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Triviality Properties of Principal Bundles on Singular Curves. II

P. Belkale and N. Fakhruddin

Abstract. For G a split semi-simple group scheme and P a principal G-bundle on a relative curve $X \to S$, we study a natural obstruction for the triviality of P on the complement of a relatively ample Cartier divisor $D \subset X$. We show, by constructing explicit examples, that the obstruction is nontrivial if G is not simply connected, but it can be made to vanish by a faithfully flat base change, if S is the spectrum of a dvr (and some other hypotheses). The vanishing of this obstruction is shown to be a sufficient condition for étale local triviality if S is a smooth curve, and the singular locus of $X - D$ is finite over S.

1 Introduction

Let $f\colon X \to S$ be a proper, flat, and finitely presented curve over an arbitrary scheme S. Let G be a split reductive group scheme over $\operatorname{Spec}(\mathbb{Z})$, base changed to S, and let B be a Borel subgroup of G. Let $D \subset X$ be a relatively ample effective Cartier divisor that is flat over S, and set $U = X \setminus D$. Generalizing results of Drinfeld and Simpson [5] for the case of smooth f, the following result was proved in [3, Theorem 1.4] without any conditions on the singularities of f:

Theorem 1.1 *Let P be a principal G-bundle on X with G semisimple and simply connected. Then, after a surjective étale base change $S' \to S$, P is trivial on $U_{S'}$.*

Now suppose G is semisimple, but not necessarily simply connected. Triviality statements similar to the above were proved in [3, Theorem 1.5] but with stronger hypotheses: For example, in characteristic zero, the Cartier divisor D is not allowed to pass through the singular locus of f, and D is also assumed to be set theoretically a union of sections of f (and some other mild conditions).

In this note, motivated in part by the article [11], we study the analogue of Theorem 1.1 for non-simply connected G. In this case, there is a natural obstruction to local triviality constructed as follows.

Let $\widetilde{G}$ be the simply connected cover of G and denote by $\pi_1(G)$ the scheme theoretic kernel of the covering map $\widetilde{G} \to G$. The central exact sequence of sheaves of groups (on the fppf site of X),

$$1 \longrightarrow \pi_1(G)_X \longrightarrow \widetilde{G}_X \longrightarrow G_X \longrightarrow 1$$

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gives rise to a coboundary map in fppf cohomology,

$$\mathrm{H}^1_{\mathrm{fl}}(X, G) \longrightarrow \mathrm{H}^2_{\mathrm{fl}}(X, \pi_1(G)). \tag{1.1}$$

Therefore, from $P \in \mathrm{H}^1_{\mathrm{fl}}(X, G)$, we get an element $\alpha_P \in \mathrm{H}^2_{\mathrm{fl}}(X, \pi_1(G))$.[1] It is clear that if P is trivial on U, then α_P maps to zero in $\mathrm{H}^2_{\mathrm{fl}}(U, \pi_1(G))$. Thus, for the generalization of Theorem 1.1 to hold for P, the following property (L) must hold:

(L) There exists a surjective étale morphism $S' \to S$ such that α_P maps to zero in $\mathrm{H}^2_{\mathrm{fl}}(U_{S'}, \pi_1(G))$.

We show that this property is nontrivial: For $G = \mathrm{PGL}(m)$, we construct principal G-bundles on families of curves $X \to S$ with nodal singularities, and D passing through the singularities of f, where property (L), so also the direct generalization of Theorem 1.1, fails (Proposition 3.1). These examples include cases when S is a smooth curve and $X \to S$ is a family of smooth curves degenerating to curve with a single nodal singularity and the divisor D passes through the node. Examples for other classical groups G can be constructed using similar methods.

Even though condition (L) is not always satisfied, we show in Lemma 2.2 that the following weaker condition (L′) often holds, *e.g.*, when S is a smooth curve and U is smooth over S:

(L′) There exists a morphism $S' \to S$ which is flat and an fpqc covering such that α_P maps to zero in $\mathrm{H}^2_{\mathrm{fl}}(U_{S'}, \pi_1(G))$.

We are thus faced with the following question.

Question 1.2 *Does condition* (L′) *always hold? If so, does there always exist a morphism $S' \to S$, which is flat and an fpqc covering such that P becomes trivial on $U_{S'}$?*

We do not know the answer to this in full generality. However, we prove the following as Theorem 2.7:

Theorem *Let $f\colon X \to S$ be flat projective curve, $D \subset X$ a relatively ample Cartier divisor which is flat over S and set $U = X \backslash D$. Let G be a semisimple group and let P be a principal G-bundle on X. Assume further that*

(i) *S is an excellent regular (purely) one dimensional scheme, and*
(ii) *U is smooth over S.*

Then there is a quasi-finite fppf morphism $S' \to S$ such that P becomes trivial on $U_{S'}$.

We also show in Proposition 2.8 that for $G = \mathrm{PGL}(m)$, and P satisfying a condition weaker than (L), and S smooth, P lifts to a principal $\mathrm{GL}(m)$-bundle on X (after an étale base change in S). This result generalizes to arbitrary groups; see Remark 2.10.

In the study of principal bundles over a nodal complex projective curve (say irreducible with a single node for simplicity here) the following technique is used [6,10,11]: For semisimple G, principal bundles on the curve restrict to trivial bundles on the

[1] We note that for a smooth group scheme, fppf cohomology is the same as étale cohomology. In particular, if $|\pi_1(G)|$ is invertible in $\mathcal{O}_S$, we may replace fppf cohomology by étale cohomology throughout this paper.

complement of the node, and also in the completion at the node. This presents the moduli-stack as an fppf double quotient [2].

Now consider a flat proper family of curves $X \to S$, S the spectrum of a complete dvr, with smooth generic fiber and a nodal special fiber. To relativize the above construction, assume that there is a section of the family passing through the node. It is therefore of interest to know when any principal bundle on the family after a base change becomes trivial on the complement of the section after a further faithfully flat base change of S, so that one can present the relative moduli stack of bundles as a double quotient. When the group is simply connected, this follows from [3, Theorem 1.5]. Theorem 1 shows that this is true for all semisimple G when we consider principal bundles on X itself; it is an interesting question whether the theorem extends to the setting of bundles on (flat) base changes $X_T \to T$ of the original family $X \to S$.

2 Consequences of Condition (L)

2.1 The Case S is Regular of Dimension One

Lemma 2.1 *Let U be a regular Noetherian scheme and let $D \subset U$ be a closed subscheme. Let $V = U \backslash D$ and let $\{D_i\}$ be the irreducible components of D of codimension one. Then for any integer $n > 0$, the kernel of the restriction map $\mathrm{H}^2_{\mathrm{fl}}(U, \mu_n) \to \mathrm{H}^2_{\mathrm{fl}}(V, \mu_n)$ is the subgroup generated by the first Chern classes of all $\mathcal{O}_U(D_i)$.*

The lemma is well known for étale cohomology, but we do not know of a reference for fppf cohomology, so we give the proof (which is essentially the same as that for étale cohomology).

Proof We first note that for any scheme U, $\mathrm{H}^1_{\mathrm{fl}}(U, \mathbb{G}_m) = \mathrm{Pic}(U)$, and if U is regular, then $\mathrm{H}^2_{\mathrm{fl}}(U, \mathbb{G}_m) = \mathrm{Br}(U)$.

Consider the commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{H}^0_{\mathrm{fl}}(U,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^0_{\mathrm{fl}}(V,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^1_{D,\mathrm{fl}}(U,\mathbb{G}_m) & \xrightarrow{g_1} & \mathrm{H}^1_{\mathrm{fl}}(U,\mathbb{G}_m) & \xrightarrow{g_2} & \mathrm{H}^1_{\mathrm{fl}}(V,\mathbb{G}_m) \\
\downarrow & & \downarrow & & \downarrow c & & \downarrow c_U & & \downarrow c_V \\
\mathrm{H}^1_{\mathrm{fl}}(U,\mu_n) & \longrightarrow & \mathrm{H}^1_{\mathrm{fl}}(V,\mu_n) & \longrightarrow & \mathrm{H}^2_{D,\mathrm{fl}}(U,\mu_n) & \longrightarrow & \mathrm{H}^2_{\mathrm{fl}}(U,\mu_n) & \longrightarrow & \mathrm{H}^2_{\mathrm{fl}}(V,\mu_n) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^1_{\mathrm{fl}}(U,\mathbb{G}_m) & \xrightarrow{f_1} & \mathrm{H}^1_{\mathrm{fl}}(V,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^2_{D,\mathrm{fl}}(U,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^2_{\mathrm{fl}}(U,\mathbb{G}_m) & \xrightarrow{f_2} & \mathrm{H}^2_{\mathrm{fl}}(V,\mathbb{G}_m)
\end{array}$$

where the rows come from the long exact sequence of cohomology with supports and the columns from the Kummer sequence.

The map f_1 is surjective, because U is regular and the map f_2 is injective by [7, Corollaire 1.10]. This implies that $\mathrm{H}^2_{D,\mathrm{fl}}(U, \mathbb{G}_m) = 0$, so the map c is surjective. The claim then follows by a simple diagram chase, noting that the map c_U gives, by definition, the first Chern class of a line bundle on U, and the kernel of g_2 (equal to the image of g_1) is precisely the subgroup of $\mathrm{Pic}(U)$ generated by the $\mathcal{O}_U(D_i)$. ∎

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Lemma 2.2 *Let $f: U \to S$ be a smooth morphism of relative dimension one with S the spectrum of a henselian dvr R with quotient field K and residue field k. Given any element $\alpha \in \mathrm{H}^2_{\mathrm{fl}}(U, \mu_n)$, there exists a faithfully flat morphism $S' \to S$, with S' also the spectrum of a dvr, such that the pullback of α in $U_{S'}$ is 0.*

Remark 2.3 If R is excellent, or n is invertible in $\mathcal{O}_S$, the proof shows that $S' \to S$ can be chosen to be finite.

Proof Let U_0 be the closed fibre of f and set $V = U \backslash U_0$. Since V is an affine curve over K, $\mathrm{H}^2_{\mathrm{fl}}(V_{\overline{K}}, \mu_n) = 0$. This follows from the Kummer sequence, and the following facts for smooth curves Y over an algebraically closed field: the vanishing of $\mathrm{Br}(Y)$ (Tsen's theorem) and, if Y is affine, the surjectivity of multiplication by n map on $\mathrm{Pic}(Y)$ for $n > 0$. It follows that there exists a finite extension K_1 of K so that the image of α in $\mathrm{H}^2_{\mathrm{fl}}(V, \mu_n)$ becomes 0 in $\mathrm{H}^2_{\mathrm{fl}}(V_{K_1}, \mu_n)$. By replacing R by its integral closure R_1 in K_1 (which is still a dvr), functoriality implies that we may assume α is in the kernel of the restriction map $r\colon \mathrm{H}^2_{\mathrm{fl}}(U, \mu_n) \to \mathrm{H}^2_{\mathrm{fl}}(V, \mu_n)$.

Since f is smooth, U is regular, so by Lemma 2.1, the kernel of r is spanned by the fundamental classes of the irreducible components of U_0. Let $R \to R'$ be a finite map, with R' a dvr, such that the ramification degree is divisible by n and set $S' = \mathrm{Spec}(R')$. The pullbacks of the Chern classes of all the components of U_0 become divisible by n, hence are all 0 in $\mathrm{H}^2_{\mathrm{fl}}(U_{S'}, \mu_n)$. We conclude that α also becomes 0 in $\mathrm{H}^2_{\mathrm{fl}}(U_{S'}, \mu_n)$. ■

Let $f: X \to S$ and let D be as in the introduction, G an arbitrary semisimple group, and P a principal G-bundle on X.

Proposition 2.4 *Assume that*

(i) *S is an excellent regular (purely) one dimensional scheme;*
(ii) *X is reduced;*
(iii) *the closure of the non-regular locus of $U = X - D$ does not intersect D;*
(iv) *α_P is zero when restricted to U.*

Then there is a surjective étale morphism $S' \to S$ such that P is trivial on $U_{S'}$.

Remark 2.5 Condition (iii) above is equivalent to assuming that the non-regular locus of U is finite over S; *e.g.*, U has isolated singularities.

Proof For the sake of clarity, we first deal with the case U is regular. Let $g: \widetilde{X} \to X$ be a resolution of singularities of X [9]. By our assumptions U is regular, so we can assume $U \subset \widetilde{X}$. Note that $\widetilde{X} \to S$ is flat, since S is regular and one-dimensional.

We first show that $\widetilde{X} - U$ supports a relatively ample Cartier divisor $\widetilde{D}$ (possibly non-effective). By assumption, D supports a relatively ample divisor D'. A resolution of singularities for X can be obtained by iterating the process of normalization and then blowing up the singular locus (*cf.* [9], and X is excellent). Let the resulting schemes be denoted by $X_0 = X, X_1, \dots, X_s = \widetilde{X}$. We build relatively ample Cartier divisors D_r at each step of this resolution X_r, $D_0 = D'$, and finally set $\widetilde{D} = D_s$. For the normalization step, we just pull back the Cartier divisor from the previous step. For a resolution step $g: X_{r+1} \to X_r$, let E_{r+1} be the exceptional divisor of the blow up g. It is easy to see that then $D_{r+1} = g^*(nD_r) - E_{r+1}$ is relatively ample for n sufficiently large (use [8, Proposition II.7.10]. We may assume that D', and hence each D_r, is actually ample, since S is affine).

Therefore, $\widetilde{X} - U$ supports a relatively ample Cartier divisor $\widetilde{D}$ (possibly non-effective). Let $L = \mathcal{O}_{\widetilde{X}}(\widetilde{D})$ be the corresponding relatively ample line bundle on $\widetilde{X}$; it is trivial on U.

Assume that S is affine. Using [3, Theorem 1.3] replace S by an étale cover such that P has a B-reduction, where B is a Borel subgroup of G. We may then also assume (as in [3, § 3.2.2]) that P is induced from an H-bundle E, where H is a maximal torus of B.

Let $\widetilde{G}$ be the simply connected cover of G and and let $Z \cong \pi_1(G)$ be the kernel of the covering map $\widetilde{G} \to G$. Let $\widetilde{H}$ be the maximal torus in $\widetilde{G}$ mapping onto H, so $Z \subset \widetilde{H}$. We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z & \longrightarrow & \widetilde{H} & \longrightarrow & H & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & Z & \longrightarrow & \widetilde{G} & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

whose rows are exact sequences of group schemes. Since we have assumed that α_P becomes 0 on U, by the commutativity of the diagram it follows that $E|_U$ lifts to a $\widetilde{H}$-bundle $\widetilde{E}_U$ on U. Since $\widetilde{H}$ is a torus and $\widetilde{X}$ is regular, $\widetilde{E}_U$ extends to a $\widetilde{H}$-bundle $\widetilde{E}$ on $\widetilde{X}$ (this follows from the fact that line bundles on U extend to $\widetilde{X}$).

Let $\widetilde{P}$ be the induced $\widetilde{G}$-bundle on $\widetilde{X}$. Since $\widetilde{G}$ is simply connected and $\widetilde{X} - U$ supports a relatively ample Cartier divisor $\widetilde{D}$, by (almost) the same argument as in [3, § 3.2.2] (see Remark 2.6), we see that $\widetilde{P}|_U$ is trivial, hence $P|_U$ is also trivial.

If U is not regular, we employ the following strategy. We choose a partial desingularization $\widetilde{X} \to X$, which is an isomorphism over U such that $\widetilde{X}$ is regular in a neighborhood of $\widetilde{X} - U$. To see that such a partial desingularization exists, first consider a full desingularization $Q \to X$. We just carry out only those blow ups with support over $X - U$, and normalize only in neighbourhoods of inverse images of $X - U$, and obtain $\widetilde{X} \to X$ which is an isomorphism over U, with $\widetilde{X}$ regular on the complement of U. ∎

Remark 2.6 [3, Theorem 1.4] can be generalized as follows. Let S be an arbitrary scheme over Spec$(\mathbb{Z})$ and let $f: X \to S$ be a proper, flat, and finitely presented curve over S. Let E be a principal G-bundle on X with G semisimple and simply connected. Let $U \subset X$ be an open subset, affine over S, such that $X - U$ supports a relatively ample Cartier divisor D for $X \to S$ (possibly non-effective, whose components need not be flat over S). Then, after a surjective étale base change $S' \to S$, E is trivial on $U_{S'}$.

To prove this, we reduce the problem to the case of SL(2) or GL(2) (with an assumption on determinants) as in [3]. The problem is then the following. Suppose both E_1 and E_2 are principal $G = \mathrm{SL}(2)$-bundles on X, or principal $G = \mathrm{GL}(2)$-bundles on X with the same determinant. Then we need to show as in [3, Proposition 3.3] that E_1 and E_2 are isomorphic on U after passing to a Zariski open cover of S. To do this, we modify the proof of [3, Proposition 3.3] as follows: Assume S is affine. We twist by tensor powers of $L = \mathcal{O}(D)$ to find subbundles $\mathcal{O} \subset E_i \otimes L^{\otimes r}$ with corresponding quotients T_i, $i = 1, 2$ after Zariski localization in S. On the affine open subset $U \subseteq X$, these extensions of vector bundles split, and L is trivial. Therefore, restricted to U, we get $E_i = \mathcal{O} \oplus T_i$, with T_i a line bundle. But T_i is the determinant of E_i, hence T_1 and T_2 are isomorphic on U.

By combining Lemma 2.2 and Proposition 2.4, we obtain the following theorem.

Theorem 2.7 *Let $f: X \to S$ be flat projective curve, $D \subset X$ a relatively ample Cartier divisor which is flat over S and set $U = X \backslash D$. Let G be a semisimple group and let P be a principal G-bundle on X. Assume further that*

(i) *S is an excellent regular (purely) one dimensional scheme, and*
(ii) *U is smooth over S.*

Then there is a quasi-finite fppf morphism $S' \to S$ such that P becomes trivial on $U_{S'}$.

Proof Since $\pi_1(G)$ is a finite group scheme of multiplicative type, by applying Lemma 2.2 we may find, using the fact that the cohomology of an inverse limit is the direct limit of the cohomology, a faithfully flat quasi-finite type cover $S_1 \to S$, with S_1 also excellent regular and one dimensional, such that α_P becomes 0 on U_{S_1}. We then apply Proposition 2.4 to the induced morphism $X_1^{\mathrm{red}} \to S_1$, where $X_1 := X \times_S S_1$, to get an étale cover $S' \to S_1$ so that P becomes trivial on $U_{S'}$. The composition of the maps $S' \to S_1 \to S$ is the desired quasi-finite fppf morphism. ∎

2.2 Lifting to Vector Bundles

A principal $\mathrm{PGL}(m)$-bundle P on X gives rise to a cohomology class $\alpha_P \in \mathrm{H}^2_{\mathrm{fl}}(X, \mu_m)$ as well a cohomology class $\beta_P \in \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m)$ $(= \mathrm{H}^2_{\mathrm{fl}}(X, \mathbb{G}_m))$, by considering the exact sequence of group schemes

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}(m) \longrightarrow \mathrm{PGL}(m) \longrightarrow 1.$$

The exact sequence

$$1 \longrightarrow \mu_m \longrightarrow \mathrm{SL}(m) \longrightarrow \mathrm{PGL}(m) \longrightarrow 1$$

maps to the exact sequence above, and hence α_P maps to β_P under the natural map $\mathrm{H}^2_{\mathrm{fl}}(X, \mu_m) \to \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m)$. It is easy to see that β_P represents the obstruction to lifting P to a principal $\mathrm{GL}(m)$-bundle, *i.e.,* a vector bundle on X.

Condition (L) implies that β_P maps to zero in $\mathrm{H}^2_{\mathrm{et}}(U, \mathbb{G}_m)$; *i.e., P* can be lifted to a vector bundle on U. In fact, under somewhat mild conditions, P can be lifted to a vector bundle on X after a surjective étale base change of S:

Proposition 2.8 *Assume S is smooth, and the smooth locus of $X \to S$ is dense in every fiber. If β_P maps to zero in $\mathrm{H}^2_{\mathrm{et}}(U, \mathbb{G}_m)$, then after an étale base change in S, P comes from a vector bundle on X, and hence $\beta_P \in \mathrm{H}^2_{\mathrm{et}}(X, \mathbb{G}_m)$ becomes zero.*

Proof After an étale base change in S, we can find sections of $X \to S$ such that their union is disjoint from D and contained in the smooth locus of $X \to S$. We can also assume that the union of these sections is relatively ample. Let $U' \subset X$ be the complement of these sections. Using [3, Theorem 1.5], we may assume that P is trivial on U' after an étale base change in S. Lift this trivial $\mathrm{PGL}(m)$-bundle to a vector bundle W on U'.

By assumption, P comes from a vector bundle V on U. Thus, we have two $\mathrm{GL}(m)$-bundles W and V on $U' \cap U$ which coincide as $\mathrm{PGL}(m)$-bundles. Let L^* be the sheaf of isomorphisms $V \to W$, which induce identity on the underlying $\mathrm{PGL}(m)$-bundle. Clearly, L^* is a $\mathbb{G}_m$-bundle, let L be the corresponding line bundle. Hence, W is isomorphic to $V \otimes L$ on $U' \cap U$. Extend L to a line bundle on U (U is smooth along $U - U' \cap U$, since S is smooth and the sections have images in the smooth locus of $X \to S$). Now glue the vector bundle $V \otimes L$ (a vector bundle on U) with W (a vector bundle on U') over $U' \cap U$, to get a vector bundle A on X. The $\mathrm{PGL}(m)$-bundle induced from A equals P, which completes the proof. ∎

Note that β_P is the obstruction to lifting the principal $\mathrm{PGL}(m)$-bundle P to a vector bundle on X, whereas the α_P is the obstruction to lifting P to a vector bundle with trivial determinant on X. The examples in Section 3 all have $\beta_P = 0$, *i.e.,* come from vector bundles on X, but not with trivial determinants.

The following lemma is well known (see, *e.g.*, [4, V, Proposition 3.1] for a stronger statement in characteristic zero), but we give a proof of the statement we need for the convenience of the reader.

Lemma 2.9 *For any semisimple group G, there exists a reductive group G' mapping surjectively to G with kernel a central torus K and such that the derived group of G' is simply connected.*

Proof Let $\widetilde{G}$ be the simply connected cover of G and let $\widetilde{T}$ be any torus in $\widetilde{G}$ containing the kernel $Z \cong \pi_1(G)$ of the covering map $\widetilde{G} \to G$. Then we may take G' to be $(\widetilde{G} \times \widetilde{T}/Z)$, where Z is embedded diagonally. There is a natural map $G' \to G$ induced by projection to the first factor, and the kernel of this is $\widetilde{T}$ (embedded in G' via the second factor). Moreover, the derived group of G' is equal to $\widetilde{G}$ (embedded via the first factor). One gets a somewhat canonical construction by taking $\widetilde{T}$ to be a maximal torus. ∎

Remark 2.10 The proof of Proposition 2.8 works more generally: For G any semisimple group (replacing $\mathrm{PGL}(m)$), let G' be as in Lemma 2.9 above. For P a principal G-bundle on X, we get, as above, elements $\alpha_P \in \mathrm{H}^2_{\mathrm{fl}}(X, \pi_1(G))$ and $\beta_P \in \mathrm{H}^2_{\mathrm{et}}(X, K)$. Suppose β_P maps to zero in $\mathrm{H}^2_{\mathrm{et}}(U, K)$, which would be the case if α_P maps to zero in $\mathrm{H}^2_{\mathrm{fl}}(U, \pi_1(G))$ (*i.e.,* if condition (L) holds). Then, after an étale base change in S, P comes from a principal G'-bundle on X and hence $\beta_P \in \mathrm{H}^2_{\mathrm{et}}(X, K)$ becomes zero.

3 The Examples

By constructing examples where property (L) does not hold, we show that Theorem 1.1 fails if G is not assumed to be simply connected.

Let S be a smooth curve over a field of characteristic zero (for simplicity) and let $X \to S$ a family of projective curves with a unique singular fibre over the point $s_0 \in S$

having an ordinary double point at the point x_0 over s_0. Locally in the étale topology at x_0, the family looks like the surface with equation $xy - z^{n+1} = 0$, for some $n > 0$, with the map given by $(x, y, z) \mapsto z$.

We assume that the family has a section $\sigma\colon S \to X$ not passing through x_0, and we also assume that there is a section $\tau\colon S \to X$ with $\tau(s_0) = x_0$. Such families with sections can be constructed by base changing any family as above by a suitable étale map $S' \to S$, with S' also a smooth curve, factoring through X. Note that the local class group at x_0 is $\mathbb{Z}/(n+1)\mathbb{Z}$ (use the method of proof of [8, Example II.6.5.2]), so $D = (n+1)\tau(S)$ is a Cartier divisor which is flat and, as is easily seen, relatively ample over S. Set $U = X - D$.

Let $L = \mathcal{O}_X(\sigma(S))$; for any positive integer m, the first Chern class of L gives a cohomology class $c_1(L) \in \mathrm{H}^2_{\mathrm{et}}(X, \mu_m)$. Let P be the PGL(m) bundle on X induced from the GL(m)-bundle $L \oplus \mathcal{O}^{\oplus(m-1)}$. Using the identification $\pi_1(\mathrm{PGL}(m)) = \mu_m$, it is easy to see that $\alpha_P = c_1(L) \in \mathrm{H}^2_{\mathrm{et}}(X, \mu_m)$.

Proposition 3.1 *Let F be the fiber of $X \to S$ over s_0. Suppose*

(i) *F is reducible and $m > n$, or*

(ii) *F is irreducible and $n + 1$ divides m.*

Then P is not trivial on $U_{S'}$ for any étale neighborhood S' of s_0.

We get finer conditions in terms of the geometry of the minimal resolution of X, see cases (a) and (b) in the proof of Claim 3.2 below.

Let S' be an étale neighbourhood of $s_0 \in S$. By functoriality of the coboundary map (1.1), Proposition 3.1 follows from:

Claim 3.2 *For suitable m and n, as in Proposition 3.1, the class $c_1(L)$ restricts to a non-zero class in $\mathrm{H}^2_{\mathrm{et}}(X_{S'} - \tau(S'), \mu_m)$.*

Remark 3.3 By the Kummer sequence, the restriction of the class of $\alpha_P = c_1(L)$ to $\mathrm{H}^2_{\mathrm{et}}(X_{S'} - \tau(S'), \mu_m)$ is zero if and only if $L = M^m$ for some line bundle M on $X_{S'} - \tau(S')$. The obstruction in property (L) in these examples can therefore be understood in terms of line bundles only.

If such an M exists, one can see that our PGL(m) bundle P, which comes from the GL(m) bundle corresponding to the vector bundle $L \oplus \mathcal{O}^{\oplus(m-1)}$, lifts to the SL$(m)$ bundle corresponding to $M^{\otimes m-1} \oplus (M^{-1})^{\oplus(m-1)}$ (which has trivial determinant). Note that (by construction) the restriction of α_P to $\mathrm{H}^2_{\mathrm{et}}(X_{S'} - \tau(S'), \mu_m)$ is the precise obstruction for lifting P to an SL(m) bundle on $X_{S'} - \tau(S')$.

Proof of Claim 3.2 It suffices to prove the claim for $S = S'$, since étale base change does not alter any of the properties of the family $X \to S$.

The singularity of X at x_0 is étale locally equivalent to $xy - z^{n+1} = 0$, so of type A_n. The exceptional divisor of the minimal resolution Q consists of a chain of n smooth rational curves $E_1, E_2, \ldots, E_n$, each with self-intersection -2 and with E_i intersecting E_{i+1} transversely, $i = 1, \ldots, n-1$; see for example, [1, III.7]. The fibre F over s_0, hence its strict transform $\widetilde{F}$, may or may not be irreducible; in the latter case we write $\widetilde{F}_1$ and $\widetilde{F}_2$ for the components. Locally, the fibre corresponds to the curve $xy = 0$, $z = 0$, so its strict transform intersects E_1 and E_n transversely (in a single point each). The strict transform $\widetilde{D}$ of $D = \tau(S)$ intersects the exceptional divisor in a single point which must be smooth (since Q is smooth), so lies on a unique exceptional divisor E_t. Note that t can be arbitrary: if S' is a smooth curve in Q which intersects E_t transversely at a point s'_0 then $X' = S' \times_S X$ has a singular fibre over s'_0 and a section which on the corresponding desingularization Q' passes through E'_t. By considering tangent directions, one sees that $\widetilde{D}$ does not intersect $\widetilde{F}$; F and D become disjoint after a single blowup.

Let $C = \sigma(S)$ and $\widetilde{C}$ its strict transform in Q. Let $E = \bigcup_i E_i$, so we have $X - D = Q - \widetilde{D} \cup E$. Note that $\mathrm{H}^2_{\widetilde{D}\cup E, et}(Q, \mu_m)$ is freely generated by the classes of $\widetilde{D}$ and all the E_i. Thus, using the Gysin sequence, it suffices to show that the class of $\widetilde{C}$ in $\mathrm{H}^2_{\mathrm{et}}(Q, \mu_m)$ is not in the span of the classes of $\widetilde{D}$ and the E_i. Assume that we have an equation

$$[\widetilde{C}] = a[\widetilde{D}] + \sum_i b_i[E_i] \in \mathrm{H}^2_{\mathrm{et}}(Q, \mu_m). \tag{3.1}$$

We will get a contradiction, in certain cases, by using elementary intersection theory. Each irreducible component G of $E \cup \widetilde{F}$ is a proper curve, so we have maps

$$\mathrm{H}^2_{\mathrm{et}}(Q, \mu_m) \longrightarrow \mathrm{H}^2_{\mathrm{et}}(G, \mu_m) \longrightarrow \mathbb{Z}/m\mathbb{Z},$$

where the first map is pullback and the second is the degree map (which is an isomorphism). Moreover, the pullback map is compatible with the restriction of line bundles or, equivalently, intersections of divisors.

We now consider the following cases.

(a) <u>F is reducible:</u> We may assume that $\widetilde{C}$ passes through $\widetilde{F}_1$, $\widetilde{F}_1$ intersects E_1, and $\widetilde{F}_2$ intersects E_n. Restricting both sides of (3.1) to each of $\widetilde{F}_1$, $\widetilde{F}_2$ and all the E_i and using the degree isomorphisms, we get $n + 2$ equations in $n + 1$ unknowns. We show below that this leads to a contradiction if m does not divide t, which will certainly be the case if $m > n$.

- Intersecting (3.1) with $\widetilde{F}_1$, we get $b_1 = 1$. Intersecting with $\widetilde{F}_2$ gives $b_n = 0$.
- By induction, we may prove that if $i \le t$, $b_i = ib_1$. These equations are obtained by intersecting the two sides of (3.1) by $E_1, \dots, E_{t-1}$. Therefore, $b_t = tb_1$.
- By descending induction from $i = n$, we can prove that $b_i = 0$ if $i \ge t$: The case $i = n$ is already known. Intersecting with E_n (if $t < n$) gives $b_{n-1} = 2b_n$. Intersect (3.1) with E_i (if $i > t$) to get $b_{i-1} - 2b_i + b_{i+1} = 0$, and hence $b_{i-1} = 0$.

Therefore, we have $b_t = tb_1 = 0 \in \mathbb{Z}/m\mathbb{Z}$, hence $t = 0 \in \mathbb{Z}/m\mathbb{Z}$.

(b) <u>F is irreducible:</u> We now intersect both sides of (3.1) with the classes of $\widetilde{F}$ and all the E_i and then apply the degree isomorphisms, getting $n + 1$ equations in $n + 1$ unknowns. As we show below, these equations imply that t is a linear combination of $n+1$ and m. Therefore, if the gcd of $n+1$ and m does not divide t (for example, if $n+1$ divides m), we reach a contradiction.

- Intersecting (3.1) with $\widetilde{F}$, we get $b_1 + b_n = 1$.
- By induction, we may prove that if $i \le t$, $b_i = ib_1$. These equations are obtained by intersecting the two sides of (3.1) by $E_1, \dots, E_{t-1}$. Therefore $b_t = tb_1$.
- By descending induction from $i = n$, we can prove that that $b_i = (n - i + 1)b_n$ if $i \ge t$: The case $i = n$ is already known. Intersecting with E_n (if $t < n$) gives $b_{n-1} = 2b_n$. Intersect (3.1) with E_i (if $i > t$) to get $b_{i-1} - 2b_i + b_{i+1} = 0$, and hence $b_{i-1} = (n - (i - 1) - 1)b_n$.

Therefore, $b_t = tb_1 = (n-t+1)b_n$, so $t(1-b_n) = (n-t+1)b_n$, hence $b_n(n+1) = t \in \mathbb{Z}/m\mathbb{Z}$. This implies that t is a $\mathbb{Z}$-linear combination of $n+1$ and m.

The claim is thus proved. ∎

Remark 3.4 Let $X \to S$ be a family of curves as at the beginning of this section, and let $f\colon Q \to S$ be the minimal desingularization of X. Let S^{sh} be the strict henselisation of S at s_0 and $Q^{\mathrm{sh}} = Q \times_S S^{\mathrm{sh}}$. By the proper base change theorem, the restriction map $\mathrm{H}^2_{\mathrm{et}}(Q^{\mathrm{sh}}, \mu_m) = \mathrm{H}^2_{\mathrm{et}}(X_0, \mu_m)$ is an isomorphism, where X_0 is the (geometric) fibre of f over s_0. Now $\mathrm{H}^2_{\mathrm{et}}(X_0, \mu_m)$ is a free $\mathbb{Z}/m\mathbb{Z}$-module with basis the irreducible components of X_0, *i.e.,* the E_i and $\widetilde{F}$ (or $\widetilde{F}_1$ and $\widetilde{F}_2$). It follows from this that for any $\alpha \in \mathrm{H}^2_{\mathrm{et}}(Q, \mu_m)$, if the numerical equations obtained by intersecting both sides of an equality

$$\alpha = a[\widetilde{D}] + \sum_i b_i[E_i] \in \mathrm{H}^2_{\mathrm{et}}(Q, \mu_m) \tag{3.2}$$

with the E_i and $\widetilde{F}$ (or $\widetilde{F}_1$ and $\widetilde{F}_2$)) can be solved, then in fact (3.2) itself can be solved (for the pullbacks) in $\mathrm{H}^2_{\mathrm{et}}(Q^{\mathrm{sh}}, \mu_m)$, so also when pulled back to some étale neighbourhood S' of $s_0 \in S$. In particular, this applies to $[\widetilde{C}]$, as above.

Remark 3.5 The referee has indicated a simple proof for the following statement that avoids our intersection theoretic computations: If $D = \tau(S)$ generates the local class group of the (normal) singularity on X, and $\gcd(m, n+1) > 1$, then $c_1(L) \neq 0 \in \mathrm{H}^2_{\mathrm{et}}(X - D, \mu_m)$, where, as before, $L = \mathcal{O}_X(\sigma(S))$. The condition on D holds when, for example, τ is given locally by $\tau(z) = (z, z^n, z)$: Use [8, Example II.6.5.2] to see that $R : x = z = 0$ generates the local class group, and note that the divisor of $x - z$ equals $R + D$ and hence $D = -R$ in the class group.

First note that under the assumption that D generates the local class group, any line bundle on the smooth variety $X - D$ extends to X: It suffices to do this for effective line bundles $\mathcal{O}(A)$ with A an effective Weil divisor on $X - D$. Let A' be the closure of A in X. Since D generates the local class group, there is an integer a such that the Weil divisor $A' + aD$ is principal at the singularity. Therefore $A' + mD$ is a Cartier divisor on X, and the associated line bundle extends the given $\mathcal{O}(A)$.

Next, assume by way of contradiction that $c_1(L) = 0 \in \mathrm{H}^2_{\mathrm{et}}(X - D, \mu_m)$. Then by Remark 3.3 there exists a line bundle M on $X - D$ so that $L|_{X-D} = M^m$. Extend M to all of X and denote the extension by M'. By construction, $L \otimes M'^{-m}$ has a non-vanishing section on $X - D$. The divisor associated to this section is supported on D, and equals a multiple of $(n+1)D$ since it is a Cartier divisor. In the divisor class group of X, we have $c_1(L) = mc_1(M') + b(n+1)D$. Restricting this to a general fiber F_{gen} of $X \to S$ and considering degrees, we obtain $1 = m\deg(M' \mid F_{\mathrm{gen}}) + b(n+1)$ which implies $\gcd(m, n+1) = 1$.

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Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599, USA
e-mail: belkale@email.unc.edu

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India
e-mail: naf@math.tifr.res.in