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A reaction–diffusion epidemic model with incubation period in almost periodic environments†

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In this paper, we propose and study an almost periodic reaction–diffusion epidemic model in which disease latency, spatial heterogeneity and general seasonal fluctuations are incorporated. The model is given by a spatially nonlocal reaction–diffusion system with a fixed time delay. We first characterise the upper Lyapunov exponent λ^* for a class of almost periodic reaction–diffusion equations with a fixed time delay and provide a numerical method to compute it. On this basis, the global threshold dynamics of this model is established in terms of λ^*. It is shown that the disease-free almost periodic solution is globally attractive if $\lambda^* < 0$, while the disease is persistent if $\lambda^* > 0$. By virtue of numerical simulations, we investigate the effects of diffusion rate, incubation period and spatial heterogeneity on disease transmission.

Key words: Almost periodicity, incubation period, spatial heterogeneity, upper Lyapunov exponent, threshold dynamics

2020 Mathematics Subject Classification: Primary: 35K57. Secondary: 37N25; 92D30.

1 Introduction

Mathematical modelling in epidemiology provides a powerful way to analyse the spread of infectious disease, and in the process, it suggests effective control strategies. One of the earliest mathematical models in epidemiology was introduced by Kermack and McKendrick [19]. The Kermack–McKendrick model is given by the following system of ordinary differential equations:

$$\begin{cases} \dfrac{dS(t)}{dt} = -\beta S(t)I(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t). \end{cases} \tag{1.1}$$

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Here, $S(t)$, $I(t)$ and $R(t)$ are the sizes of susceptible, infective and removed classes, respectively. The constant $\beta > 0$ is called the infection rate, and $\gamma > 0$ is called the recovery rate. Given $S(0) = S_0 > 0$, the analysis of model (1.1) shows that the sign of $\beta \frac{S_0}{\gamma} - 1$ completely determines the development trend of the disease. Since the work of Kermack and McKendrick, modelling of mathematical models has flourished (see, e.g., [2, 3, 6]).

Empirical evidence shows that many diseases have incubation period which differs from disease to disease (see, e.g., [3]). It is therefore necessary to incorporate the latency into the epidemic models. Cooke and van den Driessche [8] formulated and analysed an susceptible → exposed → infectious → removed → susceptible (SEIRS) epidemic model with latent and immune periods. Li and Zou [21] generalised model (1.1) to a two-patch environment with incubation period and obtained a system of delay differential equations with a fixed time delay accounting for the latency and nonlocal terms. They showed that the disease exists multiple outbreaks before it goes to extinction, which is in sharp contrast to the dynamics of classic Kermack–McKendrick susceptible → infectious → removed (SIR) model. Subsequently, Li and Zou [22] formulated an SIR model with a fixed latent period in an n-patch environment, and they investigated the threshold dynamics of the model. For the spatially continuous case, Guo et al. [14] derived and investigated a nonlocal reaction–diffusion SIR model with a fixed incubation period. By appealing to the theory of monotone dynamical systems and uniform persistence, they presented the global threshold dynamics of this model.

Seasonal variations, as mentioned in [1], are ubiquitous and can exert strong pressures on population dynamics and the spread of infectious diseases. Recently, the interaction of time delay and seasonality in epidemic models has attracted much attention. A time-periodic reaction–diffusion SIR model with latent period was proposed by Zhang et al. [38]. By using the comparison arguments and persistence theory, they investigated the global dynamics of the periodic model. Zhang and Wang [37] further considered a time-periodic reaction–diffusion epidemic model with constant infection period, which adopts the saturation incidence. More recently, Li and Zhao [20] formulated and studied a periodic SEIRS epidemic model with a time-dependent latent period. Other epidemiology models concerning the latency and the seasonality were developed in quite a few works, see, e.g., [4, 23, 36, 40].

Due to the complexity of external environments and the uncertainty in climate, the parameters in an epidemic model are not necessary to be periodic. Even if they are periodic, they are also not always share a common period. As noted in [5, 10], though one can obtain the exactly periodic parameters in controlled laboratory experiments, environmental changes in nature are hardly periodic. As a generalisation of periodicity, almost periodicity is more likely to describe natural fluctuations. Along this line, there have been some works studying the transmission dynamics for almost periodic epidemic models (see, e.g., [9, 34]). Recently, Wang et al. [33] studied the threshold dynamics of an almost periodic reaction–diffusion epidemic model. For the almost periodic reaction–diffusion epidemic model with incubation period, the global dynamics does not have an adequate characterisation. The purpose of the current paper is to formulate an SIR epidemic model which incorporates disease latency, spatial heterogeneity and general seasonal fluctuations and to study the global dynamics of the proposed model.

In this paper, we employ a reaction–diffusion system framework to model the influences of incubation period, spatial heterogeneity and natural fluctuation on the spread of infectious disease. Although the almost periodic functions preserve some properties that the periodic functions possess, the method for periodic system is not suitable for almost periodic model. Since the disease latency is taken into account, the threshold value characterised in almost periodic models without time delay (see [34, 33]) cannot be used. Our theoretical results show that there exists an upper Lyapunov exponent which serves as a threshold value for the global uniform persistence and extinction of the model, and it can be interpreted as the growth rate of infectious population in a completely susceptible population. To our best knowledge, there is no way at present of computing the upper Lyapunov exponent for reaction–diffusion equations with or without time delay. On the basis of continuous separation and comparison principle, we provide a numerical method for the computation of the upper Lyapunov exponent. Numerically, we analyse the impacts of diffusion rate, incubation period and spatial heterogeneity on the upper Lyapunov exponent. Our numerical results highlight that decreasing the difference of spatial distributions between transmission and recovery coefficients is beneficial to the control of disease.

The rest of the paper is organised as follows. In Section 2, we characterise the upper Lyapunov exponent λ^* for a class of almost periodic reaction–diffusion equations with time delay and supply a numerical method to compute it. In Section 3, we derive a model, which turns out to be an almost periodic reaction–diffusion system with nonlocal and time-delayed nonlinearity. In Section 4, we establish the threshold dynamics for the model system in terms of λ^*. In Section 5, we present some numerical simulations to interpret the obtained theoretical results and reveal the effect of parameters on λ^*. A simple discussion completes this paper.

2 The upper Lyapunov exponent

Let $(\mathbb{X}, d)$ be a metric space. A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be almost periodic if for any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$ such that every interval of $\mathbb{R}$ of length l contains at least one point of the set

$$T(f, \epsilon) = \{s \in \mathbb{R} : d(f(t+s) - f(t)) < \epsilon, \forall t \in \mathbb{R}\}.$$

Let $D \subset \mathbb{R}^n$. A function $f \in C(D \times \mathbb{R}, \mathbb{X}) : (t, x) \mapsto f(t, x)$ is said to be uniformly almost periodic in t if f is almost periodic in t for each $x \in D$, and for any compact set $E \subset D$, f is uniformly continuous on $E \times \mathbb{R}$ ([11, 12]).

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Define $X_0 := C(\overline{\Omega}, \mathbb{R})$, which is a Banach space with supremum norm $\|\cdot\|_0$. Let

$$X_0^+ := C(\overline{\Omega}, \mathbb{R}_+) = \{\psi \in X_0 : \psi(x) \geq 0, \forall x \in \overline{\Omega}\}.$$

Note that the interior of X_0^+, denoted by $\mathrm{Int}(X_0^+)$, is nonempty. For a constant $\tau > 0$, define $X := C([-\tau, 0], X_0)$ with the norm $\|\phi\| := \max_{\theta \in [-\tau, 0]} \|\phi(\theta)\|_0$, $\forall \phi \in X$. For a function $b \in C([-\tau, \rho], X_0)$ $(\rho > 0)$, define $b_t \in X$ by $b_t(\theta) := b(t+\theta)$, $\forall \theta \in [-\tau, 0]$, $t \in [0, \rho)$. Let $\hat{}$ denote the inclusion $\mathbb{R} \to X$ by $b \to \hat{b}$, $\hat{b}(\theta, x) \equiv b$, $\forall \theta \in [-\tau, 0]$, $x \in \overline{\Omega}$.

We consider the following almost periodic time-delayed nonlocal equation which comes from the equation of infectious variable in the linearisation of a given almost periodic reaction–diffusion epidemic model at a disease-free almost periodic solution (for reader's convenience, we present the detailed derivation process in the following section):

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = D_3 \Delta w(t,x) - (\gamma_I(t,x) + d(t,x))w(t,x) \\ \qquad\qquad + \displaystyle\int_\Omega \Gamma(t, t-\tau, x, y)\beta(t-\tau, y)w(t-\tau, y)dy, \quad t > 0,\ x \in \Omega, \\ \dfrac{\partial w(t,x)}{\partial \nu} = 0, \quad t > 0,\ x \in \partial\Omega. \end{cases} \tag{2.1}$$

Here, $\Gamma(t,s,x,y)$ with $t > s \geq 0$ and $x, y \in \Omega$ is the fundamental solution associated with the partial differential operator $\partial_t - D_E\Delta - d(t,\cdot) - \gamma_E(t,\cdot)$ subject to the Neumann boundary condition (see [13, Chapter 1]); $\beta(t,x)$, $\gamma_I(t,x)$, $\gamma_E(t,x)$ and $d(t,x)$ are Hölder continuous and nonnegative nontrivial on $\mathbb{R} \times \overline{\Omega}$, and uniformly almost periodic in t; D_3 and D_E are positive constants; ν denotes the outward unit normal vector on $\partial\Omega$. Since $\gamma_E(t,x)$ and $d(t,x)$ are uniformly almost periodic in t, it follows from [13, Chapter 1] that $\Gamma(t,s,x,y)$ is uniformly almost periodic in t and s, that is, for any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$ such that every interval of $\mathbb{R}$ of length l contains at least one point of the set

$$T(\Gamma,\epsilon) = \bigcap_{x,y\in\Omega} \{d \in \mathbb{R} : |\Gamma(t+d, s+d, x, y) - \Gamma(t,s,x,y)| < \epsilon,\ \forall t > s \geq 0\}.$$

For the sake of simplicity, let $D_3 = D$ and $k(t,x) = \gamma_I(t,x) + d(t,x)$, it follows that (2.1) can be rewritten as

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = D\Delta w(t,x) - k(t,x)w(t,x) \\ \qquad\qquad + \displaystyle\int_\Omega \Gamma(t, t-\tau, x, y)\beta(t-\tau, y)w(t-\tau, y)dy, \quad t > 0,\ x \in \Omega, \\ \dfrac{\partial w(t,x)}{\partial \nu} = 0, \quad t > 0,\ x \in \partial\Omega. \end{cases} \tag{2.2}$$

Define the hull of k as $H(k) = \mathrm{cls}\{k \cdot s : s \in \mathbb{R}\}$, where $(k \cdot s)(t,x) = k(t+s,x)$ and the closure is taken under the compact open topology. Similarly, we define the hulls of Γ and β, denoted by $H(\Gamma)$ and $H(\beta)$, respectively. Let $\zeta = (k, \Gamma, \beta)$ and $H(\zeta)$ be the hull of ζ. Taking $\varsigma = (\bar{k}, \bar{\Gamma}, \bar{\beta}) \in H(\zeta)$, the translation map $\sigma : \mathbb{R} \times H(\zeta) \to H(\zeta)$, $(s,\varsigma) \mapsto \varsigma \cdot s$ given by $(\varsigma \cdot s)(t,x,y) = (\bar{k}(t+s,x), \bar{\Gamma}(t+s, t-\tau+s, x, y), \bar{\beta}(t+s,x))$ ($t \in \mathbb{R}$ and $x, y \in \overline{\Omega}$) defines a compact, almost periodic minimal and distal flow (see [32, Section VI.C]). Consider

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = D_2 \Delta w(t,x) - \bar{k}(t,x)w(t,x) \\ \qquad\qquad + \displaystyle\int_\Omega \bar{\Gamma}(t, t-\tau, x, y)\bar{\beta}(t-\tau, y)w(t-\tau, y)dy, \quad t > 0,\ x \in \Omega, \\ \dfrac{\partial w(t,x)}{\partial \nu} = 0, \quad t > 0,\ x \in \partial\Omega. \end{cases} \tag{2.3}$$

By the argument to that in the proof of [38, Theorem 2.2], we get that equation (2.3) has a unique mild solution $w(t,x;\phi,\varsigma)$ with initial datum $\phi \in X^+ := C([-\tau,0], X_0^+)$. Moreover, $w(t,x;\phi,\varsigma)$ is a classical solution for $t > \tau$. Define $w_t(\phi,\varsigma)(\theta,x) := w(t+\theta, x;\phi,\varsigma)$, $\forall t \geq 0, \theta \in [-\tau,0], x \in \overline{\Omega}$. Then we can define a continuous linear skew-product semiflow:

$$\begin{aligned} \Pi : \mathbb{R}_+ \times X \times H(\zeta) &\to X \times H(\zeta), \\ (t,\phi,\varsigma) &\mapsto (w_t(\phi,\varsigma), \varsigma \cdot t). \end{aligned} \tag{2.4}$$

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It follows from [27, Corollary 1] that the skew-product semiflow (2.4) is monotone. Let $\Phi(t,\varsigma)\phi = w_t(\phi,\varsigma)$, $\forall \phi \in X$. For any $\varsigma \in H(\zeta)$, we define the Lyapunov exponent λ_ς as

$$\lambda_\varsigma = \limsup_{t\to\infty} \frac{\ln \|\Phi(t,\varsigma)\|}{t}.$$

The number,

$$\lambda^* = \sup_{\varsigma\in H(\zeta))} \lambda_\varsigma,$$

is called the upper Lyapunov exponent of (2.3) or (2.4).

To proceed further, we introduce the definition of a continuous separation of type II, which was introduced in Novo et al. [30]. We say that the skew-product semiflow (2.4) admits a continuous separation of type II if there exist subspaces $\{X_1(\varsigma)\}_{\varsigma\in H(\zeta)}$ and $\{X_2(\varsigma)\}_{\varsigma\in H(\zeta)}$ with the following properties:

(1) $X = X_1(\varsigma) \oplus X_2(\varsigma)$ $(\varsigma \in H(\zeta))$ and $X_1(\varsigma), X_2(\varsigma)$ vary continuously in $\varsigma \in H(\zeta)$;
(2) $X_1(\varsigma) = \text{span}\{w(\varsigma)\}$, where $w(\varsigma) \in \text{Int}(X^+)$ and $\|w(\varsigma)\| = 1$ for $\varsigma \in H(\zeta)$;
(3) there is a $T > 0$ such that if for some $\varsigma \in H(\zeta)$ there is a $v \in X_2(\varsigma)$ with $v > 0$, then $\Phi(t,\varsigma)v = 0$ for any $t \geq T$;
(4) for any $t > 0$, $\varsigma \in H(\zeta)$,

$$\Phi(t,\varsigma)X_1(\varsigma) = X_1(\varsigma\cdot t),$$

$$\Phi(t,\varsigma)X_2(\varsigma) \subset X_2(\varsigma\cdot t);$$

(5) there exist $\eta_2 > 0, K_2 > 0$ such that for any $\varsigma \in H(\zeta)$ and $\widehat{w} \in X_2(\varsigma)$ with $\|\widehat{w}\| = 1$, we have

$$\|\Phi(t,\varsigma)\widehat{w}\| \leq K_2 e^{-\eta_2 t}\|\Phi(t,\varsigma)w(\varsigma)\|,$$

for all $t > 0$.

It is clear that if $\phi = \hat{0}$, then $w(t,x;\phi,\varsigma) = 0$, $\forall t > 0$, $x \in \overline{\Omega}$. By the similar arguments to those in [18, Section 2], we get $w(t,x;\phi,\varsigma) > 0$, $\forall t > \tau$, $x \in \overline{\Omega}$, $\phi \in X^+$ with $\phi \neq \hat{0}$. Moreover, $\Phi(t,\varsigma)$ is compact for any $t > \tau$, $\varsigma \in H(\zeta)$ ([35, Theorem 2.1.8]). It then follows from [30, Theorem 5.4] that the following dynamical property holds (see also [29, Section 5] for the existence of a continuous separation of type II for partial functional differential equations).

Lemma 2.1 *The skew-product semiflow* (2.4) *admits a continuous separation of type II.*

Because of properties (2) and (4) of continuous separation, as mentioned in [31], we can write

$$\Phi(t,\varsigma)w(\varsigma) = \bar{r}(t,\varsigma)w(\varsigma\cdot t)$$

for a certain real coefficient $\bar{r}(t,\varsigma)$ for each $t > 0$ and $\varsigma \in H(\zeta)$, which is known to be positive. However, $\bar{r}(t,\varsigma)$ might not be always differentiable. Motivated by the argument in the proof of [7, Theorem 4.14], we can associate a one-dimensional cocycle with the same behaviour as that of $\bar{r}(t,\varsigma)$, which is further differentiable. Taking a point $x_0 \in \Omega$. Let

$$v_1(\varsigma) = \frac{w(\varsigma)}{w(\varsigma)(0,x_0)} \in X, \quad \forall \varsigma \in H(\zeta). \tag{2.5}$$

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It is easy to check that $\Phi(t, \varsigma\cdot(-2\tau))v_1(\varsigma\cdot(-2\tau)) = \tilde{r}(t, \varsigma\cdot(-2\tau))v_1(\varsigma\cdot(t-2\tau))$ for the positive coefficient

$$\tilde{r}(t, \varsigma\cdot(-2\tau)) = w(t, x_0; v_1(\varsigma\cdot(-2\tau)), \varsigma\cdot(-2\tau)), \quad \forall \varsigma\in H(\zeta),\ t\geq 0.$$

Here, $w(t, x; v_1(\varsigma\cdot(-2\tau)), \varsigma\cdot(-2\tau))$ is the solution of the following equation with initial datum $v_1(\varsigma\cdot(-2\tau))$:

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = D_2\Delta w(t,x) - \check{k}(t,x)w(t,x) \\ \qquad\qquad + \displaystyle\int_\Omega \check{\Gamma}(t, t-\tau, x, y)\check{\beta}(t-\tau, y)w(t-\tau, y)dy, \quad t>0,\ x\in\Omega, \\ \dfrac{\partial w(t,x)}{\partial \nu} = 0, \quad t>0,\ x\in\partial\Omega, \end{cases}$$

where $(\check{k}, \check{\Gamma}, \check{\beta}) = \varsigma\cdot(-2\tau)\in H(\zeta)$. Note that $w(t, x; v_1(\varsigma\cdot(-2\tau)), \varsigma\cdot(-2\tau))$ is a classical solution for $t>\tau$. It follows that $\tilde{r}(t, \varsigma\cdot(-2\tau))$ defines a one-dimensional differentiable linear cocycle for $t>\tau$.

Theorem 2.2 *There exist two almost periodic functions $a\in C(\mathbb{R}, \mathbb{R})$, $\tilde{w}\in \mathrm{Int}(C(\mathbb{R}, X_0^+))$ such that $w(t,x) = e^{\int_0^t a(s)ds}\tilde{w}(t,x)$ is a solution of equation (2.2). Furthermore,*

$$\lambda^* = \lim_{t\to\infty}\frac{1}{t}\int_0^t a(s)ds.$$

Proof. It suffices to prove the desired results for system (2.3) with $\varsigma\in H(\zeta)$. Let

$$\varphi := w_{2\tau}(x, v_1(\varsigma\cdot(-2\tau)), \varsigma\cdot(-2\tau))\in X$$

and $w(t, x; \varphi, \varsigma)$ be the solution of equation (2.3) with initial value φ. We then have $w_t(\varphi, \varsigma) = \tilde{r}(t+2\tau, \varsigma\cdot(-2\tau))v_1(\varsigma\cdot t)$. Since $\tilde{r}(t, \varsigma\cdot(-2\tau))$ is differential in t for $t>\tau$, it follows that $r(t, \varsigma) := \frac{1}{\tilde{r}(2\tau, \varsigma\cdot(-2\tau))}\tilde{r}(t+2\tau, \varsigma\cdot(-2\tau))$ is differential in t for $t\geq 0$. Thus, the map $a(\varsigma) := \frac{d}{dt}\ln r(t, \varsigma)|_{t=0}$ is well defined and continuous on $H(\zeta)$, and

$$r(t, \varsigma) = e^{\int_0^t a(\varsigma\cdot s)ds}, \quad \forall\varsigma\in H(\zeta),\ t\geq 0.$$

The continuity of $a(\varsigma)$ on $H(\zeta)$ yields that $a(\varsigma\cdot t)$ is almost periodic in t. Moreover, the properties (1) and (2) of continuous separation together with (2.5) imply that $\tilde{w}(t, x; \varsigma) := \tilde{r}(2\tau, \varsigma\cdot(-2\tau))(v_1(\varsigma\cdot t)(0, x))$ is uniformly almost periodic in t, and $\tilde{w}\in \mathrm{Int}(C(\mathbb{R}, X_0^+))$. Recall $w_t(\varphi, \varsigma) = \tilde{r}(t+2\tau, \varsigma\cdot(-2\tau))v_1(\varsigma\cdot t)$, it follows that $w(t, x; \varphi, \varsigma) = e^{\int_0^t a(\varsigma\cdot s)ds}\tilde{w}(t, x; \varsigma)$ is a solution of equation (2.3).

The existence of the continuous separation of linear skew-product semiflow (2.4) means that

$$\lambda^* = \sup_{\varsigma\in H(\zeta))}\limsup_{t\to\infty}\frac{\ln\|\Phi(t, \varsigma)\varphi\|}{t}.$$

Since $a(\varsigma\cdot t)$ is almost periodic, we obtain that $\lim_{t\to\infty}\frac{1}{t}\int_0^t a(\varsigma\cdot s)ds$ exists and is independent of $\varsigma\in H(\zeta)$ (see, e.g., [17, Lemma 3.2]). It then follows that $\lim_{t\to\infty}\frac{\ln\|\Phi(t,\varsigma)\varphi\|}{t} =$

$\lim_{t\to\infty}\frac{1}{t}\int_0^t a(\varsigma\cdot s)ds$ exists and is independent of $\varsigma\in H(\zeta)$. Hence,

$$\lambda^*=\lim_{t\to\infty}\frac{1}{t}\int_0^t a(\varsigma\cdot s)ds,$$

for any $\varsigma\in H(\zeta)$. □

Lemma 2.3 *Choose $\phi\in\mathrm{Int}(X^+)$, and let $w(t,x;\phi)$ be the solution of equation (2.2) with initial value $w_0=\phi$. Then $\frac{\ln w(t,x;\phi)}{t}$ converges uniformly to λ^* as $t\to\infty$ for all $x\in\overline{\Omega}$, that is,*

$$\lambda^*=\lim_{t\to\infty}\frac{\ln w(t,x_0;\phi)}{t},\quad \forall x_0\in\overline{\Omega}. \tag{2.6}$$

Proof. By Theorem 2.2, there exist two almost periodic functions $a(t)$ and $\tilde{w}(t,x)$ such that $w(t,x)=e^{\int_0^t a(s)ds}\tilde{w}(t,x)$ is a solution of equation (2.2). Since $\tilde{w}\in\mathrm{Int}(C(\mathbb{R},X_0^+))$, it follows that there exist two positive constants k and K such that

$$ke^{\int_0^\theta a(s)ds}\tilde{w}(\theta,x)\le\phi(\theta,x)\le Ke^{\int_0^\theta a(s)ds}\tilde{w}(\theta,x),\quad \forall\theta\in[-\tau,0],\ x\in\overline{\Omega}.$$

The comparison principle for abstract functional differential equation (see, e.g., [27, Proposition 3]) yields

$$ke^{\int_0^t a(s)ds}\tilde{w}(t,x)\le w(t,x;\phi)\le Ke^{\int_0^t a(s)ds}\tilde{w}(t,x),\quad \forall t\ge 0,\ x\in\overline{\Omega}.$$

Thus,

$$\frac{\ln k}{t}+\frac{\int_0^t a(s)ds}{t}+\frac{\ln\tilde{w}(t,x)}{t}\le\frac{w(t,x,\phi)}{t}\le\frac{\ln K}{t}+\frac{\int_0^t a(s)ds}{t}+\frac{\ln\tilde{w}(t,x)}{t},\quad \forall t>0,\ x\in\overline{\Omega}.$$

It then follows that there exist two positive constants m and M such that

$$\frac{m}{t}+\frac{\int_0^t a(s)ds}{t}\le\frac{w(t,x,\phi)}{t}\le\frac{M}{t}+\frac{\int_0^t a(s)ds}{t},\quad \forall t>0,\ x\in\overline{\Omega},$$

and hence, $\lambda^*=\lim_{t\to\infty}\frac{\ln w(t,x_0;\phi)}{t}$, $\forall x_0\in\overline{\Omega}$. □

Remark 2.4 *Lemma 2.3 provides a method to compute the upper Lyapunov exponent λ^* numerically (see Section 5 for details). The proof of Lemma 2.3 further indicates that the result remains valid for the almost periodic reaction–diffusion equations (with or without time delay) which have a solution as in Theorem 2.2 and in which the solution semiflows are monotone.*

Next we consider the perturbation equation of (2.2) with a positive parameter $\epsilon<1$:

$$\begin{cases}\dfrac{\partial w(t,x)}{\partial t}=D_2\Delta w(t,x)-k(t,x)w(t,x)\\ \qquad\qquad+(1-\epsilon)\displaystyle\int_\Omega\Gamma(t,t-\tau,x,y)\beta(t-\tau,y)w(t-\tau,y)dy,\quad t>0,\ x\in\Omega,\\ \dfrac{\partial w(t,x)}{\partial\nu}=0,\quad t>0,\ x\in\partial\Omega.\end{cases} \tag{2.7}$$

Let λ_ϵ^* be the upper Lyapunov exponent associated with (2.7).

Lemma 2.5 $\lim_{\epsilon\to 0}\lambda_\epsilon^*=\lambda^*$.

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Proof. Since $\epsilon < 1$, there exists a $\delta > 0$ such that $1-\epsilon = e^{-\delta\tau}$. Consider

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = D_2\Delta w(t,x) - (k(t,x)+\delta)w(t,x) \\ \qquad\qquad + e^{-\delta\tau}\displaystyle\int_\Omega \Gamma(t,t-\tau,x,y)\beta(t-\tau,y)w(t-\tau,y)dy, \quad t>0,\ x\in\Omega, \\ \dfrac{\partial w(t,x)}{\partial \nu} = 0, \quad t>0,\ x\in\partial\Omega. \end{cases} \tag{2.8}$$

Recall that $e^{\int_0^t a(s)ds}\tilde{w}(t,x)$ is a solution of (2.2). It then follows that $\tilde{w}_\delta(t,x) = e^{\int_0^t a(s)ds-\delta t}\tilde{w}(t,x)$ is a solution of (2.8). Let λ_δ^* denote the upper Lyapunov exponent associated with (2.8), we get $\lambda_\delta^* = \lambda^* - \delta$, and hence, $\lim_{\delta\to 0}\lambda_\delta^* = \lambda^*$. Note that $\delta\to 0$ implies that $\epsilon\to 0$. By the comparison principle, we further obtain $\lambda_\delta^* \le \lambda_\epsilon^* \le \lambda^*$. Thus, $\lim_{\epsilon\to 0}\lambda_\epsilon^* = \lambda^*$. □

3 Model formulation

In this section, we follow the ideas in [24, 38] to derive an almost periodic reaction–diffusion epidemic model with latency. Assume that the population live in the spatial habitat Ω with smooth boundary $\partial\Omega$ and the population performs an unbiased random walk. In the absence of disease, we suppose that the population changes according to a population growth equation:

$$\frac{\partial N(t,x)}{\partial t} = D_N\Delta N(t,x) + \Lambda(t,x) - d(t,x)N(t,x), \tag{3.1}$$

where D_N denotes the diffusion coefficient; $\Lambda(t,x)$ and $d(t,x)$ represent the input and natural death rates of individuals, respectively. We assume that all populations remain confined to the region Ω for all time and supplement the Neumann boundary condition to the above equation, that is,

$$\frac{\partial N(t,x)}{\partial \nu} = 0,\ \forall t>0,\ x\in\partial\Omega.$$

Let $S(t,x)$, $E(t,x)$, $I(t,x)$ and $R(t,x)$ be the numbers of susceptible, latent, infected and recovered individuals at time t and location x, respectively. Due to the mobility of the population during incubation period, we introduce a variable a represented infection age. Let $A(t,a,x)$ denote the number of infected population with infection age a at time t and location x. By the standard arguments on structured population with spatial diffusion (see, e.g., [28]), we obtain

$$\frac{\partial A(t,a,x)}{\partial t} + \frac{\partial A(t,a,x)}{\partial a} = D(a)\Delta A(t,a,x) - (\gamma(t,a,x) + d(t,x))A(t,a,x), \tag{3.2}$$

where $D(a)$ represents the diffusion rate at infection age a, and $\gamma(t,a,x)$ denotes the disease recovery rate at time t and location x with infection age a. Assume that the average incubation period is τ. It then follows that

$$E(t,x) = \int_0^\tau A(t,a,x)da, \qquad I(t,x) = \int_\tau^\infty A(t,a,x)da. \tag{3.3}$$

Let D_i and $\gamma_i(t,x)$ represent the diffusion and recovery rates of i class, $i = E, I$. We then have

$$D(a) = \begin{cases} D_E, & a\in[0,\tau], \\ D_I, & a\in[\tau,\infty], \end{cases}$$

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and

$$\gamma(t,a,x)=\begin{cases}\gamma_E(t,x), & a\in[0,\tau],\ t\geq 0,\ x\in\Omega,\\ \gamma_I(t,x), & a\in[\tau,\infty],\ t\geq 0,\ x\in\Omega.\end{cases}$$

Due to possible means such as quarantine and blocker drug, $\gamma_E(t,x)$ could be nontrivial, that is, some individuals could proceed directly from latent class to recovered class.

Integrating both sides of (3.2) from 0 to τ, it then follows from (3.3) that

$$\frac{\partial E(t,x)}{\partial t}=D_E\Delta E(t,x)-(\gamma_E(t,x)+d(t,x))E(t,x)+A(t,0,x)-A(t,\tau,x).$$

Similarly, integrating both sides of (3.2) from τ to ∞, we get

$$\frac{\partial I(t,x)}{\partial t}=D_I\Delta I(t,x)-(\gamma_I(t,x)+d(t,x))I(t,x)+A(t,\tau,x)-A(t,\infty,x).$$

Biologically, we suppose $A(t,\infty,x)=0$. It is well known that the new infections are caused by the contact between susceptible and infectious individuals, we adopt the standard incidence infection mechanism. It then follows that

$$A(t,0,x)=\beta(t,x)\frac{S(t,x)I(t,x)}{N(t,x)},$$

where $\beta(t,x)>0$ is called effective contact rate, and $N(t,x)=S(t,x)+E(t,x)+I(t,x)+R(t,x)$.

On the basis of the above assumptions, the dynamics of disease transmission is governed by the following system of partial differential equations:

$$\begin{cases}\dfrac{\partial S(t,x)}{\partial t}=D_S\Delta S(t,x)+\Lambda(t,x)-\beta(t,x)\dfrac{S(t,x)I(t,x)}{N(t,x)}-d(t,x)S(t,x),\\ \dfrac{\partial E(t,x)}{\partial t}=D_E\Delta E(t,x)+\beta(t,x)\dfrac{S(t,x)I(t,x)}{N(t,x)}-\gamma_E(t,x)E(t,x)\\ \qquad\qquad -d(t,x)E(t,x)-A(t,\tau,x),\\ \dfrac{\partial I(t,x)}{\partial t}=D_I\Delta I(t,x)-\gamma_I(t,x)I(t,x)-d(t,x)I(t,x)+A(t,\tau,x),\\ \dfrac{\partial R(t,x)}{\partial t}=D_R\Delta R(t,x)+\gamma_E(t,x)E(t,x)+\gamma_I(t,x)I(t,x)-d(t,x)R(t,x),\\ \dfrac{\partial S(t,x)}{\partial \nu}=\dfrac{\partial E(t,x)}{\partial \nu}=\dfrac{\partial I(t,x)}{\partial \nu}=\dfrac{\partial R(t,x)}{\partial \nu}=0, \quad t>0,\ x\in\partial\Omega,\end{cases} \tag{3.4}$$

where D_S and D_R denote the diffusion rates of susceptible and recovered classes. We make the following assumptions:

- **(A1)** $\Lambda(t,x)$, $\beta(t,x)$, $\gamma_E(t,x)$, $\gamma_I(t,x)$ and $d(t,x)$ are Hölder continuous and nonnegative nontrivial on $\mathbb{R}\times\overline{\Omega}$, and uniformly almost periodic in t.
- **(A2)** There exist two positive constants Λ_0, d_0 such that $\Lambda(t,x)>\Lambda_0$, $d(t,x)>d_0$ for all $t\in\mathbb{R}$ and $x\in\overline{\Omega}$, and $D_i>0$ for $i=S,E,I,R$.

It is then necessary to determine $A(t,\tau,x)$, which can be done by the integration along characteristics. Let $v(z,a,x)=A(a+z,a,x)$, $\forall z\geq 0$, and consider the solutions of (3.2) along the

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characteristic line $t = a + z$. Then for $a \in [0, \tau]$, we get

$$\begin{cases} \dfrac{\partial v(z,a,x)}{\partial a} = [\dfrac{\partial A(t,a,x)}{\partial t} + \dfrac{\partial A(z,a,x)}{\partial a}]_{t=a+z} \\ \qquad\qquad = D_a \Delta A(a+z,a,x) - (\gamma(a+z,a,x) + d(a+z,x))A(a+z,a,x) \\ \qquad\qquad = D_E \Delta v(z,a,x) - (\gamma_E(a+z,x) + d(a+z,x))v(z,a,x), \\ v(z,0,x) = E(z,0,x) = \beta(z,x)\dfrac{S(z,x)I(z,x)}{N(z,x)}. \end{cases}$$

Regarding z as a parameter and integrating the last equation, we obtain

$$v(z,a,x) = \int_\Omega \Gamma(z+a,z,x,y)\beta(z,y)\frac{S(z,y)I(z,y)}{N(z,y)}dy,$$

where $\Gamma(t,s,x,y)$ with $t > s \geq 0$ and $x, y \in \Omega$ is the fundamental solution associated with the partial differential operator $\partial_t - D_E\Delta - d(t,\cdot) - \gamma_E(t,\cdot)$ subject to the Neumann boundary condition (see [13, Chapter 1]). The uniformly almost periodicity of $d(t,x)$ and $\gamma_E(t,x)$ yields that $\Gamma(t,s,x,y)$ is uniformly almost periodic in t and s. Since $A(t,a,x) = v(t-a,a,x)$, $\forall t \geq a$, we get

$$A(t,a,x) = \int_\Omega \Gamma(t,t-a,x,y)\beta(t-a,y)\frac{S(t-a,y)I(t-a,y)}{N(t-a,y)}dy.$$

Taking $a = \tau$, we have

$$A(t,\tau,x) = \int_\Omega \Gamma(t,t-\tau,x,y)\beta(t-\tau,y)\frac{S(t-\tau,y)I(t-\tau,y)}{N(t-\tau,y)}dy. \tag{3.5}$$

For the sake of simplicity, moreover, we let $(u_1,u_2,u_3,u_4) = (S,E,I,R)$ and $(D_1,D_2,D_3,D_4) = (D_S,D_E,D_I,D_R)$. It then follows from (3.5) that model (3.4) can be rewritten as the following time-delayed and nonlocal almost periodic reaction–diffusion system with no flux boundary condition:

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = D_1\Delta u_1(t,x) + \Lambda(t,x) - \beta(t,x)\dfrac{u_1(t,x)u_3(t,x)}{\Sigma_{i=1}^4 u_i(t,x)} - d(t,x)u_1(t,x), \\ \dfrac{\partial u_2(t,x)}{\partial t} = D_2\Delta u_2(t,x) + \beta(t,x)\dfrac{u_1(t,x)u_3(t,x)}{\Sigma_{i=1}^4 u_i(t,x)} - (\gamma_E(t,x) + d(t,x))u_2(t,x) \\ \qquad\qquad - \displaystyle\int_\Omega \Gamma(t,t-\tau,x,y)\beta(t-\tau,y)\frac{u_1(t-\tau,y)u_3(t-\tau,y)}{\Sigma_{i=1}^4 u_i(t-\tau,y)}dy, \\ \dfrac{\partial u_3(t,x)}{\partial t} = D_3\Delta u_3(t,x) - (\gamma_I(t,x) + d(t,x))u_3(t,x) \\ \qquad\qquad + \displaystyle\int_\Omega \Gamma(t,t-\tau,x,y)\beta(t-\tau,y)\frac{u_1(t-\tau,y)u_3(t-\tau,y)}{\Sigma_{i=1}^4 u_i(t-\tau,y)}dy, \\ \dfrac{\partial u_4(t,x)}{\partial t} = D_4\Delta u_4(t,x) + \gamma_E(t,x)u_2(t,x) + \gamma_I(t,x)u_3(t,x) - d(t,x)u_4(t,x), \\ \dfrac{\partial u_i(t,x)}{\partial \nu} = 0, \quad t > 0,\ x \in \partial\Omega,\ 1 \leq i \leq 4. \end{cases} \tag{3.6}$$

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Model (3.6) is an almost periodic version of the classical Kermack–McKendrick epidemic model with the standard incidence infection mechanism, which incorporates disease latency, spatial heterogeneity and general seasonal fluctuations. The periodic Kermack–McKendrick model with latency has been studied extensively (see, e.g., [37, 38]).

From a biological point of view, there is an interesting question whether the outbreak of disease affects the total population size over the whole region. Note that if $D_i = D_N$, $1 \le i \le 4$, we then obtain that the change of total population size $\Sigma_{i=1}^4 u_i(t,x)$ follows equation (3.1) subject to Neumann boundary condition when there is a disease outbreak, which indicates that population dynamics are not affected by disease transmission. Moreover, in the case that natural death rate is a spatially homogeneous function $d(t)$, it is easy to verify that the total population size over the whole region $\mathcal{N}(t) = \int_{\overline{\Omega}} \Sigma_{i=1}^4 u_i(t,x)dx$ satisfies the following equation when there is a disease outbreak or no disease outbreak:

$$\frac{d\mathcal{N}(t)}{dt} = \Lambda^*(t) - d(t)\mathcal{N}(t),$$

where $\Lambda^*(t) = \int_{\overline{\Omega}} \Lambda(t,x)dx$. Thus, in the case where $D_i = D_N$ $(1 \le i \le 4)$ or $d(t,x) = d(t)$, $\forall x \in \overline{\Omega}$, the outbreak of disease does not affect the total population size over the whole region. But for more general cases, we cannot obtain concrete conclusion, and it needs to make further study.

Let $Y_0 := C(\overline{\Omega}, \mathbb{R}^4)$ be the Banach space with the supremum norm $\|\cdot\|_{Y_0}$, and let $Y_0^+ := C(\overline{\Omega}, \mathbb{R}_+^4)$. Define $Y := C([-\tau, 0], Y_0)$ and $Y^+ := C([-\tau, 0], Y_0^+)$. The norm of Y is defined by $\|\varphi\|_Y = \max_{\theta\in[-\tau,0]} \|\phi(\theta)\|_{Y_0}$, $\forall \phi \in Y$. By the arguments similar to those given in the proof of [38, Theorem 2.2], we have the following result.

Lemma 3.1 *For any $\phi \in Y^+$, system (3.6) admits a unique mild solution $u(t,x;\phi) = (u_1(t,x;\phi), u_2(t,x;\phi), u_3(t,x;\phi), u_4(t,x;\phi))$ on $[0,\infty)$ with initial value ϕ. Moreover, $u(t,x;\phi)$ is a classical solution when $t > \tau$.*

To proceed further, we need some information on the following almost periodic reaction–diffusion equation:

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = D\Delta w(t,x) + \lambda(t,x) - h(t,x)w(t,x), & t > 0,\ x \in \Omega, \\ \dfrac{\partial w(t,x)}{\partial \nu} = 0, & t > 0,\ x \in \partial\Omega, \end{cases} \tag{3.7}$$

where D is a positive constant, $\lambda(t,x)$ and $h(t,x)$ are uniformly almost periodic in t. Moreover, we assume that $\lambda(t,x)$ and $h(t,x)$ are Hölder continuous for $t \in \mathbb{R}$, $x \in \overline{\Omega}$, and there are two positive constants λ_0 and h_0 such that $\lambda(t,x) > \lambda_0$ and $h(t,x) > h_0$ for all $t \in \mathbb{R}$ and $x \in \overline{\Omega}$, respectively.

Lemma 3.2 *System (3.7) admits a unique positive almost periodic solution $w^*(t,\cdot)$ which is globally attractive in X_0^+.*

Proof. Let $H(\lambda, h)$ be the hull of (λ, h). The translation map $\mathbb{R} \times H(\lambda, h) \to H(\lambda, h)$, $(s, \mu, g) \mapsto (\mu, g)\cdot s$ given by $((\mu, g)\cdot s)(t,x) = (\mu(t+s,x), g(t+s,x))$ $(t \in \mathbb{R}$ and $x \in \overline{\Omega})$ defines a compact, almost periodic minimal and distal flow.

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For any $\phi \in X_0^+$ and $(\mu, g) \in H(\lambda, h)$. Let

$$\mu^+ = \sup_{t\in\mathbb{R}, x\in\overline{\Omega}} \mu(t,x), \quad \mu^- = \inf_{t\in\mathbb{R}, x\in\overline{\Omega}} \mu(t,x),$$
$$g^+ = \sup_{t\in\mathbb{R}, x\in\overline{\Omega}} g(t,x), \quad g^- = \inf_{t\in\mathbb{R}, x\in\overline{\Omega}} g(t,x), \quad \phi^+ = \max_{x\in\overline{\Omega}} \phi(x).$$

It then follows that any constant $M \geq \max\{\phi^+, \frac{\mu^+}{g^-}\}$ is an upper solution of the following parabolic problem:

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = D\Delta w(t,x) + \mu(t,x) - g(t,x)w(t,x), & t>0, \ x\in\Omega, \\ \dfrac{\partial w(t,x)}{\partial \nu} = 0, & t>0, \ x\in\partial\Omega, \\ w(0,x) = \phi, & x\in\overline{\Omega}. \end{cases} \tag{3.8}$$

The comparison theorem together with a priori estimates of parabolic equations (see, e.g., [16]) imply that (3.8) has a unique solution $w(t,x;\phi,\mu,g)$ on $[0,\infty)$, and for any $\delta > 0$, the set $\{w(t,\cdot;\phi,\mu,g) : t \geq \delta\}$ is relatively compact in X_0^+. We define the skew-product semiflow

$$\tilde{\Pi} : \mathbb{R}_+ \times X_0^+ \times H(\lambda, h) \to X_0^+ \times H(\lambda, h),$$
$$(t,\phi,\mu,g) \mapsto (w(t,\cdot;\phi,\mu,g), \mu\cdot t, g\cdot t).$$

We use the notation $\tilde{\Pi}_t(\phi,\mu,g) = \tilde{\Pi}(t,\phi,\mu,g)$. Then, for each $(\phi,\mu,g) \in X_0^+ \times H(\lambda,h)$, the omega limit set $\omega(\phi,\mu,g)$ of the forward orbit $\gamma^+(\phi,\mu,g) := \{\tilde{\Pi}_t(\phi,\mu,g) : t \geq 0\}$ is well defined, compact and invariant under $\tilde{\Pi}_t$, $t \geq 0$. By the standard comparison arguments, we get that for any $(\phi,\mu,g) \in X_0^+ \times H(\lambda,h)$, the omega limit set $\omega(\phi,\mu,g)$ satisfies

$$\omega(\phi,\mu,g) \subset \{\varphi \in X_0^+ : \frac{\mu^-}{g^+} \leq \varphi \leq \frac{\mu^+}{g^-}\} \times H(\lambda,h).$$

Let $w(\phi,\mu,g,t) := w(t,\cdot;\phi,\mu,g)$. It then follows from the comparison principle that $w(\cdot,\mu,g,t)$ is strongly monotone on X_0^+ for each $(\mu,g,t) \in H(\lambda,h) \times (0,\infty)$. Note that $f(t,x,w) := \lambda(t,x) - h(t,x)w$ is strictly subhomogeneous in w, that is, $f(t,x,\kappa w) > \kappa f(t,x,w)$ for any $\kappa \in (0,1)$ and $w \gg 0$, and hence, each function $m(t,x) := \mu(t,x) - g(t,x)w$, $(\mu,g) \in H(\lambda.h)$, is strictly subhomogeneous in w for any fixed $(t,x) \in \mathbb{R}_+ \times \overline{\Omega}$. By the integral version of parabolic equation (3.8) (see, e.g., [25]), we get that $w(\phi,\mu,g,t)$ is subhomogeneous on X_0^+ for each $(\mu,g,t) \in H(\lambda,h) \times \mathbb{R}_+$, and $w(\phi,\mu,g,t)$ is strictly subhomogeneous on X_0^+ for each $t > 0$. Let $\phi_0 \in X_0^+$ be fixed and $W_0 = \omega(\phi_0,\lambda,h)$. By [39, Theorem 2.3.5 and Remarks 2.3.2–2.3.3], it follows that for every $\phi \in X_0^+$, $\lim_{t\to\infty} \|w(t,\cdot;\phi,\lambda,h) - w(t,\cdot;\phi^*,\lambda,h)\|_0 = 0$, where $(\phi^*,\lambda,h) \in W_0$. Since $\tilde{\Pi}_t : W_0 \to W_0$ is an almost periodic minimal flow, $\tilde{\Pi}_t(\phi^*,\lambda,h) = (w(t,\cdot;\phi^*,\lambda,h), \lambda_t, h_t)$ is an almost periodic motion (see [32, Lemma VI.9]). Therefore, $w(t,\cdot;\phi^*,\lambda,h)$ is a unique positive almost periodic solution of (3.7), which is globally attractive in X_0^+. □

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4 Threshold dynamics

In this section, we establish a threshold-type result on the extinction and uniform persistence of system (3.6).

Letting $u_i = 0$ $(2 \le i \le 4)$ in (3.6), we get the following equation for $u_1(t,x)$ of susceptible population:

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = D_1 \Delta u_1(t,x) + \Lambda(t,x) - d(t,x)u_1(t,x), & t>0,\ x \in \Omega, \\ \dfrac{\partial u_1(t,x)}{\partial \nu} = 0, & t>0,\ x \in \partial\Omega. \end{cases} \tag{4.1}$$

According to Lemma 3.2, equation (4.1) admits a positive solution $u_1^*(t,x)$ which is globally attractive and uniformly almost periodic in $t \in \mathbb{R}$. It then follows that system (3.6) admits a unique positive disease-free almost periodic solution $E^* = (u_1^*, 0, 0, 0)$. Linearising system (3.6) at E^*, we then obtain that infectious variable u_3 satisfies equation (2.1). Now we show that λ^* is a threshold value for the global extinction and uniform persistence of the disease. Before proving the main results, we need the following Lemma.

Lemma 4.1 *Let $u(t,x;\phi) = (u_1(t,x;\phi), u_2(t,x;\phi), u_3(t,x;\phi), u_4(t,x;\phi))$ be the solution of (3.6) with initial datum $\phi \in Y^+$. There exist constants $L>0$, $T=T(\phi)$ such that*

$$u_i(t,x;\phi) < L, \quad \forall t \ge T,\ x \in \overline{\Omega},\ 1 \le i \le 4.$$

Proof. From the first equation of (3.6), we conclude that

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} \le D_1 \Delta u_1(t,x) + \Lambda(t,x) - d(t,x)u_1(t,x), & t>0,\ x \in \Omega, \\ \dfrac{\partial u_1(t,x)}{\partial \nu} = 0, & t>0,\ x \in \partial\Omega. \end{cases}$$

It then follows from the comparison theorem that for any $\phi \in Y^+$, there exist constants $L_1 > 0$ and $T_1 = T_1(\phi) > 0$ such that $u_1(t,x;\phi) < L_1$, $\forall t \ge T_1$, $x \in \overline{\Omega}$. Hence, we obtain that $u_3(t,x;\phi)$ satisfies

$$\begin{cases} \dfrac{\partial u_3(t,x)}{\partial t} \le D_3 \Delta u_3(t,x) - (\gamma_I(t,x) + d(t,x))u_3(t,x) \\ \qquad\qquad + L_1 \displaystyle\int_\Omega \Gamma(t,t-\tau,x,y)\beta(t-\tau,y)dy, \quad t>T_1,\ x \in \Omega, \\ \dfrac{\partial u_3(t,x)}{\partial \nu} = 0, \quad t>T_1,\ x \in \partial\Omega. \end{cases}$$

Thus, comparison theorem implies that there exist constants $L_3 > 0$ and $T_3 = T_3(\phi) > T_1$ such that $u_3(t,x;\phi) < L_3$, $\forall t \ge T_3$, $x \in \overline{\Omega}$. Hence, we get that $u_2(t,x;\phi)$ satisfies

$$\begin{cases} \dfrac{\partial u_2(t,x)}{\partial t} \le D_2 \Delta u_2(t,x) + L_3\beta(t,x) - (\gamma_E(t,x) + d(t,x))u_2(t,x), & t>T_3,\ x \in \Omega, \\ \dfrac{\partial u_3(t,x)}{\partial \nu} = 0, \quad t>T_3,\ x \in \partial\Omega. \end{cases}$$

By comparison theorem again, there exist $L_2 > 0$ and T_2 ($T_2 > T_3$) such that $u_2(t, x; \phi) < L_2$, $\forall t \geq T_2, x \in \overline{\Omega}$. Similarly, there exist $L_4 > 0$ and T_4 ($T_4 > T_2$) such that $u_4(t, x; \phi) < L_4$, $\forall t \geq T_4$, $x \in \overline{\Omega}$. Setting $L = \max\{L_1, L_2, L_3, L_4\}$ and $T = T_4$, we complete the proof. □

The subsequent results indicate that λ^* serves as a threshold value for the global extinction and uniform persistence of the disease.

Theorem 4.2 *Assume that* **(A1)** *and* **(A2)** *hold. Let* $u(t, x; \phi)$ *be the solution of system* (3.6) *with initial value* $\phi \in Y^+$. *If* $\lambda^* < 0$, *then* $\lim_{t\to\infty} \|u(t, \cdot; \phi) - E^*(t, \cdot)\| = 0$.

Proof. Let $A = (\Lambda, \beta, d, \gamma_E, \gamma_I, \Gamma)$ and $H(A)$ be the hull of A. For any $\vartheta = (\bar{\Lambda}, \bar{\beta}, \bar{d}, \bar{\gamma}_E, \bar{\gamma}_I, \bar{\Gamma}) \in H(A)$, the translation map $\mathbb{R} \times H(A) \to H(A)$, $(s, \vartheta) \mapsto \vartheta \cdot s$ given by $(\vartheta \cdot s)(t, x, y) = (\bar{\Lambda}(t+s, x), \bar{\beta}(t+s, x), \bar{d}(t+s, x), \bar{\gamma}_E(t+s, x), \bar{\gamma}_I(t+s, x), \bar{\Gamma}(t+s, t-\tau+s, x, y))$ $(t \in \mathbb{R},\ x, y \in \overline{\Omega})$ defines a compact, almost periodic minimal and distal flow. Consider

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = D_1 \Delta u_1(t,x) + \bar{\Lambda}(t,x) - \bar{\beta}(t,x) \dfrac{u_1(t,x) u_3(t,x)}{\Sigma_{i=1}^4 u_i(t,x)} - \bar{d}(t,x) u_1(t,x), \\ \dfrac{\partial u_2(t,x)}{\partial t} = D_2 \Delta u_2(t,x) + \bar{\beta}(t,x) \dfrac{u_1(t,x) u_3(t,x)}{\Sigma_{i=1}^4 u_i(t,x)} - (\bar{\gamma}_E(t,x) + \bar{d}(t,x)) u_2(t,x) \\ \qquad - \displaystyle\int_\Omega \bar{\Gamma}(t, t-\tau, x, y) \bar{\beta}(t-\tau, y) \dfrac{u_1(t-\tau, y) u_3(t-\tau, y)}{\Sigma_{i=1}^4 u_i(t-\tau, y)} dy, \\ \dfrac{\partial u_3(t,x)}{\partial t} = D_3 \Delta u_3(t,x) - (\bar{\gamma}_I(t,x) + \bar{d}(t,x)) u_3(t,x) \\ \qquad + \displaystyle\int_\Omega \bar{\Gamma}(t, t-\tau, x, y) \beta(t-\tau, y) \dfrac{u_1(t-\tau, y) u_3(t-\tau, y)}{\Sigma_{i=1}^4 u_i(t-\tau, y)} dy, \\ \dfrac{\partial u_4(t,x)}{\partial t} = D_4 \Delta u_4(t,x) + \bar{\gamma}_E(t,x) u_2(t,x) + \bar{\gamma}_I(t,x) u_3(t,x) - \bar{d}(t,x) u_4(t,x), \quad t > 0,\ x \in \Omega, \\ \dfrac{\partial u_i(t,x)}{\partial \nu} = 0, \quad t > 0,\ x \in \partial\Omega,\ 1 \leq i \leq 4. \end{cases} \tag{4.2}$$

By Lemma 3.1, system (4.2) admits a unique solution

$$u(t, x; \phi, \vartheta) = (u_1(t, x; \phi, \vartheta), u_2(t, x; \phi, \vartheta), u_3(t, x; \phi, \vartheta), u_4(t, x; \phi, \vartheta))$$

with initial datum $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in Y^+$. Define $u_t(\phi, \vartheta)(\theta, x) = u(t+\theta, x; \phi, \vartheta)$, $\forall t \geq 0$, $\theta \in [-\tau, 0]$, $x \in \overline{\Omega}$. The solution of (4.2) induces a skew-product semiflow:

$$\Pi^Y : \mathbb{R}_+ \times Y^+ \times H(A) \to Y^+ \times H(A),$$
$$(t, \phi, \vartheta) \mapsto (u_t(\phi, \vartheta), \vartheta \cdot t).$$

From the third equation of (4.2), $u_3(t, x; \phi, \vartheta)$ satisfies

$$\begin{cases} \dfrac{\partial u_3(t,x)}{\partial t} \leq D_3 \Delta u_3(t,x) - (\bar{\gamma}_I(t,x) + \bar{d}(t,x)) u_3(t,x) \\ \qquad + \displaystyle\int_\Omega \bar{\Gamma}(t, t-\tau, x, y) \bar{\beta}(t-\tau, y) u_3(t-\tau, y) dy, \quad t > 0,\ x \in \Omega, \\ \dfrac{\partial u_3(t,x)}{\partial \nu} = 0, \quad t > 0,\ x \in \partial\Omega. \end{cases} \tag{4.3}$$

Consider the comparison system of (4.3):

$$\begin{cases} \dfrac{\partial \tilde{u}_3(t,x)}{\partial t} = D_3\Delta \tilde{u}_3(t,x) - (\bar{\gamma}_I(t,x) + \bar{d}(t,x))\tilde{u}_3(t,x) \\ \qquad\qquad + \displaystyle\int_\Omega \bar{\Gamma}(t, t-\tau, x, y)\bar{\beta}(t-\tau, y)\tilde{u}_3(t-\tau, y)dy, \quad t > 0,\ x \in \Omega, \\ \dfrac{\partial \tilde{u}_3(t,x)}{\partial \nu} = 0, \quad t > 0,\ x \in \partial\Omega. \end{cases} \tag{4.4}$$

Due to $\lambda^* < 0$, it follows from Theorem 2.2 that there exist two almost periodic functions $a(\vartheta \cdot t)$ and $\tilde{w}(t, x; \vartheta)$ such that $w(t, x; \vartheta) = e^{\int_0^t a(\vartheta \cdot s)ds}\tilde{w}(t, x; \vartheta)$ is a solution of (4.4) and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t a(\vartheta \cdot s)ds = \lambda^* < 0.$$

Thus, the comparison principle indicates that $u_3(t, x; \phi, \vartheta) \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$. Note that $u_2(t, x; \phi, \vartheta)$ satisfies

$$\begin{cases} \dfrac{\partial u_2(t,x)}{\partial t} \le D_2\Delta u_2(t,x) + \bar{\beta}(t,x)u_3(t,x;\phi,\vartheta) - (\bar{\gamma}_E(t,x) + \bar{d}(t,x))u_2(t,x) \\ \dfrac{\partial u_2(t,x)}{\partial \nu} = 0, \quad t > 0,\ x \in \partial\Omega. \end{cases}$$

Let $\tilde{u}_2(t)$ be the solution of the following equation with initial datum $\tilde{u}_2(0) = \max_{x\in\overline{\Omega}} \phi_2(x)$,

$$\frac{du_2(t)}{dt} = p(t) - d_0u_2(t),$$

where $p(t) = \max_{x\in\overline{\Omega}}\{\bar{\beta}(t,x)u_3(t,x;\phi,\vartheta)\}$. It then follows from the comparison principle that $u_2(t, x; \phi, \vartheta) \le \tilde{u}_2(t)$, $\forall x \in \bar{\Omega}$. By [34, Theorem 2.6], we further obtain $\tilde{u}_2(t) \to 0$ as $t \to \infty$. Thus, $u_2(t, x; \phi, \vartheta) \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$. Repeating above process, we further have $u_4(t, x; \phi, \vartheta) \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$.

Let $\omega(\phi, \vartheta)$ denote the omega limit set of (ϕ, ϑ) for Π_t^Y ($\Pi_t^Y(\phi, \vartheta) = \Pi^Y(t, \phi, \vartheta)$), that is,

$$\omega(\phi, \vartheta) = \{(\phi^*, \vartheta^*) \in Y \times H(A) : \exists t_n \to \infty \text{ such that } \lim_{t_n\to\infty}(u_{t_n}(\phi, \vartheta), \vartheta_{t_n}) = (\phi^*, \vartheta^*)\}.$$

Since $\lim_{t\to\infty} u_i(t, \cdot, \phi, \vartheta) = 0, 2 \le i \le 4$, we get $\omega(\phi, \vartheta) \subset \{(\omega_1, \hat{0}, \hat{0}, \hat{0}, \vartheta) : \omega_1 \in X^+, \vartheta \in H(A)\}$. Note that $u_1(t, x; \phi, \vartheta)$ satisfies

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} \ge D_1\Delta u_1(t,x) + \bar{\Lambda}(t,x) - (\bar{\beta}(t,x) + \bar{d}(t,x))u_1(t,x), & t > 0,\ x \in \Omega, \\ \dfrac{\partial u_1(t,x)}{\partial \nu} = 0, & t > 0,\ x \in \partial\Omega, \end{cases}$$

which implies that $\lim_{t\to\infty} u_1(t, x; \phi, \vartheta) \ge \frac{\Lambda_0}{M_0}$, where $M_0 = \sup_{t\in\mathbb{R}, x\in\overline{\Omega}}(\beta(t, x) + d(t, x))$. Then, we have $\omega_1 \in \text{Int}(X^+)$ for every $(\omega_1, \hat{0}, \hat{0}, \hat{0}, \vartheta) \in \omega(\phi, \vartheta)$.

According to Lemma 3.2, the following equation:

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = D_1\Delta w(t,x) + \bar{\Lambda}(t,x) - \bar{d}(t,x)w(t,x), & t > 0,\ x \in \Omega, \\ \dfrac{\partial w(t,x)}{\partial \nu} = 0, & t > 0,\ x \in \partial\Omega, \end{cases}$$

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admits a solution $u_1^*(t,x;\vartheta)$, which is globally attractive and uniformly almost periodic in $t\in\mathbb{R}$. Define $u_1^*(\vartheta)(\theta,x)=u_1^*(\theta,x;\vartheta)$, $\forall\theta\in[-\tau,0]$, $x\in\overline{\Omega}$. Since every trajectory of the omega limit set has backward extension, it follows from Lemma 3.2 that $\omega_1=u_1^*(\vartheta)$ for every $(\omega_1,\hat{0},\hat{0},\hat{0},\vartheta)\in\omega(\phi,\vartheta)$. Thus,

$$\lim_{t\to\infty}\|(u_1(t,\cdot;\phi,\vartheta),u_2(t,\cdot;\phi,\vartheta),u_3(t,\cdot;\phi,\vartheta),u_4(t,\cdot;\phi,\vartheta))-(u_1^*(t,\cdot;\vartheta),0,0,0)\|=0.$$

We complete the proof. □

Theorem 4.3 *Assume that* **(A1)** *and* **(A2)** *hold. Let $u(t,x;\phi)$ be the solution of* (3.6) *with initial value $\phi\in Y^+$. If $\lambda^*>0$, then there exists an $\epsilon>0$ such that for any $\phi\in Y^+$ with $\phi_3(0,\cdot)\not\equiv 0$, we have*

$$\liminf_{t\to\infty}u_3(t,x;\phi)\ge\epsilon,$$

uniformly on $x\in\overline{\Omega}$.

Proof. Define

$$U_0:=\{\phi\in Y^+:\phi_3(0,\cdot)\not\equiv 0\},$$
$$\partial U_0:=Y^+\backslash U_0=\{\phi\in Y^+:\phi_3(0,\cdot)\equiv 0\},$$

and

$$P=Y^+\times H(A),\quad P_0=U_0\times H(A),\quad \partial P_0=\partial U_0\times H(A).$$

It is clear that P_0 and ∂P_0 are relatively open and closed in P, respectively. Let $u(t,x;\phi,\vartheta)=(u_1(t,x;\phi,\vartheta),u_2(t,x;\phi,\vartheta),u_3(t,x;\phi,\vartheta),u_4(t,x;\phi,\vartheta))$ be the unique solution of (4.2) with initial value $\phi=(\phi_1,\phi_2,\phi_3,\phi_4)\in Y^+$, and let Π_t^P denote the skew-product semiflow induced by the solution of (4.2), that is,

$$\Pi_t^P:P\to P,$$
$$(\phi,\vartheta)\mapsto(u_t(\phi,\vartheta),\vartheta\cdot t),$$

where $u_t(\phi,\vartheta)(x,\theta)=u(t+\theta,x;\phi,\vartheta)$, $\forall t\ge 0$, $\theta\in[-\tau,0]$ and $x\in\overline{\Omega}$. It is easy to see that $\Pi_t^P(P_0)\in P_0$ and $\Pi_t^P(\partial P_0)\in\partial P_0$, $\forall t\ge 0$. Lemma 4.1 means that Π_t^P is continuous and point dissipative. Moreover, Π_t^P is compact for any $t>\tau$ ([35, Theorem 2.1.8]). It then follows from [15, Theorem 3.4.8] that $\Pi_t^P:P\to P$ admits a global attractor $\mathcal{A}$. Let M_∂ be the maximal positively invariant set for Π_t^P in ∂P_0, that is,

$$M_\partial:=\{(\phi,\vartheta)\in\partial P_0:\Pi_t^P(\phi,\vartheta)\in\partial P_0,\ \forall t\ge 0\}.$$

Let $\omega(\phi,\vartheta)$ represent the omega limit set for Π_t^P and define $\mathcal{M}:=\{(u_1^*(\vartheta),\hat{0},\hat{0},\hat{0},\vartheta):\vartheta\in H(A)\}$. It is clear that for any $(\psi,\vartheta)\in\partial P_0$, $u_3(t,x;\psi,\vartheta)=0$, $\forall t\ge 0$, $\lim_{t\to\infty}u_i(t,\cdot;\psi,\vartheta)=0$, $i=2,4$, and $\lim_{t\to\infty}\|u_1(t,\cdot;\psi,\vartheta)-u_1^*(t,\cdot;\vartheta)\|=0$. Hence, we have $\cup_{(\psi,\vartheta)\in\partial P_0}\omega(\psi,\vartheta)=\mathcal{M}$. It then follows that $\omega(M_\partial)=\cup_{(\psi,\vartheta)\in\partial M_\partial}\omega(\psi,\vartheta)=\mathcal{M}$, $\mathcal{M}$ is a compact and isolated invariant set, and no subset of $\mathcal{M}$ forms a cycle for Π_t^P in ∂P_0.

Consider the following perturbation system with a positive parameter η:

$$\begin{cases} \dfrac{\partial u_3(t,x)}{\partial t} = D_3\Delta u_3(t,x) - (\bar{\gamma}_I(t,x) + \bar{d}(t,x))u_3(t,x) \\ \qquad + \displaystyle\int_\Omega \bar{\Gamma}(t,t-\tau,x,y)\bar{\beta}(t-\tau,y)\frac{u_1^*(t,x;\vartheta)-\eta}{u_1^*(t,x;\vartheta)+4\eta}u_3(t-\tau,y)dy, \quad t>0,\ x\in\Omega, \\ \dfrac{\partial u_3(t,x)}{\partial \nu} = 0, \quad t>0,\ x\in\partial\Omega. \end{cases} \tag{4.5}$$

Let λ_η^* be the upper Lyapunov exponent associated with (4.5). Due to $\lambda^* > 0$, the continuity of upper Lyapunov exponent (see Lemma 2.5) means that we can take a sufficiently small number $\eta > 0$ such that $\lambda_\eta^* > 0$. Furthermore, we have the following claim.
Claim. $\mathcal{M}$ is a uniform weak repeller for Π_t^P, that is,

$$\limsup_{t\to\infty} d(\Pi_t^P(\phi,\vartheta),\mathcal{M}) \ge \eta, \quad \forall(\phi,\vartheta)\in P_0.$$

Suppose, by contradiction, there exists some $(\phi^0,\vartheta^0)\in P_0$ such that

$$\limsup_{t\to\infty} d(\Pi_t^P(\phi^0,\vartheta^0),\mathcal{M}) < \eta.$$

Then there exists $t_1 > 0$ such that $\limsup_{t\to\infty} d(\Pi_t^P(\phi^0,\vartheta^0),\mathcal{M}) < \eta$, $\forall t \ge t_1$, which implies that $u_1^*(t,x;\vartheta^0) - \eta \le u_1(t,x;\phi^0,\vartheta^0) \le u_1^*(t,x;\vartheta^0) + \eta$ and $u_i(t,x;\phi^0,\vartheta^0) \le \eta, \forall t \ge t_1, x\in\overline{\Omega}, 2\le i\le 4$. It follows that $u_3(t,x;\phi^0,\vartheta^0)$ satisfies

$$\begin{cases} \dfrac{\partial u_3(t,x)}{\partial t} \ge D_3\Delta u_3(t,x) - (\tilde{\gamma}_I(t,x) + \tilde{d}(t,x))u_3(t,x) \\ \qquad + \displaystyle\int_\Omega \tilde{\Gamma}(t,t-\tau,x,y)\tilde{\beta}(t-\tau,y)\frac{u_1^*(t,x;\vartheta^0)-\eta}{u_1^*(t,x;\vartheta^0)+4\eta}u_3(t-\tau,y)dy,\ t>t_1,\ x\in\Omega, \\ \dfrac{\partial u_3(t,x)}{\partial \nu} = 0, \quad t>t_1,\ x\in\partial\Omega, \end{cases}$$

where $(\tilde{\Lambda},\tilde{\beta},\tilde{d},\tilde{\gamma}_E,\tilde{\gamma}_I,\tilde{\Gamma}) = \vartheta^0$. By Theorem 2.2, there exist two almost periodic functions $a(\vartheta^0\cdot t)$ and $\tilde{w}(t,x;\vartheta^0)$ such that $w(t,x;\vartheta^0) = e^{\int_0^t a(\vartheta^0\cdot s)ds}\tilde{w}(t,x;\vartheta^0)$ is a solution of (4.5) and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t a(\vartheta^0\cdot s)ds = \lambda_\eta^* > 0.$$

Recall $(\phi^0,\vartheta^0)\in P_0$, a similar argument to that in [38, Lemma 4.2 (i)] means that $u_3(t,x;\phi^0,\vartheta^0) > 0$ for $t>0$. Thus, there exist $t_2>0$ and a sufficiently small number $\delta>0$ such that $u_3(t_2+\theta,x;\phi^0,\vartheta^0) \ge \delta w(t_2+\theta,x,\vartheta^0)$, $\forall\theta\in[-\tau,0]$, $x\in\overline{\Omega}$. It then follows from the comparison principle that

$$u_3(t,x;\phi^0,\vartheta^0) \ge \delta w(t,x;\vartheta^0) = \delta e^{\int_0^t a(\vartheta^0\cdot s)ds}\tilde{w}(t,x;\vartheta^0), \quad \forall t\ge t_3,\ x\in\overline{\Omega},$$

where $t_3 = \max\{t_1,t_2\}$. Note that $\tilde{w}(t,x;\vartheta^0)$ is almost periodic and

$$\lim_{t\to\infty} e^{\int_0^t a(\vartheta^0\cdot s)ds} = \lim_{t\to\infty}(e^{\frac{1}{t}\int_0^t a(\vartheta^0\cdot s)ds})^t = \infty,$$

we have $\lim_{t\to\infty} u_3(t,x;\phi^0,\vartheta^0) = \infty$, a contradiction.

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Since $\mathcal{M}$ is an isolated invariant set for Π_t^P in ∂P_0, the above claim implies that $\mathcal{M}$ is also an isolated invariant set for Π_t^P on P. The claim also shows that $W^s(\mathcal{M}) \cap P_0 = \emptyset$, where $W^s(\mathcal{M})$ is the stable set of $\mathcal{M}$ for Π_t^P, that is,

$$W^s(\mathcal{M}) = \{(\phi, \vartheta) \in P : \omega(\phi, \vartheta) \neq \emptyset, \omega(\phi, \vartheta) \subset \mathcal{M}\}.$$

By the continuous-time version of [39, Theorem 1.3.1 and Remark 1.3.1], the skew-product semiflow $\Pi_t^P : P \to P$ is uniformly persistent with respect to $(P_0, \partial P_0)$. Since Π_t^P is compact for any $t > \tau$, it follows that Π_t^P is asymptotically smooth. By [26, Theorem 3.7 and Remark 3.10], $\Pi_t^P : P_0 \to P_0$ admits a global attractor $\mathcal{A}_0$.

It remains to prove the practical uniform persistence. Since $\mathcal{A}_0 \in P_0$ and $\Pi_t^P(\mathcal{A}_0) = \mathcal{A}_0$, it follows that $\mathcal{A}_0 \in \mathrm{Int}(P)$. Define a function $c : P \to [0, \infty)$ by

$$c(\phi, \vartheta) = \min_{x \in \overline{\Omega}} \{\phi_3(0, x)\}, \quad \forall (\phi, \vartheta) \in P.$$

It is easy to see that c is continuous and $c(\phi, \vartheta) > 0$ for all $(\phi, \vartheta) \in \mathcal{A}_0$. The compactness of $\mathcal{A}_0$ implies that $\inf_{(\phi,\vartheta)\in\mathcal{A}_0} c(\phi, \vartheta) = \min_{(\phi,\vartheta)\in\mathcal{A}_0} c(\phi, \vartheta) > 0$. Consequently, we conclude that there exists a number $\epsilon > 0$ such that

$$\liminf_{t\to\infty} u_3(t, \cdot; \phi, \vartheta) \geq \epsilon, \quad \forall (\phi, \vartheta) \in P_0.$$

This completes the proof. □

5 Numerical simulations

In this section, we carry out numerical simulations to illustrate the theoretical results obtained in previous sections and numerically analyse the influence of the incubation period, the spatial heterogeneity and the diffusion on the upper Lyapunov exponent and disease transmission. The numerical computation of upper Lyapunov exponent is supported by Lemma 2.3. Choose $\phi \in \mathrm{Int}(X^+)$, Lemma 2.3 indicates that for every $\epsilon > 0$, there exists $t(\epsilon) > 0$ such that $t \geq t(\epsilon)$ implies $|\lambda^* - \frac{\ln w(t,x_0;\phi)}{t}| < \epsilon$, $\forall x_0 \in \overline{\Omega}$. In our numerical simulations, we mimic (2.6) on a finite interval. For given $x_0 \in \overline{\Omega}$ and $\phi \in \mathrm{Int}(X^+)$, we specify a value T and compute $\lambda_T^* = \frac{\ln w(T,x_0;\phi)}{T}$, which will provide an approximation to λ^*.

For the sake of convenience, we concentrate on one-dimensional domain $\Omega = (0, 2)$. Considering the seasonality and the spatial heterogeneity, we choose $\beta = 0.6 + 0.2(\sin(\frac{\pi t}{6}) + \cos(\frac{\sqrt{2}\pi t}{3}))$ Year^{-1} and $\gamma_I = K + k \sin(\pi x)$ Year^{-1}, where $k \leq K$ are two positive constants. Take $d = 0.14$ Year^{-1}, $D_1 = D_2 = D_3 = D_4 = 0.1$ Km$^2\cdot$ Year^{-1}, $\gamma_E = 0.01$ Year^{-1}, $\Lambda = 2$ Km$\cdot$ Year^{-1} and $\tau = 0.5$ Year. For this set of parameters, we can compute the upper Lyapunov exponent numerically and get $\lambda^* = -0.305 < 0$ if we choose $K = 1$ and $k = 0.5$. The graph of $\frac{\ln w(t,x_0;\phi)}{t}$ is shown in Figure 1. We mimic $\frac{\ln w(t,x_0;\phi)}{t}$ on finite interval $[0, 2000]$ and take $\phi(\theta, x) = 1$, $\forall \theta \in [-\tau, 0]$, $x \in [0, 2]$, it is clear that λ^* is approximated very well (see Figure 1(a)). We truncate the time interval by $[1900, 2000]$, Figure 1(b) indicates that $\frac{\ln w(t,x_0;\phi)}{t}$ converges to $[-0.306, -0.3045]$. In the case where $D_1 = D_2 = D_3 = D_4 = D_0$ for some positive number D_0, system (3.6) can be rewritten as

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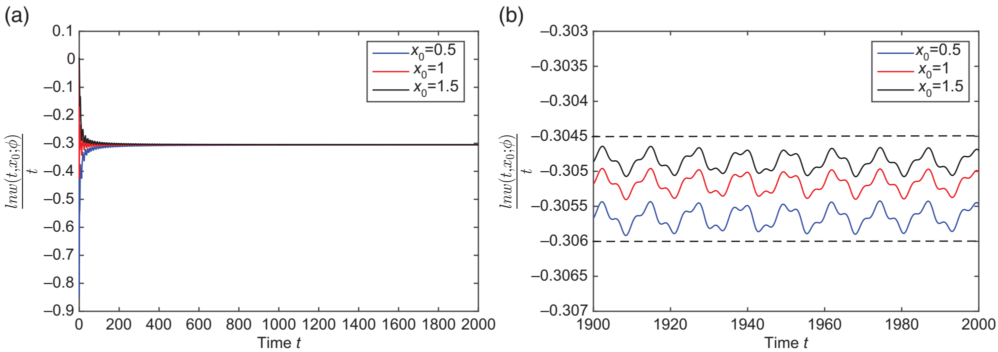

FIGURE 1. The graph of $\frac{\ln w(t,x_0;\phi)}{t}$. Baseline parameter values: $D_2 = D_3 = 0.1$, $\gamma_E = 0.01$, $d = 0.14$, $\tau = 0.5$, $\beta = 0.6 + 0.2(\sin(\frac{\pi t}{6}) + \cos(\frac{\sqrt{2}\pi t}{3}))$, $\gamma_I = 1 + 0.5\sin(\pi x)$ and $\phi(\theta, x) = 1$, $\forall\theta \in [-\tau, 0]$, $x \in [0, 2]$.

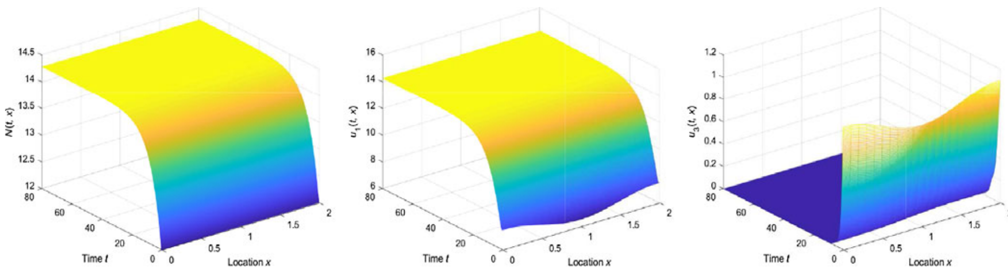

FIGURE 2. The long-term behaviour of the solution of system (5.1) when $\lambda^* = -0.305 < 0$. Baseline parameter values: $D_0 = 0.1$, $\gamma_E = 0.01$, $\Lambda = 2$, $d = 0.14$, $\tau = 0.5$, $\beta = 0.6 + 0.2(\sin(\frac{\pi t}{6}) + \cos(\frac{\sqrt{2}\pi t}{3}))$ and $\gamma_I = 1 + 0.5\sin(\pi x)$.

$$\begin{cases} \dfrac{\partial N(t,x)}{\partial t} = D_0 \Delta N(t,x) + \Lambda(t,x) - d(t,x)N(t,x), \\ \dfrac{\partial u_1(t,x)}{\partial t} = D_0 \Delta u_1(t,x) + \Lambda(t,x) - \beta(t,x)\dfrac{u_1(t,x)u_3(t,x)}{N(t,x)} - d(t,x)u_1(t,x), \\ \dfrac{\partial u_3(t,x)}{\partial t} = D_0 \Delta u_3(t,x) - (\gamma_I(t,x) + d(t,x))u_3(t,x) \\ \qquad + \displaystyle\int_\Omega \Gamma(t, t-\tau, x, y)\beta(t-\tau, y)\dfrac{u_1(t-\tau, y)u_3(t-\tau, y)}{N(t-\tau, y)}dy, \quad t > 0,\ x \in \Omega, \\ \dfrac{\partial N(t,x)}{\partial \nu} = \dfrac{\partial u_1(t,x)}{\partial \nu} = \dfrac{\partial u_3(t,x)}{\partial \nu} = 0, \quad t > 0,\ x \in \partial\Omega. \end{cases} \tag{5.1}$$

Figure 2 shows the corresponding long-term behaviour of the solution of system (5.1) in the case of $\lambda^* = -0.305$, with initial data

$$u(\theta, x) = \begin{pmatrix} N(\theta, x) \\ u_1(\theta, x) \\ u_3(\theta, x)) \end{pmatrix} = \begin{pmatrix} 12 \\ 7 + 0.5\cos(\pi x) \\ 1 + 0.1\cos(\pi x) \end{pmatrix}, \quad \forall\theta \in [-\tau, 0],\ x \in [0, 2].$$

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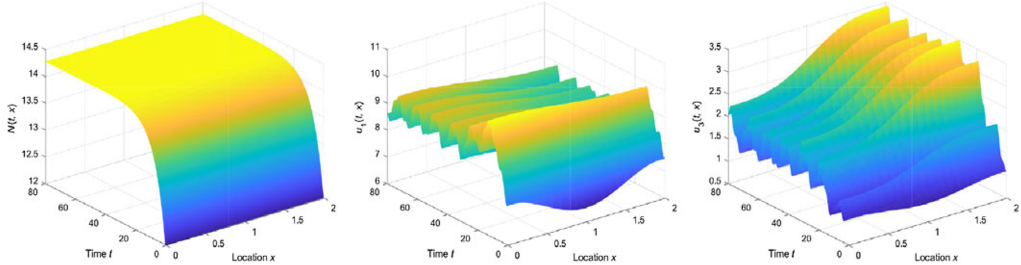

FIGURE 3. The long-term behaviour of the solution of system (5.1) when $\lambda^* = 0.173 > 0$. Baseline parameter values: $D_0 = 0.1$, $\gamma_E = 0.01$, $\Lambda = 2$, $d = 0.14$, $\tau = 0.5$, $\beta = 0.6 + 0.2(\sin(\frac{\pi t}{6}) + \cos(\frac{\sqrt{2}\pi t}{3}))$ and $\gamma_I = 0.2 + 0.1\sin(\pi x)$.

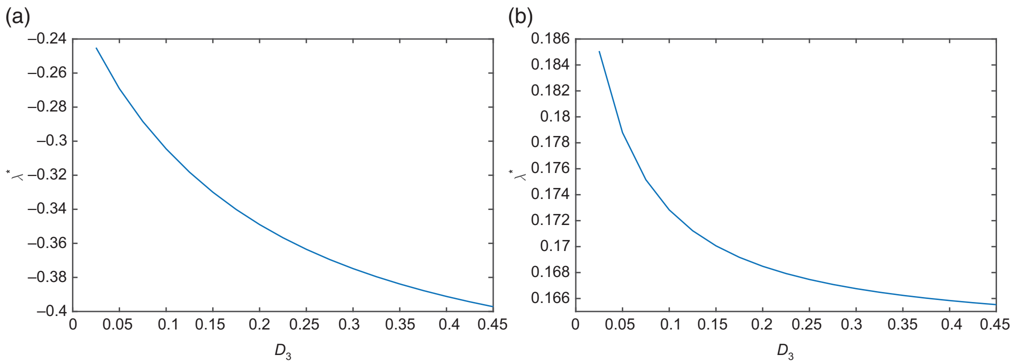

FIGURE 4. λ^* versus D_3. Baseline parameter values: $D_2 = 0.1$, $\gamma_E = 0.01$, $d = 0.14$, $\beta = 0.6 + 0.2(\sin(\frac{\pi t}{6}) + \cos(\frac{\sqrt{2}\pi t}{3}))$ and $\tau = 0.5$ ($\gamma_I = 1 + 0.5\sin(\pi x)$ in (a), $\gamma_I = 0.2 + 0.1\sin(\pi x)$ in (b)).

It is coincident with the result in Theorem 4.2. Decreasing the recovery rate to $0.2 + 0.1\sin(\frac{2\pi x}{2})$ Year^{-1}, and other parameters remain unchanged, then $\lambda^* = 0.173$. The corresponding long-term behaviour of system (5.1) in this case is shown in Figure 3, where the initial data are the same to the previous case. It shows that the disease is persistent, which is coincident with the result in Theorem 4.3.

Note that the sign of λ^* completely determines the development trend of the disease, and the larger the λ^*, the greater the disease risk. We are interested in the dependence between λ^* and the model parameters. Figure 4 indicates that λ^* is a decreasing function of D_3, and the rate of decline becomes slow when D_3 is large. It seems that increasing the diffusion rate D_3 is beneficial to disease control when D_3 is small. Biologically, the lager the diffusion rate of infected individuals possess, the more opportunities that access to medical resources the infected individuals obtain. Figure 5 shows that λ^* is also a decreasing function of incubation period τ, which implies that prolonging the length of incubation period could help to control the disease.

Motivated by [36], we observe that the population density of infectious individuals at the area $[0, 1]$ is less than that in the area $[1, 2]$ in Figure 3, which is corresponding to the spatial distribution of recovery rate γ_I. It motivates us to investigate the effect of γ_I on λ^* in a spatially

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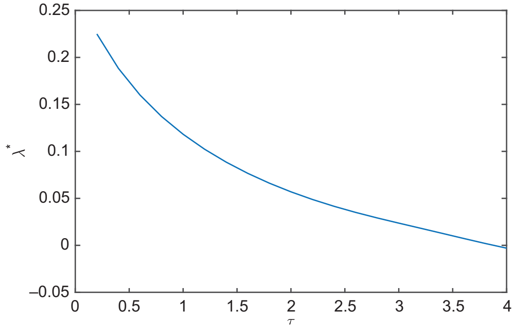

FIGURE 5. λ^* versus τ. Baseline parameter values: $D_2 = 0.1$, $D_3 = 0.1$, $\gamma_E = 0.01$, $d = 0.14$, $\beta = 0.6 + 0.2(\sin(\frac{\pi t}{6}) + \cos(\frac{\sqrt{2}\pi t}{3}))$ and $\gamma_I = 1 + 0.5\sin(\pi x)$.

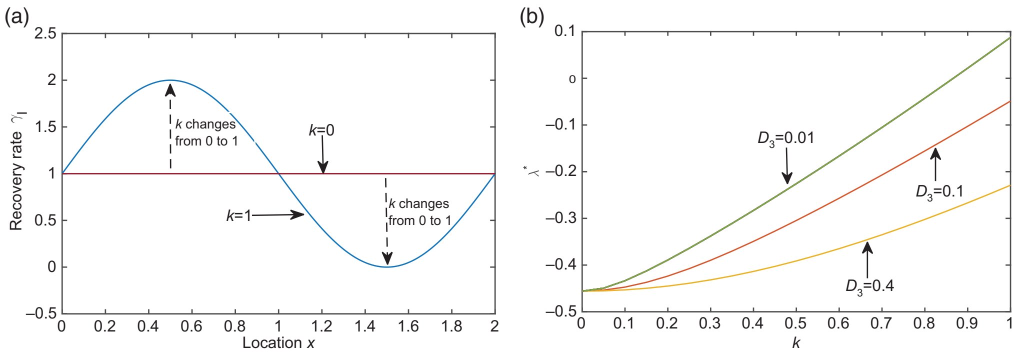

FIGURE 6. The effect of the spatial distribution of recovery rate γ_I on λ^*. (a) The spatial distribution of γ_I. (b) The graph of λ^* versus k. Baseline parameter values: $D_2 = 0.1$, $\gamma_E = 0.01$, $d = 0.14$, $\tau = 0.5$, $\beta = 0.6 + 0.2(\sin(\frac{\pi t}{6}) + \cos(\frac{\sqrt{2}\pi t}{3}))$ and $\gamma_I = 1 + k\sin(\pi x)$ ($k \in [0, 1]$).

heterogeneous environment. We take $K = 1$ and $k \in [0, 1]$. As k changing from 0 to 1, the integration $\int_0^2 \gamma_I(x)dx$ remains unchanged, which means that the total medical resources stay the same. When $k = 0$, the spatial distribution of recovery rate is uniform and more uneven with the increase of k (see Figure 6(a)). The numerical simulations show that λ^* is increasing with respect to k, and the smaller the diffusion rate D_3 becomes, the more obvious the increase is (see Figure 6(b)). This seems to indicate that keeping balance of the distribution of medical resources could help to control the disease. It is worth noting that, however, the spatial distribution of disease transmission rate β is homogeneous. We choose $\beta = 0.8 + 0.5\sin(\pi x)$, and other parameters remain unchanged. Let $F = \int_0^2 |k\sin(\pi x) - 0.5\sin(\pi x)|dx$. It is clear that the smaller the F is, the smaller the spatial heterogeneity difference between γ_I and β becomes. Figure 7(a) gives the relationship between F and k, and the dependence of λ^* on k is presented in Figure 7(b). Figure 7 shows that λ^* reaches a minimum when the minimum of $F = \int_0^2 |k\sin(\pi x) - 0.5\sin(\pi x)|dx$ is reached, which demonstrates that decreasing the difference of spatial distributions between γ_I and β could help to control the disease, especially when D_3 is small.

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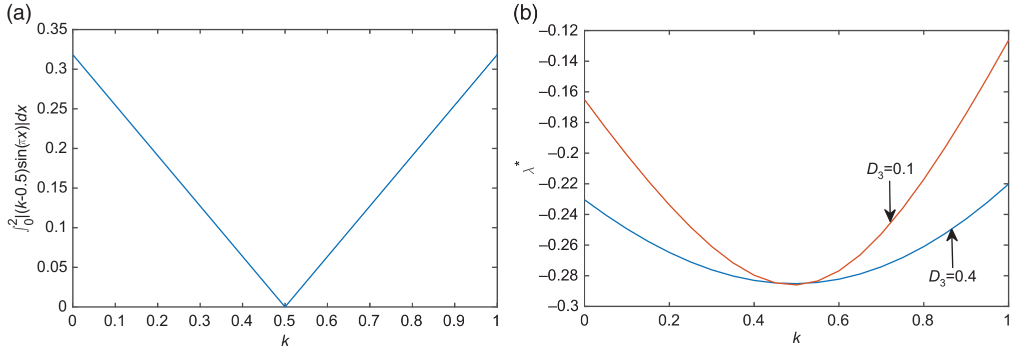

FIGURE 7. The effect of the spatial distribution of recovery rate γ_I on λ^*. (a) The difference of spatial distributions between γ_I and β. (b) The graph of λ^* versus k. Baseline parameter values: $D_2 = 0.1$, $\gamma_E = 0.01$, $d = 0.14$, $\tau = 0.5$, $\beta = 0.8 + 0.5\sin(\pi x)$, $\gamma_I = 1 + k\sin(\pi x)$ ($k \in [0, 1]$).

6 Discussion

It is widely known in epidemiology that environmental heterogeneity and climate variations exhibit complex effects on disease transmission. With the combination of environmental factors and incubation period of disease, we formulate and investigate a nonlocal almost periodic reaction–diffusion epidemic model. The almost periodicity for time reflects the effect of certain seasonal changes which are approximatively but not exactly periodic and allows one to take into account general seasonal fluctuations. For this mathematical model, we introduce a threshold index, the upper Lyapunov exponent λ^*, which can be regarded as the growth rate of infectious population in a completely susceptible population. Moreover, we present a numerical method to compute λ^*. By the skew-product semiflow, comparison arguments and persistence theory, we show that λ^* serves as a threshold parameter for the extinction and persistence of the disease. More precisely, the disease will be eliminated if $\lambda^* < 0$, while the disease persists in the population if $\lambda^* > 0$.

In the simulation section, we numerically analyse the impacts of the incubation period, the diffusion and the spatial heterogeneity on the upper Lyapunov exponent and disease transmission. Numerical simulations indicate that increasing diffusion rate of infected population has positive effect on reducing λ^* (see Figure 4(a)), since the infected individuals could access to more medical resources when the diffusion rate of infected population is large. Moreover, numerical results show that prolonging the length of incubation period has a good influence on disease control (see Figure 5), which might be realised by drugs. Figure 6 seems to indicate that keeping balance of the distribution of medical resources could help to control the disease. It is worth mentioning that the transmission coefficient in the numerical simulations is assumed to be spatially homogeneous. By choosing a spatially heterogeneous transmission coefficient, our numerical results further demonstrate that keeping the coincidence of spatial distributions between the recovery and transmission coefficients has a beneficial effect on disease control, instead of keeping balance of resources distribution.

Conflict of interest

None.

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