Q_{\ell,a}$ acting on $\mathcal{B}$ by

$$Q_{\ell,a}(h):=P^\ell(\mathbf{1}_{\{S_\ell=a\}}h)=\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}e^{-i\langle u,a\rangle}P_u^\ell(h)\,du,$$

where we used (3.1) to establish the second formula. For $1\le q'\le q$, we write

$$D_{q'}:=\prod_{j=1}^{q'}(\mathbf{1}_{\{S_{\ell_j}=a_j-a_{j-1}\}}\circ T^{n_{j-1}}).$$

Recall that $P^k(g\circ T^k\cdot h)=g\,P^k(h)$. Hence, by induction,

$$\begin{aligned}P^{n_q}(D_q)&=P^{n_q}(\mathbf{1}_{\{S_{\ell_q}=a_q-a_{q-1}\}}\circ T^{n_{q-1}}\cdot D_{q-1})\\&=P^{n_q-n_{q-1}}(\mathbf{1}_{\{S_{\ell_q}=a_q-a_{q-1}\}}P^{n_{q-1}}(D_{q-1}))\\&=Q_{\ell_q,a_q-a_{q-1}}(P^{n_{q-1}}(D_{q-1}))\\&=\cdots\\&=Q_{\ell_q,a_q-a_{q-1}}\cdots Q_{\ell_1,a_1-a_0}(\mathbf{1}).\end{aligned}$$

Plugging this identity into equation (3.12) yields

$$A_{n;q;\mathbf{N}}=\sum_{\mathbf{a}\in(\mathbb{Z}^d)^q}\bigg[\bigg(\prod_{j=1}^{q}\beta(a_j)^{N_j}\bigg)\sum_{\boldsymbol{\ell}\in E_{q,n}}\mathbb{E}_\mu[Q_{\ell_q,a_q-a_{q-1}}\cdots Q_{\ell_1,a_1-a_0}(\mathbf{1})]\bigg]. \tag{3.13}$$

We further split the operators $Q_{\ell,a}$. Let us write

$$Q_{\ell,a}=Q_{\ell,a}^{(0)}+Q_{\ell,a}^{(1)}, \tag{3.14}$$

with

$$\begin{aligned}Q_{\ell,a}^{(0)}&:=\Phi(0)\frac{\Pi_0^\ell}{\mathfrak{a}_\ell^d}\\Q_{\ell,a}^{(1)}&=\varepsilon_{\ell,a}+\frac{\Phi(a/\mathfrak{a}_\ell)-\Phi(0)}{\mathfrak{a}_\ell^d}\Pi_0^\ell\quad\text{with }\|\varepsilon_{\ell,a}\|=o(\mathfrak{a}_\ell^{-d}),\end{aligned}$$

where Π_0 is as in Hypothesis 3.1, which we know is possible thanks to Lemma 3.6.

† If β has a non-zero integral, different terms dominate, and the moments grow faster. It is thus essential to cancel out these 'first-order terms'.

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We introduce these operators $Q^{(0)}_{\ell,a}$ and $Q^{(1)}_{\ell,a}$ into (3.13), creating new data we need to track: the index of the operator we use at each point in the weighted path. Fix n, q and $\mathbf{N}$. Given $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_q) \in \{0, 1\}^q$ and $s \in \mathbb{Z}^d$, write

$$\begin{aligned}
B^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}} &:= \sum_{\substack{a_0,\ldots,a_q\in\mathbb{Z}^d\\ a_0=s}} \bigg(\prod_{i=1}^{q} \beta(a_i)^{N_i}\bigg) Q^{(\varepsilon_q)}_{\ell_q,a_q-a_{q-1}} \cdots Q^{(\varepsilon_1)}_{\ell_1,a_1-a_0},\\
b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(\cdot) &:= \mathbb{E}_\mu[B^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(\cdot)]\\
&= \sum_{\substack{a_0,\ldots,a_q\in\mathbb{Z}^d\\ a_0=s}} \mathbb{E}_\mu[\beta(a_q)^{N_q} Q^{(\varepsilon_q)}_{\ell_q,a_q-a_{q-1}} \beta(a_{q-1})^{N_{q-1}} \ldots \beta(a_1)^{N_1} Q^{(\varepsilon_1)}_{\ell_1,a_1-s}(\cdot)],\\
A^{\boldsymbol{\varepsilon}}_{n;q;\mathbf{N}} &:= \sum_{\boldsymbol{\ell}\in E_{q,n}} b^{\boldsymbol{\varepsilon}}_{0,\boldsymbol{\ell},\mathbf{N}}(\mathbf{1}),
\end{aligned} \tag{3.15}$$

so that

$$A_{n;q;\mathbf{N}} = \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^q} A^{\boldsymbol{\varepsilon}}_{n;q;\mathbf{N}} = \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^q} \sum_{\boldsymbol{\ell}\in E_{q,n}} b^{\boldsymbol{\varepsilon}}_{0,\boldsymbol{\ell},\mathbf{N}}(\mathbf{1}).$$

The datum $s \in \mathbb{Z}^d$ is the starting point of a weighted path. Later on, we shall estimate $B^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}$ and $b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}$ on pieces of the weighted path $(a_0, \ldots, a_q)$, in which case the point s will not always be 0. The quantity $A^{\boldsymbol{\varepsilon}}_{n;q;\mathbf{N}}$ is related to $A_{n;q;\mathbf{N}}$ and therefore shall only be estimated for the whole path $(a_0, \ldots, a_q)$ with, as before, the convention $a_0 = 0$, so in its definition we always take $s = 0$.

Now, the main question is: for which data $(q, \mathbf{N}, \boldsymbol{\varepsilon})$ do the coefficients $A^{\boldsymbol{\varepsilon}}_{n;q;\mathbf{N}}$, seen as functions of n, grow the fastest? One would want to use the larger operator $Q^{(0)}_{\ell,a}$ whenever possible, and to use the lowest possible weights whenever possible (because lower weights means larger value of q, so a faster combinatorial growth). *A priori*, the best possible choice would be $\boldsymbol{\varepsilon} = (0, \ldots, 0)$ and $\mathbf{N} = (1, \ldots, 1)$. That is indeed true for observables β with a non-zero integral. However, in our case, the fact that $\sum_{a\in\mathbb{Z}^d} \beta(a) = 0$ induces a cancellation, which makes the corresponding coefficient vanish. This can be seen with the following elementary properties.

PROPERTIES 3.3. *Consider a single linear form $b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}$. For all $1 \le i \le q$, the terms on the right side of $Q^{(\varepsilon_i)}_{\ell_i,a_i-a_{i-1}}$ in equation (3.15) depend only on $a_1, \ldots, a_{i-1}$, and the terms on its left side only depend on $a_i, \ldots, a_q$. Hence, the following is true.*

(i) *Since $Q^{(0)}_{\ell,a}$ does not depend on a, the value of $b^{(0,\boldsymbol{\varepsilon})}_{s,(\ell_0,\boldsymbol{\ell}),(N_0,\mathbf{N})}$ does not depend on s. Without loss of generality, we shall choose s to be 0 when $\varepsilon_1 = 0$.*

(ii) $b^{(0)}_{s,(\ell),(1)}(\cdot) = \Phi(0)\mathfrak{a}_\ell^{-d} \sum_{a\in\mathbb{Z}^d} \beta(a)\mathbb{E}_\mu[\cdot] = 0$ *and* $b^{(0)}_{s,(\ell),(N)}(\cdot) = \Phi(0)\mathfrak{a}_\ell^{-d} \sum_{a\in\mathbb{Z}^d} \beta(a)^N\mathbb{E}_\mu[\cdot]$ *for all ℓ, $N \ge 1$.*

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(iii) $b^{(\boldsymbol{\varepsilon},0,\boldsymbol{\varepsilon}')}_{s,(\boldsymbol{\ell},\ell_0,\boldsymbol{\ell}'),(\mathbf{N},N_0,\mathbf{N}')} = \sum_{j\in\mathbb{Z}/M\mathbb{Z}} \mathbb{E}_\mu[\mathbf{1}_{A_j} B^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(\cdot)]\mathbb{E}_\mu[B^{(0,\boldsymbol{\varepsilon}')}_{0,(\ell_0,\boldsymbol{\ell}'),(N_0,\mathbf{N}')}(\mathbf{1}_{A_{j+\ell_0}})]$, *i.e.*

$$b^{(\boldsymbol{\varepsilon},0,\boldsymbol{\varepsilon}')}_{s,(\boldsymbol{\ell},\ell_0,\boldsymbol{\ell}'),(\mathbf{N},N_0,\mathbf{N}')}(\cdot) = \sum_{j\in\mathbb{Z}/M\mathbb{Z}} b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(\mathbf{1}_{A_j}\cdot) b^{(0,\boldsymbol{\varepsilon}')}_{0,(\ell_0,\boldsymbol{\ell}'),(N_0,\mathbf{N}')}(\mathbf{1}_{A_{j+|\boldsymbol{\ell}|_1}}),$$

since $Q^{(\varepsilon_i)}_{\ell_i,a_i-a_{i-1}}(\mathbf{1}_{A_j}\cdot) = \mathbf{1}_{A_{j+\ell_i}} Q^{(\varepsilon_i)}_{\ell_i,a_i-a_{i-1}}(\cdot)$.

(iv) *In particular,* $b^{(\boldsymbol{\varepsilon},0)}_{s,(\boldsymbol{\ell},\ell_0),(\mathbf{N},1)} = 0$, *and*

$$\begin{aligned} &b^{(\boldsymbol{\varepsilon},0,0,\boldsymbol{\varepsilon}')}_{s,(\boldsymbol{\ell},\ell_0,\ell_0',\boldsymbol{\ell}'),(\mathbf{N},1,N_0',\mathbf{N}')} \\ &\quad= \sum_{j\in\mathbb{Z}/M\mathbb{Z}} b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(\mathbf{1}_{A_j}\cdot) b^{(0)}_{0,(\ell_0),(1)}(\mathbf{1}_{A_{j+|\boldsymbol{\ell}|_1}}) b^{(0,\boldsymbol{\varepsilon}')}_{0,(\ell_0',\boldsymbol{\ell}'),(N_0',\mathbf{N}')}(\mathbf{1}_{A_{j+|\boldsymbol{\ell}|_1+\ell_0}}) = 0. \end{aligned}$$

(v)
$$\begin{aligned} &b^{(0,1,\ldots,1)}_{s,(\ell_1,\ldots,\ell_q),(N_1,N_2,\ldots,N_q)}(\mathbf{1}_{A_j}) \\ &\quad= \Phi(0)\mathfrak{a}_{\ell_1}^{-d} \sum_{a_1\in\mathbb{Z}^d} \beta(a_1)^{N_1} b^{(1,\ldots,1)}_{a_1,(\ell_2,\ldots,\ell_q),(N_2,\ldots,N_q)}(\mathbf{1}_{A_{j+\ell_1}}). \end{aligned}$$

(vi) *Applying point* (v) *and the fact that* $\sum_{a,b\in\mathbb{Z}^d} \beta(a)\beta(b)\mathbb{E}_\mu[Q^{(0)}_{\ell',a-b}(\mathbf{1})] = 0$, *we get*

$$\begin{aligned} b^{(0,1)}_{s,(\ell,\ell'),(1,1)}(\mathbf{1}_{A_j}) &= \Phi(0)\mathfrak{a}_\ell^{-d} \sum_{a,b\in\mathbb{Z}^d} \beta(a)\beta(b)\mathbb{E}_\mu[Q_{\ell',a-b}(\mathbf{1}_{A_{j+\ell}})] \\ &= \Phi(0)\mathfrak{a}_\ell^{-d} \sum_{a,b\in\mathbb{Z}^d} \beta(a)\beta(b)\mu(A_{j+\ell};\, S_{\ell'} = a-b). \end{aligned}$$

Given a sequence $\boldsymbol{\varepsilon} \in \{0,1\}^q$, we can iterate point (iii) above to cut $b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}$ into smaller pieces, for which 0 may only appear at the beginning of the associated sequences of indices, and then use point (v) to transform the heading $\varepsilon_i = 0$. Write $m_1 < m_2 < \cdots < m_K$ for the indices $i \in \{1,\ldots,q\}$ such that $\varepsilon_i = 0$. We use the conventions that $m_{K+1} := q+1$ and $\varepsilon_{q+1} := 0$, that $b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}} \equiv 1$ if $q = 0$, and that an empty product is also equal to 1. Then

$$\begin{aligned} b^{\boldsymbol{\varepsilon}}_{s,\boldsymbol{\ell},\mathbf{N}}(\mathbf{1}) &= \sum_{j\in\mathbb{Z}/M\mathbb{Z}} b^{(1,\ldots,1)}_{s,(\ell_1,\ldots,\ell_{m_1-1}),(N_1,\ldots,N_{m_1-1})}(\mathbf{1}_{A_j}) \\ &\quad\times \prod_{i=1}^K b^{(0,1,\ldots,1)}_{0,(\ell_{m_i},\ldots,\ell_{m_{i+1}-1}),(N_{m_i},\ldots,N_{m_{i+1}-1})}(\mathbf{1}_{A_{j+\ell_1+\cdots+\ell_{m_i}}}) \\ &= (M\Phi(0))^K \sum_{j\in\mathbb{Z}/M\mathbb{Z}} b^{(1,\ldots,1)}_{s,(\ell_1,\ldots,\ell_{m_1-1}),(N_1,\ldots,N_{m_1-1})}(\mathbf{1}_{A_j}) \\ &\quad\times \prod_{i=1}^K \mathfrak{a}_{\ell_{m_i}}^{-d} \sum_{a\in\mathbb{Z}^d} \beta(a)^{N_{m_i}} b^{(1,\ldots,1)}_{a,(\ell_{m_i+1},\ldots,\ell_{m_{i+1}-1}),(N_{m_i+1},\ldots,N_{m_{i+1}-1})}(\mathbf{1}_{A_{j+\ell_1+\cdots+\ell_{m_i}}}). \end{aligned}$$

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We sum over $\boldsymbol{\ell} \in E_{q,n}$, and get

$$|A^{\boldsymbol{\varepsilon}}_{n,q,\mathbf{N}}| \leq \sum_{\boldsymbol{\ell}\in\{1,\ldots,n\}^q} |b^{\boldsymbol{\varepsilon}}_{0,\boldsymbol{\ell},\mathbf{N}}(\mathbf{1})|$$

$$\leq (M\Phi(0))^K \Bigg(\sum_{\substack{(\ell_1,\ldots,\ell_{m_1-1})\\ \in\{1,\ldots,n\}^{m_1-1}}} \sup_{j\in\mathbb{Z}/M\mathbb{Z}} |b^{(1,\ldots,1)}_{0,(\ell_1,\ldots,\ell_{m_1-1}),(N_1,\ldots,N_{m_1-1})}(\mathbf{1}_{A_j})| \Bigg) \quad (3.16)$$

$$\times \prod_{i=1}^{K} \Bigg(\mathfrak{A}_n^2 \sum_{\substack{(\ell_{m_i+1},\ldots,\ell_{m_{i+1}-1})\\ \in\{1,\ldots,n\}^{m_{i+1}-m_i-1}}} \sup_{j\in\mathbb{Z}/M\mathbb{Z}}$$

$$\times \Bigg| \sum_{a\in\mathbb{Z}^d} \beta(a)^{N_{m_i}} b^{(1,\ldots,1)}_{a,(\ell_{m_i+1},\ldots,\ell_{m_{i+1}-1}),(N_{m_i+1},\ldots,N_{m_{i+1}-1})}(\mathbf{1}_{A_j}) \Bigg| \Bigg). \quad (3.17)$$

Fix $\omega \in (0, 1]$ such that $(\alpha - d)/2 < \omega < (\alpha - d)/2 + \varepsilon$, and $\eta \in (0, \omega]$ such that $\omega + \eta \leq (\alpha - d)/2 + \varepsilon$. The control of (3.16) and of (3.17) shall be done with the following technical lemma, the proof of which is postponed until §3.4.

LEMMA 3.4. *Under the assumptions of Theorem 2.4 and with the previous notation, for every $q \geq 1$ and $\mathbf{N} = (N_1, \ldots, N_q) \in \mathbb{N}_+^q$, for every $j \in \mathbb{Z}/M\mathbb{Z}$,*

$$\sup_{a\in\mathbb{Z}^d} \frac{1}{1+|a|^\eta} \sum_{\boldsymbol{\ell}\in\{1,\ldots,n\}^q} |b^{(1,\ldots,1)}_{a,\boldsymbol{\ell},\mathbf{N}}(\mathbf{1}_{A_j})| = o(\mathfrak{A}_n^{N_1+\cdots+N_q}), \quad (3.18)$$

$$\sum_{\ell=1}^{n} \Bigg| \sum_{a\in\mathbb{Z}^d} \beta(a) b^{(1)}_{a,(\ell),(N)}(\mathbf{1}_{A_j}) \Bigg| = \begin{cases} O(1) & \text{if } q = 1,\ N = 1, \\ o(\mathfrak{A}_n) & \text{if } q = 1,\ N \geq 2, \end{cases} \quad (3.19)$$

$$\sum_{\boldsymbol{\ell}\in\{1,\ldots,n\}^q} \Bigg| \sum_{a\in\mathbb{Z}^d} \beta(a) b^{(1,\ldots,1)}_{a,\boldsymbol{\ell},\mathbf{N}}(\mathbf{1}_{A_j}) \Bigg| = o(\mathfrak{A}_n^{N_1+\cdots+N_q-1}) \quad \text{if } q \geq 2. \quad (3.20)$$

Equation (3.19) implies in particular that $\sum_{\ell=1}^n |\sum_{a,b\in\mathbb{Z}^d} \beta(a)\beta(b)\mathbb{E}_\mu[Q^{(1)}_{\ell,a-b}(\mathbf{1})]|$ is bounded independently of n. Since $\sum_{a,b\in\mathbb{Z}^d} \beta(a)\beta(b)\mathbb{E}_\mu[Q^{(0)}_{\ell,a-b}(\mathbf{1})] = 0$ for all ℓ, we get that $\sum_{\ell\geq 0}\sum_{a,b\in\mathbb{Z}^d} \beta(a)\beta(b)\mu(S_\ell = b - a)$ is absolutely convergent in ℓ, as claimed in Theorem 2.4.

We consider the following condition on the sequences $\boldsymbol{\varepsilon}$ and $\mathbf{N}$:

$$m_1 = 1,$$
$$\text{for all } i \in \{1, \ldots, K\}, \begin{cases} N_{m_i} \in \{1, 2\} & \\ N_{m_i} = 1 & \Rightarrow \quad m_{i+1} = m_i + 2,\ N_{1+m_i} = 1, \\ N_{m_i} = 2 & \Rightarrow \quad m_{i+1} = m_i + 1. \end{cases} \quad (3.21)$$

Note that this condition implies that $m = 2K$.

COROLLARY 3.5. *Use the assumptions of Theorem 2.4 and the previous notation. Let $m \geq 1$, $q \geq 1$ and $N_1, \ldots, N_q \in \mathbb{N}_+$ be such that $N_1 + \cdots + N_q = m$. If condition (3.21) holds, then*

$$|A^{\boldsymbol{\varepsilon}}_{n,q,\mathbf{N}}| = O(\mathfrak{A}_n^m);$$

otherwise,

$$|A^{\boldsymbol{\varepsilon}}_{n,q,\mathbf{N}}| = o(\mathfrak{A}_n^m).$$

In particular,

$$\begin{cases}\mathbb{E}_\mu[\mathcal{Z}_n(\beta)^m] = O(\mathfrak{A}_n^m) & \textit{for all } m \in 2\mathbb{N},\\ \mathbb{E}_\mu[\mathcal{Z}_n(\beta)^m] = o(\mathfrak{A}_n^m) & \textit{for all } m \in 2\mathbb{N}+1.\end{cases} \tag{3.22}$$

Proof. Due to equation (3.18), the term (3.16) is an $o(\mathfrak{A}_n^{N_1+\cdots+N_{m_1-1}})$ if $m_1 \neq 1$, and an $O(1) = O(\mathfrak{A}_n^{N_1+\cdots+N_{m_1-1}})$ if $m_1 = 1$.

Let us now estimate the term (3.17). Due to equation (3.18), since $\sum_{a\in\mathbb{Z}^d} |a|^\eta |\beta(a)| < +\infty$, for all $N_0 \geq 2$ and $q \geq 1$, for all $N_1, \ldots, N_q \geq 1$,

$$\mathfrak{A}_n^2 \sum_{\boldsymbol{\ell}\in\{1,\ldots,n\}^q} \sup_{j\in\mathbb{Z}/M\mathbb{Z}} \left| \sum_{a\in\mathbb{Z}^d} \beta(a)^{N_0} b^{(1,\ldots,1)}_{a,\ell,(N_1,\ldots,N_q)}(\mathbf{1}_{A_j})\right| = o(\mathfrak{A}_n^{N_0+N_1+\cdots+N_q}).$$

Due to equation (3.19), this estimate holds when $N_0 \geq 3$ and $q = 0$; due to equation (3.20), this estimate holds when $N_0 = 1$ and $q \geq 2$.

The two remaining cases are $N_0 = 2$, $q = 0$ and $N_0 = q = 1$. When $N_0 = 2$ and $q = 0$, we have a upper bound in $O(\mathfrak{A}_n^2) = O(\mathfrak{A}_n^{N_0+\cdots+N_q})$. When $q = N_0 = N_1 = 1$, the same upper bound is given by equation (3.19). If $q = N_0 = 1$ and $N_1 \geq 2$, then equation (3.19) yields a upper bound in $o(\mathfrak{A}_n^3) = o(\mathfrak{A}_n^{N_0+\cdots+N_q})$.

Hence, the term (3.17) is in $O(\mathfrak{A}_n^{2K}) = O(\mathfrak{A}_n^{N_{m_1}+\cdots+N_q})$ if, for every $i \in \{1, \ldots, K\}$, we are in one of two cases:

- $N_{m_i} = 1$, $m_{i+1} = m_i + 2$ and $N_{1+m_i} = 1$;
- $N_{m_i} = 2$, $m_{i+1} = m_i + 1$.

Otherwise, (3.17) is in $o(\mathfrak{A}_n^{N_{m_1}+\cdots+N_q})$. In particular,

$$|A^{\boldsymbol{\varepsilon}}_{n,q,\mathbf{N}}| = O(\mathfrak{A}_n^m).$$

Furthermore, if condition (3.21) is not satisfied, either (3.16) is an $o(\mathfrak{A}_n^{N_1+\cdots+N_{m_1-1}})$ or one of the terms in (3.17) is an $o(\mathfrak{A}_n^{N_{m_1}+\cdots+N_{m_{i+1}-1}})$, so $|A^{\boldsymbol{\varepsilon}}_{n,q,\mathbf{N}}| = o(\mathfrak{A}_n^m)$. This is the case, in particular, if m is odd. □

Condition (3.21) can be rewritten:

- $\max_i N_i \leq 2$;
- $\varepsilon_i = 0$ as soon as $N_i = 2$;
- there exists $\mathcal{J} \subset \{1, \ldots, q\}$ such that $\{i : N_i = 1\} = \bigsqcup_{j\in\mathcal{J}} \{j, j+1\}$;
- $\varepsilon_j = 0$ and $\varepsilon_{j+1} = 1$ for all $j \in \mathcal{J}$.

Assume now that $m \geq 0$ is even. Let us write $\mathcal{G}(q)$ for the set of $\mathbf{N} = (N_1, \ldots, N_q) \in \{1, 2\}^q$ such that $N_1 + \cdots + N_q = m$ and $\{i \in \{1, \ldots, q\} : N_i = 1\}$ is the disjoint union of pairs of the form $\{j, j+1\}$. Given $\mathbf{N} \in \mathcal{G}(q)$, there exists a unique $\boldsymbol{\varepsilon}(\mathbf{N}) \in \{0, 1\}^q$ such that $(\boldsymbol{\varepsilon}(\mathbf{N}), \mathbf{N})$ satisfies condition (3.21). Note that $q = |\{i : \varepsilon_i = 0, N_i = 2\}| + 2|\{i : \varepsilon_i = 0, N_i = 1\}|$ and $m/2 = |\{i : \varepsilon_i = 0, N_i = 2\}| + |\{i : \varepsilon_i = 0, N_i = 1\}|$, so that

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$|\{i : \varepsilon_i = 0,\ N_i = 1\}| = q - m/2$ and $|\{i : N_i = 2\}| = m - q$. Then

$$b^{\boldsymbol{\varepsilon}(\mathbf{N})}_{0;\boldsymbol{\ell},\mathbf{N}}(\mathbf{1}) = \sum_{j\in\mathbb{Z}/M\mathbb{Z}} \bigg(\prod_{i:N_i=2} b^{(0)}_{(\ell_i),(2)}(\mathbf{1}_{A_{j+\ell_1+\cdots+\ell_i}})\bigg) \times\bigg(\prod_{i:N_i=1,\varepsilon_i=0} b^{(0,1)}_{(\ell_i,\ell_{i+1}),(1,1)}(\mathbf{1}_{A_{j+\ell_1+\cdots+\ell_i}})\bigg).$$

Let $\tilde{E}_{q,n}$ be the set of q-uplets of integers $(\ell_1, \ldots, \ell_q) \in \{1, \ldots, n\}^q$ such that $M\sum_{i=1}^q \lceil \ell_i/M\rceil \le n$. Using points (ii) and (vi) in properties 3.3, we get

$$\begin{aligned}
A^{\boldsymbol{\varepsilon}(\mathbf{N})}_{n;q;\mathbf{N}} &= (M\Phi(0))^{m/2} \sum_{j\in\mathbb{Z}/M\mathbb{Z}} \sum_{\boldsymbol{\ell}\in E_{q,n}} \bigg(\prod_{i:N_i=2} \frac{\sum_{a\in\mathbb{Z}^d}\beta(a)^2}{M\mathfrak{a}^d_{\ell_i}}\bigg) \\
&\quad\times\bigg(\prod_{i:N_i=1,\varepsilon_i=0} \frac{\sum_{a,b\in\mathbb{Z}^d}\beta(a)\beta(b)\mu(A_{j+\ell_1+\cdots+\ell_i};\ S_{\ell_{i+1}} = a-b)}{\mathfrak{a}^d_{\ell_i}}\bigg) \\
&= o(\mathfrak{A}^m_n) + \Phi(0)^{m/2} \sum_{j\in\mathbb{Z}/M\mathbb{Z}} \sum_{\boldsymbol{\ell}\in\tilde{E}_{q,n}} \bigg(\prod_{i:N_i=2} \frac{\sum_{a\in\mathbb{Z}^d}\beta(a)^2}{\mathfrak{a}^d_{\ell_i}}\bigg) \\
&\quad\times\bigg(\prod_{i:N_i=1,\varepsilon_i=0} \frac{\sum_{a,b\in\mathbb{Z}^d}\beta(a)\beta(b)\sum_{k=1}^M \mu(A_{j+\ell_1+\cdots+\lfloor \ell_i/M\rfloor M+k};\ S_{\ell_{i+1}} = a-b)}{\mathfrak{a}^d_{\lceil \ell_j/M\rceil M}}\bigg) \\
&= o(\mathfrak{A}^m_n) + \Phi(0)^{m/2} \sum_{\boldsymbol{\ell}\in\tilde{E}_{q,n}} \bigg(\prod_{i:N_i=2} \frac{\sum_{a\in\mathbb{Z}^d}\beta(a)^2}{\mathfrak{a}^d_{\ell_i}}\bigg) \\
&\quad\times\bigg(\prod_{i:N_i=1,\varepsilon_i=0} \frac{\sum_{a,b\in\mathbb{Z}^d}\beta(a)\beta(b)\mu(S_{\ell_{i+1}} = a-b)}{\mathfrak{a}^d_{\lceil \ell_i/M\rceil M}}\bigg) \\
&= o(\mathfrak{A}^m_n) + \Phi(0)^{m/2}\bigg(\sum_{a\in\mathbb{Z}^d}\beta(a)^2\bigg)^{m-q} \sum_{\ell_1,\ldots,\ell_{q-m/2}\ge 1} \\
&\quad\times\bigg[\bigg(\prod_{i=1}^{q-m/2}\sum_{a,b\in\mathbb{Z}^d}\beta(a)\beta(b)\mu(S_{\ell_i} = a-b)\bigg) \\
&\quad\times\bigg(\sum_{\boldsymbol{\ell}'\in E_{m/2,n-\sum_{i=1}^{q-m/2}\ell_i}} \prod_{i=1}^{m/2}\frac{1}{\mathfrak{a}^d_{\ell'_i}}\bigg)\bigg].
\end{aligned}$$

The sequence $(\mathfrak{A}_n)$ has regular variation. Due to Lemma 3.7, for all $\ell_1, \ldots, \ell_{q-m/2} \ge 1$,

$$\sum_{\ell'\in E_{m/2,n-\sum_{i=1}^{q-m/2}\ell_i}} \prod_{j=1}^{m/2} \mathfrak{a}^{-d}_{\ell'_j} \sim \mathfrak{A}^m_n \frac{\Gamma(1+(\alpha-d)/\alpha)^{m/2}}{\Gamma(1+m/2(\alpha-d)/\alpha)} \quad \text{as } n\to+\infty.$$

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Hence, by the dominated convergence theorem,

$$A_{n;q;\mathbf{N}}^{\boldsymbol{\varepsilon}(\mathbf{N})} \sim \mathfrak{A}_n^m \Phi(0)^{m/2} \bigg(\sum_{a \in \mathbb{Z}^d} \beta(a)^2 \bigg)^{m-q} \times \bigg(\sum_{\ell \geq 1} \sum_{a,b \in \mathbb{Z}^d} \beta(a)\beta(b)\mu(S_\ell = a-b) \bigg)^{q-m/2} \frac{\Gamma(1+(\alpha-d)/\alpha)^{m/2}}{\Gamma(1+(m)/2(\alpha-d)/\alpha)}.$$

If $\mathbf{N} \notin \mathcal{G}(q)$, or $\mathbf{N} \in \mathcal{G}(q)$ but $\boldsymbol{\varepsilon} \neq \boldsymbol{\varepsilon}(\mathbf{N})$, we have already seen that $A_{n;q;\mathbf{N}}^{\boldsymbol{\varepsilon}} = o(\mathfrak{A}_n^m)$. Therefore, by equation (3.11),

$$\mathbb{E}_\mu[\mathcal{Z}_n(\beta)^m] \sim \sum_{q=1}^m \sum_{\mathbf{N} \in \mathcal{G}(q)} c_{\mathbf{N}} A_{n;q;\mathbf{N}}^{\boldsymbol{\varepsilon}(\mathbf{N})}.$$

For fixed q, the value of $c_{\mathbf{N}}$ does not depend on $\mathbf{N} \in \mathcal{G}(q)$, as the multiset of weights is the same. There are $2^{-(m-q)}m!$ maps from $\{1, \ldots, m\}$ to $\{1, \ldots, q\}$ such that $1, \ldots, m-q$ each have two preimages, and $m-q+1, \ldots, q$ each have one preimage. Thus,

$$\text{for all } m \in 2\mathbb{Z}, \quad \mathbb{E}_\mu[\mathcal{Z}_n(\beta)^m] \sim m! \sum_{q=1}^m 2^{-(m-q)} \sum_{\mathbf{N} \in \mathcal{G}(q)} A_{n;q;\mathbf{N}}^{\boldsymbol{\varepsilon}(\mathbf{N})}.$$

For fixed q, there are $\binom{m/2}{q-m/2}$ sequences $\mathbf{N} \in \mathcal{G}(q)$: each such sequence is the concatenation of $m/2$ blocs of two different kinds, with $r := q - m/2$ blocs of one kind. Thus, for even m,

$$\begin{aligned}
\mathbb{E}_\mu[\mathcal{Z}_n(\beta)^m] &\sim \mathfrak{A}_n^m m! \Phi(0)^{m/2} \frac{\Gamma(1+(\alpha-d)/\alpha)^{m/2}}{\Gamma(1+m/2(\alpha-d)/\alpha)} \\
&\quad \times \sum_{r=0}^{m/2} \binom{m/2}{r} \bigg(\frac{\sum_{a \in \mathbb{Z}^d} \beta(a)^2}{2} \bigg)^{m/2-r} \\
&\quad \times \bigg(\sum_{\ell \geq 1} \sum_{a,b \in \mathbb{Z}^d} \beta(a)\beta(b)\mu(S_\ell = a-b) \bigg)^r \\
&= \mathfrak{A}_n^m m! \frac{\Gamma(1+(\alpha-d)/\alpha)^{m/2}}{\Gamma(1+(m/2)(\alpha-d)/\alpha)} \\
&\quad \times \bigg[\frac{\Phi(0)}{2} \bigg(\sum_{a \in \mathbb{Z}^d} \beta(a)^2 + 2 \sum_{\ell \geq 1} \sum_{a,b \in \mathbb{Z}^d} \beta(a)\beta(b)\mu(S_\ell = a-b) \bigg) \bigg]^{m/2} \\
&= \mathfrak{A}_n^m \frac{m!\Gamma(1+(\alpha-d)/\alpha)^{m/2}}{2^{m/2}\Gamma(1+(m/2)(\alpha-d)/\alpha)} \Phi(0)^{m/2} \sigma_{GK}(\beta, \widetilde{A}, \tilde{\mu}, \widetilde{T})^m.
\end{aligned}$$

Let $\mathcal{Y}$ be a random variable with a standard $MLGM(1-\alpha/d)$ distribution. Its distribution function is even, so all its odd moments are 0. Let Y have a standard Mittag–Leffler distribution of parameter $1-\alpha/d$ and N be a standard Gaussian random variable. Then the even moments of $\mathcal{Y}$ are

$$\begin{aligned}
\mathbb{E}[\mathcal{Y}^m] = \mathbb{E}[Y^{m/2}]\mathbb{E}[N^m] &= \frac{(m/2)!\Gamma(1+(\alpha-d)/\alpha)^{m/2}}{\Gamma(1+(m/2)(\alpha-d)/\alpha)} \frac{m!}{2^{m/2}(m/2)!} \\
&= \frac{m!\Gamma(1+(\alpha-d)/\alpha)^{m/2}}{2^{m/2}\Gamma(1+(m/2)(\alpha-d)/\alpha)},
\end{aligned}$$

so that, for even m:

$$\mathbb{E}_\mu[\mathcal{Z}_n(\beta)^m] \sim \mathfrak{A}_n^m \mathbb{E}[(\sqrt{\Phi(0)}\sigma_{GK}(\beta, \widetilde{A}, \tilde{\mu}, \widetilde{T})\mathcal{Y})^m].$$

We already know that $\mathbb{E}_\mu[\mathcal{Z}_n(\beta)^m] = o(\mathfrak{A}_n^m)$ for odd m. Hence, all the moments of $(\mathcal{Z}_n(\beta)/\mathfrak{A}_n)_n$ converge to the moments of $\sqrt{\Phi(0)}\sigma_{GK}(\beta, \widetilde{A}, \tilde{\mu}, \widetilde{T})\mathcal{Y}$. Since,

$$\sum_{m\geq 0}\left[\frac{\Gamma(1+m/2(\alpha-d)/\alpha)}{m!\Gamma(1+(\alpha-d)/\alpha)^{m/2}}\right]^{1/2m} = +\infty,$$

Carleman's criterion is satisfied [**24**, Ch. XV.4], so $(\mathcal{Z}_n(\beta)/\mathfrak{A}_n)_n$ converges in distribution to $\sqrt{\Phi(0)}\sigma_{GK}(\beta, \widetilde{A}, \tilde{\mu}, \widetilde{T})\mathcal{Y}$, when $A \times \mathbb{Z}$ is endowed with the probability measure $\mu \times \delta_0$.

Finally, remark that

$$\left|\frac{\mathcal{Z}_n(\beta)}{\mathfrak{A}_n}\circ \widetilde{T} - \frac{\mathcal{Z}_n(\beta)}{\mathfrak{A}_n}\right| \leq \frac{2\|\beta\|_\infty}{\mathfrak{A}_n} \to_{n\to+\infty} 0,$$

so by [**72**, Theorem 1], the sequence $(\mathcal{Z}_n(\beta)/\mathfrak{A}_n)_n$ converges strongly in distribution to $\sqrt{\Phi(0)}\sigma_{GK}(\beta, \widetilde{A}, \tilde{\mu}, \widetilde{T})\mathcal{Y}$.

3.4. *Technical lemmas.* In the previous section, we used three technical lemmas, whose proofs would have been too long to include in our main line of reasoning. Their statements and proofs follow.

We begin with Lemma 3.6, which we used to control each part of the decomposition $Q_{\ell,a} = Q_{\ell,a}^{(0)} + Q_{\ell,a}^{(1)}$. Recall that Φ is the continuous version of the density function of the stable distribution with characteristic function $e^{-\psi(\sqrt{\Sigma}\cdot)}$. Since $\mu(S_\ell = a) = \mathbb{E}_\mu[Q_{\ell,a}(\mathbf{1})]$ for $a \in \mathbb{Z}^d$, the following lemma can be understood as a strong form of the the local limit theorem for $(S_\ell)_{\ell\geq 1}$.

LEMMA 3.6. *Assume Hypothesis 3.1. Let $a \in \mathbb{Z}^d$. For every positive integer ℓ,*

$$Q_{\ell,a}(h) = \frac{\Phi(a/\mathfrak{a}_\ell)}{\mathfrak{a}_\ell^d}\Pi_0^\ell(h) + \varepsilon_{\ell,a}(h),$$

with $\sup_{a\in\mathbb{Z}^d}\|\varepsilon_{\ell,a}\|_{\mathcal{B}\to\mathcal{B}} = o(\mathfrak{a}_\ell^{-d})$.

Moreover, for every $\omega \in (0, 1]$,

$$\sup_{\substack{a,p\in\mathbb{Z}^d\\ p\neq 0}} |p|^{-\omega}\|Q_{\ell,a} - Q_{\ell,a-p}\| = O(\mathfrak{a}_k^{-(d+\omega)}), \tag{3.23}$$

and

$$\|Q_{\ell,a-p} - Q_{\ell,a} - Q_{\ell,-p} + Q_{\ell,0}\| = O((|a|\,|p|)^\omega \mathfrak{a}_\ell^{-(d+2\omega)}). \tag{3.24}$$

Proof of Lemma 3.6. Recall that $Q_{\ell,a}(h) = 1/(2\pi)^d \int_{\mathbb{T}^d} e^{-i\langle u,a\rangle} P_u^\ell(h)\,du$. From Hypothesis 3.1, and up to taking a smaller neighbourhood U, there exist constants $C_0, c_0 > 0$ such that $\|P_u\|_{L(\mathcal{B})} \leq C_0$ and

$$\max\{|\lambda_u|, |e^{-\ell\psi(\sqrt{\Sigma}u)L(|\sqrt{\Sigma}u|^{-1})}|\} \leq e^{-c_0|u|^\alpha L(|u|^{-1})},$$

for all $u \in U$.

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Let $\varepsilon \in (0, \alpha)$. Since L is slowly varying at infinity and Σ is invertible, by Karamata [**35**] (or Potter's bound [**9**, Theorem 1.5.6]), there exists $\ell_0 \geq 0$ such that, for every $\ell \geq \ell_0$ and $v \in U$,

$$\frac{2}{|v|^\varepsilon} \leq \left| \frac{L(\mathfrak{a}_\ell/|v|)}{L(\mathfrak{a}_\ell)} \right| \leq \frac{|v|^\varepsilon}{2}.$$

Since $nL(\mathfrak{a}_n) \sim \mathfrak{a}_n^\alpha$, up to choosing a larger ℓ_0, for every $\ell \geq \ell_0$ and $v \in U$,

$$|v|^{\alpha-\varepsilon} \leq \ell \frac{|v|^\alpha}{\mathfrak{a}_\ell^\alpha} L\left(\frac{\mathfrak{a}_\ell}{|v|}\right) \leq |v|^{\alpha+\varepsilon}. \tag{3.25}$$

We begin with the first point of the lemma. Let $a \in \mathbb{Z}^d$ and $\ell \geq \ell_0$ be an integer. By Hypothesis 3.1,

$$Q_{\ell,a} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-i\langle u,a\rangle} P_u^\ell \, du = \frac{1}{(2\pi)^d} \int_U e^{-i\langle u,a\rangle} \lambda_u^\ell \Pi_u^\ell \, du + O(r^\ell), \tag{3.26}$$

and, for every $u \in U$,

$$\begin{aligned}
&\|\lambda_u^\ell \Pi_u^\ell - e^{-\ell\psi(\sqrt{\Sigma}u)L(|\sqrt{\Sigma}u|^{-1})} \Pi_0^\ell\| \\
&\quad \leq |\lambda_u|^\ell \|\Pi_u^\ell - \Pi_0^\ell\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} + |\lambda_u^\ell - e^{-\ell\psi(\sqrt{\Sigma}u)L(|\sqrt{\Sigma}u|^{-1})}| \|\Pi_0^\ell\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \\
&\quad \leq C(1 + \ell|u|^\alpha L(|u|^{-1})) e^{-c_0\ell|u|^\alpha L(|u|^{-1})} \xi(u) \|h\|_{\mathcal{B}},
\end{aligned} \tag{3.27}$$

where ξ is bounded and $\lim_{u\to 0} \xi(u) = 0$, due to the asymptotic expansion of $u \mapsto \lambda_u$, to the continuity of $u \mapsto \Pi_u$ at 0 and since $\Pi_u^\ell = \Pi_u^{\{\ell/M\}M}$ (see equation (3.4)). Hence

$$\begin{aligned}
&\left\| \frac{1}{(2\pi)^d} \int_U e^{-i\langle u,a\rangle} \lambda_u^\ell \Pi_u^\ell \, du - \frac{1}{(2\pi)^d} \int_U e^{-i\langle u,a\rangle} e^{-\ell\psi(\sqrt{\Sigma}u)L(|\sqrt{\Sigma}u|^{-1})} \Pi_0^\ell \, du \right\| \\
&\quad \leq C \int_U (1 + \ell|u|^\alpha L(|u|^{-1})) e^{-c_0\ell|u|^\alpha L(|u|^{-1})} \xi(u) \, du \\
&\quad \leq C\mathfrak{a}_\ell^{-d} \int_{\mathfrak{a}_\ell U} \left(1 + \ell \frac{|v|^\alpha}{\mathfrak{a}_\ell^\alpha} L\left(\frac{\mathfrak{a}_\ell}{|v|}\right)\right) e^{-c_0\ell(|v|^\alpha/\mathfrak{a}_\ell^\alpha)L(\mathfrak{a}_\ell/|v|)} \xi\left(\frac{v}{\mathfrak{a}_\ell}\right) dv \\
&\quad \leq C\mathfrak{a}_\ell^{-d} \int_{\mathfrak{a}_\ell U} (1 + |v|^{\alpha+\varepsilon}) e^{-c_0|v|^{\alpha-\varepsilon}} \xi\left(\frac{v}{\mathfrak{a}_\ell}\right) dv \\
&\quad = o(\mathfrak{a}_\ell^{-d}),
\end{aligned} \tag{3.28}$$

due to (3.25) and to the Lebesgue dominated convergence theorem. Finally,

$$\begin{aligned}
&\left| \frac{1}{(2\pi)^d} \int_U e^{-i\langle u,a\rangle} e^{-\ell\psi(\sqrt{\Sigma}u)L(|\sqrt{\Sigma}u|^{-1})} \, du - \frac{1}{\mathfrak{a}_\ell^d} \Phi\left(\frac{a}{\mathfrak{a}_\ell}\right) \right| \\
&= \left| \frac{1}{(2\pi)^d \mathfrak{a}_\ell^d} \int_{\mathfrak{a}_\ell U} e^{-i\langle v,a\rangle/\mathfrak{a}_\ell} e^{-\ell\psi(v/\mathfrak{a}_\ell)L(\mathfrak{a}_\ell/|v|)} - \frac{1}{(2\pi)^d \mathfrak{a}_\ell^d} \int_{\mathbb{R}^d} e^{-i\langle v,a\rangle/\mathfrak{a}_\ell} e^{-\psi(v)} \, dv \right| \\
&= \left| \frac{1}{(2\pi)^d \mathfrak{a}_\ell^d} \int_{\mathfrak{a}_\ell U} e^{-i\langle v,a\rangle/\mathfrak{a}_\ell} (e^{-\ell\psi(v/\mathfrak{a}_\ell)L(\mathfrak{a}_\ell/|v|)} - e^{-\psi(v)}) \, dv \right| + o(\mathfrak{a}_\ell^{-d}) \\
&\leq \frac{1}{(2\pi)^d \mathfrak{a}_\ell^d} \int_{\mathfrak{a}_\ell U} |e^{-\ell(\psi(v)/\mathfrak{a}_\ell^\alpha)L(\mathfrak{a}_\ell/|v|)} - e^{-\psi(v)}| \, dv + o(\mathfrak{a}_\ell^{-d}) \\
&= o(\mathfrak{a}_\ell^{-d}),
\end{aligned} \tag{3.29}$$

using again the Lebesgue dominated convergence theorem (with (3.25) for the necessary upper bound). Note that the upper bounds we used are independent of a, whence

$$\sup_{a\in\mathbb{Z}^d}\left|\frac{1}{(2\pi)^d}\int_U e^{-i\langle u,a\rangle}e^{-\ell\psi(\sqrt{\Sigma}u)L(|\sqrt{\Sigma}u|^{-1})}\,du-\frac{1}{\mathfrak{a}_\ell^d}\Phi\left(\frac{a}{\mathfrak{a}_\ell}\right)\right|=o(\mathfrak{a}_\ell^{-d}).$$

This ends the proof of the first point.

Let $\beta>-1$. Let $F:\mathbb{T}^d\to\mathbb{C}$ be a measurable function, with $|F(u)|\le K|u|^\beta$ for all $u\in U$. Then, for all large enough ℓ,

$$\begin{aligned}\left\|\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}F(u)P_u^\ell\,du\right\| &\le \left\|\frac{1}{(2\pi)^d}\int_U F(u)\lambda_u^\ell\Pi_u^\ell\,du\right\|+\|F\|_{\mathbb{L}^1}O(r^\ell)\\ &\le \frac{KC_0}{(2\pi)^d}\int_U |u|^\beta e^{-c_0\ell|u|^\alpha L(|u|^{-1})}\,du+K\,O(r^\ell)\\ &\le \frac{KC_0}{(2\pi)^d\mathfrak{a}_\ell^{d+\beta}}\int_{\mathfrak{a}_\ell U}|v|^\beta e^{-c_0\ell(|v|^\alpha/\mathfrak{a}_\ell^\alpha)L(\mathfrak{a}_\ell/|v|)}du+K\,O(r^\ell)\\ &\le \frac{KC_0}{(2\pi)^d\mathfrak{a}_\ell^{d+\beta}}\int_{\mathbb{R}}|v|^\beta e^{-c_0|v|^{\alpha-\varepsilon}}\,du+K\,O(r^\ell)\\ &= K.O(\mathfrak{a}_\ell^{-(d+\beta)}),\end{aligned}\tag{3.30}$$

where the $O(\mathfrak{a}_\ell^{-(d+\beta)})$ depends on β but not on K.

With $|F(u)|=|e^{-i\langle u,a\rangle}-e^{-i\langle u,a-p\rangle}|\le\min(2,|u|\,|p|)\le 2^{1-\omega}|p|^\omega|u|^\omega$, equation (3.30) yields

$$\sup_{a\in\mathbb{Z}^d}\|Q_{\ell,a}-Q_{\ell,a-p}\|=O(|p|^\omega\mathfrak{a}_\ell^{-(d+\omega)}),$$

which is equation (3.23).

With $|F(u)|=|e^{-i\langle u,a\rangle}-1||e^{i\langle u,p\rangle}-1|\le\min(2,|u|\,|p|)\cdot\min(2,|u|\,|a|)\le 4^{1-\omega}|a|^\omega|p|^\omega|u|^{2\omega}$, equation (3.30) yields:

$$\|Q_{\ell,a-p}-Q_{\ell,a}-Q_{\ell,-p}+Q_{\ell,0}\|=O(|a|^\omega|p|^\omega\mathfrak{a}_\ell^{-(d+2\omega)}),$$

which is equation (3.24). □

We now give a proof of Lemma 3.4, which was stated in the previous section. This lemma allowed us to control various sums involving the coefficients $b_{a,\boldsymbol{\ell},\mathbf{N}}^{(1,\ldots,1)}$, depending on $\mathbf{N}$, and was central in the proof of the main theorem. For the convenience of the reader, what we have to prove is reformulated at the beginning of the proof.

Proof of Lemma 3.4. Let us introduce the following operators on $\mathcal{B}$:

$$C_{b,a,(\ell_1,\ldots,\ell_q),(N_1,\ldots,N_{q-1})}:=\sum_{\substack{a_0,\ldots,a_q\in\mathbb{Z}^d\\ a_0=a,a_q=b}}Q^{(1)}_{\ell_q,a_q-a_{q-1}}\beta(a_{q-1})^{N_{q-1}}\cdots Q^{(1)}_{\ell_2,a_2-a_1}\beta(a_1)^{N_1}Q^{(1)}_{\ell_1,a_1-a_0},$$

and

$$D_{a,(\ell_1,\ldots,\ell_q),(N_1,\ldots,N_q)}:=\sum_{\substack{a_0,\ldots,a_q\in\mathbb{Z}^d\\ a_0=a}}\beta(a_q)^{N_q}Q^{(1)}_{\ell_q,a_q-a_{q-1}}\beta(a_{q-1})^{N_{q-1}}\cdots Q^{(1)}_{\ell_2,a_2-a_1}\beta(a_1)^{N_1}Q^{(1)}_{\ell_1,a_1-a_0}.$$

Note that

$$b^{(1,\dots,1)}_{a;\boldsymbol{\ell},\mathbf{N}}(\cdot)=\mathbb{E}_\mu[D_{a,\boldsymbol{\ell},\mathbf{N}}(\cdot)] \quad \text{and} \quad \sum_{a\in\mathbb{Z}^d}\beta(a)b^{(1,\dots,1)}_{a,\boldsymbol{\ell},\mathbf{N}}(\cdot)=\sum_{a\in\mathbb{Z}^d}\beta(a)\mathbb{E}_\mu[D_{a,\boldsymbol{\ell},\mathbf{N}}(\cdot)].$$

Hence, it is sufficient to prove that

$$\sup_{a\in\mathbb{Z}^d}(1+|a|^\eta)^{-1}\sum_{\boldsymbol{\ell}\in\{1,\dots,n\}^q}\|D_{a,\boldsymbol{\ell},\mathbf{N}}\|=o(\mathfrak{A}_n^{N_1+\cdots+N_q}), \tag{3.31}$$

$$\sum_{\ell=1}^n\bigg\|\sum_{a\in\mathbb{Z}^d}\beta(a)D_{a,(\ell),(N)}\bigg\|=\begin{cases}O(1) & \text{if } q=1,\ N=1,\\ o(\mathfrak{A}_n) & \text{if } q=1,\ N\geq 2,\end{cases} \tag{3.32}$$

$$\sum_{\boldsymbol{\ell}\in\{1,\dots,n\}^q}\bigg\|\sum_{a\in\mathbb{Z}^d}\beta(a)D_{a,\boldsymbol{\ell},\mathbf{N}}\bigg\|=o(\mathfrak{A}_n^{N_1+\cdots+N_q-1}) \quad \text{if } q\geq 2. \tag{3.33}$$

Restriction of the problem. We first observe that we can restrict our study to the case where all the N_j's are equal to 1. The price to pay will be that we will have to consider both $D_{a,\boldsymbol{\ell},(1,\dots,1)}$ and $C_{b,a,\boldsymbol{\ell},(1,\dots,1)}$. Equation (3.32) shall be proved separately with the next step (Case $q=1$).

We shall prove the estimates (3.31) and (3.33) in the particular case where $(N_1,\dots,N_q)=(1,\dots,1)$ (or equivalently $N_1+\cdots+N_q=q$), that is

$$\sup_{a\in\mathbb{Z}^d}(1+|a|^\eta)^{-1}\sum_{\boldsymbol{\ell}\in\{1,\dots,n\}^q}\|D_{a,\boldsymbol{\ell},(1,\dots,1)}\|=o(\mathfrak{A}_n^q), \tag{3.34}$$

$$\sum_{\boldsymbol{\ell}\in\{1,\dots,n\}^q}\bigg\|\sum_{a\in\mathbb{Z}^d}\beta(a)D_{a,\boldsymbol{\ell},(1,\dots,1)}\bigg\|=o(\mathfrak{A}_n^{q-1}) \quad \text{if } q\geq 2, \tag{3.35}$$

together with the following estimates:

$$\sup_{a\in\mathbb{Z}^d}(1+|a|^\eta)^{-1}\sum_{\boldsymbol{\ell}\in\{1,\dots,n\}^q}\sum_{b\in\mathbb{Z}^d}|\beta(b)|\|C_{b,a,\boldsymbol{\ell},(1,\dots,1)}\|=o(\mathfrak{A}_n^{q+1}), \tag{3.36}$$

and

$$\sum_{\boldsymbol{\ell}\in\{1,\dots,n\}^q}\sum_{b\in\mathbb{Z}^d}|\beta(b)|\bigg\|\sum_{a\in\mathbb{Z}^d}\beta(a)C_{b,a,\boldsymbol{\ell},(1,\dots,1)}\bigg\|=o(\mathfrak{A}_n^q). \tag{3.37}$$

Assume these estimates to be proved. If $(N_1,\dots,N_q)\neq(1,\dots,1)$, let j be the largest index such that $N_j\neq 1$. Then

$$\begin{aligned}&\|D_{a,\boldsymbol{\ell},(N_1,\dots,N_q)}\|\\ &\quad\leq\sum_{a_j\in\mathbb{Z}^d}|\beta(a_j)|^{N_j}\|D_{a_j,(\ell_{j+1},\dots,\ell_q),(1,\dots,1)}\|\ \|C_{a_j,a,(\ell_1,\dots,\ell_j),(N_1,\dots,N_{j-1})}\|,\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}&\|C_{b,a,\boldsymbol{\ell},(N_1,\dots,N_{q-1})}\|\\ &\quad\leq\sum_{a_j\in\mathbb{Z}^d}|\beta(a_j)|^{N_j}\|C_{b,a_j,(\ell_{j+1},\dots,\ell_q),(1,\dots,1)}\|\ \|C_{a_j,a,(\ell_1,\dots,\ell_j),(N_1,\dots,N_{j-1})}\|.\end{aligned} \tag{3.39}$$

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Let us iterate this decomposition. Given $(N_1, \ldots, N_{q-1}) \neq (1, \ldots, 1)$, let $\mathcal{J} := \{1 \leq j < q : N_j \geq 2\} = \{j_1, \ldots, j_J\}$, with $j_1 < \cdots < j_J$ and $J = |\mathcal{J}|$. We also use the convention $j_0 = 0$. Iterating equations (3.38) and (3.39) then yields

$$\begin{aligned}
&\sup_{a_0 \in \mathbb{Z}^d} (1 + |a_0|^\eta)^{-1} \sum_{\boldsymbol{\ell} \in \{1,\ldots,n\}^q} \|D_{a_0, \boldsymbol{\ell}, (N_1,\ldots,N_{q-1})}\| \\
&\quad \leq \sum_{a_J \in \mathbb{Z}^d} \sum_{\ell_{1+j_J},\ldots,\ell_q = 1}^{n} |\beta(a_J)|^{N_{j_J}} \|D_{a_{j_J}, (\ell_{1+j_J},\ldots,\ell_q),(1,\ldots,1)}\| \\
&\quad \times \prod_{k=2}^{J} \Bigg(\sum_{a_{j_{k-1}} \in \mathbb{Z}^d} \sum_{\ell_{1+j_{k-1}},\ldots,\ell_{j_k} = 1}^{n} \\
&\quad \times \sum_{a_{j_k} \in \mathbb{Z}^d} |\beta(a_{j_k})|^{N_{j_k}} |\beta(a_{j_{k-1}})|^{N_{j_{k-1}}} \|C_{a_{j_k}, a_{j_{k-1}}, (\ell_{1+j_{k-1}},\ldots,\ell_{j_k}),(1,\ldots,1)}\| \Bigg) \\
&\quad \times \Bigg(\sup_{a_0 \in \mathbb{Z}^d} \sum_{\ell_1,\ldots,\ell_{j_1} = 1}^{n} (1 + |a_0|^\eta)^{-1} \sum_{a_{j_1} \in \mathbb{Z}^d} |\beta(a_{j_1})|^{N_{j_1}} \|C_{a_{j_1}, a_0, (\ell_1,\ldots,\ell_{j_k}),(1,\ldots,1)}\| \Bigg).
\end{aligned} \tag{3.40}$$

Recall that, since $\eta < (\alpha - d)/2 + \varepsilon$ and β is bounded, $|\beta(a)|^x = O(|\beta(a)|) = O((1 + |a|^\eta)^{-1})$ for all $x \geq 1$. Using (3.34) on the first term and (3.36) on the others, we get (3.31)

$$\begin{aligned}
&\sup_{a_0 \in \mathbb{Z}^d} (1 + |a_0|^\eta)^{-1} \sum_{\boldsymbol{\ell} \in \{1,\ldots,n\}^q} \|D_{a_0, \boldsymbol{\ell}, (N_1,\ldots,N_{q-1})}\| \\
&\quad = o(\mathfrak{A}_n^{q - j_J}) \prod_{k=1}^{J} o(\mathfrak{A}_n^{j_k - j_{k-1} + 1}) = o(\mathfrak{A}_n^{q+J}) = o(\mathfrak{A}_n^{N_1 + \cdots + N_q}).
\end{aligned}$$

We use the same decomposition to get (3.33). The only difference is that the last term in the decomposition becomes

$$\sum_{\ell_1,\ldots,\ell_{j_1} = 1}^{n} \sum_{a_{j_1} \in \mathbb{Z}^d} |\beta(a_{j_1})|^{N_{j_1}} \Bigg\| \sum_{a_0 \in \mathbb{Z}^d} \beta(a_0) C_{a_{j_1}, a_0, (\ell_1,\ldots,\ell_{j_1}),(1,\ldots,1)} \Bigg\|,$$

which by (3.37) is an $o(\mathfrak{A}_n^{j_1})$. The exponent in the estimate is improved by 1, which is what we wanted.

First estimates. We first provide some general inequalities. From Lemma 3.6 and the definition of $Q_{\ell,a}^{(1)}$,

$$\|Q_{\ell,a}^{(1)}\| = o(\mathfrak{a}_\ell^{-d}) + O\left(\frac{\Phi(a/\mathfrak{a}_\ell) - \Phi(0)}{\mathfrak{a}_\ell^d} \right).$$

Since Φ is proportional to the Fourier transform of $e^{-\psi(\sqrt{\Sigma}\cdot)}$, it is η-Hölder for all $\eta \in (0, 1]$, whence

$$\|Q_{\ell,a}^{(1)}\| = o(\mathfrak{a}_\ell^{-d}) + O(|a|^\eta \mathfrak{a}_\ell^{-d-\eta}) = o((1 + |a|^\eta)\mathfrak{a}_\ell^{-d}). \tag{3.41}$$

Due to (3.23),

$$\sup_{b\neq 0} |b|^{-\omega} \| Q^{(1)}_{\ell,b-a} - Q^{(1)}_{\ell,-a}\| = \sup_{b\neq 0} |b|^{-\omega} \| Q_{\ell,b-a} - Q_{\ell,-a}\| = O(\mathfrak{a}_\ell^{-(d+\omega)}). \qquad (3.42)$$

In particular, since $\sum_{b\in\mathbb{Z}^d} |b|^{\omega}|\beta(b)| < +\infty$,

$$\sup_{a\in\mathbb{Z}^d} \left\| \sum_b \beta(b) Q^{(1)}_{\ell,b-a} \right\| = \sup_{a\in\mathbb{Z}^d} \left\| \sum_b \beta(b)(Q^{(1)}_{\ell,b-a} - Q^{(1)}_{\ell,-a}) \right\| = O(\mathfrak{a}_\ell^{-(d+\omega)}). \qquad (3.43)$$

Due to (3.24), and since $\sum_{a\in\mathbb{Z}^d} \beta(a) = 0$,

$$\begin{aligned}
&\sup_{a\neq 0}(|a|\,|b|)^{-\omega} \left\| \sum_{b\in\mathbb{Z}^d} \beta(b)(Q^{(1)}_{\ell,b-a} - Q^{(1)}_{\ell,b}) \right\| \\
&\quad = \sup_{a\neq 0}(|a|\,|b|)^{-\omega} \left\| \sum_{b\in\mathbb{Z}^d} \beta(b)(Q_{\ell,b-a} - Q_{\ell,b}) \right\| \\
&\quad = \sup_{a\neq 0}(|a|\,|b|)^{-\omega} \left\| \sum_{b\in\mathbb{Z}^d} \beta(b)(Q_{\ell,b-a} - Q_{\ell,b} - Q_{\ell,-a} + Q_{\ell,0}) \right\| \\
&\quad = O(\mathfrak{a}_\ell^{-(d+2\omega)}). \qquad (3.44)
\end{aligned}$$

In particular, using again the fact that $\sum_{b\in\mathbb{Z}^d} |b|^{\omega}|\beta(b)| < +\infty$,

$$\begin{aligned}
\left\| \sum_{a,b\in\mathbb{Z}^d} \beta(a)\beta(b) Q^{(1)}_{\ell,b-a} \right\| &= \left\| \sum_{a,b\in\mathbb{Z}^d} \beta(a)\beta(b)(Q_{\ell,b-a} - Q_{\ell,b}) \right\| \\
&= O(\mathfrak{a}_\ell^{-(d+2\omega)}). \qquad (3.45)
\end{aligned}$$

We will also repeatedly use the two following facts:

$$\sum_{\ell\geq 1} \mathfrak{a}_\ell^{-(d+2\omega)} < +\infty \quad \text{and} \quad \sum_{\ell=1}^{n} \mathfrak{a}_\ell^{-(d+\omega)} = o(\mathfrak{A}_n). \qquad (3.46)$$

In order to get the first upper bound, notice that $(\mathfrak{a}_\ell)_{\ell\geq 0}$ is $1/\alpha$-regularly varying and $d + 2\omega > d + 2(\alpha - d)/2 = \alpha \geq 1$, so the sequence $(\mathfrak{a}_\ell^{-(d+2\omega)})_\ell$ is summable.

If $\alpha > d$, then $(\mathfrak{A}_n)_n = (\sqrt{\sum_{\ell=1}^n \mathfrak{a}_\ell^{-d}})_n$ is $(\alpha - d)/2\alpha$-regularly varying, whereas $\sum_{\ell=1}^n \mathfrak{a}_\ell^{-(d+\omega)}$ is $(\alpha - d - \omega)/\alpha$-regularly varying. The condition $\omega > (\alpha - d)/2$ implies that $(\alpha - d - \omega)/\alpha < (\alpha - d)/2\alpha$. We get the second upper bound of equation (3.46) in the case $\alpha > d$.

Finally, if $\alpha = d$, then $(\mathfrak{A}_n)_n$ is slowly varying and goes to $+\infty$, whereas $d + \omega > 1$ and so $\sum_{\ell=1}^{+\infty} \mathfrak{a}_\ell^{-(d+\omega)} < \infty$. We get the second part of equation (3.46) in the case $\alpha = d$.

Case $q = 1$. We prove separately the case $q = 1$, which either involves different inequalities, or shall provide the base case for a recursion. We have to prove four estimates, which shall be in order: (3.32), (3.34), (3.36) and (3.37).

We begin with (3.32). Due to (3.45), if $N = 1$,

$$\begin{aligned}
\sum_{\ell=1}^{n} \left\| \sum_{a\in\mathbb{Z}^d} \beta(a) D_{a,(\ell),(1)} \right\| &= \sum_{\ell=1}^{n} \left\| \sum_{a,b\in\mathbb{Z}^d} \beta(a)\beta(b) Q^{(1)}_{\ell,b-a} \right\| \\
&= \sum_{\ell=1}^{n} O(\mathfrak{a}_\ell^{-(d+2\omega)}) = O(1).
\end{aligned}$$

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If $N \geq 2$, we use (3.43) instead:

$$\sum_{\ell=1}^{n} \bigg\| \sum_{a\in\mathbb{Z}^d} \beta(a) D_{a,(\ell),(N)} \bigg\| \leq \sum_{\ell=1}^{n} \sum_{b\in\mathbb{Z}^d} |\beta(b)|^N \bigg\| \sum_{a\in\mathbb{Z}^d} \beta(a) Q_{\ell,b-a}^{(1)} \bigg\|$$
$$= \bigg(\sum_{b\in\mathbb{Z}^d} |\beta(b)|^N \bigg) \sum_{\ell=1}^{n} O(\mathfrak{a}_\ell^{-(d+\omega)}) = o(\mathfrak{A}_n).$$

Now, consider (3.34) for $q = 1$. Using (3.43) and (3.46),

$$\sup_{a\in\mathbb{Z}^d} (1 + |a|^\eta)^{-1} \sum_{\ell=1}^{n} \|D_{a,(\ell),(1)}\| \leq \sum_{\ell=1}^{n} \sup_{a\in\mathbb{Z}^d} \bigg\| \sum_{b\in\mathbb{Z}^d} \beta(b) Q_{\ell,b-a}^{(1)} \bigg\|$$
$$= \sum_{\ell=1}^{n} O(\mathfrak{a}_\ell^{-(d+\omega)}) = o(\mathfrak{A}_n). \tag{3.47}$$

Next, we prove (3.36) for $q = 1$. Note that $C_{b,a,(\ell),\emptyset} = Q_{\ell,b-a}^{(1)}$, and that $(1 + |b - a|^\eta) \leq (1 + |a|^\eta)(1 + |b|^\eta)$ since $\eta \leq 1$. Hence, by (3.41),

$$\sum_{\ell=1}^{n} \sum_{b\in\mathbb{Z}^d} |\beta(b)| \|C_{b,a,(\ell),\emptyset}\| = \sum_{\ell=1}^{n} \sum_{b\in\mathbb{Z}^d} |\beta(b)| o((1 + |b - a|^\eta)\mathfrak{a}_\ell^{-d})$$
$$= \bigg(\sum_{b\in\mathbb{Z}^d} |\beta(b)|(1 + |b|^\eta) \bigg)(1 + |a|^\eta) \sum_{\ell=1}^{n} o(\mathfrak{a}_\ell^{-d})$$
$$= o((1 + |a|^\eta)\mathfrak{A}_n^2).$$

Finally, we deal with (3.37) for $q = 1$. Due to (3.43) and (3.46),

$$\sum_{\ell=1}^{n} \sum_{b\in\mathbb{Z}^d} |\beta(b)| \bigg\| \sum_{a\in\mathbb{Z}^d} \beta(a) C_{b,a,\ell,\emptyset} \bigg\| = \bigg(\sum_{b\in\mathbb{Z}^d} |\beta(b)| \bigg) \sum_{\ell=1}^{n} O(\mathfrak{a}_\ell^{-(d+\omega)}) = o(\mathfrak{A}_n).$$

Case $q \geq 2$. It remains to check four estimates, which shall be in order: (3.34), (3.35), (3.36) and (3.37), for $q \geq 2$. To simplify the notation, we omit $(1, \ldots, 1)$ in indices, and use the convention $D_{a,\ell,\emptyset} = 1$ for all a and ℓ.

We shall prove (3.34) and (3.35) with recursive bounds involving the functions:

$$u_{q,n}(a) := \sum_{\ell_1,\ldots,\ell_q=1}^{n} \|D_{a,(\ell_1,\ldots,\ell_q)}\|$$

and

$$v_{q,n}(a) := \sum_{\ell_1,\ldots,\ell_q=1}^{n} \|D_{a,(\ell_1,\ldots,\ell_q)} - D_{0,(\ell_1,\ldots,\ell_q)}\|.$$

Note that (3.34) is equivalent to the statement that $u_{q,n}(a) = o((1 + |a|^\eta)\mathfrak{A}_n^q)$, while (3.35) is implied by the bound $v_{q,n}(a) = o(|a|^\omega \mathfrak{A}_n^{q-1})$ for $q \geq 2$ (since $\sum_{a\in\mathbb{Z}^d} \beta(a) = 0$). We shall express $u_{q,n}$ and $v_{q,n}$ in terms of $u_{q-1,n}$, $v_{q-1,n}$, $u_{q-2,n}$ and $v_{q-2,n}$.

We start with the sequence $(u_{q,n})$. For all $q \geq 2$,

$$
\begin{aligned}
D_{a_0,(\ell_1,\ldots,\ell_q)} &= \sum_{a_1,a_2} \beta(a_1)\beta(a_2) D_{a_2,(\ell_3,\ldots,\ell_q)} Q^{(1)}_{\ell_2,a_2-a_1} Q^{(1)}_{\ell_1,a_1-a_0} \\
&= \sum_{a_1,a_2} \beta(a_2)[D_{0,(\ell_3,\ldots,\ell_q)} + D_{a_2,(\ell_3,\ldots,\ell_q)} - D_{0,(\ell_3,\ldots,\ell_q)}] \\
&\quad \times (\beta(a_1) Q^{(1)}_{\ell_2,a_2-a_1}[Q^{(1)}_{\ell_1,-a_0} + Q^{(1)}_{\ell_1,a_1-a_0} - Q^{(1)}_{\ell_1,0-a_0}]) \\
&= D_{0,(\ell_3,\ldots,\ell_q)}\Bigg[\Bigg(\sum_{a_1,a_2} \beta(a_1)\beta(a_2) Q^{(1)}_{\ell_2,a_2-a_1}\Bigg) Q^{(1)}_{\ell_1,-a_0} \\
&\quad + \sum_{a_1,a_2} \beta(a_1)\beta(a_2)(Q^{(1)}_{\ell_2,a_2-a_1} - Q^{(1)}_{\ell_2,-a_1})(Q^{(1)}_{\ell_1,a_1-a_0} - Q^{(1)}_{\ell_1,0-a_0})\Bigg] \\
&\quad + \sum_{a_2} \beta(a_2)(D_{a_2,(\ell_3,\ldots,\ell_q)} - D_{0,(\ell_3,\ldots,\ell_q)}) \\
&\quad \times\Bigg[\Bigg(\sum_{a_1} \beta(a_1) Q^{(1)}_{\ell_2,a_2-a_1}\Bigg) Q^{(1)}_{\ell_1,-a_0} + \sum_{a_1} \beta(a_1) Q^{(1)}_{\ell_2,a_2-a_1}(Q^{(1)}_{\ell_1,a_1-a_0} - Q^{(1)}_{\ell_1,0-a_0})\Bigg]
\end{aligned}
\tag{3.48}
$$

since $\sum_{a_2} \beta(a_2) Q^{(1)}_{\ell_2,-a_1} = 0$. Note that $\sum_p |p|^{\eta+\omega}|\beta(p)| < +\infty$ and $(1 + |a_2 - a_1|^\eta)|a_1|^\omega \leq 2(1 + |a_1|^{\eta+\omega})(1 + |a_2|^\eta)$. Therefore, using in addition (3.41), (3.42), (3.44) and (3.45), we get that, for all $q \geq 2$,

$$
\begin{aligned}
\|D_{a_0,(\ell_1,\ldots,\ell_q)}\| &= \|D_{0,(\ell_3,\ldots,\ell_q)}\| O((1 + |a_0|^\eta)\mathfrak{a}_{\ell_2}^{-(d+2\omega)} o(\mathfrak{a}_{\ell_1}^{-d}) + (\mathfrak{a}_{\ell_1}\mathfrak{a}_{\ell_2})^{-(d+\omega)}) \\
&\quad + \sum_{a_2} |\beta(a_2)| \|D_{a_2,(\ell_3,\ldots,\ell_q)} - D_{0,(\ell_3,\ldots,\ell_q)}\| O((1 + |a_0|^\eta)\mathfrak{a}_{\ell_2}^{-(d+\omega)} o(\mathfrak{a}_{\ell_1}^{-d}) \\
&\quad + (1 + |a_2|^\eta) o(\mathfrak{a}_{\ell_2}^{-d})\mathfrak{a}_{\ell_1}^{-(d+\omega)}),
\end{aligned}
$$

uniformly in a_0. If $q = 2$, this simplifies to

$$
\|D_{a_0,(\ell_1,\ell_2)}\| = O((1 + |a_0|^\eta)\mathfrak{a}_{\ell_2}^{-(d+2\omega)} o(\mathfrak{a}_{\ell_1}^{-d}) + (\mathfrak{a}_{\ell_1}\mathfrak{a}_{\ell_2})^{-(d+\omega)}).
$$

These estimates, combined with (3.46), yield for all $q \geq 3$

$$
\begin{aligned}
u_{q,n}(a) = O\Bigg((1 + |a|^\eta)\Bigg(& u_{q-2,n}(0) o(\mathfrak{A}_n^2) \\
&+ \sum_{a_2 \in \mathbb{Z}^d} |\beta(a_2)|(1 + |a_2|^\eta) v_{q-2,n}(a_2) o(\mathfrak{A}_n^3)\Bigg)\Bigg),
\end{aligned}
\tag{3.49}
$$

and, for $q = 2$,

$$
u_{2,n}(a) = o((1 + |a|^\eta)\mathfrak{A}_n^2).
\tag{3.50}
$$

Now, let us consider the sequence $(v_{q,n})$. For all $q \geq 2$,

$$
\begin{aligned}
D_{a_0,(\ell_1,\ldots,\ell_q)} - D_{0,(\ell_1,\ldots,\ell_q)} &= \sum_{a_1} \beta(a_1) D_{a_1,(\ell_2,\ldots,\ell_q)}(Q^{(1)}_{\ell_1,a_1-a_0} - Q^{(1)}_{\ell_1,a_1}) \\
&= D_{0,(\ell_2,\ldots,\ell_q)} \sum_{a_1} \beta(a_1)(Q^{(1)}_{\ell_1,a_1-a_0} - Q^{(1)}_{\ell_1,a_1}) \\
&\quad + \sum_{a_1} \beta(a_1)(D_{a_1,(\ell_2,\ldots,\ell_q)} - D_{0,(\ell_2,\ldots,\ell_q)})(Q^{(1)}_{\ell_1,a_1-a_0} - Q^{(1)}_{\ell_1,a_1}).
\end{aligned}
\tag{3.51}
$$

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From (3.44) and (3.42), we get that, for all $q \geq 2$,

$$\begin{aligned}\|D_{a_0,(\ell_1,\dots,\ell_q)} - D_{0,(\ell_1,\dots,\ell_q)}\| &= \|D_{0,(\ell_2,\dots,\ell_q)}\| O(|a_0|^\omega \mathfrak{a}_{\ell_1}^{-(d+2\omega)}) \\ &\quad + \sum_{a_1\in\mathbb{Z}^d} |\beta(a_1)| \|D_{a_1,(\ell_2,\dots,\ell_q)} - D_{0,(\ell_2,\dots,\ell_q)}\| \\ &\quad \times O(|a_0|^\omega \mathfrak{a}_{\ell_1}^{-(d+\omega)}),\end{aligned}$$

so that, using (3.46),

$$v_{q,n}(a) = O\bigg(|a|^\omega \bigg(u_{q-1,n}(0) + \sum_{a_1\in\mathbb{Z}^d} |\beta(a_1)| v_{q-1,n}(a_1) o(\mathfrak{A}_n)\bigg)\bigg). \tag{3.52}$$

From (3.44) and (3.46), we also obtain

$$v_{1,n}(a) = O(|a|^\omega). \tag{3.53}$$

Equation (3.34) can be reformulated as $u_{q,n}(a) = o((1+|a|^\eta)\mathfrak{A}_n^q)$ for $q \geq 1$, while equation (3.35) is a straightforward consequence of the fact that $\sum_{a\in\mathbb{Z}^d} |\beta(a)| v_{q,n}(a) = o(\mathfrak{A}_n^{q-1})$ for $q \geq 1$ (since $\sum_{a\in\mathbb{Z}^d} \beta(a) = 0$). We prove these two identities recursively, and more precisely that

$$u_{q,n}(a) = o((1+|a|^\eta)\mathfrak{A}_n^q) \quad \text{and} \quad \sup_{a\neq 0} |a|^{-\omega} v_{q,n}(a) = \begin{cases} O(1) & \text{if } q = 1, \\ o(\mathfrak{A}_n^{q-1}) & \text{if } q \geq 2. \end{cases}$$

This follows from (3.49) and (3.52) by an induction of degree two for $u_{q,n}$ and of degree one for $v_{q,n}$. The initialization is given by (3.47), (3.50) and (3.53) (for, respectively, $u_{1,n}$, $u_{2,n}$ and $v_{1,n}$).

It remains to prove equations (3.36) and (3.37). Note that (3.48) and (3.51) hold true if we replace $D_{\dots}$ by $C_{a_q,\dots}$. Hence (3.49) and (3.52) also hold if we replace $u_{q,n}$ and $v_{q,n}$ by, respectively, $\tilde{u}_{q,n}$ and $\tilde{v}_{q,n}$, which are given by

$$\tilde{u}_{q,n}(a) := \sum_{\ell_1,\dots,\ell_q=1}^{n} \sum_{a_q\in\mathbb{Z}^d} |\beta(a_q)| \|C_{a_q,a,(\ell_1,\dots,\ell_q)}\|,$$

$$\tilde{v}_{q,n}(a) := \sum_{\ell_1,\dots,\ell_q=1}^{n} \sum_{a_q\in\mathbb{Z}^d} |\beta(a_q)| \|C_{a_q,a,(\ell_1,\dots,\ell_q)} - C_{a_q,0,(\ell_1,\dots,\ell_q)}\|.$$

Note that (3.36) is equivalent to the statement that $\tilde{u}_{q,n}(a) = o((1+|a|^\eta)\mathfrak{A}_n^{q+1})$, while (3.37) is implied by the bound $\tilde{v}_{q,n}(a) = o(|a|^\omega \mathfrak{A}_n^q)$ for $q \geq 2$.

The first terms are the following. For $\tilde{u}_{1,n}(a)$, we get

$$\begin{aligned}\tilde{u}_{1,n}(a) &= \sum_{\ell=1}^{n} \sum_{b\in\mathbb{Z}^d} |\beta(b)| \|Q_{\ell,b-a}^{(1)}\| \\ &= \sum_{\ell=1}^{n} \sum_{b\in\mathbb{Z}^d} |\beta(b)| (1+|b|^\eta) o((1+|a|^\eta)\mathfrak{a}_\ell^{-d}) \\ &= o((1+|a|^\eta)\mathfrak{A}_n^2).\end{aligned}$$

For $\tilde{u}_{2,n}(a)$, we get

$$\begin{aligned}
\tilde{u}_{2,n}(a) &= \sum_{\ell_1,\ell_2=1}^{n} \sum_{b\in\mathbb{Z}^d} |\beta(b)| \bigg\| \sum_{a_1\in\mathbb{Z}^d} \beta(a_1) Q^{(1)}_{\ell_2,b-a_1} Q^{(1)}_{\ell_1,a_1-a} \bigg\| \\
&\leq \sum_{\ell_1,\ell_2=1}^{n} \sum_{b\in\mathbb{Z}^d} |\beta(b)| \bigg\| Q^{(1)}_{\ell_2,b} \sum_{a_1\in\mathbb{Z}^d} \beta(a_1) Q^{(1)}_{\ell_1,a_1-a} \bigg\| \\
&\quad + \bigg\| \sum_{a_1\in\mathbb{Z}^d} \beta(a_1)(Q^{(1)}_{\ell_2,b-a_1} - Q^{(1)}_{\ell_2,b}) Q^{(1)}_{\ell_1,a_1-a} \bigg\| \\
&= \sum_{\ell_1,\ell_2=1}^{n} \sum_{b\in\mathbb{Z}^d} |\beta(b)| \bigg[o((1+|b|^{\eta})\mathfrak{a}_{\ell_1}^{-(d+\omega)} \mathfrak{a}_{\ell_2}^{-d}) \\
&\quad + \sum_{a_1\in\mathbb{Z}^d} |\beta(a_1)| o((1+|a|^{\eta})(1+|a_1|^{\omega+\eta}) \mathfrak{a}_{\ell_1}^{-d} \mathfrak{a}_{\ell_2}^{-(d+\omega)}) \bigg] \\
&= o((1+|a|^{\eta})\mathfrak{A}_n^3),
\end{aligned}$$

where we used (3.42) and (3.44) for the first part, (3.42) and (3.43) for the second part, and (3.46) to finish. Finally, for $\tilde{v}_{1,n}(a)$, we get

$$\tilde{v}_{1,n}(a) = \sum_{\ell=1}^{n} \sum_{b\in\mathbb{Z}^d} |\beta(b)| \| Q^{(1)}_{\ell_1,b-a} - Q^{(1)}_{\ell_1,b} \| = o(|a|^{\omega}\mathfrak{A}_n),$$

due to (3.42) and (3.46).

By induction, we obtain

$$\tilde{u}_{q,n}(a) = o((1+|a|^{\eta})\mathfrak{A}_n^{q+1}) \quad \text{and} \quad \sup_{a\neq 0} |a|^{-\omega} \tilde{v}_{q,n}(a) = o(\mathfrak{A}_n^{q}),$$

which ends the proof of Lemma 3.4. □

The third and last lemma of this sub-section gives a simple formula for the asymptotic growth of the quantity $\sum_{\boldsymbol{\ell}\in E_{q,n}} \prod_{j=1}^{q} \mathfrak{a}_{k_j}^{-d}$.

LEMMA 3.7. *Let $1 \leq d \leq \alpha \leq 2$ be an integer and a real number, respectively. Recall that, for every $q \geq 1$,*

$$E_{q,n} = \bigg\{ \boldsymbol{\ell} \in \{1, \ldots, n\}^q : \sum_{j=1}^{q} \ell_j \leq n \bigg\}.$$

Let $(\mathfrak{a}_\ell)_{\ell\geq 0}$ be a sequence of positive real numbers with regular variation of index $1/\alpha$, and $\mathfrak{A}_n := \sqrt{\sum_{\ell=1}^{n} \mathfrak{a}_\ell^{-d}}$. Assume that $\lim_{n\to+\infty} \mathfrak{A}_n = +\infty$.

For every $q \geq 1$, as n goes to infinity,

$$\sum_{\boldsymbol{\ell}\in E_{q,n}} \prod_{j=1}^{q} \mathfrak{a}_{\ell_j}^{-d} \sim \mathfrak{A}_n^{2q} \frac{\Gamma(1+(\alpha-d)/\alpha)^q}{\Gamma(1+q(\alpha-d)/\alpha)}.$$

Proof. We deal separately with the cases $d = \alpha$ (where $(\mathfrak{A}_n)$ has slow variation) and $d < \alpha$.

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Case $d = \alpha$. If $d = \alpha \in \{1, 2\}$, then $\mathfrak{a}_\ell^d$ is 1-regularly varying, so $\mathfrak{A}_n$ has slow variation. By the pigeonhole principle, for all $\boldsymbol{\ell} \in \{1, \ldots, n\}^q \backslash E_{q,n}$, there is always one ℓ_i such that $\ell_i \geq \lceil n/q \rceil$. Hence

$$\left| \sum_{\boldsymbol{\ell} \in E_{q,n}} \prod_{j=1}^q \mathfrak{a}_{\ell_j}^{-d} - \sum_{\boldsymbol{\ell} \in \{1,\ldots,n\}^q} \prod_{j=1}^q \mathfrak{a}_{\ell_j}^{-d} \right| \leq q \Bigg(\sum_{\ell_1 = \lceil n/q \rceil}^n \mathfrak{a}_{\ell_1}^{-d} \Bigg) \Bigg(\sum_{\ell_2,\ldots,\ell_q=1}^n \prod_{j=2}^q \mathfrak{a}_{\ell_j}^{-d} \Bigg)$$
$$= o(\mathfrak{A}_n^2) \cdot O(\mathfrak{A}_n^{2(q-1)}),$$

and thus $\sum_{\boldsymbol{\ell} \in E_{q,n}} \prod_{j=1}^q \mathfrak{a}_{\ell_j}^{-d} \sim \mathfrak{A}_n^{2q}$.

Case $d < \alpha$. If $d = 1 < \alpha$,

$$\sum_{\boldsymbol{\ell} \in E_{q,n}} \prod_{j=1}^q \left(\frac{\mathfrak{a}_{\ell_j}}{\mathfrak{a}_n} \right)^{-1} = \int_{\left\{ \substack{(u_1,\ldots,u_q) \in [0,1]^q: \\ \lceil nu_1 \rceil + \cdots + \lceil nu_q \rceil \leq n} \right\}} \prod_{j=1}^q \left(\frac{\mathfrak{a}_{\lceil nu_j \rceil}}{\mathfrak{a}_n} \right)^{-1} du_1 \cdots du_q.$$

The sequence $(\mathfrak{a}_n)_n$ is $1/\alpha$-regularly varying; by the dominated convergence theorem (the domination coming e.g. from [**9**, Theorem 1.5.6]),

$$\lim_{n \to +\infty} \frac{1}{n^q} \sum_{\boldsymbol{\ell} \in E_{q,n}} \prod_{j=1}^q \left(\frac{\mathfrak{a}_{\ell_j}}{\mathfrak{a}_n} \right)^{-1} = \int_{\Delta_q} \prod_{j=1}^q u_j^{-(1/\alpha)} \, du_1 \cdots du_q,$$

where $\Delta_q = \{(u_1, \ldots, u_q) \in (0, 1)^q : \sum_{j=1}^q u_j \leq 1\}$. Finally, $n\mathfrak{a}_n^{-1} \sim (1 - \alpha^{-1})\mathfrak{A}_n^2$ by Karamata's theorem [**35**] or [**9**, Proposition 1.5.8], so that, as n goes to $+\infty$:

$$\sum_{\boldsymbol{\ell} \in E_{q,n}} \prod_{j=1}^q \mathfrak{a}_{\ell_j}^{-1} = (n\mathfrak{a}_n^{-1})^q \left(n^{-q} \sum_{\boldsymbol{\ell} \in E_{q,n}} \prod_{j=1}^q \left(\frac{\mathfrak{a}_{\ell_j}}{\mathfrak{a}_n} \right)^{-1} \right)$$
$$\sim \mathfrak{A}_n^{2q} \left(\frac{\alpha - 1}{\alpha} \right)^q \int_{\Delta_q} \prod_{j=1}^q u_j^{-(1/\alpha)} \, du_1 \cdots du_q. \tag{3.54}$$

All that remains is to estimate this later integral. Note that, for all $t \geq 0$,

$$\int_{t\Delta_q} \prod_{j=1}^q u_j^{-(1/\alpha)} \, du_1 \cdots du_q = t^{q(\alpha-1)/\alpha} \int_{\Delta_q} \prod_{j=1}^q u_j^{-(1/\alpha)} \, du_1 \cdots du_q.$$

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Hence, using Fubini–Tonnelli's theorem,

$$\begin{aligned}
&\int_{\Delta_q} \prod_{j=1}^{q} u_j^{-(1/\alpha)}\, du_1 \cdots du_q \\
&= \frac{1}{\Gamma(1+q(\alpha-1)/\alpha)} \\
&\quad \times \int_0^{+\infty} t^{((\alpha-1)/\alpha)q} e^{-t} \int_{\mathbb{R}_+^q} \prod_{j=1}^{q} u_j^{-(1/\alpha)} \mathbf{1}_{\{\sum_{j=1}^q u_j \le 1\}}\, du_1 \cdots du_q\, dt \\
&= \frac{1}{\Gamma(1+q(\alpha-1)/\alpha)} \int_{\mathbb{R}_+^q} \prod_{j=1}^{q} u_j^{-(1/\alpha)} \int_0^{+\infty} e^{-t} \mathbf{1}_{\{\sum_{j=1}^q u_j \le t\}}\, dt\, du_1 \cdots du_q \\
&= \frac{1}{\Gamma(1+q(\alpha-1)/\alpha)} \left(\int_0^{+\infty} u^{-(1/\alpha)} e^{-u}\, du \right)^q \\
&= \frac{\Gamma(1-(1/\alpha))^q}{\Gamma(1+q(\alpha-1)/\alpha)}.
\end{aligned}$$

Finally, using the identity $\Gamma(z+1) = z\Gamma(z)$,

$$\sum_{\boldsymbol{\ell} \in E_{q,n}} \prod_{j=1}^{q} \mathfrak{a}_{\ell_j}^{-1} \sim \mathfrak{A}_n^{2q} \frac{\Gamma(1+(\alpha-1)/\alpha)^q}{\Gamma(1+q(\alpha-1)/\alpha)}. \qquad \square$$

3.5. *Renewal properties.* The goal of this sub-section is to prove Proposition 2.6. We assume without loss of generality that the function L appearing in Hypothesis 3.1 is continuous on $(x_0^{-1}, +\infty)$ for some $x_0 > 0$, and that $u \mapsto uL(u^{-1})$ is increasing on this set [**9**, Theorem 1.5.3]. When $\alpha = d$, we set for all $x \in (0, x_0)$:

$$I(x) := \int_x^{x_0} \frac{1}{tL(t^{-1})}\, dt. \tag{3.55}$$

We compute the asymptotics of $g(p)$ according to the method in [**61**, Ch. III.12, P3], which yields Proposition 2.6. Before starting the proof, though, we use the Fourier transform to represent g in an integral form.

LEMMA 3.8. *For all $u \in \mathbb{T}^d$, let $\Psi(u) := \sum_{n \ge 0} \mathbb{E}_\mu[e^{i\langle u, S_n\rangle}]$. Under Hypothesis 3.1, the function Ψ is continuous on $\mathbb{T}^d \setminus \{0\}$, and, for every $p \in \mathbb{Z}^d$,*

$$g(p) = \frac{2}{(2\pi)^d} \int_{\mathbb{T}^d} (1 - \cos(\langle u, p\rangle)) \Psi(u)\, du. \tag{3.56}$$

In addition, for all small enough neighbourhoods U of 0*,*

$$\sup_{p \in \mathbb{Z}^d} \left| g(p) - \frac{2}{(2\pi)^d} \Re \int_U \frac{1 - \cos(\langle u, p\rangle)}{1 - \lambda_u^M} \sum_{k=0}^{M-1} \lambda_u^k \mathbb{E}_\mu[\Pi_u^k(\mathbf{1})]\, du \right| < +\infty. \tag{3.57}$$

Proof. Using the Fourier transform, we know that

$$\begin{aligned}
g(p) &= 2\mu(S_n = 0) - \mu(S_n = p) - \mu(S_n = -p) \\
&= \frac{2}{(2\pi)^d} \int_{\mathbb{T}^d} (1 - \cos(\langle u, p\rangle)) \mathbb{E}_\mu[e^{i\langle u, S_n\rangle}]\, du.
\end{aligned}$$

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Thanks to the Lebesgue dominated convergence theorem, it is then enough to prove that

$$\sum_{n\geq 0}\int_{\mathbb{T}^d}|1-\cos(\langle u,\,p\rangle)|\,|\mathbb{E}_\mu[e^{i\langle u,S_n\rangle}]|\,du<+\infty.$$

Note that $\mathbb{E}_\mu[e^{i\langle u,S_n\rangle}]=\mathbb{E}_\mu[P_u^n\mathbf{1}]$. Hence, for any small enough neighbourhood U of 0,

$$\sup_{u\in U^c}\sum_{n\geq 0}|\mathbb{E}_\mu[e^{i\langle u,S_n\rangle}]|\leq\sum_{n\geq 0}Cr^n\|\mathbf{1}\|_{\mathcal{B}}=O(1),$$

which proves the continuity of Ψ on $\mathbb{T}^d\backslash\{0\}$, as it is the uniform limit of a sequence of continuous functions. In addition, for every $u\in U$,

$$\sum_{n\geq 0}|\mathbb{E}_\mu[e^{i\langle u,S_n\rangle}]|=\sum_{n\geq 0}(|\lambda_u|^n|\mathbb{E}_\mu[\Pi_u^n(\mathbf{1})]|+Cr^n\|\mathbf{1}\|_{\mathcal{B}})\leq\frac{C'}{1-|\lambda_u|}+O(1).$$

Finally,

$$\frac{|1-\cos(\langle u,\,p\rangle)|}{1-|\lambda_u|}\leq C''\frac{|p|^2|u|^{2-\alpha}}{L(|\sqrt{\Sigma}u|^{-1})},$$

since $1-|\lambda_u|\sim\vartheta|\sqrt{\Sigma}u|^\alpha L(|\sqrt{\Sigma}u|^{-1})$ as u goes to 0, and $|1-\cos(\langle u,\,p\rangle)|\leq|u|^2|p|^2$. Since $\alpha\in[1,\,2]$ and since L is slowly varying, this yields equation (3.56). Moreover, due to (3.4),

$$\sum_{n\geq 0}\lambda_u^n\Pi_u^n=\sum_{n\geq 0}\lambda_u^{Mn}\sum_{k=0}^{M-1}\lambda_u^k\Pi_u^k=\frac{1}{1-\lambda_u^M}\sum_{k=0}^{M-1}\lambda_u^k\Pi_u^k.$$

As can be seen in this proof, the error terms that come from integrating over U (instead of $\mathbb{T}^d$) and using Π_u instead of P_u are uniformly bounded in p, so that

$$\sup_{p\in\mathbb{Z}^d}\left|g(p)-\frac{2}{(2\pi)^d}\int_U\frac{1-\cos(\langle u,\,p\rangle)}{1-\lambda_u^M}\sum_{k=0}^{M-1}\lambda_u^k\mathbb{E}_\mu[\Pi_u^k(\mathbf{1})]\,du\right|<+\infty.$$

This equation stays true *a fortiori* if we take its real part, which yields equation (3.57). □

Now, let us begin the proof of Proposition 2.6 in earnest.

Proof of Proposition 2.6. We use the same conventions as in the proof of Lemma 3.8. For all small enough $\delta>0$, put $U(\delta):=\sqrt{\Sigma}^{-1}B(0,\,\delta)$. By equation (3.57), for any small enough neighbourhood U of 0,

$$\sup_{p\in\mathbb{Z}^d}\left|g(p)-\frac{2}{(2\pi)^d}\Re\int_U\frac{1-\cos(\langle u,\,p\rangle)}{1-\lambda_u^M},\sum_{k=0}^{M-1}\lambda_u^k\mathbb{E}_\mu[\Pi_u^k(\mathbf{1})]\,du\right|<+\infty. \tag{3.58}$$

Fix $\varepsilon\in(0,\,1)$. Under Hypothesis 3.1, for all small enough $\delta>0$, for all $u\in U(\delta)$,

$$|\lambda_u-1+\psi(\sqrt{\Sigma}u)L(|\sqrt{\Sigma}u|^{-1})|\leq\varepsilon|\psi(\sqrt{\Sigma}u)|L(|\sqrt{\Sigma}u|^{-1}),$$

and $\max_{0\leq k\leq M-1}\|\Pi_u^k-\Pi_0^k\|_{L(\mathcal{B})}\leq\varepsilon$. Note also that $\sum_{k=0}^{M-1}\lambda_u^k=(1-\lambda_u^M)/(1-\lambda_u)$. Then

$$\left|\frac{1}{1-\lambda_u}-\frac{1}{\psi(\sqrt{\Sigma}u)L(|\sqrt{\Sigma}u|^{-1})}\right|\leq\frac{\varepsilon}{|1-\lambda_u|}\leq\frac{\varepsilon}{1-\varepsilon}\frac{1}{|\psi(\sqrt{\Sigma}u)|L(|\sqrt{\Sigma}u|^{-1})},$$

and

$$\left| \int_{U(\delta)} \frac{1-\cos(\langle u, p\rangle)}{1-\lambda_u^M} \sum_{k=0}^{M-1} \lambda_u^k \mathbb{E}[\Pi_u^k(\mathbf{1})]\, du - \int_{U(\delta)} \frac{1-\cos(\langle u, p\rangle)}{\psi(\sqrt{\Sigma}u)L(|\sqrt{\Sigma}u|^{-1})}\, du \right|$$
$$\leq \varepsilon \left(\frac{(1+\varepsilon)\|\mathbf{1}\|_{\mathcal{B}}}{1-\varepsilon} + 1 \right) \int_{U(\delta)} \frac{1-\cos(\langle u, p\rangle)}{|\psi(\sqrt{\Sigma}u)|L(|\sqrt{\Sigma}u|^{-1})}\, du$$
$$\leq 2\varepsilon \left(\frac{(1+\varepsilon)\|\mathbf{1}\|_{\mathcal{B}}}{1-\varepsilon} + 1 \right) \sqrt{1+\zeta^2} H_\delta(p),$$

where

$$H_\delta(p) := \frac{2}{(2\pi)^d} \int_{U(\delta)} \Re\left(\frac{1-\cos(\langle u, p\rangle)}{\psi(\sqrt{\Sigma}u)L(|\sqrt{\Sigma}u|^{-1})} \right) du.$$

Hence,

$$\lim_{\varepsilon\to 0} \sup_{p\in\mathbb{Z}^d} H_\delta(p)^{-1} \left| \int_{U(\delta)} \frac{1-\cos(\langle u, p\rangle)}{1-\lambda_u^M} \sum_{k=0}^{M-1} \lambda_u^k \mathbb{E}[\Pi_u^k(\mathbf{1})]\, du - H_\delta(p) \right| = 0. \quad (3.59)$$

Assume that there exists a function $h : \mathbb{Z}^d \to \mathbb{R}$ such that, for all $\delta > 0$ small enough, $H_\delta(p) \sim h(p)$ as p goes to infinity. If in addition $\lim_\infty h = +\infty$, then equations (3.58) and (3.59) imply that $g(p) \sim h(p)$.

Now, let us simplify those integrals. First, note that

$$\Re\left(\frac{1}{\psi(\sqrt{\Sigma}u)} \right) = \frac{1}{\vartheta(1+\zeta^2)|\sqrt{\Sigma}u|^\alpha}.$$

Let $e_1 := (1, 0, \ldots, 0)$. Then

$$H_\delta(p) = \frac{2}{(2\pi)^d \vartheta(1+\zeta^2)} \int_{\sqrt{\Sigma}^{-1}B(0,\delta)} \frac{1-\cos(\langle u, p\rangle)}{|\sqrt{\Sigma}u|^\alpha L(|\sqrt{\Sigma}u|^{-1})}\, du$$
$$= \frac{2}{(2\pi)^d \vartheta(1+\zeta^2)\sqrt{\det(\Sigma)}} \int_{B(0,\delta)} \frac{1-\cos(\langle v, \sqrt{\Sigma}^{-1}p\rangle)}{|v|^\alpha L(|v|^{-1})}\, dv$$
$$= \frac{2}{(2\pi)^d \vartheta(1+\zeta^2)\sqrt{\det(\Sigma)}} \int_{B(0,\delta)} \frac{1-\cos(|\sqrt{\Sigma}^{-1}p|\langle v, e_1\rangle)}{|v|^\alpha L(|v|^{-1})}\, dv$$
$$= \frac{2|\sqrt{\Sigma}^{-1}p|^{\alpha-1}}{(2\pi)^d \vartheta(1+\zeta^2)\sqrt{\det(\Sigma)}} \int_{B(0,|\sqrt{\Sigma}^{-1}p|\delta)} \frac{1-\cos(\langle w, e_1\rangle)}{|w|^\alpha L(|\sqrt{\Sigma}^{-1}p|\, |w|^{-1})}\, dw.$$

We shall now distinguish between three sub-cases: $d = 1$ and $\alpha \in (1, 2]$, then $d = \alpha = 1$ (in the basin of Cauchy distributions), and finally $d = \alpha = 2$.

Case $d = 1$, $\alpha \in (1, 2]$. In this case, most of the mass in the integral representation of $g(p)$ is present in a small neighbourhood of 0, of size roughly $1/|p|$.

Let $\eta \in (0, \alpha - 1)$. By Potter's bound [**9**, Theorem 1.5.6], if δ is small enough, there exists a constant C such that, for all $p \in \mathbb{Z}$ with a large enough absolute value, for all $|w| < |p|\delta$,

$$C^{-1} \min\{|w|^\eta, |w|^{-\eta}\} \leq \left| \frac{L(|p|\, |w|^{-1})}{L(|p|)} \right| \leq C \max\{|w|^\eta, |w|^{-\eta}\}.$$

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For $w \in [-1, 1]$, we get

$$L(p)\frac{1-\cos(w)}{|w|^\alpha L(|p|\,|w|^{-1})} \leq \frac{C|w|^{2-\alpha-\eta}}{2},$$

while for $1 < |w| < |\sqrt{\Sigma}^{-1} p|\delta$

$$L(p)\frac{1-\cos(w)}{|w|^\alpha L(|p|\,|w|^{-1})} \leq \frac{2C}{|w|^{\alpha-\eta}}.$$

In addition, $L(p)(1-\cos(w))|w|^{-\alpha}/L(|p|\,|w|^{-1})$ converges pointwise, as p goes to infinity, to $(1-\cos(w))|w|^{-\alpha}$. By the Lebesgue dominated convergence theorem

$$H_\delta(p) \sim_{p\to\infty} \frac{2|p|^{\alpha-1}}{\pi\vartheta(1+\zeta^2)L(|p|)} \int_0^{+\infty} \frac{1-\cos(w)}{w^\alpha}\,dw.$$

Since $\lim_{p\to\infty} H_\delta(p) = +\infty$ and the right-hand side does not depend on δ, by (3.59),

$$g(p) \sim_{p\to\infty} \frac{2|p|^{\alpha-1}}{\pi\vartheta(1+\zeta^2)L(|p|)} \int_0^{+\infty} \frac{1-\cos(w)}{w^\alpha}\,dw.$$

Finally, using an integration by parts and [**24**, Ch. XVII.4, (4.11)], we get

$$\int_0^{+\infty} \frac{1-\cos(w)}{w^\alpha}\,dw = \frac{\pi}{2\Gamma(\alpha)\sin((\alpha-1)\pi/2)}.$$

Case $d = \alpha = 1$. First, by the same computations and the monotone convergence theorem

$$\begin{aligned}\sum_{n\geq 0} \mu(S_n = 0) &= \frac{1}{2\pi}\int_{\mathbb{T}} \Psi(u)\,du \\ &= \frac{1}{2\pi\vartheta(1+\zeta^2)} \int_U \frac{1}{|u|L(|u|^{-1})}\,du(1+o(1)) + O(1),\end{aligned}$$

where U is any small neighbourhood of 0 in $\mathbb{T}$. But Halmos' recurrence theorem [**29**] and the conservativity of $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ implies that the left-hand side is infinite, so the right-hand side is also infinite, and $\lim_0 I = +\infty$.

Let us go back to the study of g. If $\alpha = d = 1$, a neighbourhood of size $1/|p|$ of the origin makes for a negligible part of the mass of $g(p)$. We must look at a larger scale, where the oscillations makes the cosine ultimately vanish (much as with Riemann–Lebesgue's lemma).

Let $R > 0$. using again Potter's bound (in the same way as in the previous case), we get that

$$\sup_{p\in\mathbb{Z}} L(|p|) \int_{B(0,R)} \frac{1-\cos(w)}{|w|L(|p|\,|w|^{-1})}\,dw < +\infty,$$

whence

$$\begin{aligned}\frac{\pi\vartheta(1+\zeta^2)}{2} H_\delta(p) &= \int_{R/|p|}^{\delta} \frac{1-\cos(|p|v)}{|v|L(|v|^{-1})}\,dv + O(L(|p|)^{-1}) \\ &= I(R/|p|) - I(\delta) - \int_{R/|p|}^{\delta} \frac{\cos(|p|v)}{|v|L(|v|^{-1})}\,dv + O(L(|p|)^{-1}).\end{aligned}$$

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By [**9**, Theorem 1.5.9a], $I(R/|p|) \gg L(|p|)^{-1}$. Set $F(v) := 1/(vL(v^{-1}))$, which is monotonous on a neighbourhood of 0. Remark that, by [**9**, Theorem 1.5.9a] again, $|p|^{-1}F(\delta) \ll |p|^{-1}F(R/|p|) \ll I(R/|p|)$ as p goes to infinity. Then, using the Riemann–Stieltjes version of the integration by parts [**6**, Theorem 7.6]:

$$\left|\int_{R/|p|}^{\delta} \frac{\cos(|p|v)}{|v|L(|v|^{-1})}\,dv\right| = \left|\frac{1}{|p|}[\sin(|p|v)F(v)]_{R/|p|}^{\delta} - \frac{1}{|p|}\int_{R/|p|}^{\delta} \sin(|p|v)\,dF(v)\right| \leq \frac{2}{|p|}(F(R/|p|) + F(\delta)). \tag{3.60}$$

Hence, $H_\delta(p) \sim 2/\pi\vartheta(1+\zeta^2)I(R/|p|) \sim (2/\pi\vartheta)I(|p|^{-1})$ as p goes to infinity. Since $\lim_{p\to\infty} H_\delta(p) = +\infty$ and $I(|p|^{-1})$ does not depend on δ, using the remark following (3.59), we get the claim of the proposition.

Case $d = \alpha = 2$. The method is much the same as for $d = \alpha = 1$, but the oscillations happen along one axis in the plane. Hence, there is cancellation in almost all directions, but not uniformly. Using again Potter's bound, we get that

$$\begin{aligned}
&\frac{\pi\sqrt{\det(\Sigma)}}{2} H_\delta(p) \\
&= \frac{1}{2\pi}\int_{B(0,\delta)\setminus B(0,R/|\sqrt{\Sigma}^{-1}p|)} \frac{1-\cos(|\sqrt{\Sigma}^{-1}p|\langle v, e_1\rangle)}{|v|^2 L(|v|^{-1})}\,dv + O(L(|p|)^{-1}) \\
&= I(R/|\sqrt{\Sigma}^{-1}p|) - I(\delta) \\
&\quad - \frac{1}{2\pi}\int_0^{2\pi}\int_{R/|\sqrt{\Sigma}^{-1}p|}^{\delta} \frac{\cos(|\sqrt{\Sigma}^{-1}p| r\cos(t))}{rL(r^{-1})}\,dr\,dt + O(L(|p|)^{-1}).
\end{aligned}$$

Fix $\eta > 0$. On $\{|\cos(t)| > \eta\}$, as in (3.60),

$$\begin{aligned}
&\left|\int_0^{2\pi}\int_{R/|\sqrt{\Sigma}^{-1}p|}^{\delta} \frac{\cos(|\sqrt{\Sigma}^{-1}p| v\cos(t))}{vL(v^{-1})}\,dv\,dt\right| \\
&\quad \leq \frac{2\,\mathrm{Leb}(|\cos(t)| > \eta)}{\eta|\sqrt{\Sigma}^{-1}p|}(F(R/|\sqrt{\Sigma}^{-1}p|) + F(\delta)) \\
&\qquad + \mathrm{Leb}(|\cos(t)| < \eta) I(|\sqrt{\Sigma}^{-1}p|/R).
\end{aligned}$$

Since this holds for all $\eta > 0$, and since $F(\delta)/|\sqrt{\Sigma}^{-1}p| \ll F(R/|\sqrt{\Sigma}^{-1}p|)/|\sqrt{\Sigma}^{-1}p| \ll I(|\sqrt{\Sigma}^{-1}p|^{-1})$, we get that $H_\delta(p) \sim (2/(\pi\sqrt{\det(\Sigma)}))I(R/|\sqrt{\Sigma}^{-1}p|) \sim (2/(\pi\sqrt{\det(\Sigma)}))I(|p|^{-1})$ as p goes to infinity. Again, this is what we claimed in the proposition. □

4. *Theorems 2.7 and 2.11: context and proofs*

Theorem 2.4 yields a limit theorem using only spectral methods. If the factor (A, μ, T) is Gibbs–Markov, then we also have the limit theorems from [**66**, **67**]. Comparing the expressions of the limits yields Corollary 2.9.

Using the structure of the Gibbs–Markov map, we can leverage Corollary 2.9 to get an estimate of the probability that an excursion from $A \times \{0\}$ hits $A \times \{p\}$, with $p \in \mathbb{Z}^d$. This is the content of Theorem 2.7. Finally, Theorem 2.7 allows us to improve the main theorems from [**67**], yielding Theorem 2.11. In turn, this improves Corollary 2.9, yielding Corollary 2.13.

We present our strategy in §4.1. In §4.2, we present Gibbs–Markov maps, and their main properties of interest. Sections 4.3 and 4.4 deal with the tightness and convergence in distribution of the (renormalized) number of hits of $A \times \{p\}$ by an excursion, and from there the convergence in moments. Finally, Theorem 2.7 and Theorem 2.11 are proved in §§4.5 and 4.6, respectively.

4.1. *Strategy: working with excursions.* Our end goal is Theorem 2.11. Let us describe the strategy behind our proof.

The method used in [**67**] to get a distributional limit theorem for observables of a Markovian $\mathbb{Z}^d$-extension was the following. To keep things simple, we ignore Lévy stable distributions and stay in dimension $d = 1$. Let $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ be an ergodic and conservative Markovian $\mathbb{Z}$-extension of a Gibbs–Markov map (A, μ, T) with a square integrable step function F with asymptotic variance $\sigma_{GK}^2(F, A, \mu, T) := \mathbb{E}_\mu[F^2] + 2\sum_{k\geq 1} \mathbb{E}_\mu[F \cdot F \circ T^k]$.

As in §2.3, let $\varphi_{\{0\}}$ be the first return time to $A \times \{0\}$, and $\widetilde{T}_{\{0\}}$ be the induced map on $A \times \{0\} \simeq A$. Recall that, for any measurable function $f : A \times \mathbb{Z}^d \to \mathbb{R}$ and almost every $x \in A$, we define

$$f_{\{0\}}(x) := \sum_{k=0}^{\varphi_{\{0\}}(x)-1} f \circ \widetilde{T}^k(x, 0),$$

that is, $f_{\{0\}}(x)$ is the sum of f along the excursion from $A \times \{0\}$ starting from $(x, 0)$.

For every $n \geq 0$ and $x \in A$, let $\tau_n(x)$ be the number of visits of $(\widetilde{T}^k(x, 0))_{k\geq 1}$ to $A \times \{0\}$ before time n. Then, for x in A

$$S_n^{\widetilde{T}} f(x, 0) = \sum_{k=0}^{n-1} f \circ \widetilde{T}^k(x, 0) \simeq \sum_{k=0}^{\tau_n(x)-1} f_{\{0\}} \circ \widetilde{T}_{\{0\}}^k(x),$$

where, under reasonable assumptions on f and on the extension, the error terms are negligible for large n. If f is integrable and has zero integral, then so does $f_{\{0\}}$. If in addition $|f|_{\{0\}}$ belongs to $\mathbb{L}^p$ for some $p > 2$ and if f is regular enough, then (τ_n) and $(\sum_{k=0}^{N-1} f_{\{0\}} \circ \widetilde{T}_{\{0\}}^k)$ are asymptotically independent [**67**, Theorem 1.7], and we have a generalized central limit theorem [**68**, Corollary 6.9], which has the following form when $d = 1$:

$$\lim_{n\to+\infty} \frac{S_n^{\widetilde{T}} f}{n^{1/4}} = \left(\frac{2}{\pi\sigma_{GK}^2(F, A, \mu, T)}\right)^{1/4} \sigma_{GK}(f_{\{0\}}, A, \mu, \widetilde{T}_{\{0\}})L,$$

where the convergence is strong in distribution and where L is a parameter $1/\sqrt{2}$ centered Laplace random variable and where†

$$\sigma_{GK}^2(f_{\{0\}}, A, \mu, \widetilde{T}_{\{0\}}) = \mathbb{E}_\mu[f_{\{0\}}^2] + 2\mathbb{E}_\mu[f_{\{0\}} \cdot f_{\{0\}} \circ \widetilde{T}_{\{0\}}^n]. \tag{4.1}$$

Similar limit theorems hold in dimension two or when the jumps are in the basin of attraction of a Lévy stable distribution.

† Assuming $(A, \mu, \widetilde{T}_{\{0\}})$ is mixing, otherwise the formula differs slightly.

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Due to [**67**, Theorem 1.11], we already know that the limit theorem holds for observables f, which are Hölder and such that $|f|_{\{0\}} \in \mathbb{L}^q$ for some $q > 2$. However, this condition is hard to check, and we would like to get a condition that may be stronger, but more manageable. Our idea is to leverage what we know about the observables $f_p : \widetilde{A} \to \{\pm 1\}$, which we recall are defined for $p \in \mathbb{Z}^d$ by $f_p(x, q) := (\mathbf{1}_{\{p\}} - \mathbf{1}_{\{0\}})(q)$.

Note that $f_{p,\{0\}}(x) = N_p(x) - 1$, where $N_p(x)$ is the number of visits to $A \times \{p\}$ starting from $(x, 0)$ before coming back to $A \times \{0\}$. In addition, for any observable f and any $q \geq 1$,

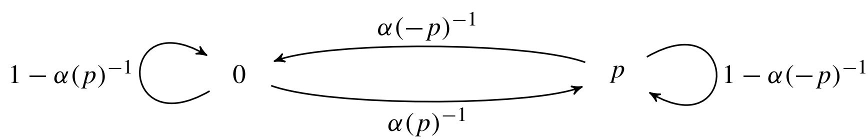
$$\| |f|_{\{0\}} \|_{\mathbb{L}^q(A,\mu)} \leq \sum_{p \in \mathbb{Z}^d} \|f(\cdot, p)\|_{\mathbb{L}^\infty(A,\mu)} \|N_p\|_{\mathbb{L}^q(A,\mu)}. \tag{4.2}$$

Hence, we are led to the study of $\|N_p\|_{\mathbb{L}^q(A,\mu)}$ for $q > 2$. Note that $\|N_p\|_{\mathbb{L}^q(A,\mu)} = \|f_{p,\{0\}}\|_{\mathbb{L}^q(A,\mu)} + O(1)$.

First, we will see that $\|N_p\|_{\mathbb{L}^2(A,\mu)} \sim \sigma_{GK}(f_{p,\{0\}}, A, \mu, \widetilde{T}_{\{0\}})$. Moreover, comparing the conclusions of Theorem 2.4 of the present paper with a previous result, we obtain that $\sigma^2_{GK}(f_{p,\{0\}}, A, \mu, \widetilde{T}_{\{0\}}) = 2(g(p) - 1)$ for every p. The control on higher moments ($q > 2$) of $f_{p,\{0\}}$ helps us to extend Theorem 2.4 to a wider class of observables, thanks to the argument in [**18**].

Our main issue is then to control $\|N_p\|_{\mathbb{L}^q(A,\mu)}$ for any $q > 2$ with the weaker norm $\|N_p\|_{\mathbb{L}^2(A,\mu)}$. For random walks, there is a simple argument, which we will replicate in the context of Gibbs–Markov maps. Recall that $\alpha(p)^{-1} := \mu(N_p > 0)$ is the probability to visit $A \times \{p\}$ before coming back to $A \times \{0\}$, when starting from 0.

To identify the distribution of N_p, it is enough to consider the Markov chain corresponding to the times at which the random walk is in $\{0, p\}$, which is given by



Since the random walk spends as much time in $A \times \{0\}$ and $A \times \{p\}$, we get $\alpha(p) = \alpha(-p)$. Hence, the random variable N_p conditioned on $\{N_p > 0\}$ has a geometric distribution of parameter $\alpha(p)^{-1}$. So $\|f_{p,\{0\}}\|_{\mathbb{L}^q}$ is determined by $\|f_{p,\{0\}}\|_{\mathbb{L}^2}$ for all q.

In the context of Markovian $\mathbb{Z}^d$-extensions of Gibbs–Markov maps, we cannot expect to know the explicit distribution of N_p; however, the same results will hold asymptotically, which is enough for our purposes. The main idea is that $N_p - 1$ conditioned on $\{N_p > 0\}$ is the hitting time of the event $\{\widetilde{T}_{\{0,p\}} \in A \times \{0\}\}$, which becomes small as p goes to infinity, so $\alpha(-p)^{-1} N_p$ conditioned on $\{N_p > 0\}$ converges in distribution to an exponential random variable of parameter 1. Exponential tightness gives the convergence of the moments of N_p, which is what we want.

4.2. *Recalls on Gibbs–Markov maps.* Throughout this section, $(A, \pi, \lambda, \mu, T)$ denotes a Gibbs–Markov map. These models provide a large enough family of dynamical systems, including many important examples, most notably inductions of Markov maps with respect to a stopping time. Together with the construction of Young towers [**70**], Gibbs–Markov

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maps appear in a variety of subjects, including intermittent chaos [**26**, **46**, **56**, **71**], Anosov flows [**12**], or hyperbolic billiards [**70**]. Their definition is flexible enough to allow $\mathbb{Z}^d$-extensions with large jumps [**4**]. Yet, Gibbs–Markov maps have a very strong structure, which makes them tractable. We refer the reader to [**1**, Ch. 4] and [**26**, Ch. 1] for more general references on Gibbs–Markov maps, and to §4.2 for some more specialized results. Let us recall their definition.

Definition 4.1. (Measure-preserving Gibbs–Markov maps) Let $(A, d, \mathcal{B}, \mu)$ be a probability, metric, bounded Polish space. Let π be a partition of A in sub-sets of positive measure (up to a null set). Let $T : A \to A$ be a μ-preserving map, Markov with respect to the partition π, and such that π is generating. Such a map is said to be Gibbs–Markov if it also satisfies the following.

- Big image property: $\inf_{a\in\pi} \mu(Ta) > 0$.
- Expansivity: there exists $\lambda > 1$ such that $d(Tx, Ty) \geq \lambda d(x, y)$ for all $a \in \pi$ and $(x, y) \in a \times a$.
- Bounded distortion: there exists $C > 0$ such that, for all $a \in \pi$, for almost every $(x, y) \in a \times a$:

$$\left|\frac{d\mu}{d\mu\circ T}(x) - \frac{d\mu}{d\mu\circ T}(y)\right| \leq C d(Tx, Ty)\frac{d\mu}{d\mu\circ T}(x). \tag{4.3}$$

A measure-preserving Gibbs–Markov map is thus the data (A, π, d, μ, T) of five objects: a topological space, a partition, a distance, a measure and a measure-preserving transformation. We will sometimes abuse the notation, and say, for instance, that (A, μ, T) is a Gibbs–Markov map.

Later on, we shall use liberally many fine properties of Gibbs–Markov maps. We put them together in this sub-section, which is divided in three parts.

- *Fundamental definitions and facts*: what is a Gibbs–Markov map, and what are stopping times?
- *Good Banach spaces*: the Banach spaces we work with, and the properties of the transfer operator.
- *Extensions and induction*: what happens when we induce a Markovian $\mathbb{Z}^d$-extension $(\widetilde{A}, \tilde{\mu}, \widetilde{A})$ on a nice set, and a distortion estimate?

4.2.1. *Fundamental definitions and facts.* Let (A, π, d, μ, T) be a Gibbs–Markov map. For all x and y in A, we define the *separation time* of x and y as

$$s(x, y) := \inf\{n \geq 0 : \forall a \in \pi,\ T^n x \notin a \text{ or } T^n y \notin a\}.$$

Let λ be the expansion constant of a Gibbs–Markov map. Then $(A, \pi, \tau^{-s}, \mu, T)$ is Gibbs–Markov for all $\tau \in (1, \lambda]$. Without loss of generality, we assume that the distance d belongs to this family of distances, and (if needed) we specify the parameter τ instead of the distance d. This simplifies greatly the induction processes.

For $n \geq 0$, a *cylinder* of length n is a non-trivial element of $\pi_n := \bigvee_{k=0}^{n-1} T^{-k}\pi$. It is given by a unique finite sequence $(a_k)_{0\leq k<n}$ of elements of π such that $T(a_k) \cap a_{k+1}$ is non-negligible for all $0 \leq k < n-1$. Such a cylinder shall be denoted by $[a_0, \ldots, a_{n-1}]$.

With any Markov maps comes a natural filtration given by $\mathcal{F}_n := \sigma(\pi_n)$ for all $n \geq 0$. From this filtration we define *stopping times*.

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Definition 4.2. (Stopping time) Let (A, π, d, μ, T) be a Gibbs–Markov map. Let $\varphi : A \to \mathbb{N} \cup \{+\infty\}$ be measurable. We say that φ is a stopping time if $\{\varphi \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$.

If φ is a stopping time, which is almost surely positive and finite, the associated countable partition of A is given by

$$\pi_\varphi := \bigcup_{n \geq 1} \{\overline{a} \in \pi_n : \mu(\overline{a}) > 0 \text{ and } \overline{a} \subset \{\varphi = n\}\},$$

and the associated transformation is

$$T_\varphi(x) := T^{\varphi(x)}(x),$$

which is well-defined almost everywhere if φ is finite almost everywhere.

One of the great advantages of the class of Gibbs–Markov maps is that it is stable by induction, and that ergodic Gibbs–Markov maps admit some iterate that is mixing on ergodic components, as the following results assert.

LEMMA 4.3. [**1**, Proposition 4.6.2] *Let $(A, \pi, \lambda, \mu, T)$ be a Gibbs–Markov map, and φ be a stopping time for the associated filtration $(\mathcal{F}_n)_{n \geq 0}$.*

Assume that φ is almost surely positive and finite, and that T_φ preserves μ. Then $(A, \pi_\varphi, \lambda, \mu, T_\varphi)$ is a measure-preserving Gibbs–Markov map.

PROPOSITION 4.4. [**26**, Proposition 1.3.14] *Let $(A, \pi, \lambda, \mu, T)$ be an ergodic Gibbs–Markov map. Then there exists an integer $M \geq 1$ and a $\sigma(\pi)$-measurable partition $(A_k)_{k \in \mathbb{Z}/M\mathbb{Z}}$ of A such that:*
- *$T(A_k) = A_{k+1}$ for all $k \in \mathbb{Z}/M\mathbb{Z}$;*
- *each $(A_k, \pi_M, \lambda, \mu(\cdot | A_k), T^M)$ is a mixing Gibbs–Markov map.*

4.2.2. *Good Banach spaces.* Let $P : \mathbb{L}^1(A, \mu) \to \mathbb{L}^1(A, \mu)$ be the transfer operator associated with T. For any bounded measurable function $h : A \to \mathbb{R}$, let

$$P_h : \begin{cases} \mathbb{L}^1(A, \mu) & \to \quad \mathbb{L}^1(A, \mu), \\ f & \mapsto \quad P(hf). \end{cases}$$

For any $a \in \pi$ and any measurable function $f : A \to \mathbb{R}$, we define the Lipschitz semi-norm of f on a by

$$|f|_{\mathrm{Lip}(a,d)} := \mathrm{Einf}_{\{x,y \in a\}}\{C \geq 0 : |f(x) - f(y)| \leq C d(x, y)\},$$

where Einf denotes the essential infimum.

Definition 4.5. Let us define the following two norms:

$$\|f\|_{\mathrm{Lip}^1(A,\pi,d,\mu)} := \|f\|_{\mathbb{L}^1(A,\mu)} + \sum_{a \in \pi} \mu(a) |f|_{\mathrm{Lip}(a,d)};$$
$$\|f\|_{\mathrm{Lip}^\infty(A,\pi,d,\mu)} := \|f\|_{\mathbb{L}^\infty(A,\mu)} + \sup_{a \in \pi} |f|_{\mathrm{Lip}(a,d)}.$$

The spaces $\mathrm{Lip}^1(A, \pi, d, \mu)$ and $\mathrm{Lip}^\infty(A, \pi, d, \mu)$ are the spaces of measurable functions whose respective norms are finite. The space Lip^∞ is the space of all *globally Lipschitz functions*, while Lip^1 is the space of all *summably locally Lipschitz functions*.

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A family of observables is uniformly globally (respectively, summably locally) Lipschitz if the Lip^∞ norm (respectively, the Lip^1 norm) is bounded on this family.

Let $\theta \in (0, 1]$. If we replace d by d^θ, we get spaces of globally or summably locally θ-Hölder observables. Any result about Lipschitz observables can be generalized freely to θ-Hölder observables.

The transfer operator P acts quasi-compactly on Lip^∞. If the transformation is mixing, then the transfer operator has a spectral gap, which implies an exponential decay of correlation for Lipschitz (and, by extension, Hölder) observables [**26**, Corollaire 1.1.21].

PROPOSITION 4.6. (Exponential decay of correlations) *Let (A, π, d, μ, T) be a mixing Gibbs–Markov map. Then there exist constants $C, \kappa > 0$ such that, for all $n \geq 0$, for all $g \in \mathrm{Lip}^\infty(A)$,*

$$\|P^n g - \int_A g \, d\mu \cdot \mathbf{1}\|_{\mathrm{Lip}^\infty(A)} \leq C e^{-\kappa n} \|g\|_{\mathrm{Lip}^\infty(A)}.$$

In addition, P maps continuously Lip^1 into Lip^∞ [**26**, Lemme 1.1.13]. This feature (that P maps a large space of integrable functions into a space of bounded functions) is specific to Gibbs–Markov maps.

4.2.3. *Extensions and induction.* Let $(A, \pi, \lambda, \mu, T)$ be a measure-preserving Gibbs–Markov map. Let G be a discrete countable Abelian group with counting measure ν. Let $F : A \to G$ be $\sigma(\pi)$-measurable. If $(A \times G, \mu_{A\times G}, \widetilde{T})$ is conservative and ergodic, then for any non-empty sub-set $S \subset G$ and any $p \in G$, the function:

$$\varphi_{p,S} : \begin{cases} A \times S & \to \quad \mathbb{N}_+, \\ x & \mapsto \quad \inf\{n \geq 1 : \widetilde{T}^n(x, p) \in A \times S\}, \end{cases}$$

is a stopping time, which is almost surely positive and finite.

Let $S \subset G$ be non-empty and finite. Set:

- a partition $\pi_S := \{a \times \{p\} : p \in S, a \in \pi_{\varphi_{p,S}}\}$;
- a measure $\mu_S := \nu(S)^{-1} \mu \otimes \nu_{|S}$;
- a transformation:

$$T_S : \begin{cases} A \times S & \to \quad A \times S, \\ (x, p) & \mapsto \quad \widetilde{T}^{\varphi_{p,S}(x)}(x). \end{cases}$$

PROPOSITION 4.7. (Inductions of extensions of Gibbs–Markov maps are Gibbs–Markov) *Let $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ be an ergodic and conservative Markovian $\mathbb{Z}^d$-extension of a Gibbs–Markov map $(A, \pi, \lambda, \mu, T)$.*

Then, for any non-empty finite sub-set $S \subset G$, the dynamical system $(A \times S, \pi_S, \lambda, \mu_S, T_S)$ is a measure-preserving ergodic Gibbs–Markov map.

Proof. Up to straightforward modifications, the proof is the same as in [**1**, Proposition 4.6.2]. □

Given any non-trivial and finite $S \subset G$, we can then define the transfer operator P_S associated with the system $(A \times S, \mu_S, T_S)$.

For the remainder of the section, we assume that $(A, \pi, \lambda, \mu, T)$ is a measure-preserving and ergodic Gibbs–Markov map, G a discrete countable Abelian group with counting measure ν, and $F : A \to G$ a $\sigma(\pi)$-measurable function. We assume that $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ is conservative and ergodic.

In our proof, we will sometimes have to control the distortion of $T_\varphi : x \mapsto T^{\varphi(x)}(x)$ for various stopping times φ. This is done with the next lemma, which generalizes [**26**, Lemme 1.1.13]. We write $P^{(\varphi)}$ for the transfer operator associated with T_φ.

LEMMA 4.8. *Let $(A, \pi, \lambda, \mu, T)$ be a measure-preserving Gibbs–Markov map. Then there exists a constant $K > 0$ with the following property. Let φ be a stopping time, which is finite with positive probability as well as almost surely positive. Let $A \subset \{\varphi < +\infty\}$ be $\sigma(\pi_\varphi)$-measurable and non-trivial. Then*

$$\left\| \frac{P^{(\varphi)} \mathbf{1}_A}{\mu(A)} \right\|_{\mathrm{Lip}^\infty(A,\pi,\lambda)} \leq K.$$

Proof. Let $n \geq 1$, and let $\overline{a}$ be a cylinder of length n for the Gibbs–Markov map $(A, \pi, \lambda, \mu, T)$. By a strengthening of the distortion lemma, e.g. [**26**, Lemme 1.1.13], there is a constant K independent of n and $\overline{a}$ such that

$$\|P^n(\mathbf{1}_{\overline{a}})\|_{\mathrm{Lip}^\infty(A,\pi,\lambda)} \leq K\mu(\overline{a}).$$

By additivity, this inequality remains true whenever $\overline{a}$ is $\sigma(\pi_n)$-measurable. For all $n \geq 1$, let $A_n := A \cap \{\varphi = n\}$. Then $(A_n)_{n\geq 1}$ is a partition of A. In addition, each A_n is $\sigma(\pi_n)$-measurable, and $P^{(\varphi)} = P^n$ for functions supported by A_n, so that

$$\frac{P^{(\varphi)} \mathbf{1}_A}{\mu(A)} = \mu(A_\varphi)^{-1} \sum_{n\geq 1} P^n \mathbf{1}_{A_n}.$$

By additivity again, the Lip^∞ norm of the right-hand side is at most

$$K\mu(A_\varphi)^{-1} \sum_{n\geq 1} \mu(A_n) = K\mu(A)^{-1}\mu(A) = K. \qquad \square$$

4.2.4. *Fulfilment of the spectral hypotheses.* The spectral hypotheses 3.1 are used in our main theorems, and Gibbs–Markov maps appear in a variety of applications. We provide here a simple sufficient criterion to ensure that the spectral hypotheses are satisfied for Gibbs–Markov maps. The hypothesis of aperiodicity will be central.

Definition 4.9. (Aperiodic extensions) Let (A, μ, T) be a dynamical system preserving a probability measure. Let $d \geq 0$ and $F : A \to \mathbb{Z}^d$ be a measurable function. The corresponding extension $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ is said to be *aperiodic* if the coboundary equation:

$$F = k + \theta \circ T - \theta \mod \Lambda \tag{4.4}$$

has no solution, where Λ is a proper sub-lattice of $\mathbb{Z}^d$, $k \in (\mathbb{Z}^d)_{/\Lambda}$, and $\theta : A \to (\mathbb{Z}^d)_{/\Lambda}$ is measurable.

For Gibbs–Markov maps, aperiodicity translates nicely into spectral properties.

LEMMA 4.10. *Let $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ be a Markovian $\mathbb{Z}^d$-extension of an ergodic Gibbs–Markov map (A, μ, T). Then the following can be said.*

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- *This extension is aperiodic if and only if the operator P_u acting on* Lip^∞ *has spectral radius strictly smaller than* 1 *for all* $u \in \mathbb{T}^d \setminus \{0\}$.
- *If P_u has an eigenvalue of modulus* 1 *for some u, then this eigenvalue is simple.*
- *The set* $\{u \in \mathbb{T}^d : \rho(P_u) = 1\}$ *is a closed sub-group of* $\mathbb{T}^d$. *If* $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ *is ergodic, then this sub-group is discrete.*

Proof. The fact that the aperiodicity of the extension implies that $\rho(P_u) < 1$ for $u \neq 0$ is [**69**, Lemma 2.6]. We point out that the later lemma includes the hypotheses that T be mixing and F integrable. However, mixing can be replaced by ergodicity without changing the proof. The integrability of F only matters to show that P_u has spectral radius strictly smaller that 1 for u close to, but different from, 0. However, $u \mapsto P_u$ is continuous in the operator norm by the discussion following [**26**, Corollaire 4.1.3], without restriction on the integrability of F. The argument in the proof of [**69**, Lemma 2.6] then implies that if $\rho(P_u) = 1$ for some u with at least one irrational coordinate, then P_v has an eigenvalue of modulus 1 for all v on a non-trivial sub-torus of $\mathbb{T}^d$, and hence for some v with rational coordinates. The same conclusion ensues.

Conversely, assume that the extension is not aperiodic. Let Λ, k and θ be as in equation (4.4). Since Λ is a proper sub-lattice, there exists $u \in \mathbb{T}^d \setminus \{0\}$ such that $e^{i\langle u, \cdot\rangle} \equiv 1$ on Λ. Then

$$\mathbf{1} = P(\mathbf{1}) = P(e^{i\langle u, F-k-\theta\circ T+\theta\rangle}\mathbf{1}) = e^{-i\langle u,k\rangle}e^{-i\langle u,\theta\rangle}P_u(e^{i\langle u,\theta\rangle}),$$

so $P_u(e^{i\langle u,\theta\rangle}) = e^{i\langle u,k\rangle}e^{i\langle u,\theta\rangle}$.

In addition, F is constant on the elements of the Markov partition. By [**26**, Théorème 1.2.1], the function θ is locally Lipschitz, so $e^{i\langle u,\theta\rangle} \in \mathrm{Lip}^\infty$. Hence, $e^{i\langle u,k\rangle}$ belongs to the spectrum of P_u. This ends the proof of the first point.

Let $H := \{u \in \mathbb{T}^d : \rho(P_u) = 1\}$. This sub-set is closed by continuity of $u \mapsto P_u$. We have $0 \in H$. If $P_u(f) = \lambda f$, then $P_{-u}(\overline{f}) = \overline{\lambda} \cdot \overline{f}$, so H is invariant under inversion. Now, let $u_1, u_2 \in H$, and let λ_1 and λ_2 be eigenvalue of modulus 1 of P_{u_1} and P_{u_2} respectively. Let f_1 and f_2 be corresponding eigenfunctions, chosen to have modulus 1. Then, by [**69**, Equation 2.10], we have $P_{u_1+u_2}(f_1 f_2) = \lambda_1\lambda_2 f_1 f_2$, so $u_1 + u_2 \in H$. Hence we have proved that H is a closed sub-group of $\mathbb{T}^d$.

Let $u \in \mathbb{T}^d$, and let f and g be two eigenfunctions for the same eigenvalue λ of P_u. Then:

$$P(f\overline{g}) = P_{u-u}(f\overline{g}) = \lambda f \cdot \overline{\lambda g} = f\overline{g}.$$

By ergodicity, $f\overline{g}$ is constant, so f and g are colinear. This ends the proof of the second point.

Assume that H is not discrete. Let M be the periodicity of the Gibbs–Markov map (A, μ, T), so that the peripherical spectrum of P_0 is the set of Mth roots of the unit. For $u \in H$ close to 0, choose an eigenvalue η_u of modulus 1 and an eigenfunction f_u of modulus 1. Then $P_{Mu}(f_u^M) = \eta_u^M f_u^M$. Any limit point of (η_u) as u goes to 0 being a Mth root of the unit, η_u^M converges to 1 as u goes to 0. Let U be a small neighbourhood of 0, and let $(\lambda_u)_{u\in U}$ be the unique continuous family of eigenvalues of $(P_u)_{u\in U}$ such that $\lambda_0 = 1$. Then $\lambda_u = \eta_{u/M}^M$ for all small enough $u \in H$, so $|\lambda_u| = 1$ for all small enough $u \in H$.

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Let (f_u) be a continuous family of eigenfunctions of P_u for the eigenvalue λ_u, chosen with modulus 1. Then $\lim_{u\to 0} f_u = \mathbf{1}$ in Lip^∞ norm. Let $i\theta_u$ (respectively, R_u) be the main determination of the logarithm of f_u (respectively, λ_u).

As $u \mapsto \lambda_u^2$ is continuous and λ_u^2 is an eigenvalue of P_{2u} for all small enough $u \in H$, we have $\lambda_u^2 = \lambda_{2u}$ for all small enough $u \in H$, so $R_{2u} = 2R_u$ and $\theta_{2u} = 2\theta_u$. Fix some small enough non-zero $u \in H$. By [**69**, equation 2.10],

$$\theta_{2^{-n}u} \circ T - \theta_{2^{-n}u} = R_{2^{-n}u} + \langle 2^{-n}u, F\rangle \quad [2\pi\mathbb{Z}^d],$$

whence, multiplying by 2^n

$$\theta_u \circ T - \theta_u = R_u + \langle u, F\rangle \quad [2^{n+1}\pi\mathbb{Z}^d].$$

As n goes to infinity, we get $\theta_u \circ T - \theta_u = R_u + \langle u, F\rangle$ almost everywhere. Note that θ_u is bounded. If R_u is non-zero, then $(\langle u, S_n\rangle)_{n\geq 0}$ diverges almost surely, so $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ is not recurrent. If $R_u = 0$, then $(\langle u, S_n\rangle)_{n\geq 0}$ is bounded almost surely with $u \neq 0$, so $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ is not ergodic, which ends the proof of the last point. □

We prove the following.

PROPOSITION 4.11. *Let $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ be an aperiodic Markovian $\mathbb{Z}^d$-extension of a Gibbs–Markov map $(A, \pi, \lambda, \mu, T)$ with step function F. Assume that the extension is ergodic, conservative, and either of the following hypotheses:*

- *$d = 1$ and F is in the domain of attraction of an α-stable distribution, with $\alpha \in (1, 2]$.*
- *$d = 1$ and $\int_A e^{iuF}\, d\mu = e^{-\vartheta|u|[1-i\zeta\,\mathrm{sgn}(u)]L(|u|^{-1})} + o(|u|L(|u|^{-1}))$ at 0, for some real numbers $\vartheta > 0$ and $\zeta \in \mathbb{R}$ and some function L with slow variation.*
- *$d = 2$ and F is in the domain of attraction of a non-degenerate Gaussian random variable.*

Then Hypothesis 3.1 is satisfied with $\mathcal{B} := \mathrm{Lip}^\infty$.

Proof. The recurrence of the extension is among the hypotheses. Since the extension is ergodic, so is (A, μ, T). The existence of an integer $M \geq 1$ and a decomposition of A into M measurable sub-sets $(A_i)_{i\in\mathbb{Z}/M\mathbb{Z}}$ on which T^M is mixing follows [**26**, Théorème 1.1.8].

We choose the Banach space $\mathrm{Lip}^\infty \subset \mathbb{L}^\infty(A, \mu) \subset \mathbb{L}^1(A, \mu)$. Then $\mathbf{1} \in \mathrm{Lip}^\infty$, and P acts continuously on Lip^∞. In addition, the sub-sets A_i are $\sigma(\pi)$-measurable, so for all $f \in \mathrm{Lip}^\infty$:

$$\|\mathbf{1}_{A_i} f\|_{\mathrm{Lip}^\infty(A,\pi,d,\mu)} = \|\mathbf{1}_{A_i} f\|_{\mathbb{L}^\infty(A,\mu)} + \sup_{a\in\pi} |\mathbf{1}_{A_i} f|_{\mathrm{Lip}(a,d)} \leq \|f\|_{\mathrm{Lip}^\infty(A,\pi,d,\mu)},$$

so the multiplication by $\mathbf{1}_{A_i}$ acts continuously on Lip^∞.

We use Proposition 3.2 to check the third item. The function F is constant on elements of the partition π, so, with the notation of [**26**] $D_\tau f(a) \equiv 0$. Hence, by [**26**, Corollaire 4.1.3], the application $u \mapsto P_u$, as a function with values in $\mathcal{L}(\mathrm{Lip}^\infty, \mathrm{Lip}^\infty)$, is continuous in 0. But multiplication by $e^{i\langle u,F\rangle}$ is continuous on Lip^∞, and $P_v(f) - P_u(f) = (P_{v-u} - P)(e^{i\langle u,F\rangle} f)$. Hence, $u \mapsto P_u$ is continuous for all u.

The action of P on Lip^∞ is quasi-compact: the spectrum of P is included in the closed unit ball, its intersection with the unit circle is exactly the set of Mth roots of the unity, and the remainder of the spectrum lies in a ball of smaller radius. Hence, the eigendecomposition of P is continuous for small parameters u. The hypotheses of Proposition 3.2 follow, except for the last one (that P_u has no eigenvalue of modulus one for $u \neq 0$).

Since the extension is assumed to be aperiodic, the spectral radius of P_u acting on Lip^∞ is strictly smaller than 1 for $u \neq 0$, by Lemma 4.10. We have checked all the assumptions of Proposition 3.2, and thus the third item of Hypothesis 3.1.

The expansion of the main eigenvalue for Gibbs–Markov maps is done in [**4**] in the one-dimensional case. If $F \in \mathbb{L}^2$, then it is an instance of the central limit theorem by spectral methods, as in [**48**]; otherwise, the expansion ultimately satisfies

$$1 - \lambda_u \sim 1 - \int_A e^{i\langle u, F\rangle}\, d\mu, \tag{4.5}$$

and the formulas come from [**24**].

Note that, if $\alpha \in (1, 2]$, Birkhoff's theorem and the conservativity of the extension imply that F has no drift, which finishes this case. For $\alpha = 1$, the expansion of $\int_A e^{iuF}\, d\mu$ is part of the hypothesis. □

4.3. *Tightness.* In this sub-section and the next, for any metric space (E, d), any $x \in E$ and any $R > 0$, we write $\overline{B}_E(x, R)$ for the closed ball in (E, d) of center x and radius R, and $S_E(x, R)$ for the corresponding sphere.

Recall that, for all $p \in G$, for all $x \in A$, we put $N_p(x) = |\{0 \leq k < \varphi_{\{0\}}(x) : \widetilde{T}^k(x, 0) \in A \times \{p\}\}|$ and $N_{0,p}(x) = \inf\{n \geq 0 : T^n_{\{0\}}(x) \in \{N_p > 0\}\}$. The goal of this section is to obtain an upper bound for the tail distribution of $N_{0,p}$. This estimate will be used later to prove the tightness of $\alpha(p)^{-1} N_p$ given $\{N_p > 0\}$.

Since T is ergodic, we consider $M \in \mathbb{N}_+$ and $(A_k)_{k \in \mathbb{Z}/M\mathbb{Z}}$ as in Proposition 4.4. For all $k \in \mathbb{Z}/M\mathbb{Z}$ and $f \in \mathbb{L}^1(A, \mu)$, let Π_k be the projection $f \mapsto \int_{A_k} f\, d\mu \cdot \mathbf{1}_{A_k}$. For all $K > 0$, we set

$$\mathcal{S}_K := \{h : A \to [0, 1]\} \cap \overline{B}_{\mathrm{Lip}^\infty(A)}(\mathbf{0}, K).$$

PROPOSITION 4.12. *Let $(A, \pi, \lambda, \mu, T)$ be an ergodic Gibbs–Markov map. Then for all $K > 0$, there exist constants $C, \kappa > 0$ such that for all $h \in \mathcal{S}_K$, for all $n \geq 0$,*

$$\| P_h^n(\mathbf{1}) \|_{\mathbb{L}^1(A,\mu)} \leq C e^{-\kappa \|\mathbf{1} - h\|_{\mathbb{L}^1(A,\mu)} n}. \tag{4.6}$$

Proof. First, let us assume that (A, μ, T) is mixing. We only need to prove the assertion for $K \geq 1$. Let $h \in \mathcal{S}_K$.

If $h < 1/2$ somewhere, since $\mathcal{S}_K$ is convex and $\mathbf{1} \in \mathcal{S}_K$, the function $h' := (\mathbf{1} + h)/2$ also belongs to $\mathcal{S}_K$ and satisfies $h' \geq 1/2$. In addition, $P_h^n(\mathbf{1}) \leq P_{h'}^n(\mathbf{1})$ for all n, so any upper bound for $\| P_{h'}^n(\mathbf{1}) \|_{\mathbb{L}^1(A,\mu)}$ is also an upper bound for $\| P_h^n(\mathbf{1}) \|_{\mathbb{L}^1(A,\mu)}$. Moreover, $\|\mathbf{1} - h\|_{\mathbb{L}^1(A,\mu)} = 2\|\mathbf{1} - h'\|_{\mathbb{L}^1(A,\mu)}$. Hence, if we get the bound (4.6) for h', up to dividing κ by 2, we also get the bound (4.6) for h. Hence, without loss of generality, we assume from now on that $h \geq 1/2$.

Let $f \in \overline{B}_{\mathrm{Lip}^\infty(A)}(\mathbf{1}, 1/2) \cap S_{\mathbb{L}^1(A,\mu)}(\mathbf{0}, 1)$. Then, on the one hand, for all $h \in \mathcal{S}_K$,

$$\left| \int_A P_h(f)\, d\mu \right| = \left| \int_A hf\, d\mu \right| \geq \int_A h\, d\mu - \int_A h|\mathbf{1} - f|\, d\mu \geq \frac{\|h\|_{\mathbb{L}^1(A,\mu)}}{2}. \tag{4.7}$$

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On the other hand,

$$\left| \int_A P_h(f)\, d\mu \right| \le \int_A |f|\, d\mu + \int_A (\mathbf{1}-h)|\mathbf{1}-f|\, d\mu - \int_A (\mathbf{1}-h)\, d\mu$$
$$\le \|f\|_{\mathbb{L}^1(A,\mu)} - \frac{\|\mathbf{1}-h\|_{\mathbb{L}^1(A,\mu)}}{2}$$
$$= 1 - \frac{\|\mathbf{1}-h\|_{\mathbb{L}^1(A,\mu)}}{2}. \tag{4.8}$$

From (4.7), we compute

$$\left\| \frac{P_h(f)}{\int_A P_h(f)\, d\mu} \right\|_{\mathrm{Lip}^\infty(A)} \le \frac{2\|P\|_{L(\mathrm{Lip}^\infty(A))} \|hf\|_{\mathrm{Lip}^\infty(A)}}{\|h\|_{\mathbb{L}^1(A,\mu)}}$$
$$\le \frac{2\|P\|_{L(\mathrm{Lip}^\infty(A))} K}{\|h\|_{\mathbb{L}^1(A,\mu)}} \|f\|_{\mathrm{Lip}^\infty(A)}$$
$$\le 4\|P\|_{L(\mathrm{Lip}^\infty(A))} K \|f\|_{\mathrm{Lip}^\infty(A)}$$
$$\le 6\|P\|_{L(\mathrm{Lip}^\infty(A))} K.$$

Due to Proposition 4.6, there exists $m \ge 1$ such that, for any h fitting our assumptions, for all $f \in \overline{B}_{\mathrm{Lip}^\infty(A)}(\mathbf{1}, 1/2)$,

$$\left\| \frac{P^{m-1} P_h(f)}{\int_A P_h(f)\, d\mu} - \mathbf{1} \right\|_{\mathrm{Lip}^\infty(A)} \le \frac{1}{2}.$$

We fix such a value of m. Then, the following map is well-defined:

$$F : \begin{cases} \overline{B}_{\mathrm{Lip}^\infty(A)}(\mathbf{1}, 1/2) \cap S_{\mathbb{L}^1(A,\mu)}(\mathbf{0}, 1) & \to \quad \overline{B}_{\mathrm{Lip}^\infty(A)}(\mathbf{1}, 1/2) \cap S_{\mathbb{L}^1(A,\mu)}(\mathbf{0}, 1) \\ f & \mapsto \quad \dfrac{P^{m-1} P_h(f)}{\int_A P_h(f)\, d\mu}. \end{cases}$$

Furthermore, by virtue of (4.8), for all $n \ge 0$,

$$\left| \int_A (P^{m-1} P_h)^n f\, d\mu \right| = \left| \int_A F^n(f)\, d\mu \cdot \prod_{k=0}^{n-1} \int_A P_h(F^k(f))\, d\mu \right|$$
$$\le \left(1 - \frac{\|\mathbf{1}-h\|_{\mathbb{L}^1(A,\mu)}}{2} \right)^n.$$

Remark that $0 \le P_h^n f \le P^n f$ for all non-negative $f \in \mathbb{L}^1$, for all $n \ge 0$ and all $h \in \mathcal{S}_K$. In addition, F preserves the sub-set of real-valued functions. Fix $h \in \mathcal{S}_K$. Then, for all $n \ge 0$,

$$0 \le \int_A P_h^{nm}(\mathbf{1})\, d\mu \le \int_A (P^{m-1} P_h)^n (\mathbf{1})\, d\mu$$
$$\le \left(1 - \frac{\|\mathbf{1}-h\|_{\mathbb{L}^1(A,\mu)}}{2} \right)^n \le e^{-(\|\mathbf{1}-h\|_{\mathbb{L}^1(A,\mu)}/2)n},$$

so that

$$\|P_h^n(\mathbf{1})\|_{\mathbb{L}^1(A,\mu)} \le \sqrt{e} \max_{0 \le k < m} \sup_{h \in \mathcal{S}_K} \|P_h^k\|_{L(\mathrm{Lip}^\infty(A))} e^{-(\|\mathbf{1}-h\|_{\mathbb{L}^1(A,\mu)}/2m)n}.$$

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We have proved that the conclusion of the lemma holds if (A, μ, T) is assumed to be mixing.

Finally, assume that (A, μ, T) is ergodic but not necessarily mixing. Let $(A_k)_{k\in\mathbb{Z}/M\mathbb{Z}}$ be its decomposition in components on which T^M is mixing, and write $\mu_k := M\mu_{|A_k}$. Let $K \geq 0$, and let $h \in \mathcal{S}_K$. Let k_0 be such that $\|\mathbf{1} - h\|_{\mathbb{L}^1(A_{k_0},\mu_{k_0})} \geq \|\mathbf{1} - h\|_{\mathbb{L}^1(A,\mu)}$. Note that $h \cdot \mathbf{1}_{A_{k_0}}$ is in $\mathcal{S}_K$ when we replace $\mathrm{Lip}^\infty(A, \pi, \lambda)$ by $\mathrm{Lip}^\infty(A_{k_0}, \pi_M, \lambda)$. Let $\tilde{P}_h(f) := P^M(hf)$ for $f \in \mathrm{Lip}^\infty(A_{k_0}, \pi_M, \lambda)$. Then, there exist positive constants C_0, κ_0 depending only on K such that, for all $n \geq 0$,

$$\|\tilde{P}_h^n(\mathbf{1})\|_{\mathbb{L}^1(A_{k_0},\mu_{k_0})} \leq C_0 e^{-\kappa_0 \|\mathbf{1}-h\|_{\mathbb{L}^1(A_{k_0},\mu_{k_0})} n} \leq C_0 e^{-\kappa_0 \|\mathbf{1}-h\|_{\mathbb{L}^1(A,\mu)} n}.$$

But then, for all $k \in \mathbb{Z}/M\mathbb{Z}$, for all $n \geq 1$,

$$\begin{aligned}\|P_h^{nM}(\mathbf{1})\|_{\mathbb{L}^1(A_k,\mu_k)} &\leq \|P_h^{(n-1)M}(\mathbf{1})\|_{\mathbb{L}^1(A_{k_0},\mu_{k_0})} \\ &\leq \|\tilde{P}_h^{n-1}(\mathbf{1})\|_{\mathbb{L}^1(A_{k_0},\mu_{k_0})} \\ &\leq C_0 e^{-\kappa_0 \|\mathbf{1}-h\|_{\mathbb{L}^1(A,\mu)}(n-1)} \\ &\leq C_0 e^{\kappa_0} e^{-\kappa_0 \|\mathbf{1}-h\|_{\mathbb{L}^1(A,\mu)} n},\end{aligned}$$

so that, for all $n \geq 0$:

$$\|P_h^n(\mathbf{1})\|_{\mathbb{L}^1(A,\mu)} = \frac{1}{M} \sum_{k\in\mathbb{Z}/M\mathbb{Z}} \|P_h^n(\mathbf{1})\|_{\mathbb{L}^1(A_k,\mu_k)} \leq C_0 e^{2\kappa_0} e^{-(\kappa_0/M)\|\mathbf{1}-h\|_{\mathbb{L}^1(A,\mu)} n}. \qquad \square$$

Proposition 4.12 yields an upper bound on the probability that the orbits do not visit a given sub-set of A before a given time.

COROLLARY 4.13. *Let $(A, \pi, \lambda, \mu, T)$ be an ergodic Gibbs–Markov map. Let G be a set, and $(a_p)_{p\in G}$ be a family of non-trivial $\sigma(\pi)$-measurable sub-sets. Let $C, \kappa > 0$ be constants associated with $\mathcal{S}_1$ in Proposition 4.12. Let $K > 0$. Let $(\mu_p)_{p\in G}$ be a family of probability measures on A such that $\mu_p \ll \mu$ and $\|d\mu_p/d\mu\|_{\mathrm{Lip}^\infty(A)} \leq K$ for all p. Then, for all $n \geq 0$ and all $p \in G$,*

$$\mu_p\left(\bigcap_{k=0}^{n-1}\{T^k(x) \notin a_p\}\right) \leq KCe^{-\kappa\mu(a_p)n}. \tag{4.9}$$

Proof. We compute

$$\begin{aligned}\mu_p\left(\bigcap_{k=0}^{n-1}\{T^k(x) \notin a_p\}\right) &= \int_A \prod_{k=0}^{n-1} \mathbf{1}_{a_p^c} \circ T^k \cdot \frac{d\mu_p}{d\mu}\, d\mu \\ &\leq K \int_A P^n_{\mathbf{1}-\mathbf{1}_{a_p}}(\mathbf{1})\, d\mu.\end{aligned}$$

But $\mathbf{1} - \mathbf{1}_{a_p} \in \mathcal{S}_1$ for all p. All remains is to use Proposition 4.12. $\square$

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4.4. *Convergence in distribution.* Let (A, μ, T) be a sufficiently hyperbolic measure-preserving dynamical system, and let (A_p) be a family of measurable sub-sets such that $\lim_{p\to\infty} \mu(A_p) = 0$. Let φ_p be the first hitting time of A_p. As p goes to infinity, hitting this set becomes a rare event. Knowing that a trajectory has not hit the set until some time gives us little information about later times, which implies that any limit distribution exhibits a loss of memory characteristic of the exponential distributions. Hence, one can usually prove that $\mu(A_p)\varphi_p$ converges in distribution to a exponential random variable of parameter 1. There is an extensive literature on the subject; we refer the interested reader to the reviews [**17**, **30**, **58**]. Note that this family of results can usually be strengthened, for instance to show convergence to a Poisson process [**57**, Théorème 3.6]. More promisingly, there are also ways to get a rate of convergence [**25**], which may be adapted to get rates of convergence in Theorem 2.7.

In the previous sub-section, we showed that, under any probability measure with uniformly bounded density, the tail of the hitting time of a $\sigma(\pi)$-measurable set decays exponentially, at a speed which is at most inversely proportional to the size of the set. Now, we shall prove that, as the size of the sets goes to 0, the distribution of the renormalized hitting time is asymptotically exponential. This is the content of Proposition 4.14. Due to some specificities of our situation (the hitting sets are not exactly cylinders, and the measure changes with the sets), we prove the convergence ourselves, instead of using some already established theorem.

Afterwards, we shall prove Lemma 4.16, which is useful in the proof of Theorem 2.11 and whose proof uses ideas very similar to the proof of Proposition 4.14.

PROPOSITION 4.14. *Let $(A, \pi, \lambda, \mu, T)$ be an ergodic Gibbs–Markov map. Let G be a locally compact space, and $(a_p)_{p\in G}$ be a family of non-trivial $\sigma(\pi)$-measurable sub-sets such that* $\lim_{p\to\infty} \mu(a_p) = 0$. *For all $p \in G$ and $x \in A$, let $N_p(x) := \inf\{k \geq 0 : T^k(x) \in a_p\}$. Let $(\mu_p)_{p\in G}$ be a family of probability measures on A such that $\mu_p \ll \mu$ for all p, and*

$$\sup_{p\in G} \left\| \frac{d\mu_p}{d\mu} \right\|_{\mathrm{Lip}^\infty(A)} < +\infty.$$

Then the family of random variables $(\mu(a_p)N_p)_{p\in G}$ defined on the probability space (A, μ_p) converges in distribution to an exponential random variable of parameter 1.

Proof. At first, we assume that the system is mixing. We work with the distribution function $\mathbf{1} - F_{p,f}$ of N_p under the distribution $f\, d\mu$, that is, for all $t \geq 0$:

$$F_{p,f}(t) = \int_A \mathbf{1}_{\{N_p\geq t\}} f\, d\mu.$$

In a first step, we prove that $F_{p,f}$ does not depend too much on the density f. This will imply the loss of memory: in the second step, we prove that any limit distribution of $\mu(a_p)N_p$ is exponential, and that the limit points do not depend on the choice of f. Then, we have to identify the parameter of the limit distribution, which is done in the third and fourth steps. In the third step, we prove that some $\mathbb{Z}/2\mathbb{Z}$-extension of the system is ergodic, at least for large p's and, in the fourth step, we use Kac's formula to prove that, for a good choice of f (depending on p), the expectation of $\mu(a_p)N_p$ is 1. Finally, in the last step we

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extend this result to dynamical systems, which are merely ergodic. We assume in the first four steps that (A, μ, T) is mixing.

Step 1 (mixing case): Loss of memory. First, let us prove that $F_{p,f}$ does not depend on f as p goes to infinity. Let $h_p := \mathbf{1} - \mathbf{1}_{a_p}$. Then $h_p \in \mathcal{S}_1$ for all p. Let $K \geq 1$. Let $f \in \overline{B}_{\mathrm{Lip}^\infty(A)}(\mathbf{0}, K)$ with $f \geq 0$ and $\int_A f \, d\mu = 1$. Let $n, k \in \mathbb{N}$ and $p \in G$. Note that $F_{p,f}(n) = \|P^n_{h_p}(f)\|_{\mathbb{L}^1(A,\mu)}$. Since each P_{h_p} is a weak contraction when acting on $\mathbb{L}^1(A, \mu)$,

$$|F_{p,f}(n+k) - F_{p,P^k f}(n)| = \left| \int_A P^{n+k}_{h_p}(f) - P^n_{h_p}(P^k f) \, d\mu \right|$$
$$\leq \left| \int_A P^k_{h_p}(f) - P^k f \, d\mu \right|$$
$$\leq \mathbf{1} - F_{p,f}(k).$$

In addition,

$$|F_{p,P^k f}(n) - F_{p,1}(n)| \leq \|P^k f - 1\|_{\mathbb{L}^\infty} \leq K \|P^k - \Pi_0\|_{L(\mathrm{Lip}^\infty(A))},$$

and

$$\mathbf{1} - F_{p,f}(k) \leq k \|f\|_{\mathbb{L}^\infty(A,\mu)} \mu(a_p) \leq K k \mu(a_p).$$

Hence, we finally get

$$|F_{p,f}(n+k) - F_{p,1}(n)| \leq K k \mu(a_p) + K \|P^k - \Pi_0\|_{L(\mathrm{Lip}^\infty(A))}.$$

Since (A, μ, T) is a mixing Gibbs–Markov map, $\|P^k - \Pi_0\|_{L(\mathrm{Lip}^\infty(A))}$ converges to 0 as k goes to infinity (Proposition 4.6). Taking $n = \lfloor \mu(a_p)^{-1} t \rfloor$ and $k := \lfloor \sqrt{\mu(a_p)^{-1}} \rfloor$ yields

$$F_{p,f}\left(\lfloor \mu(a_p)^{-1} t \rfloor + \left\lfloor \sqrt{\mu(a_p)} \right\rfloor \right) = F_{p,1}(\lfloor \mu(a_p)^{-1} t \rfloor) + o(1) \quad \text{as } p \to \infty, \tag{4.10}$$

uniformly for f in $\overline{B}_{\mathrm{Lip}^\infty(A)}(0, K)$ and $t \geq 0$.

Step 2 (mixing case): Limit distributions. Now, we prove that any limit distribution of $\mu(a_p) N_p$ is δ_0 or exponential, and that the limit distributions do not depend on the choice of the measures μ_p. For every $p \in G$, we set g_p for the density of μ_p with respect to μ. By Corollary 4.13, there exist positive constants C, κ such that, for all $t \geq 0$ and for all $p \in G$,

$$\mu_p(\mu(a_p) N_p \geq t) = \int_A \mathbf{1}_{\{\mu(a_p) N_p \geq t\}} g_p \, d\mu \leq C K e^{-\kappa t}.$$

Hence, the sequence $(\mu(a_p) N_p)_{p \in G}$ defined on (A, μ_p) is tight. Let F be the tail distribution function of one of its limit points, and let $G_F \subset G$ be such that the distribution function of $\mu(a_p) N_p$ converges to F for $p \in G_F$. By equation (4.10), F does not depend on f. Note that F is non-increasing and càdlàg.

If $F(t) = 0$ for all $t > 0$, then the limit distribution is δ_0, and we are done. Let us assume that there exists $T > 0$ with $F(T) > 0$, and let $t \in [0, T)$. Then $F_{p,1}(\lceil \mu(a_p)^{-1} t \rceil) > 0$ for all large enough $p \in G_F$. We apply Lemma 4.8 with the stopping time $n_p(t) := \lceil \mu(a_p)^{-1} t \rceil$ and the event $A := \bigcap_{k=0}^{n_p(t)-1} T^{-k} a_p^c$, which has positive probability if p is

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large enough. There exists a constant K' such that $P_{h_p}^{n_p(t)}(\mathbf{1})/F_{p,\mathbf{1}}(n_p(t))$ belongs to $\overline{B}_{\mathrm{Lip}^\infty(A)}(\mathbf{0}, K')$ for all large enough p. But then, for all $k \in \mathbb{N}_+$ and for all $p \in G_F$:

$$F_{p,\mathbf{1}}\Big(n_p(t) + \Big\lfloor \sqrt{\mu(a_p)^{-1}} \Big\rfloor + k\Big)$$
$$= F_{p,\mathbf{1}}(n_p(t)) \cdot F_{p,(P_{h_p}^{n_p(t)}(\mathbf{1}))/(F_{p,\mathbf{1}}(n_p(t)))}\Big(\Big\lfloor \sqrt{\mu(a_p)^{-1}} \Big\rfloor + k\Big).$$

Let $t' \geq 0$ and $k = \lceil \mu(a_p)^{-1} t' \rceil$. Letting p go to infinity in G_F, by equation (4.10),

$$F(t + t') = F(t)F(t').$$

In addition, trivially, $F = 1$ on $\mathbb{R}_-$. Hence, $\mathbf{1} - F$ is the distribution function of an exponential random variable with parameter in $[0, \infty]$.

Step 3 (mixing case): Ergodicity of a $\mathbb{Z}/2\mathbb{Z}$-extension. We have proved that any limit distribution of $(\mu(a_p)N_p)_{p\in G}$ is exponential; now, we show that its parameter must be 1. To this end, we first prove that a certain $\mathbb{Z}/2\mathbb{Z}$-extension is ergodic. This fact shall allow us to apply Kac's formula in the next step, and from there to identify the parameter of the limit exponential distribution.

Consider the dynamical system:

$$T_p : \begin{cases} A \times \mathbb{Z}/2\mathbb{Z} & \to \quad A \times \mathbb{Z}/2\mathbb{Z} \\ (x, q) & \mapsto \quad \begin{cases} (T(x), q) & \text{if } x \notin a_p, \\ (T(x), q+1) & \text{otherwise.} \end{cases} \end{cases}$$

Let π_p be the canonical projection from $A \times \mathbb{Z}/2\mathbb{Z}$ onto A, which is a factor map. We shall prove that this extension is ergodic for all large enough p. The idea is that otherwise, we could divide A into two sub-sets, which communicate only through a_p; as the a_p get smaller, this would make the communication more difficult, and the mixing arbitrarily slow, which is absurd.

Assume that $(A \times \mathbb{Z}/2\mathbb{Z}, \mu \otimes (\delta_0 + \delta_1)/2, T_p)$ is not ergodic. Let I_p be a T_p-invariant non-trivial measurable sub-set. Then, since $\pi_p(I_p) = \pi_p \circ T_p(I_p) = T \circ \pi_p(I_p)$, we see that $\pi_p(I_p)$ is a non-trivial T-invariant sub-set, so $\pi_p(I_p) = A$. Doing the same with I_p^c, we see that there exists a measurable partition $(I_{p,0}, I_{p,1})$ of A such that $I_p = I_{p,0} \times \{0\} \cup I_{p,1} \times \{1\}$. In addition, neither $A \times \{0\}$ nor $A \times \{1\}$ are T_p-invariant, so I_p cannot be either, and neither $I_{p,0}$ nor $I_{p,1}$ are trivial. Finally, since the $\mathbb{Z}/2\mathbb{Z}$-extension is still Gibbs–Markov, its partition into ergodic components is coarser than its underlying partition, so both $I_{p,0}$ and $I_{p,1}$ are $\sigma(\pi)$-measurable.

The map T_p sends $I_{p,0} \cap a_p$ into $I_{p,1}$ and $I_{p,1} \cap a_p$ into $I_{p,0}$. By the big image property of Gibbs–Markov maps, there exists a constant $m > 0$ such that $\mu(I_{p,i}) \geq m$ for all $p \in G$ and $i \in \mathbb{Z}/2\mathbb{Z}$. Let $f_p := \mu(I_{p,0})^{-1}\mathbf{1}_{I_{p,0}}$. Then $(f_p)_{p\in G}$ is uniformly bounded in $\mathrm{Lip}^\infty(A)$ by m^{-1}. Hence, there exist constants $C', \kappa' > 0$ such that $\|P^n f_p - \mathbf{1}\|_{\mathrm{Lip}^\infty(A)} \leq C' e^{-\kappa' n}$ for all p, n. Hence,

$$\int_{I_{p,1}} P^n f_p \, d\mu \geq m(1 - C' e^{-\kappa' n}).$$

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But we know that

$$\int_{I_{p,1}} P^n f_p \, d\mu = \mu(T^{-n} I_{p,1} | I_{p,0}) \leq \sum_{k=0}^{n-1} \mu(T^{-(k+1)} I_{p,1} \mid T^{-k} I_{p,0})$$
$$\leq n\mu(T^{-1} I_{p,1} | I_{p,0}) \leq n \frac{\mu(a_p \cap I_{p,0})}{\mu(I_{p,0})} \leq m^{-1} \mu(a_p) n.$$

There is a contradiction for some $n \geq 0$ and all large enough $p \in G$.

Step 4 (mixing case): Computation of the parameter of the exponential distribution. Now, let us apply Kac's formula. For all large enough p, the system $(A \times \mathbb{Z}/2\mathbb{Z}, \mu \otimes (\delta_0 + \delta_1)/2, T_p)$ is ergodic. Let φ_p be the first return time for T_p to $A \times \{0\}$ starting from $A \times \{0\}$. By Kac's formula,

$$\int_A \varphi_p \, d\mu = 2.$$

But $\varphi_p \equiv 1$ on a_p^c, and $\varphi_p \equiv 1 + N_p \circ T$ on a_p. Hence,

$$1 = \int_{a_p} N_p \circ T \, d\mu = \int_A N_p \cdot P(\mathbf{1}_{a_p}) \, d\mu = \int_A (\mu(a_p) N_p) \cdot \frac{P(\mathbf{1}_{a_p})}{\mu(a_p)} \, d\mu.$$

Let X be a limit in distribution of $(\mu(a_p)N_p)_{p \in G}$, and let $G_X \subset G$ be such that $(\mu(a_p)N_p)_{p \in G_X}$ converges to X in distribution. We already know that X has an exponential distribution of parameter at most κ. By Lemma 4.8, using the stopping time 1, there exists a constant K such that, for all $p \in G$, the density $P(\mathbf{1}_{a_p})/\mu(a_p)$ lies in $\overline{B}_{\mathrm{Lip}^\infty(A)}(0, K)$. Hence, due to (4.10), the limit distribution of $(\mu(a_p)N_p)$ on $(A, \mu(a_p)^{-1} P(\mathbf{1}_{a_p}))$ is the limit distribution of $(\mu(a_p)N_p)$ on (A, μ), that is, the distribution of X. Furthermore, the tail of $(\mu(a_p)N_p)$ on $(A, \mu(a_p)^{-1} P(\mathbf{1}_{a_p}))$ is dominated by a decaying exponential, so all the moments converge to those of X. In particular, $\mathbb{E}[X] = 1$, so X follows an exponential distribution of parameter 1.

Step 5: General case. We have proved the proposition under the assumption that (A, μ, T) is mixing. Now, let us assume that the system is only ergodic, but not mixing. Let $M \geq 1$ and $(A_k)_{k \in \mathbb{Z}/M\mathbb{Z}}$ be as in Proposition 4.4. Let (a_p, ν_p) be a sequence satisfying the hypotheses of the proposition. Let $k \in \mathbb{Z}/M\mathbb{Z}$, and let $(\overline{\nu}_p)$ be a sequence of probability measures on A_k, absolutely continuous with respect to $\mu_k := \mu(\cdot | A_k) = M\mu(\cdot \cap A_k)$, and with densities uniformly bounded in $\mathrm{Lip}^\infty(A_k, \pi_M, \lambda)$. We define

$$\overline{a}_p := \{x \in A_k : \exists 0 \leq i < M, T^i(x) \in a_p\} \in \pi_M.$$

Note that $\mu_k(\overline{a}_p) \leq M \sum_{i=0}^{M-1} \mu(A_{k+i} \cap a_p) = M\mu(a_p)$. Let $0 \leq i_1 < i_2 < M$. Then, by [**26**, Lemme 1.1.13], $P^{i_2 - i_1}$ maps continuously Lip^1 into Lip^∞, and

$$\mu_k(T^{-i_1}(a_p \cap A_{k+i_1}) \cap T^{-i_2}(a_p \cap A_{k+i_2}))$$
$$\leq \int_{A_{k+i_1}} P^{i_2 - i_1} \mathbf{1}_{a_p \cap A_{k+i_1}} \cdot \mathbf{1}_{a_p \cap A_{k+i_2}} \, d\mu_{k+i_1}$$
$$\leq C\mu_{k+i_1}(a_p)\mu_{k+i_2}(a_p).$$

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and so, by Bonferroni's inequality,

$$\begin{aligned}\mu_k(\overline{a}_p) &\geq \sum_{0\leq i<M} \mu_k(T^{-i}(a_p \cap A_{k+i})) \\ &\quad - \sum_{0\leq i_1<i_2<M} \mu_k(T^{-i_1}(a_p \cap A_{k+i_1}) \cap T^{-i_2}(a_p \cap A_{k+i_2})) \\ &\geq M\mu(a_p) - \frac{CM^4}{2}\mu(a_p)^2.\end{aligned}$$

Hence, $\mu_k(\overline{a}_p) \sim M\mu(a_p)$.

Let $\overline{N}_p$ be the first hitting time of $\overline{a}_p$ for T^M. Note that $|N_p - M\overline{N}_p| \leq M-1$ on A_k. Since Proposition 4.14 holds for mixing transformations, the sequence $(\mu_k(\overline{a}_p)\overline{N}_p)_{p\in G}$ defined on $(A_k, \overline{\nu}_p)$ converges in distribution to an exponential random variable of parameter 1. But $\mu_k(\overline{a}_p) \sim M\mu(a_p)$ and $M\overline{N}_p = N_p + O(1)$, so $(\mu(a_p)N_p)_{p\in G}$ defined on $(A_k, \overline{\nu}_p)$ converges in distribution to the same exponential random variable of parameter 1.

Finally, let $(\nu_p)_p$ be a sequence of probability measures on A whose densities $(h_p)_p$ with respect to μ are bounded in $\mathrm{Lip}^\infty(A, \pi, \lambda)$. For any $x \in A$, let $0 \leq i < M$ be such that $T^i(x) \in A_k$, and set $P(x) := (x, T^i(x)) \in A \times A_k$. Then $\nu_p \mapsto \bar{\nu}_p := P_*\nu_p$ is a transference plan between ν_p and a probability measure $\overline{\nu}_p$ on A_k, with density:

$$\overline{h}_p := \frac{d\overline{\nu}_p}{d\mu_k} = \frac{1}{M}\sum_{i=0}^{M-1} P^i(\mathbf{1}_{A_{k-i}} h_p).$$

This transference plan yields a coupling between N_p (seen as a random variable on (A, ν_p)) and N_p (seen as a random variable on $(A_k, \overline{\nu}_p)$). For the sake of clarity, we shall call the second random variable $\widetilde{N}_p$.

The sequence $(\overline{h}_p)$ is bounded in $\mathrm{Lip}^\infty(A_k, \pi_M, \lambda)$. Hence, $(\mu(a_p)\widetilde{N}_p)_{p\in G}$ converges in distribution to an exponential random variable of parameter 1.

Let $x \in A$. Let $0 \leq i < M$ be such that $T^i(x) \in A_k$. If $N_p(x) \geq i$, then $N_p(x) = i + N_p(T^i(x))$, so $N_p = i + \widetilde{N}_p$. The event $\{N_p < i\}$ has probability $O(\mu(a_p))$, and $|\mu(a_p)N_p - \mu(a_p)\tilde{N}_p| \leq M\mu(a_p)$ outside of this event, so $(\mu(a_p)N_p)_{p\in G}$ has the same limit in distribution as $(\mu(a_p)\widetilde{N}_p)_{p\in G}$. □

Remark 4.15. In our applications, a_p will be the set of points $x \in A$ such that the trajectory $(S_nF(x))_{n\geq 0}$ of $(x, 0)$ under the action of $\widetilde{T}$ goes to $A \times \{p\}$ before coming back to $A \times \{0\}$. If the $\mathbb{Z}^d$-extension is ergodic, then the $\mathbb{Z}/2\mathbb{Z}$-extension used in the proof is also automatically ergodic, as it is the induced system on $A \times \{0, p\}$. Hence, the stage in the proof above where we proved that such a $\mathbb{Z}/2\mathbb{Z}$-extension is ergodic for all large enough n is not necessary for our applications. This detour however made for a cleaner and more general statement in the proposition.

The following lemma allows us to control the $\mathbb{L}^q(A, \mu)$ norm of the Birkhoff sum of an observable until N_p.

LEMMA 4.16. *Let $(A, \pi, \lambda, \mu, T)$ be an ergodic Gibbs–Markov map. Let G be a locally compact space, and $(a_p)_{p\in G}$ be a family of non-trivial $\sigma(\pi)$-measurable sub-sets such* *that* $\lim_{p\to\infty}\mu(a_p)=0$. *For all* $p\in G$ *and* $x\in A$, *let* $N_p(x):=\inf\{k\ge 0: T^k(x)\in a_p\}$. *Let* $(\mu_p)_{p\in G}$ *be a family of probability measures on* A *such that* $\mu_p \ll \mu$ *for all* p. *Let* $C>1$. *Then for all* $q\in[1,\infty)$, *for all* $f\in\mathbb{L}^q(A,\mu)$, *for all large enough* $p\in G$,

$$\left\|\sum_{k=0}^{N_p-1} f\circ T^k\right\|_{\mathbb{L}^q(A,\mu_p)} \le Cq\alpha(p)\sup_{p\in G}\left\|\frac{d\mu_p}{d\mu}\right\|_{\mathbb{L}^\infty(A,\mu)}\|f\|_{\mathbb{L}^q(A,\mu)}. \tag{4.11}$$

Proof. Let $\varepsilon>0$. Let $f\in\mathbb{L}^q(A,\mu)$, which we can assume without loss of generality to be non-negative. Fix $p\in G$ and $\mathcal{N}>(1+\varepsilon)N>0$, such that εN is a multiple of the period M of the Gibbs–Markov map. Define $N'_p(x):=\inf\{n\ge 0: n\notin[N,2N),\ T^n(x)\in a_p\}\ge N_p$. Then

$$\begin{aligned}\left\|\sum_{k=0}^{\mathcal{N}\wedge N_p-1} f\circ T^k\right\|_{\mathbb{L}^q(A,\mu)} &\le \left\|\sum_{k=0}^{\mathcal{N}\wedge N'_p-1} f\circ T^k\right\|_{\mathbb{L}^q(A,\mu)}\\ &\le \left\|\sum_{k=0}^{((1+\varepsilon)N)\wedge N'_p-1} f\circ T^k\right\|_{\mathbb{L}^q(A,\mu)} + \left\|\sum_{k=(1+\varepsilon)N}^{\mathcal{N}\wedge N'_p-1} f\circ T^k\right\|_{\mathbb{L}^q(A,\mu)}\\ &\le (1+\varepsilon)N\|f\|_{\mathbb{L}^q(A,\mu)} + \left\|\sum_{k=2N}^{(2N+\mathcal{N})\wedge N'_p-1} f\circ T^k\right\|_{\mathbb{L}^q(A,\mu)}.\end{aligned}$$

Now, focus on the right-hand side. We get

$$\mathbb{E}_\mu\left[\left(\sum_{k=2N}^{((1+\varepsilon)N+\mathcal{N})\wedge N'_p-1} f\circ T^k\right)^p\right] = \mathbb{E}_\mu\left[\left(\sum_{k=0}^{\mathcal{N}\wedge N_p-1} f\circ T^k\right)^p P^{(1+\varepsilon)N}(\mathbf{1}_{N_p\ge N})\right].$$

By Lemma 4.8, applied to the stopping time whose value is $N-1$ if $N_p<N$ (and $+\infty$ otherwise), and to the set $A:=\{N_p<N\}$, we get $\|P^{N-1}(\mathbf{1}_{N_p\ge N})\|_{\mathrm{Lip}^\infty(A)}\le K$. Hence

$$\begin{aligned}\|P^{(1+\varepsilon)N}(\mathbf{1}_{N_p\ge N})\|_{\mathbb{L}^\infty(A,\mu)} &\le (1+KC\rho^{-(\varepsilon N/M)})\|P^{N-1}(\mathbf{1}_{N_p\ge N})\|_{\mathrm{Lip}^1(A)}\\ &= (1+KC\rho^{-(\varepsilon N/M)})\|\mathbf{1}_{N_p\ge N}\|_{\mathbb{L}^1(A,\mu)}\\ &= (1+KC\rho^{-(\varepsilon N/M)})\mu(N_p\ge N),\end{aligned}$$

whence

$$\begin{aligned}\left\|\sum_{k=0}^{\mathcal{N}\wedge N_p-1} f\circ T^k\right\|_{\mathbb{L}^q(A,\mu)} &\le (1+\varepsilon)N\|f\|_{\mathbb{L}^q(A,\mu)}\\ &\quad + (1+KC\rho^{-(\varepsilon N/M)})^{1/q}\mu(N_p\ge N)^{1/q}\left\|\sum_{k=0}^{\mathcal{N}\wedge N_p-1} f\circ T^k\right\|_{\mathbb{L}^q(A,\mu)}.\end{aligned}$$

We choose $N(p)\sim\varepsilon\alpha(p)$. Then $\rho^{-(\varepsilon^2 N(p)/M)}$ converges to 0, while by Proposition 4.14, $\mu(N_p\ge N(p))$ converges to $e^{-\varepsilon}<1$. For all large enough p, this yields

$$\left\|\sum_{k=0}^{\mathcal{N}\wedge N_p-1} f\circ T^k\right\|_{\mathbb{L}^q(A,\mu)} \le \frac{\varepsilon\alpha(p)(1+\varepsilon+o(1))}{1-e^{-(\varepsilon/q)}}\|f\|_{\mathbb{L}^q(A,\mu)}.$$

The $o(1)$ is independent from $\mathcal{N}$. We choose ε small enough that $\varepsilon(1+2\varepsilon) < Cq(1-e^{-(\varepsilon/q)})$, and then take the limit as $\mathcal{N}$ goes to infinity. Finally, notice that $d\mu_p/d\mu$ is uniformly bounded (in $\mathbb{L}^\infty(A,\mu)$ norm and in p), so that this inequality, up to the constant $\sup_{p\in G} \|d\mu_p/d\mu\|_{\mathbb{L}^\infty(A,\mu)}$, extends to $\| \sum_{k=0}^{N_p-1} f\circ T^k\|_{\mathbb{L}^q(A,\mu_p)}$. □

4.5. *Hitting probabilities and limit theorems.* In this sub-section, we work with ergodic, discrete Abelian, Markovian $\mathbb{Z}^d$-extension of Gibbs–Markov maps. Let G be an infinite countable Abelian group. Let (A,π,λ,μ,T) be a Gibbs–Markov map, and let $F: A\to G$ be $\sigma(\pi)$-measurable. We shall assume that the associated extension $(\widetilde{A},\tilde{\mu},\widetilde{T})$ is conservative and ergodic.

First, we shall relate the probability that an excursion from 0 hits a specific point p with the moments of the time spent in p. This is where the results from §§4.3 and 4.4 are used directly.

For $p\in G$, let $A_p := \{x\in A : \widetilde{T}_{\{0,p\}}(x,0)\in A\times\{p\}\}$ be the set of points x such that the excursion starting from $(x,0)$ reaches $A\times\{p\}$ before $A\times\{0\}$. Let $\alpha(p) := \mu(A_p)^{-1}$. The function α is well-defined because the extension is conservative and ergodic. The next lemma asserts that it converges to infinity as p goes to infinity.

LEMMA 4.17. *Let G be an infinite countable Abelian group. Let $(\widetilde{A},\tilde{\mu},\widetilde{T})$ be a conservative and ergodic Markovian G-extension of a measure-preserving dynamical system (A,μ,T). Then $\lim_{p\to\infty}\alpha(p) = +\infty$.*

Proof. Let $(K_n)_{n\geq 0}$ be an exhaustion of G by an increasing sequence of finite sub-sets of G. For all $x\in A$ such that $\varphi_{\{0\}}(x)$ is finite, set

$$N(x) := \max_{0\leq k<\varphi_{\{0\}}(x)} \min\{n\geq 0 : \widetilde{T}^k(x,0)\in A\times K_n\}.$$

Then $A=\bigcup_{n\geq 0} N^{-1}(n)$ up to set of measure 0, so that $\lim_{n\to+\infty}\mu(N>n) = 0$. But, if $p\notin K_n$, then $A_p\subset\{N>n\}$, so $\lim_{n\to+\infty}\sup_{p\in K_n^c}\mu(A_p)=0$, i.e. $\lim_{n\to+\infty}\inf_{p\in K_n^c}\alpha(p) = +\infty$. □

Let us go back to the study of the local time. Recall that, for $p\in G$ and $x\in A$, we set

$$f_{p,\{0\}}(x) := N_p(x) - 1 = \left(\sum_{k=0}^{\varphi_{\{0\}}(x)-1} \mathbf{1}_{\{S_kF(x)=p\}}\right) - 1,$$

which is the difference between the time spent in $A\times\{p\}$ and $A\times\{0\}$ in the excursion starting from $(x,0)$. Our next goal in this sub-section is to evaluate the tail and moments of $f_{p,\{0\}}$ as p goes to infinity.

PROPOSITION 4.18. *Let (A,π,d,μ,T) be a Gibbs–Markov map, and G be a countable Abelian group. Let $(\widetilde{A},\tilde{\mu},\widetilde{T})$ be a conservative and ergodic Markovian G-extension of (A,π,d,μ,T).*

The conditional distributions of $\alpha(-p)^{-1}N_p$ given $\{N_p>0\}$ have exponential tails, uniformly in p. In addition, $\alpha(-p)^{-1}N_p$, seen as a random variable on $(A,\mu(\cdot|A_p))$, converges in distribution and in moments to an exponential distribution of parameter 1.

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Proof. The random variable $N_p(x)$ counts the time the process starting from $(x, 0)$ spends in p before going back to 0. On A_p, it is positive. For x in A_p, let $T_p(x)$ be such that $\widetilde{T}_{\{0,p\}}(x, 0) = (T_p(x), p)$. Then, on A_p,

$$\begin{aligned} N_p(x) &= \inf\{k \geq 1 : \widetilde{T}^k_{\{0,p\}}(T_p(x), p) \in A \times \{0\}\} \\ &= 1 + \inf\{k \geq 0 : \widetilde{T}^k_{\{0,p\}}(T_p(x), p) \in A_{-p} \times \{p\}\}. \end{aligned}$$

But, if $y \notin A_{-p}$, then the first return time of $(\widetilde{T}^k(y, p))$ to $A \times \{0, p\}$ is the first return time of $(\widetilde{T}^k(y, p))$ to $A \times \{p\}$. Hence, $\widetilde{T}_{\{0,p\}}(y, p) = (\widetilde{T}_{\{0\}}(y), p)$, and

$$N_p(x) = 1 + \inf\{k \geq 0 : \widetilde{T}^k_{\{0\}}(T_p(x)) \in A_{-p}\}.$$

Let $N_p^{(0)}$ be the hitting time of A_{-p} for the process $(\widetilde{T}^k_{\{0\}}(x))_{k\geq 0}$. Then the random variable N_p seen on $(A, \alpha(p)\mathbf{1}_{A_p}\, d\mu)$ has the same distribution as the random variable $\mathbf{1} + N_p^{(0)}$ seen on $(A, \alpha(p)P_{\{0,p\}}\mathbf{1}_{A_p}\, d\mu)$. We write $\pi_{\{0\}} := \pi_{\varphi_{\{0\}}}$. In addition, each A_{-p} is non-trivial (as the extension is conservative and ergodic), and each A_{-p} is $\sigma(\pi_{\{0\}})$-measurable (because $\sigma(\pi_{\{0\}})$ contains all the information about the sites visited in an excursion, and in particular whether $-p$ is visited or not).

Due to Lemma 4.8 with the stopping time $\varphi_{\{0,p\}}$, the sequence of densities $(\alpha(p)P_{\{0,p\}}\mathbf{1}_{A_p})_{p\in G\setminus\{0\}}$ is uniformly bounded in $\mathrm{Lip}^\infty(A, \pi_{\{0,p\}}, \lambda)$. Since $\pi_{\{0\}} \leq \pi_{\{0,p\}}$, it is also uniformly bounded in $\mathrm{Lip}^\infty(A, \pi_{\{0\}}, \lambda)$. By Proposition 4.14, the sequence of random variables $\mu(A_{-p})N_p(\cdot)$ seen on $(A, \alpha(p)P_{\{0,p\}}\mathbf{1}_{A_p}\, d\mu)$ converges in distribution to an exponential random variable of parameter 1. By Corollary 4.13, this sequence of random variables is also exponentially tight, so it converges in moments, which proves the first part of Proposition 4.18. Since $(\alpha(-p))_{p\in G}$ goes to infinity as p goes to infinity, $(\alpha(-p)^{-1}N_p)_{p\in G}$, with respect to $(\mu(\cdot|A_p))_{p\in G}$, converges in distribution and in moments to an exponential random variable of parameter 1. □

Proposition 4.18 yields directly a rough description of the distribution of $f_{p,\{0\}}$ for large p's: it is -1 with probability $1 - \alpha(p)^{-1}$, and an exponential random variable of parameter $\alpha(-p)$ on the remaining set. This is part of Theorem 2.7.

Proof of Theorem 2.7. Let $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ be a conservative and ergodic Markovian $\mathbb{Z}^d$-extension of a Gibbs–Markov map (A, μ, T). We prove the second item, then the third, and we finish by the first item.

Let $p \in \mathbb{Z}^d\setminus\{0\}$. Set $\mu_{\{0,p\}} := \mu \otimes (\delta_0 + \delta_p)/2$. The dynamical system $(A \times \{0, p\}, \mu_{\{0,p\}}, T_{\{0,p\}})$ is measure-preserving, whence

$$\begin{aligned} \frac{1}{2} &= \mu_{\{0,p\}}(A \times \{0\}) = \mu_{\{0,p\}}(T^{-1}_{\{0,p\}}A \times \{0\}) \\ &= \mu_{\{0,p\}}(A \times \{0\} \cap T^{-1}_{\{0,p\}}A \times \{0\}) + \mu_{\{0,p\}}(A \times \{p\} \cap T^{-1}_{\{0,p\}}A \times \{0\}) \\ &= \frac{1 - \alpha(p)}{2} + \frac{\alpha(-p)}{2}, \end{aligned}$$

and thus $\alpha(p) = \alpha(-p)$. Together with Proposition 4.18, this yields the second item of Theorem 2.7.

Let $q > 1$, and apply Proposition 4.18 to the moments of order q of N_p. This yields

$$\begin{aligned}\|f_{p,\{0\}} + 1\|^q_{\mathbb{L}^q(A,\mu)} &= \int_{A_p} N_p^q \, d\mu \\ &= \alpha(p)^{-1}\alpha(-p)^q \|\alpha(-p)^{-1} N_p\|^q_{\mathbb{L}^q(A,\mu(\cdot|A_p))} \sim \alpha(p)^{q-1}\mathbb{E}[\mathcal{E}^q],\end{aligned}$$

where $\mathcal{E}$ is a random variable with an exponential distribution of parameter 1. Finally, we use the fact that $\mathbb{E}[\mathcal{E}^q] = \Gamma(1+q)$ to get the third item of Theorem 2.7.

We have proved that $\alpha(p) = \alpha(-p) \sim \mathbb{E}_\mu[f^2_{p,\{0\}}]/2$. Due to Proposition 4.18, $\alpha(p)^{-1}\mathbb{E}_\mu[N_p | N_p > 0] \to_{p\to\infty} 1$. Due to Propositions 4.14 and 4.12, the random variable $N_{0,p}$, which is the first hitting time of A_p for $T_{\{0\}}$, once divided by $\alpha(p)$, converges in distribution and in moments to an exponential random variable of parameter 1. Hence,

$$\mathbb{E}_\mu[N_{0,p}] \sim \alpha(p).$$

Now let us prove the link with $\sigma^2_{GK}(f_{p,\{0\}}, A, \mu, T_{\{0\}})$. Note that $f_{p,\{0\}}$ is constant on elements on $\pi_{\{0\}}$, and that $\|f_{p,\{0\}}\|_{\mathbb{L}^1(A,\mu)} \leq 1 + \|N_p\|_{\mathbb{L}^1(A,\mu)} \leq 2$. Hence, $\|f_{p,\{0\}}\|_{\mathrm{Lip}^1(A,\pi_{\{0\}},\lambda,\mu)}$, as a function of p, is bounded. Since $P_{\{0\}}$ sends $\mathrm{Lip}^1(A, \pi_{\{0\}}, \lambda, \mu)$ continuously into $\mathrm{Lip}^\infty(A, \pi_{\{0\}}, \lambda, \mu)$, and $P^M_{\{0\}}$ contracts exponentially fast on the subspace of functions in $\mathrm{Lip}^\infty(A, \pi_{\{0\}}, \lambda, \mu)$ with zero average on each of the M ergodic components of T^M, all the terms $\int_A f_{p,\{0\}} \circ \widetilde{T}^k_{\{0\}} \cdot f_{p,\{0\}} \circ \widetilde{T}^\ell_{\{0\}} \, d\mu$ with $k \neq \ell$ have a bounded contribution. Hence,

$$\sup_{p\in G} |\sigma^2_{GK}(f_{p,\{0\}}, A, \mu, \widetilde{T}_{\{0\}}) - \mathbb{E}_\mu[f^2_{p,\{0\}}]| < +\infty.$$

Note that, if the system is a random walk, then $P_{\{0\}}$ sends any function, which is constant on elements of the partition to its average, which is 0 for $f_{p,\{0\}}$. In this case, the supremum above is actually 0. □

4.6. *Proof of Theorem 2.11.* In this section we prove Theorem 2.11. Our goal is mostly to get a more explicit integrability condition in the statement of [**68**, Theorem 6.8]. We first give a lemma, which gives a good sufficient condition for this integrability condition to hold.

LEMMA 4.19. *Let G be a countable Abelian group. Let $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ be a conservative and ergodic Markovian G-extension of a Gibbs–Markov map (A, μ, T). Let $f : A \times G \to \mathbb{R}$ be measurable. Let $q \in [1, \infty)$. Assume that*

$$\sum_{p\in G} \alpha(p)^{1-(1/q)} \|f(\cdot, p)\|_{\mathbb{L}^q(A,\mu)} < +\infty. \tag{4.12}$$

Then $f_{\{0\}} \in \mathbb{L}^q(A, \mu)$.

Proof. Now, consider a function f satisfying the condition (4.12). Without loss of generality, we can assume f to be non-negative. Note that

$$\|f_{\{0\}}\|_{\mathbb{L}^q(A,\mu)} = \bigg\| \sum_{p\in\mathbb{Z}^d} (f\mathbf{1}_p)_{\{0\}} \bigg\|_{\mathbb{L}^q(A,\mu)} \leq \sum_{p\in\mathbb{Z}^d} \|(f\mathbf{1}_p)_{\{0\}}\|_{\mathbb{L}^q(A,\mu)}.$$

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Then, for all $p \in G\backslash\{0\}$,

$$\|(f\mathbf{1}_p)_{\{0\}}\|_{\mathbb{L}^q(A,\mu)}^q = \int_{A_p}\left(\sum_{k=0}^{N_{-p}-1} f\circ \widetilde{T}_{\{0\}}^k \circ \widetilde{T}_{\{0,p\}}\right)^q d\mu$$
$$= \frac{1}{\alpha(p)}\int_A \left(\sum_{k=0}^{N_{-p}-1} f\circ \widetilde{T}_{\{0\}}^k\right)^q d\widetilde{T}_{\{0,p\}*}\mu(\cdot|A_p).$$

By Lemma 4.8, $\widetilde{T}_{\{0,p\}*}\mu(\cdot|A_p) \ll \mu$, with a density, which is bounded in Lip^∞ norm, and *a fortiori* in $\mathbb{L}^\infty(A,\mu)$ norm. We can thus apply Lemma 4.16: there exists a constant C, independent from p, such that

$$\|(f\mathbf{1}_p)_{\{0\}}\|_{\mathbb{L}^q(A,\mu)}^q \leq C^q\frac{\alpha(-p)^q}{\alpha(p)}\|f\mathbf{1}_p\|_{\mathbb{L}^q(A,\mu)}^q.$$

Since $\alpha(p)\sim_{p\to\infty}\alpha(-p)$ by Theorem 2.7, up to taking a larger value of C,

$$\|(f\mathbf{1}_p)_{\{0\}}\|_{\mathbb{L}^q(A,\mu)} \leq C\alpha(p)^{1-(1/q)}\|f(\cdot,p)\|_{\mathbb{L}^q(A,\mu)},$$

whence

$$\|f_{\{0\}}\|_{\mathbb{L}^q(A,\mu)} \leq C\sum_{p\in\mathbb{Z}^d}\alpha(p)^{1-(1/q)}\|f(\cdot,p)\|_{\mathbb{L}^q(A,\mu)}. \qquad \square$$

Finally, we prove Theorem 2.11.

Proof of Theorem 2.11. Let (A,π,λ,μ,T) be an ergodic Gibbs–Markov map. Let $F: A\to\mathbb{Z}^d$ be $\sigma(\pi)$-measurable, integrable, and such that $\int_A F\,d\mu = 0$. Assume that the distribution of F with respect to μ is in the domain of attraction of an α-stable distribution, and that the Markovian $\mathbb{Z}^d$-extension $(\widetilde{A},\tilde{\mu},\widetilde{T})$ is conservative and ergodic.

We first assume that the extension $(\widetilde{A},\tilde{\mu},\widetilde{T})$ is aperiodic.

Aperiodic case. By Proposition 4.11, this extension satisfies Hypothesis 3.1. We can thus apply Theorem 2.4. Let $\beta:\mathbb{Z}^d\to\mathbb{R}$ be such that:

- β has finite support;
- $\sum_{p\in\mathbb{Z}^d}\beta(p) = 0$.

Let $f(x,p) := \beta(p)$. Then

$$\frac{S_n^{\widetilde{T}}f}{\sqrt{\sum_{k=0}^{n-1}\mu(S_k=0)}} \Rightarrow \sigma_{GK}(f,\widetilde{A},\tilde{\mu},\widetilde{T})\mathcal{Y},$$

where $\mathcal{Y}$ is a standard $MLGM(1-d/\alpha)$ random variable and the convergence is strong in distribution.

We can also apply [**68**, Theorem 6.8], with $r\equiv 1$. The regularity conditions are satisfied, since f and r are constant on the sub-sets of the Markov partition. The integrability condition '$|f|_{\{0\}}\in\mathbb{L}^p(A,\mu)$ for some $p>2$' is satisfied thanks to [**68**, Lemma 6.6]. Hence,

$$\frac{S_n^{\widetilde{T}}f}{\sqrt{\sum_{k=0}^{n-1}\mu(S_k=0)}} \Rightarrow \sigma(f)\mathcal{Y},$$

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where $\mathcal{Y}$ is a standard $MLGM(1-d/\alpha)$ random variable and the convergence is strong in distribution, and where

$$\sigma(f) = \lim_{N\to+\infty} \frac{1}{N} \int_A \bigg(\sum_{k=0}^{n-1} f_{\{0\}} \circ \widetilde{T}_{\{0\}}^k\bigg)^2 \, d\mu.$$

Following the proof of Lemma A.2, $\sigma(f_{\{0\}}) = \sigma_{GK}(f_{\{0\}}, A, \mu, \widetilde{T}_{\{0\}})$.

Hence, for any function β on $\mathbb{Z}^d$ with finite support and that sums to 0,

$$\sigma_{GK}(f, \widetilde{A}, \tilde{\mu}, \widetilde{T}) = \sigma_{GK}(f_{\{0\}}, A, \mu, \widetilde{T}_{\{0\}}).$$

Take $\beta := \mathbf{1}_p - \mathbf{1}_0$. Then, for all $q \in (1, \infty)$,

$$g(p) \sim_{p\to\infty} \frac{\sigma_{GK}^2(f_p, \widetilde{A}, \tilde{\mu}, \widetilde{T})}{2} = \frac{\sigma_{GK}^2(f_{p,\{0\}}, A, \mu, \widetilde{T}_{\{0\}})}{2} \sim_{p\to\infty} \alpha(p),$$

where we used Theorem 2.7 to get the last equivalence. Note that we already obtain Corollary 2.9.

Let $\varepsilon > 0$. Let $\delta > 0$ and $q > 2$ be small enough such that

$$(\alpha - d + \delta)\bigg(2 - \frac{2}{q}\bigg) \leq \alpha - d + 2\varepsilon. \tag{4.13}$$

By Proposition 2.6 and Potter's bound, $g(p) = O((1+|p|)^{\alpha-d+\delta})$, so $\alpha(p) = O((1+|p|)^{\alpha-d+\delta})$.

We are now ready to apply again [**68**, Theorem 6.8]. Let $f : \widetilde{A} \to \mathbb{R}$ be such that:

- the family of function $(f(\cdot, p))_{p\in\mathbb{Z}^d}$ is uniformly locally η-Hölder for some $\eta > 0$;
- $\int_{\widetilde{A}} (1+|p|)^{(\alpha-d)/2+\varepsilon} \|f(\cdot, p)\|_{\mathbb{L}^q(A,\mu)} \, d\tilde{\mu}(x, p) < +\infty$ for some $\varepsilon > 0$ and $q > 2$;
- $\int_{\widetilde{A}} f \, d\tilde{\mu} = 0$.

To apply [**68**, Theorem 6.8], we only need to check that:

- $\mathbb{E}_\mu(\sup_{p\in\mathbb{Z}^d} D(f(\cdot, p)) < +\infty$;
- $|f|_{\{0\}} \in \mathbb{L}^q(A, \mu)$.

$D(f)(x)$ is the Lipschitz norm of f restricted to the Markov sub-set to which x belongs.

Without loss of generality, we can use the metric d^η on A, so that $(f(\cdot, p))_{p\in\mathbb{Z}^d}$ is uniformly locally Lipschitz. Then $D(f(\cdot, p))$ is, by hypothesis, bounded uniformly in p. Hence, $\sup_{p\in\mathbb{Z}^d} D(f(\cdot, p))$ is bounded, and *a fortiori* integrable: the first point holds.

The only thing left to check is the second point. We adapt an argument by Csáki, Csörgő, Földes and Révész [**18**, Lemma 3.1] to control the norm of $|f|_{\{0\}}$. Up to choosing a smaller value of q, there exists $\delta > 0$, which satisfies the condition (4.13). Then

$$\begin{aligned}\sum_{p\in\mathbb{Z}^d} \alpha(p)^{1-(1/q)} \|f(\cdot, p)\|_{\mathbb{L}^q(A,\mu)} &\leq C \sum_{p\in\mathbb{Z}^d} (1+|p|)^{(\alpha-d+\delta)(1-(1/q))} \|f(\cdot, p)\|_{\mathbb{L}^q(A,\mu)} \\ &\leq C \sum_{p\in\mathbb{Z}^d} (1+|p|)^{(\alpha-d)/2+\varepsilon} \|f(\cdot, p)\|_{\mathbb{L}^q(A,\mu)} < +\infty.\end{aligned}$$

By Lemma 4.19, $|f|_{\{0\}} \in \mathbb{L}^q(A, \mu)$. This proves the theorem for aperiodic extensions.

Non-aperiodic case, 1: Construction of an aperiodic extension. For the remainder of this proof, we do not assume that the extension is aperiodic.

By Lemma 4.10, there exists a non-trivial closed sub-group $H \subset \mathbb{T}^d$ such that $\rho(P_u) = 1$ if and only if $u \in H$. Let $\Lambda \subset \mathbb{Z}^d$ be the lattice dual to H. As H is discrete, Λ has full rank. Let $B := \mathbb{Z}^d_{/\Lambda}$, and

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- $A_B := A \times B$;
- $\mu_B := |B|^{-1}\mu \otimes \sum_{b\in B} \delta_b$;
- $T_B(x, b) = (T(x), b + F(x)[\Lambda])$.

Since $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ is ergodic, (A_B, μ_B, T_B) is a measure-preserving ergodic dynamical system, which is Gibbs–Markov. Choose a fundamental box B_0 for B (by chosing a representant of every element of B). Observe that $\mathbb{Z}^d = B_0 + \Lambda$. We consider the projections $\pi_\Lambda : \mathbb{Z}^d \to \Lambda$ and $\pi_B : \mathbb{Z}^d \to B_0$ associated to this decomposition. Using this decomposition to identify $(x, q) \in \widetilde{A}$ with $(x, q[\Lambda], \pi_\Lambda(q)) \in A_B \times \Lambda =: \widetilde{A}_B$, we obtain that $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ is isomorphic to the extension $(\widetilde{A}_B, \tilde{\mu}_B, \widetilde{T}_B)$ with step function $F_B : A_B \to \Lambda$ given by $F_B(x, q[\Lambda]) := \pi_\Lambda(q + F(x)) - \pi_\Lambda(q)$ and with $\tilde{\mu}_B := \mu_B \otimes \sum_{b\in\Lambda} \delta_b$. Note that F_B is constant on the elements of the Gibbs–Markov partition of A_B. The extension $(\widetilde{A}_B, \tilde{\mu}_B, \widetilde{T}_B)$ is a conservative and ergodic Markovian $\mathbb{Z}^d$-extension of a Gibbs–Markov map. Let us show that this extension is aperiodic.

Assume that $(\widetilde{A}_B, \tilde{\mu}_B, \widetilde{T}_B)$ is not aperiodic. Let $P_{B,u}(\cdot) := P_B(e^{i\langle u, F_B\rangle}\cdot)$, and $H_B := \{u \in \widehat{\Lambda} : \rho(P_{B,u}) = 1\}$. Choose $u \in H_B\backslash\{0\}$.

Let us extend F to a function on A_B by making it depend only on the first coordinate. Observe that $F_B - F = \theta - \theta \circ T_B$ with $\theta(x, q[\Lambda]) = \pi_B(q)$ is a T_B-coboundary; indeed,

$$\begin{aligned} F_B(x, q[\Lambda]) - F(x, q[\Lambda]) &= \pi_\Lambda(q + F(x)) - \pi_\Lambda(q) - F(x) \\ &= q + F(x) - \pi_{B_0}(q + F(x)) - q + \pi_{B_0}(q) - F(x) \\ &= -\pi_B(q + F(x)) + \pi_B(q). \end{aligned}$$

Due to Lemma 4.10, there exist $\lambda_{B,u} \in \mathbb{S}_1$ and $f_{B,u} \in \mathrm{Lip}^\infty(A_B)$ with modulus 1 such that $P_B(e^{i\langle u, F_B\rangle} f_{B,u}) = \lambda_{B,u} f_{B,u}$. Then $P_B(e^{i\langle u, F\rangle} e^{i\langle u, \theta\rangle} f_{B,u}) = \lambda_{B,u} e^{i\langle u, \theta\rangle} f_{B,u}$. Therefore, $g_{B,u} := e^{i\langle u, \theta\rangle} f_{B,u}$ is an eigenfunction of $Q_{B,u} := P_B(e^{i\langle u, F\rangle}\cdot)$ associated to the eigenvalue $\lambda_{B,u}$.

For $r \in B$, let us define the translation $\pi_r : A_B \to A_B$ by $\pi(x, r) = (x, q + r)$. Then π_r commutes with T_B, and $F \circ \pi_r = F$, so that

$$\lambda_{B,u} g_{B,u} \circ \pi_r = P_B(e^{i\langle u, F\rangle} g_{B,u}) \circ \pi_r = P_B(e^{i\langle u, F\circ\pi_r\rangle} g_{B,u} \circ \pi_r) = Q_{B,u}(g_{B,u} \circ \pi_r).$$

Hence, $g_{B,u} \circ \pi_r$ is also an eigenfunction of $Q_{B,u}$ for the eigenvalue $\lambda_{B,u}$. By Lemma 4.10, there exists $\chi(r) \in \mathbb{S}_1$ such that $g_{B,u} \circ \pi_r = \chi(r) g_{B,u}$. Set $g(\cdot) := g_{B,u}(\cdot, 0)$. The function $\chi : B \to \mathbb{S}_1$ is a character, so there exists $v \in \widehat{B} = H$ such that $g_{B,u}(x, q[\Lambda]) = e^{-i\langle v, q\rangle} g(x)$.

By the definition of T_B, for all $(x, q) \in A \times \mathbb{Z}^d$,

$$\lambda_{B,u} e^{-i\langle v, q\rangle} g(x) = Q_{B,u}(g_{B,u})(x, q[\Lambda]) = e^{-i\langle v, q\rangle} Q_{u+v}(g)(x),$$

since the function $\chi(x, q) := e^{-i\langle v, q\rangle}$ satisfies $\chi \circ T_B = \chi e^{-i\langle v, F\rangle}$. Hence, $u + v \in H$. But v already belongs to H, so $u \in H$. This contradicts the fact that u is non-zero in $\widehat{\Lambda} = \mathbb{T}^d_{/H}$. Hence (A_B, μ_B, T_B) is aperiodic.

Non-aperiodic case, 2: Reduction to the aperiodic case. The function f still satisfies our assumptions for the new system $(\widetilde{A}_B, \tilde{\mu}_B, \widetilde{T}_B)$ (it is uniformly locally Hölder, decays at a sufficient rate at infinity, and has zero integral). Thus, we can apply the version of Theorem 2.11 for aperiodic systems; this yields

$$\frac{S_n^{\widetilde{T}} f}{\sqrt{\sum_{k=0}^{n-1} \mu(S_k \in B)}} \Rightarrow \sigma_{GK}(f_B, A_B, \mu_B, \widetilde{T}_{B,\{0\}})\mathcal{Y},$$

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where $\mathcal{Y}$ is a standard $MLGM(1-d/\alpha)$ random variable, the convergence is strong in distribution, and

$$\sigma_{GK}^2(f_B, A_B, \mu_B, \widetilde{T}_{B,\{0\}}) := \lim_{n\to+\infty} \int_{A_B} f_B^2 \, d\mu_B + 2\sum_{k=1}^{n} \int_{A_B} f_B \cdot f_B \circ \widetilde{T}_{B,\{0\}}^k \, d\mu_B,$$

where the limit is taken in the Cesàro sense.

The proof of [**1**, Lemma 3.7.4] can be adapted to ergodic Gibbs–Markov maps (instead of continued fraction mixing maps), by replacing $\widetilde{T}_A^k$ with $M^{-1}\sum_{k=0}^{M-1}\widetilde{T}_A^k$, which can be done up to a uniformly bounded error term. As $\widetilde{T}_B$ is an ergodic Gibbs–Markov map, $A\times B$ is thus also a Darling–Kac set, and a set on which Rényi's inequality is satisfied. By [**1**, Theorem 3.3.1],

$$\lim_{n\to+\infty} \frac{\sum_{k=0}^{n-1}\mu(S_k\in B)}{\sum_{k=0}^{n-1}\mu(S_k=0)} = \lim_{n\to+\infty} \frac{\sum_{k=0}^{n-1}\tilde{\mu}(A\times\{0\}\cap\widetilde{T}^{-k}(A\times B))}{\sum_{k=0}^{n-1}\tilde{\mu}(A\times\{0\}\cap\widetilde{T}^{-k}(A\times\{0\}))} = |B|. \quad (4.14)$$

Using the induction invariance of the Green–Kubo formula (Lemma A.2) with the observable f_B on $(A_B, \mu_B, \widetilde{T}_{B,\{0\}})$, noting that the induced transformation on $A\times\{0\}$ is $\widetilde{T}_{\{0\}}$, we get

$$\begin{aligned}\sigma_{GK}^2(f_B, A_B, \mu_B, \widetilde{T}_{B,\{0\}}) &= \lim_{n\to+\infty} \int_A f^2\, d\mu_B + 2\sum_{k=1}^{n}\int_A f\cdot f\circ \widetilde{T}_{\{0\}}^k \, d\mu_B \\ &= |B|^{-1}\sigma_{GK}^2(f, A, \mu, \widetilde{T}_{\{0\}}), \qquad (4.15)\end{aligned}$$

where the limit is taken in the Cesàro sense. Equations (4.14) and (4.15) together yield the claim. □

5. *Applications*

In this section, we prove our claims of §2.4, starting with the geodesic flow and finishing with the billiards.

5.1. *Periodic planar billiard in finite horizon.*

Recall that the billiard table is $\mathbb{R}^2\setminus\bigcup_{i\in\mathcal{I},\, p\in\mathbb{Z}^2}(p+O_i)$, where $(O_i)_{i\in\mathcal{I}}$ corresponds to a finite family of open convex sub-sets of $\mathbb{T}^2$, whose boundaries are non-overlapping, $\mathcal{C}^3$, and with non-vanishing curvature. For the collision map, the phase space is $\Omega := \partial Q\times[-\pi/2, \pi/2]$. The invariant measure is the Liouville measure $\cos(\phi)\, dx\, d\phi$ in (x, ϕ), where x is the curvilinear coordinate on ∂Q.

A particle has configuration (x, ϕ, i, p) if it is located in $p+\partial O_i$, with curvilinear coordinate x on ∂O_i (for some counterclockwise curvilinear parametrization of ∂O_i) and if its reflected vector V makes the angle ϕ with the inward normal vector to ∂O_i. The billiard map $\widetilde{T}_0 : \Omega\to\Omega$ maps a configuration in Ω to the configuration corresponding to the next collision time. This transformation preserves the Liouville measure $\tilde{\nu}$, which has infinite mass.

We consider a particle starting from the *original cell* $\mathcal{C}_0 = \bigcup_{i\in\mathcal{I}} O_i$ with initial distribution $\nu := \tilde{\nu}(\cdot|\mathcal{C}_0)$.

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The associated compact billiard is the system (M, ν, T_0), with $M := \mathcal{C}_0$ and $\widetilde{T}_0(x, \phi, i, p) = (T_0(x, \phi, i), p + H(x, \phi, i))$. Then $(\Omega, \tilde{\nu}/\tilde{\nu}(\mathcal{C}_0), \widetilde{T}_0)$ is the $\mathbb{Z}^2$-extension of (M, ν, T_0) with step function $H : M \to \mathbb{Z}^2$ corresponding to the change of cells. The quantity $S_n^T H(y) := \sum_{k=0}^{n-1} H \circ T_0^k(y)$ corresponds to the index of the cell containing $\widetilde{T}_0^n(y)$, for all $y \in \mathcal{C}_0$.

Let $\varepsilon > 0$, and $\beta : \mathbb{Z}^2 \to \mathbb{R}$ be such that $\sum_{p \in \mathbb{Z}^2} \beta(p) = 0$. We associate the value $\beta(p)$ to the cell $\mathcal{C}_p$, and put for $y \in \mathcal{C}_0$

$$\mathcal{Y}_n(y) := \sum_{k=1}^{n} \beta(S_k^{T_0} H(y)).$$

Proof of Corollaries 2.14 and 2.15. Due to Young's towers [**70**], we know that there exists a dynamical system (A, μ, T) such that (A, μ, T) and (M, ν, T_0) are both factors of another dynamical system $(\widehat{A}, \hat{\mu}, \widehat{T})$. This means that there exist two maps $\hat{\pi} : (\widehat{A}, \hat{\mu}, \widehat{T}) \to (A, \mu, T)$ and $\pi : (\widehat{A}, \hat{\mu}, \widehat{T}) \to (M, \nu, T_0)$ such that

$$\begin{aligned} \hat{\pi} \circ \hat{T} &= T \circ \hat{\pi}, \\ \pi \circ \widehat{T} &= T_0 \circ \pi, \\ \hat{\pi}_* \hat{\mu} &= \mu, \\ \pi_* \hat{\mu} &= \nu. \end{aligned}$$

Moreover, there exist $F : A \to \mathbb{Z}^d$ and $\beta : A \to \mathbb{Z}^2$ such that $F \circ \hat{\pi} = H \circ \pi$ and $\hat{\beta} \circ \hat{\pi} = \beta \circ \pi$.

The properties of the family of transfer operators $P_u = P(e^{i\langle u, F\rangle})$ for such step function F have been studied: see for instance [**22**, **51**, **52**, **62**], in which local limit theorems with various remainder terms have been established. The matrix Σ corresponds to the asymptotic variance matrix of $(S_n^T F/\sqrt{n})_{n \geq 1}$ with respect to μ, which is the same as the asymptotic variance matrix of $(S_n^{T_0} H/\sqrt{n})_{n \geq 1}$ with respect to ν, and is given by

$$\Sigma = \sum_{k \in \mathbb{Z}} C(H, H \circ T^k),$$

where $C(H, H \circ T^k)$ denotes the covariance matrix of H and $H \circ T^k$ with respect to ν. Recall that $(S_n^{T_0} H/\sqrt{n})_{n \geq 1}$ converges in distribution to a centered Gaussian random variable with variance matrix Σ.

Let $\mathcal{Z}_n : M \to \mathbb{R}$ be defined by $\mathcal{Z}_n(x) := \sum_{k=0}^{n-1} \hat{\beta}(S_k^T F(x))$. This function satisfies $\mathcal{Z}_n \circ \hat{\pi} = \mathcal{Y}_n \circ \pi$ on $\hat{A}$. Applying Theorem 2.4 to the dynamical system (A, μ, T), step function F (respectively, the first coordinate $F_1 : A \to \mathbb{Z}$ of F) and $\hat{\beta}$ (respectively, $p \mapsto \hat{\beta}(p, 0)$), we obtain Corollary 2.15 (respectively, Corollary 2.14). □

5.2. *Geodesic flow on periodic hyperbolic manifolds.* We recall that M is a compact, connected manifold with a Riemannian metric of negative sectional curvature, and $\varpi : N \to M$ be a connected $\mathbb{Z}^d$-cover of M, with $d \in \{1, 2\}$. The manifold $T^1 N$ is endowed with the σ-finite lift μ_N of a Gibbs measure μ_M corresponding to a reversible Hölder potential. The geodesic flow on $T^1 N$ is denoted by $(g_t)_{t \in \mathbb{R}}$.

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Let (A, μ, T) be a Markov section for the geodesic flow on $T^1 M$, as constructed by Bowen [**12**], [**13**, Theorem 3.12]. The section A is constructed by carefully choosing a finite number of pieces of strong unstable manifolds $(W^u(a))_{a \in \pi}$, then, for all $x \in W^u(a)$, adding a piece of strong stable manifold $W^s(x)$ to get rectangles. We shall denote by p_+ the projection onto unstable manifolds, defined by $p_+(y) = x$ whenever $y \in W^s(x)$ and $x \in W^u(a)$. Let r be the return time to A; by Bowen's construction, $r(x)$ depends only on the future (the non-negative coordinates) of x. Finally, we put $A_+ := \bigcup_{a \in \pi} W^u(a)$ as the state space of the one-sided transformation.

The set $\widetilde{A} := \varpi^{-1}(A)$ is a section for the geodesic flow on $T^1 N$, with return time $\tilde{r} = r \circ \varpi$. The induced map on $\widetilde{A}$ is the $\mathbb{Z}^d$-extension of the natural extension of a Gibbs–Markov map, with step function F. Without loss of generality, we may refine the Markov partition on A so that F depends only on the first coordinate of the shift; then, the $\mathbb{Z}^d$-extension $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ is Markovian. The geodesic flow on $T^1 N$ is thus isomorphic to the suspension flow over $(\widetilde{A}, \tilde{\mu}, \widetilde{T})$ with roof function $\tilde{r}$. In particular, $T^1 N \simeq \{(x, q, t) : x \in A, q \in \mathbb{Z}^d, t \in [0, r(x))\}$.

Let $f : T^1 N \to \mathbb{R}$ be Hölder. The following lemma asserts that, up to adding a coboundary, we can assume that f depends only on the future, which allows us to work with Gibbs–Markov maps instead of their natural extension. While this lemma is classic [**13**, Lemma 1.6], we give a statement, which is valid in the context of $\mathbb{Z}^d$-extensions.

LEMMA 5.1. *Let $(A, \pi, \lambda, \mu, T)$ be the natural extension of an ergodic Gibbs–Markov map*†. *Let $(A \times \mathbb{Z}^d, \tilde{\mu}, \widetilde{T})$ be a Markovian $\mathbb{Z}^d$-extension with step function F. Let f be a measurable real-valued function on $A \times \mathbb{Z}^d$. Assume that*

$$\|D(f)\|_\infty := \|f\|_\infty + \sup_{q \in \mathbb{Z}^d} \sup_{a \in \pi} |f|_{\mathrm{Lip}(a \times \{q\})} < +\infty.$$

Then there exists a function u which is bounded by $\lambda(\lambda - 1)^{-1} \|D(f)\|_\infty$, uniformly 1/2-Hölder, and such that the function $f_+ := f + u \circ T - u$ is $\mathcal{B}_+$-measurable, with $\mathcal{B}_+ := (\bigvee_{n \geq 0} T^{-n}\pi) \otimes \mathcal{P}(\mathbb{Z}^d)$.

Proof. Let $\tilde{p}_+(x, q) := (p_+(x), q)$ be defined on $\widetilde{A}$. We put

$$u := \sum_{n=0}^{+\infty} f \circ \widetilde{T}^n - f \circ \widetilde{T}^n \circ \tilde{p}_+.$$

The proof then proceeds as in [**68**, Lemma 6.11]: the function u satisfies the conclusion of the lemma. Most changes in the proof of [**68**, Lemma 6.11] are straightforward; the only observation needed is that, if x and y are in the same cylinder of length n in A, then $\widetilde{T}^k(x, q)$ and $\widetilde{T}^k(y, q)$ are in the same set $A \times \{S_k F(x)\}$ for $|k| \leq n$, so that we can use the Lipschitz estimate for each $f(\cdot, S_k F(x))$. □

We are now ready to prove Proposition 2.16.

† The metric being defined by λ^{-s}, where s is the two-sided separation time.

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Proof of Proposition 2.16. The proof follows the one in [**68**, Proposition 6.12], with a few significant modifications. The first step is to eliminate the past, that is, add a coboundary to get an observable, which depends only on the future, to be able to use [**68**, Proposition 6.1]. Let $\eta \in (0, 1]$. Let $f : T^1 N \to \mathbb{R}$ be a η-Hölder observable, which satisfies the hypotheses of the proposition. We put:

- $f_{\widetilde{A}}(x, q) := \int_0^{r(x)} f(x, q, s)\, ds$;
- $u_{\widetilde{A}}$ the function obtained from $f_{\widetilde{A}}$ by the construction of Lemma 5.1;
- $f_{+,\widetilde{A}} := f_{\widetilde{A}} + u_{\widetilde{A}} \circ \widetilde{T} - u_{\widetilde{A}}$;
- $f_+(x, q, t) := r(x)^{-1} f_{+,\widetilde{A}}(x, q)$.

By Lemma 5.1, the function $u_{\widetilde{A}}$ is $\eta/2$-Hölder and bounded. Then, using the fact that $f_{+,\widetilde{A}} - f_{\widetilde{A}}$ is a coboundary,

$$\sup_{t\geq 0}\left\| \int_0^t f \circ g_s\, ds - \int_0^t f_+ \circ g_s\, ds \right\|_\infty \leq 2\|u_{\widetilde{A}}\|_\infty + 2\|f_{+,\widetilde{A}}\|_\infty + 2\|r\|_\infty \|f\|_\infty < +\infty. \tag{5.1}$$

Hence, it is enough to prove the limit theorem for f_+. Note that f_+ is a coboundary if and only if f is a coboundary.

Let $\varphi_{A\times\{0\}}$ be the first return time to $A \times \{0\}$ for the geodesic flow, and $\overline{\varphi}_{A\times\{0\}}$ the first return time to $A \times \{0\}$ for $\widetilde{T}$. The proof then proceeds as in [**68**], with the same weakened criterion: we only need to check that, for some $\delta > 0$,

$$\sup_{0\leq t\leq \varphi_{A\times\{0\}}} \left| \int_0^t f_+ \circ g_s\, ds \right| \in \mathbb{L}^{2+\delta}(A \times \{0\}). \tag{5.2}$$

Now, we shall go back to the initial (invertible) system to use the integrability assumption on f. Equations (5.1) and (5.2) together yield

$$\sup_{0\leq t\leq \varphi_{A\times\{0\}}} \left| \int_0^t f \circ g_s\, ds \right| \in \mathbb{L}^{2+\delta}(A \times \{0\}). \tag{5.3}$$

Finally, once again, we go to the non-invertible factor. Let $\overline{f}_{\widetilde{A}}(x, q) := \|\int_0^{r(\cdot)} |f|(\cdot, q, t)\, dt\|_\infty$. Then

$$\sup_{0\leq t\leq \varphi_{A\times\{0\}}} \left| \int_0^t f \circ g_s(x, 0, 0)\, ds \right| \leq \sum_{n=0}^{\overline{\varphi}_{A\times\{0\}}(x,0)-1} \overline{f}_{\widetilde{A}} \circ \widetilde{T}^n(x).$$

The function $\overline{f}_{\widetilde{A}}$ is an upper bound on $|f|_{\widetilde{A}}$, which depends only on q, and thus not on the past. Hence, it factorizes as a function of $A_+ \times \mathbb{Z}^d$. In addition, the integrability assumptions yields

$$\sum_{q\in\mathbb{Z}^d} |q|^{1-(d/2)+\varepsilon} \|\overline{f}_{\widetilde{A}}(\cdot, q)\|_\infty < +\infty.$$

By Lemma 4.19, $\overline{f}_{\widetilde{A}}$ belongs to $\mathbb{L}^{2+\delta}(A \times \{0\})$ for all small enough $\delta > 0$, which yields equation (5.3). □

A. *Appendix. About Green–Kubo's formula*

The spirit behind Corollary 2.13, and thus of our alternative proof of Spitzer's theorem [**61**, Ch. III.11, P5], is that Green–Kubo's formula satisfies an invariance by induction,

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which is reminiscent of Kac's theorem. We shall draw this parallel here, as well as prove a specific instance of this phenomenon, which is useful in the proof of Theorem 2.11. In what follows, the measure may be finite or σ-finite.

Given an ergodic, conservative, measure-preserving dynamical system (A, μ, T) and a measurable sub-set $B \subset A$ such that $\mu(B) > 0$, one may define the system induced on B by $(B, \mu_{|B}, T_B)$. Given any measurable observable $f : A \to \mathbb{C}$, we also define the induced observable f_B by

$$f_B(x) = \sum_{k=0}^{\varphi_B(x)-1} f(T^k(x)),$$

where φ_B is the first return time to B. Then, a generalization of Kac's theorem [**33**] asserts that the integral is invariant by induction.

THEOREM A.1. (Kac's theorem: induction invariance of the integral) *Let (A, μ, T) be an ergodic, conservative, measure-preserving dynamical system. Let $B \subset A$ be a measurable sub-set with $0 < \mu(B) < +\infty$. Then, for all $f \in \mathbb{L}^1(A, \mu)$,*

$$\int_A f \, d\mu = \int_B f_B \, d\mu. \tag{A.1}$$

A consequence is that the map $f \mapsto f_B$ is a weak contraction from $\mathbb{L}^1(A, \mu)$ to $\mathbb{L}^1(B, \mu)$. There are two different ways to prove this theorem.

- Using the fact that the system is measure-preserving [**34**]: up to going to the natural extension, we can define $\varphi_{-1,B}(x) := \inf\{n \geq 0 : \ T^{-n}(x) \in B\}$, and then use, for all $n \geq 0$,
$$\int_A f \mathbf{1}_{\varphi_{-1,B}=n} \, d\mu = \int_B f \circ T^n \mathbf{1}_{\varphi_B \geq n} \, d\mu.$$
- Using a convergence theorem, such as Hopf's ergodic theorem [**32**, §14, Individueller Ergodensatz für Abbildungen], and the preservation of the measure for the induced system. Setting $g := \mathbf{1}_B$, one can identify the almost sure limit of $(S_n^T f)/(S_n^T g)$ with that of $(S_n^{T_B} f_B)/n$, and conclude.

Green–Kubo's formula†, at least at a formal level, behaves the same. For any square-integrable function f with zero integral, we can ask whether

$$\int_A f^2 \, d\mu + 2 \sum_{n=1}^{+\infty} \int_A f \cdot f \circ T^n \, d\mu = \int_B f_B^2 \, d\mu + 2 \sum_{n=1}^{+\infty} \int_B f_B \cdot f_B \circ T_B^n \, d\mu. \tag{A.2}$$

The reader may compare equations (A.1) and (A.2). As with Kac's theorem, we may choose different strategies to prove rigorously such an identity. Using the fact that the system is measure-preserving, and cutting in a well-chosen way the integrals above, one can see that they are formally the same. However, to get a rigorous proof, one would have to use Fubini's theorem, which fails in this case. This is not surprising, as even the definition of these sums requires some care: if there is some periodicity in the dynamical system, the convergence may have to be in the Cesàro sense, as in Theorem 2.11, or in the Abel sense if T is an irrational rotation and f is analytic.

† The discussion can be generalized by taking two different observables: what is invariant is actually the underlying bilinear form.

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Another strategy is to use a distributional limit theorem: for sufficiently hyperbolic systems and nice enough observables, Green–Kubo's formula is the asymptotic variance in a central limit theorem. Working at two different time scales (with the initial system and with the induced system), one can prove that this invariance holds. A very simple example is given by the following lemma.

LEMMA A.2. *Let $(A, \pi, \lambda, \mu, T)$ be a Gibbs–Markov map. Let $f \in \mathbb{L}^2(A, \mu)$ be a real-valued function such that:*

- *f is summably locally Lipschitz: $\sum_{a\in\pi} \mu(a)|f|_{\mathrm{Lip}(a)} < +\infty$;*
- *$\int_A f\, d\mu = 0$.*

Let $B \subset A$, with $\mu(B) > 0$, be $\sigma(\pi)$-measurable. Assume that φ_B is essentially constant. Then

$$\lim_{n\to+\infty} \int_A f^2\, d\mu + 2\sum_{k=1}^{n} \int_A f \cdot f \circ T^k\, d\mu$$
$$= \lim_{n\to+\infty} \int_B f_B^2\, d\mu + 2\sum_{k=1}^{n} \int_B f_B \cdot f_B \circ T_B^k\, d\mu,$$

where both limits are taken in the Cesàro sense.

Proof. Let $M := \varphi_B$ almost everywhere. Under the assumptions, the Birkhoff sums (for T) of f satisfy a central limit theorem (see e.g. [**26**, Théorème 4.1.4], and use the Taylor expansion of $(I - P)^{-1}$):

$$\frac{S_n^T f}{\sqrt{n}} \to \sigma\mathcal{N},$$

where the convergence is in distribution on (A, μ), $\mathcal{N}$ follows a standard Gaussian distribution, and

$$\sigma^2 = \lim_{n\to+\infty} \frac{1}{n} \int_A (S_n^T f)^2\, d\mu.$$

By [**72**, Theorem 1], the same central limit theorem holds strongly in distribution, that is, when the initial measured space is (A, ν), with $\nu \ll \mu$. This holds in particular on $(B, M\mu_{|B})$.

Under the same assumptions, the Birkhoff sums (for T_B) of f_B satisfy a central limit theorem. Then

$$\frac{S_n^{T_B} f_B}{\sqrt{n}} \to \sigma'\mathcal{N},$$

where the convergence is in distribution on $(B, M\mu_{|B})$, $\mathcal{N}$ follows a standard Gaussian distribution, and

$$(\sigma')^2 = \lim_{n\to+\infty} \frac{M}{n} \int_B (S_n^{T_B} f_B)^2\, d\mu.$$

Note that $S_n^{T_B} f_B = S_{Mn}^T f$, whence $\sigma' = \sqrt{M}\sigma$. This yields

$$\lim_{n\to+\infty} \frac{1}{n} \int_A (S_n^T f)^2\, d\mu = \lim_{n\to+\infty} \frac{1}{n} \int_B (S_n^{T_B} f_B)^2\, d\mu.$$

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Finally, note that

$$\frac{1}{N}\int_A (S_N^T f)^2\, d\mu = \frac{1}{N}\sum_{k,n=0}^{N-1}\int_A f\circ T^k \cdot f\circ T^n\, d\mu$$
$$= \frac{1}{N}\sum_{n=0}^{N-1}\bigg[\int_A f^2 d\mu + 2\sum_{k=1}^{n}\int_A f\cdot f\circ T^k\, d\mu\bigg].$$

Hence, σ^2 is the Cesàro-limit of $(\int_A f^2\, d\mu + 2\sum_{k=1}^n \int_A f\cdot f\circ T^k\, d\mu)_{n\geq 1}$. The same manipulation with $1/n\int_B (S_n^{T_B} f_B)^2\, d\mu$ yields the lemma. □

The assumption on φ_B can be relaxed, as long as one can prove a central limit theorem both for f on (A, μ, T) and f_B on $(B, \mu(\cdot|B), T_B)$, and ensure that the limits coincide up to a change in time. This can be done for instance with an almost sure invariance principle [**27**].

In this article, the proof of Corollary 2.13 relies on this approach: we obtain two distributional limit theorems by working at two different time scales, and then identify the limits. However, as can be seen, obtaining these limit theorems gets much more challenging when working with null recurrent processes.

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