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Twisted Alexander invariants of complex hypersurface complements

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We define and study twisted Alexander-type invariants of complex hypersurface complements. We investigate torsion properties for the twisted Alexander modules and extend the local-to-global divisibility results of Maxim and of Dimca and Libgober to the twisted setting. In the process, we also study the splitting fields containing the roots of the corresponding twisted Alexander polynomials.

Keywords: twisted Alexander polynomial; twisted Alexander module; complex hypersurface; singularities; Hopf link

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1. Introduction

The classical Alexander polynomial from knot theory has proven to be a powerful and versatile tool in the study of complements of plane algebraic curves. As noted by Zariski [26] (see also [13]), the Alexander polynomial of a plane curve complement is sensitive to the local type and position of singularities of the curve, and it can be used to detect Zariski pairs (i.e. pairs of plane curves that have homeomorphic tubular neighbourhoods but non-homeomorphic complements). The study of Alexander polynomials of complements of higher-dimensional complex hypersurfaces was initiated by Libgober [14], and pursued in greater generality (for arbitrary singularities) in [4, 17, 19].

A twisted version of the Alexander polynomial (based on the extra datum of a representation of the fundamental group) was introduced by Lin [16], Wada [25] and Kirk and Livingston [12] in the 1990s, and has proved its worth, for instance, in the works of Friedl and Vidussi (see, for example, [9] and the references therein). Of course, the classical Alexander invariants correspond to the trivial rank-1 representation.

The twisted Alexander polynomial was adapted to the study of plane algebraic curves by Cogolludo and Florens [2], who used it to refine Libgober's divisibility results from [13], and showed that these twisted Alexander polynomials can detect the new Zariski pairs that were undistinguishable by the classical Alexander polynomial. Moreover, twisted Alexander invariants associated with rank-1 representations are closely related to the so-called characteristic varieties of the complement.

In this paper, we extend the Cogolludo–Florens construction to high dimensions and arbitrary singularities, and establish some of the basic properties of the twisted Alexander invariants in this algebro-geometric setting. More concretely, we investigate torsion properties for the twisted Alexander modules, and extend the local-to-global divisibility results of [4, 19] to the twisted setting. In the process, we also study the splitting fields containing the roots of the corresponding twisted Alexander polynomials.

1.1. Main results

In what follows, we establish our notation and give a brief overview of our results.

Let $V \subset \mathbb{CP}^{n+1}$ be a projective complex hypersurface defined globally as the zero set of a degree-d homogeneous polynomial, and fix a hyperplane H in $\mathbb{CP}^{n+1}$ that we call the hyperplane at infinity. Let

$$\mathcal{U} := \mathbb{CP}^{n+1} \setminus (V \cup H) = \mathbb{C}^{n+1} \setminus V^a$$

denote the (affine) hypersurface complement, with $V^a := V \setminus H$ the affine part of V. Alternatively, if $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ is a degree-d polynomial, then $V^a := \{f = 0\} \subset \mathbb{C}^{n+1}$ and $V \subset \mathbb{CP}^{n+1}$ is the projectivization of V^a.

Fix a field $\mathbb{F}$, and let $\mathbb{V} \cong \mathbb{F}^\ell$ be a finite ℓ-dimensional $\mathbb{F}$-vector space. To a pair (ε, ρ) of an epimorphism $\varepsilon\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ and a representation $\rho\colon \pi_1(\mathcal{U}) \to \mathrm{GL}(\mathbb{V})$, we associate *(co)homological (global) twisted Alexander modules* $H_i^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm1}])$ and $H^i_{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm1}])$, respectively, which are $\mathbb{F}[t^{\pm1}]$-modules of finite type and, moreover, homotopy invariants of the complement $\mathcal{U}$.

In *all* our results below, we assume in addition that the epimorphism ε is *positive*, in the sense that it takes positive values on the meridian generators of $H_1(\mathcal{U}, \mathbb{Z})$.

DEFINITION 1.1. We say that the projective hypersurface V (or its affine part V^a) is *in general position at infinity* if the reduced hypersurface V_{red} underlying V is transversal to H in the stratified sense.

One of our first results describes torsion properties of the (global) twisted Alexander modules (see theorems 3.1 and 4.1 and corollary 4.4).

THEOREM 1.2. *Let $V \subset \mathbb{CP}^{n+1}$ be a hypersurface in general position at infinity, let $\varepsilon\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ be a positive epimorphism and let $\rho\colon \pi_1(\mathcal{U}) \to \mathrm{GL}(\mathbb{V})$ be an arbitrary representation on the ℓ-dimensional $\mathbb{F}$-vector space $\mathbb{V}$. Then the twisted Alexander modules $H_i^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm1}])$ are torsion $\mathbb{F}[t^{\pm1}]$-modules for any $0 \leqslant i \leqslant n$; they vanish for $i > n+1$, and $H_{n+1}^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm1}])$ is a free $\mathbb{F}[t^{\pm1}]$-module of rank $(-1)^{n+1} \cdot \ell \cdot \chi(\mathcal{U})$.*

This is a far-reaching generalization of results in [4, 17, 19], which only dealt with the case of the linking number homomorphism and the trivial representation defined on complements of *reduced* hypersurfaces (i.e. defined by square-free polynomials).

For any point $x \in V$, let $\mathcal{U}_x = \mathcal{U} \cap B_x$ denote the local complement at x for B_x a small ball about x in $\mathbb{CP}^{n+1}$. Then (ε, ρ) induces via the inclusion map $i_x\colon \mathcal{U}_x \hookrightarrow \mathcal{U}$ a pair (ε_x, ρ_x) on $\mathcal{U}_x$, so that *local twisted Alexander modules* of $(\mathcal{U}_x, \varepsilon_x, \rho_x)$ can be defined. Proposition 4.9 asserts that for any pair (ε, ρ) as above, with ε a positive epimorphism, we have the following local torsion (or acyclicity) property.

PROPOSITION 1.3. *If V is in general position at infinity, then the local twisted Alexander modules $H_i^{\varepsilon_x,\rho_x}(\mathcal{U}_x, \mathbb{F}[t^{\pm 1}])$ are torsion $\mathbb{F}[t^{\pm 1}]$-modules for any $x \in V$.*

This local torsion property removes a technical assumption used by Cogolludo and Florens [2] in the proof of their main divisibility result for twisted Alexander polynomials of plane curve complements.

Since $\mathbb{F}[t^{\pm 1}]$ is a principal ideal domain, torsion $\mathbb{F}[t^{\pm 1}]$-modules of finite type have orders (called *Alexander polynomials*) associated with them (see, for example, [21]). For $\varepsilon\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ a positive epimorphism and $\rho\colon \pi_1(\mathcal{U}) \to \mathrm{GL}(\mathbb{V})$ an arbitrary representation, we let $\Delta_{i,\mathcal{U}}$ and $\Delta^i_{\mathcal{U}}$ (for $0 \leqslant i \leqslant n$) and $\Delta_{k,x}$ and Δ^k_x (for $k \in \mathbb{Z}$), respectively, be the corresponding *global* and *local* twisted Alexander polynomials associated with the above (co)homological twisted Alexander modules (which are torsion $\mathbb{F}[t^{\pm 1}]$-modules by theorem 1.2 and proposition 1.3). In theorem 4.13 we indicate how to estimate the global twisted Alexander polynomials from the local topological information at points on the hypersurface. This relationship can be roughly formulated as follows (see theorem 4.13 for the precise formulation).

THEOREM 1.4. *For a projective hypersurface V in general position at infinity, the zeros of the global twisted Alexander polynomials of the complement $\mathcal{U}$ are among those of the local twisted Alexander polynomials at points in the affine part of some irreducible component of V.*

This result is a generalization to the twisted setting of the local-to-global analysis for the classical Alexander polynomials initiated by Libgober [13,14] in the isolated singularities case, and extended to arbitrary singularities in [19] (see also [4,17]).

Let us briefly comment on our working assumption of hypersurfaces in general position (i.e. transversality) at infinity. Firstly, such an assumption is needed to conclude that the link at infinity of the hypersurface is *fibred*, and this is the key feature behind the torsion property of theorem 1.2. The $\mathbb{F}[t^{\pm 1}]$-ranks of the classical (untwisted) Alexander modules without the transversality assumption were computed in [6] in terms of vanishing cycles. However, in the case of hypersurfaces with only isolated singularities, *including at infinity* (e.g. in the case of plane curves), the only relevant classical (untwisted) Alexander module (i.e. in degree n) is still torsion (see [14]), but the divisibility statement also includes local contributions coming from the singular points at infinity. For more instances when the torsion property for the classical (untwisted) Alexander modules still holds (below the middle degree), see [18, proposition 6.8]. Furthermore, as a consequence of the proof of our theorem 4.13, we remark that the torsion property for the local twisted Alexander modules at points in $V \cap H$ is enough to conclude that the global twisted Alexander modules are torsion $\mathbb{F}[t^{\pm 1}]$-modules in the desired range. For hypersurfaces in general position at infinity, such a local torsion property at points in $V \cap H$ is a consequence of transversality and the Künneth formula (see the proof of proposition 4.9 and corollary 4.10), but there may be other instances (e.g. for various choices of (ε, ρ)) when it is satisfied. Secondly, since $\mathcal{U} := \mathbb{C}^{n+1} \setminus V^a$ is defined only in terms of the affine hypersurface V^a, it is desirable to understand its global invariants (such as twisted Alexander polynomials) only in terms of information encoded by the singularities of V^a, independently of the hyperplane at infinity, H; this is achieved here under the assumption of transversality at infinity, since complements of links at points in $V \cap H$ are in this case determined by those at nearby points in V (or V^a) (but see also [14], where isolated singularities at infinity are taken into account). Dealing with singularities at infinity in the non-isolated context is certainly much more challenging, and most methods used in this paper (and other papers on the subject) break down.

We also single out the contribution of the meridian at infinity (i.e. a meridian loop about H) to the global twisted Alexander polynomials (see theorem 4.11 for the precise formulation). For the case of the linking number homomorphism and trivial representation, theorem 4.11 reduces to the zeros of the classical Alexander polynomials of $\mathcal{U}$ being roots of unity of order $d = \deg(V)$: a fact shown in [4, 19] for reduced hypersurfaces.

In the case of reduced plane curves and for ε the linking number homomorphism, we explicitly identify splitting fields containing the roots of the corresponding global twisted Alexander polynomials. Similar results were obtained by Libgober [15] using Hodge-theoretic methods. More precisely, in theorem 3.5 we prove the following.

Theorem 1.5. *Let C be a reduced curve of degree d and in general position at infinity, and assume that $\varepsilon\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ is the linking number homomorphism. Suppose $\mathbb{F} = \mathbb{C}$, and let $\rho\colon \pi_1(\mathcal{U}) \to \mathrm{GL}_\ell(\mathbb{C})$ be an arbitrary representation. Denote by x_0 the (homotopy class of the) meridian about the line H at infinity, and let $\lambda_1, \ldots, \lambda_\ell$ be the eigenvalues of $\rho(x_0)^{-1}$. Then the roots of $\Delta^{\varepsilon,\rho}_{1,\mathcal{U}}(t)$ lie in the splitting field $\mathbb{S}$ of $\prod_{i=1}^{\ell}(t^d - \lambda_i)$ over $\mathbb{Q}$, which is cyclotomic over $\mathbb{K} = \mathbb{Q}(\lambda_1, \ldots, \lambda_\ell)$.*

This result is based on our calculation of the twisted Alexander polynomial for the Hopf link on d components (see proposition 2.9), which in our geometric situation can be identified with the link of C 'at infinity'.

2. Twisted chain complexes and twisted Alexander invariants

2.1. Definitions

In this section, we recall the definitions of twisted chain complexes, twisted Alexander modules and twisted Alexander polynomials of path-connected finite CW-complexes. For more details, see [2, 12].

Let X have the homotopy type of a path-connected finite CW-complex, with $\pi = \pi_1(X)$, and fix a group homomorphism

$$\varepsilon\colon \pi_1(X) \to \mathbb{Z}.$$

Note that ε extends to an algebra homomorphism

$$\varepsilon\colon \mathbb{F}[\pi] \to \mathbb{F}[\mathbb{Z}] \cong \mathbb{F}[t^{\pm 1}].$$

Fix a field $\mathbb{F}$, and let

$$\rho\colon \pi \to \mathrm{GL}(\mathbb{V})$$

be a linear representation of π on a finite ℓ-dimensional $\mathbb{F}$-vector space $\mathbb{V}$. For future reference, we fix an isomorphism $\mathbb{V} \cong \mathbb{F}^\ell$. For simplicity, this representation will also be denoted by $\mathbb{V}_\rho$.

Let $\tilde{X}$ be the universal cover of X. The cellular chain complex $C_*(\tilde{X}, \mathbb{F})$ of $\tilde{X}$ is a complex of free left $\mathbb{F}[\pi]$-modules, generated by lifts of the cells of X. For notational

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convenience, we follow [12] and regard $\mathbb{V}$ as a *right* $\mathbb{F}[\pi]$-module, i.e. with the right π-action for $v \in \mathbb{V}$ and $\alpha \in \pi$ given by

$$v \cdot \alpha = v\rho(\alpha),$$

where we view the elements of $\mathbb{V} \cong \mathbb{F}^\ell$ as row vectors. Also consider the right $\mathbb{F}[\pi]$-module $\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V}$, with $\mathbb{F}[\pi]$-multiplication induced by $\varepsilon \otimes \rho$ as

$$(p \otimes v) \cdot \alpha = pt^{\varepsilon(\alpha)} \otimes v \cdot \alpha = pt^{\varepsilon(\alpha)} \otimes v\rho(\alpha), \quad \alpha \in \pi.$$

Let the chain complex of (X, ε, ρ) be defined as the complex of left $\mathbb{F}[t^{\pm 1}]$-modules:

$$C_*^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}]) := (\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V}) \otimes_{\mathbb{F}[\pi]} C_*(\tilde{X}, \mathbb{F}),$$

where the (left) $\mathbb{F}[t^{\pm 1}]$-action is given by

$$t^n((p \otimes v) \cdot c) = (t^n \cdot p \otimes v) \cdot c.$$

It is a complex of free $\mathbb{F}[t^{\pm 1}]$-modules.

Definition 2.1. The *ith homological twisted Alexander module* $H_i^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$ of the triple (X, ε, ρ) is the $\mathbb{F}[t^{\pm 1}]$-module defined by

$$H_i^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}]) := H_i(C_*^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])).$$

Similarly, the *ith cohomological twisted Alexander module* $H^i_{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$ of (X, ε, ρ) is the $\mathbb{F}[t^{\pm 1}]$-module given by

$$H^i_{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}]) := H^i(\mathrm{Hom}_{\mathbb{F}[\pi]}(C_*(\tilde{X}, \mathbb{F}), \mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V})),$$

where $\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V}$ is now regarded as a *left* $\mathbb{F}[\pi]$-module with π-action defined by using the involution on $\mathbb{F}[\pi]$, i.e.

$$\alpha \cdot (p \otimes v) := (p \otimes v) \cdot \alpha^{-1} = pt^{-\varepsilon(\alpha)} \otimes v\rho(\alpha)^{-1}, \quad \alpha \in \pi.$$

The twisted Alexander modules are homotopy invariants of X.

Remark 2.2.

(i) The classical Alexander modules correspond to the case of the trivial representation $\rho = \mathrm{triv}$, i.e. $\mathbb{V} = \mathbb{F} = \mathbb{Q}$ and $\rho(\alpha) = 1$ for all $\alpha \in \pi$.

(ii) In [5, 18], cohomological Alexander-type invariants were considered via the cohomology of the dual complex $\mathrm{Hom}_{\mathbb{F}[t^{\pm 1}]}(C_*^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}])$. These are directly related to the homological Alexander modules via the universal coefficient theorem applied to the principal ideal domain $\mathbb{F}[t^{\pm 1}]$, namely

$$\begin{aligned} &H^i(\mathrm{Hom}_{\mathbb{F}[t^{\pm 1}]}(C_*^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}])) \\ &\quad \cong \mathrm{Hom}_{\mathbb{F}[t^{\pm 1}]}(H_i^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]) \oplus \mathrm{Ext}_{\mathbb{F}[t^{\pm 1}]}(H_{i-1}^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]). \end{aligned}$$

On the other hand, the relationship between the cohomological twisted Alexander modules of definition 2.1 and the homological twisted Alexander modules is explicitly described in [12, pp. 638–639], as we shall now explain. Let

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$\mathbb{W} := \mathbb{V}^* = \mathrm{Hom}_{\mathbb{F}}(\mathbb{V}, \mathbb{F})$ be endowed with the dual representation $\rho^* \colon \pi \to \mathrm{GL}(\mathbb{W})$:

$$(w \cdot \alpha)(v) = w(v \cdot \alpha^{-1}), \quad w \in \mathbb{W},\ v \in \mathbb{V},\ \alpha \in \pi,$$

which induces a corresponding right $\mathbb{F}[\pi]$-module structure on $\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V}^*$ by

$$(p \otimes w) \cdot \alpha = pt^{\varepsilon(\alpha)} \otimes w \cdot \alpha.$$

Then the co-chain complexes

$$\mathrm{Hom}_{\mathbb{F}[t^{\pm 1}]}(C_*^{\varepsilon,\rho^*}(X, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]) \quad \text{and} \quad \mathrm{Hom}_{\mathbb{F}[\pi]}(C_*(\tilde{X}, \mathbb{F}), \mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V})$$

are anti-isomorphic, i.e. isomorphic as co-chain complexes of $\mathbb{F}[t^{\pm 1}]$-modules, provided one of them is given the conjugate $\mathbb{F}[t^{\pm 1}]$-module structure $(p \cdot h)(z) = \bar{p} \cdot h(z)$, which is obtained by composing all $\mathbb{F}[t^{\pm 1}]$-module structures with the involution $\bar{\cdot} \colon \mathbb{F}[t^{\pm 1}] \to \mathbb{F}[t^{\pm 1}]$, $t \mapsto \bar{t} := t^{-1}$. In particular, if we denote by $\bar{H}^i_{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$ the group $H^i_{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$ with the conjugate $\mathbb{F}[t^{\pm 1}]$-module structure, then

$$\bar{H}^i_{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}]) \cong H^i(\mathrm{Hom}_{\mathbb{F}[t^{\pm 1}]}(C_*^{\varepsilon,\rho^*}(X, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}])).$$

Therefore, the universal coefficient theorem yields

$$\bar{H}^i_{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}]) \cong \mathrm{Hom}_{\mathbb{F}[t^{\pm 1}]}(H_i^{\varepsilon,\rho^*}(X, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]) \oplus \mathrm{Ext}_{\mathbb{F}[t^{\pm 1}]}(H_{i-1}^{\varepsilon,\rho^*}(X, \mathbb{F}[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]). \tag{2.1}$$

Furthermore, in the case when ρ and ρ^* are conjugate representations (e.g. $\mathbb{V}$ is a real orthogonal representation of π), one can take $\mathbb{W} = \mathbb{V}$ and use ρ on both sides of (2.1).

An equivalent definition of the twisted chain complex of (X, ε, ρ) was given in [12]. Let X_∞ be the infinite cyclic cover of X associated with $\pi' = \ker \varepsilon$. The chain complex

$$C_*(X_\infty, \mathbb{V}_\rho) := \mathbb{V} \otimes_{\mathbb{F}[\pi']} C_*(\tilde{X}),$$

defined via the restricted actions to π', can be regarded as a complex of $\mathbb{F}[t^{\pm 1}]$-modules via the action $t^n \cdot (v \otimes c) = v \cdot \gamma^{-n} \otimes \gamma^n c$, where γ is an element in π such that $\varepsilon(\gamma) = 1$. Then [12, theorem 2.1] states that $C_*(X_\infty, \mathbb{V}_\rho)$ and $C_*^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$ are isomorphic as left $\mathbb{F}[t^{\pm 1}]$-modules (and the isomorphism is independent of the choice of γ).

Definition 2.3. Denote by $\mathbb{F}(t)$ the field of fractions of $\mathbb{F}[t^{\pm 1}]$, and define

$$C_*^{\varepsilon,\rho}(X, \mathbb{F}(t)) = \mathbb{F}(t) \otimes_{\mathbb{F}[t^{\pm 1}]} C_*^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}]).$$

We say that (X, ε, ρ) is *acyclic* if the chain complex $C_*^{\varepsilon,\rho}(X, \mathbb{F}(t))$ is acyclic over $\mathbb{F}(t)$.

Remark 2.4. Since $\mathbb{F}[t^{\pm 1}]$ is a principal ideal domain, $\mathbb{F}(t)$ is flat over $\mathbb{F}[t^{\pm 1}]$. So, (X, ε, ρ) is acyclic if and only if $H_*^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$ are torsion $\mathbb{F}[t^{\pm 1}]$-modules.

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Since $\mathbb{F}[t^{\pm 1}]$ is a principal ideal domain and $\mathbb{V}$ is finite dimensional over $\mathbb{F}$, the twisted Alexander modules $H_*^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$ are finitely generated modules over $\mathbb{F}[t^{\pm 1}]$. Thus, they have a direct sum decomposition into cyclic modules. Similar considerations apply for the cohomological invariants.

Definition 2.5. The order of the torsion part of $H_i^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$ is called the *ith homological twisted Alexander polynomial* of (X, ε, ρ), and is denoted by $\Delta_{i,X}^{\varepsilon,\rho}(t)$. Similarly, we define the *ith cohomological twisted Alexander polynomial* of (X, ε, ρ) to be the order $\Delta^i_{\varepsilon,\rho,X}(t)$ of the torsion part of the $\mathbb{F}[t^{\pm 1}]$-module $H^i_{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$.

The twisted Alexander polynomials are well defined up to units in $\mathbb{F}[t^{\pm 1}]$. Moreover, it follows from (2.1) that

$$\bar{\Delta}^i_{\varepsilon,\rho,X}(t) = \Delta^{\varepsilon,\rho^*}_{i-1,X}(t).$$

For further use, we also recall here the following fact.

Proposition 2.6 (Kirk and Livingston [12]). *If ε is non-trivial, $H_0^{\varepsilon,\rho}(X, \mathbb{F}[t^{\pm 1}])$ is a torsion $\mathbb{F}[t^{\pm 1}]$-module.*

2.2. Examples

In this subsection, we compute the twisted Alexander invariants on several examples with geometric significance.

2.2.1. *Hopf link with d components*

This example has important consequences in the study of twisted Alexander invariants of plane curve complements. More precisely, for a degree-d plane curve C with regular behaviour at infinity, the Hopf link with d components is what we call 'the link of C at infinity'.

Recall that a link in S^3 is an embedding of a disjoint union of circles (link components) into S^3. Throughout this section, let K be the d-component Hopf link in S^3, consisting of d fibres of the Hopf fibration.

Lemma 2.7. *If $K \subset S^3$ is the d-component Hopf link, then*

$$\pi_1(S^3 \setminus K) \cong \mathbb{Z} \times F_{d-1} \cong \langle x_0, x_1, \ldots, x_{d-1} \mid x_0 x_i x_0^{-1} x_i^{-1}, i = 1, \ldots, d-1 \rangle, \quad (2.2)$$

with F_{d-1} the free group on $d-1$ generators.

Proof. First note that $S^3 \setminus K$ is homotopy equivalent to the link exterior associated with the singularity $\{x^d = y^d\} \subset \mathbb{C}^2$. Equivalently, if $\mathcal{A} = \{x^d = y^d\}$ is the central line arrangement of d lines in $\mathbb{C}^2$, then $S^3 \setminus K \simeq \mathbb{C}^2 \setminus \mathcal{A}$.

On the other hand, it can easily be seen that

$$\mathbb{C}^2 \setminus \mathcal{A} \simeq \mathbb{C}^* \times (\mathbb{CP}^1 \setminus \{d \text{ points}\}).$$

Indeed, the Hopf fibration $\mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$ restricts to a $\mathbb{C}^*$-locally trivial fibration $\mathbb{C}^2 \setminus \mathcal{A} \to \mathbb{CP}^1 \setminus \{d \text{ points}\}$. Moreover, the latter fibration is trivial, since it can be seen as a restriction of the trivial fibration $\mathbb{C}^2 \setminus H \to \mathbb{CP}^1 \setminus \{\text{one point}\} = \mathbb{C}$

obtained from the Hopf fibration by first restricting to the complement of only one line H of $\mathcal{A}$.

Altogether, we have

$$S^3 \setminus K \simeq \mathbb{C}^2 \setminus \mathcal{A} \simeq S^1 \times \left(\bigvee_{d-1} S^1 \right),$$

which yields the desired presentation for $\pi_1(S^3 \setminus K)$. □

Remark 2.8. An equivalent presentation of $\pi_1(S^3 \setminus K)$ can be obtained by using the van Kampen theorem (see, for example, [3, theorem 4.2.17, proposition 4.2.21] and the references therein). More precisely, $\pi_1(S^3 \setminus K)$ is called $G(d,d)$ in [3], and has the presentation

$$\pi_1(S^3 \setminus K) \cong \langle x_0, x_1, \dots, x_d \mid x_d x_{d-1} \cdots x_1 x_0^{-1}, x_0 x_i x_0^{-1} x_i^{-1}, i = 1, \dots, d \rangle,$$

where the generators $x_1, \dots, x_d$ correspond to meridian loops about the d lines of $\mathcal{A}$.

We can now compute the twisted Alexander invariants of $S^3 \setminus K$ (see also [8,10]).

Proposition 2.9. *Let $K \subset S^3$ be the Hopf link with d components. Let*

$$\varepsilon \colon \pi_1(S^3 \setminus K) \to \mathbb{Z}$$

be an epimorphism with

$$\varepsilon(x_0) \neq 0,$$

and

$$\rho \colon \pi_1(S^3 \setminus K) \to \mathrm{GL}(\mathbb{V}) = \mathrm{GL}_\ell(\mathbb{F})$$

be a linear representation of rank ℓ. Then the following hold:

(a) *$H_i^{\varepsilon,\rho}(S^3 \setminus K, \mathbb{F}[t^{\pm 1}])$ are torsion $\mathbb{F}[t^{\pm 1}]$-modules for $i = 0, 1$;*

(b) *$H_i^{\varepsilon,\rho}(S^3 \setminus K, \mathbb{F}[t^{\pm 1}]) = 0$ for $i \geqslant 2$;*

(c) *$\Delta_0^{\varepsilon,\rho}(t)$ is the greatest common divisor of the $\ell \times \ell$ minors of the column matrix*

$$(t^{\varepsilon(x_i)} \rho(x_i) - \mathrm{Id})_{i=0,\dots,d-1};$$

(d) *$\Delta_1^{\varepsilon,\rho}(t)/\Delta_0^{\varepsilon,\rho}(t) = (\det(t^{\varepsilon(x_0)} \rho(x_0) - \mathrm{Id}))^{d-2}$.*

Proof. Recall from lemma 2.7 that the link complement $S^3 \setminus K$ has the homotopy type of a (central) line arrangement complement, namely $S^3 \setminus K \simeq \mathbb{C}^2 \setminus \mathcal{A}$. Thus, it has a minimal cell structure (i.e. so that the number of i-cells equals its ith Betti number b_i for all $i \geqslant 0$); see, for example, [7, 23]. Moreover, since $\mathbb{C}^2 \setminus \mathcal{A}$ is a complex two-dimensional smooth affine variety, it follows by Morse theory [20] (see also [11]) that it has the homotopy type of a finite real two-dimensional CW-complex. Therefore, $H_i^{\varepsilon,\rho}(S^3 \setminus K, \mathbb{F}[t^{\pm 1}]) = 0$ for $i \geqslant 3$.

We next note that $S^3 \setminus K$ is a $K(\pi, 1)$-space, since $\mathbb{C}^2 \setminus \mathcal{A}$ is a $K(\pi, 1)$-space, with $\pi = \pi_1(S^3 \setminus K)$. Indeed, since $\mathcal{A}$ is defined by a homogeneous polynomial, there is a global Milnor fibration

$$F \hookrightarrow \mathbb{C}^2 \setminus \mathcal{A} \to \mathbb{C}^*$$

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whose fibre F has the homotopy type of a wedge of circles. The long exact sequence of homotopy groups for this fibration then yields that $\pi_i(\mathbb{C}^2 \setminus \mathcal{A}) = 0$ for all $i \geqslant 2$.

Since $S^3 \setminus K$ is a $K(\pi, 1)$-space, its (twisted) homology can be computed from its (twisted) group homology using Fox calculus (this was the starting point for Wada's construction of twisted Alexander invariants [25]). So the twisted chain complex of $S^3 \setminus K$ can be identified with the complex of Fox derivatives for the presentation

$$\pi_1(S^3 \setminus K) \cong \langle x_0, x_1, \ldots, x_{d-1} \mid x_0 x_i x_0^{-1} x_i^{-1}, i = 1, \ldots, d-1 \rangle$$

in lemma 2.7, and it has the form

$$0 \to \mathbb{F}[t^{\pm 1}]^{\ell(d-1)} \xrightarrow{\partial_2} \mathbb{F}[t^{\pm 1}]^{\ell d} \xrightarrow{\partial_1} \mathbb{F}[t^{\pm 1}]^{\ell} \to 0.$$

In particular, as in [12, § 4], we have that ∂_1 is the column matrix with ith entry given by

$$t^{\varepsilon(x_i)}\rho(x_i) - \mathrm{Id},$$

which yields the desired description of $\Delta_0^{\varepsilon,\rho}(t)$. Similarly, ∂_2 is a $(d-1) \times d$ matrix with entries in $M_\ell(\mathbb{F}[t^{\pm 1}])$ given by the matrix of Fox derivatives of the relations, tensored with $\mathbb{F}[t^{\pm 1}]^\ell$. Therefore, ∂_2 equals

$$\begin{pmatrix} \mathrm{Id} - t^{\varepsilon(x_1)}\rho(x_1) & t^{\varepsilon(x_0)}\rho(x_0) - \mathrm{Id} & 0 & \cdots & 0 \\ \mathrm{Id} - t^{\varepsilon(x_2)}\rho(x_2) & 0 & t^{\varepsilon(x_0)}\rho(x_0) - \mathrm{Id} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathrm{Id} - t^{\varepsilon(x_{d-2})}\rho(x_{d-2}) & 0 & \cdots & t^{\varepsilon(x_0)}\rho(x_0) - \mathrm{Id} & 0 \\ \mathrm{Id} - t^{\varepsilon(x_{d-1})}\rho(x_{d-1}) & 0 & \cdots & 0 & t^{\varepsilon(x_0)}\rho(x_0) - \mathrm{Id} \end{pmatrix}.$$

Since, by our assumption, $\varepsilon(x_0) \neq 0$, this yields that $\ker(\partial_2) = 0$. Therefore,

$$H_2^{\varepsilon,\rho}(S^3 \setminus K, \mathbb{F}[t^{\pm 1}]) = 0.$$

Also, since ε is non-trivial, we get by proposition 2.6 that $H_0^{\varepsilon,\rho}(S^3 \setminus K, \mathbb{F}[t^{\pm 1}])$ is a torsion $\mathbb{F}[t^{\pm 1}]$-module. So, by using the fact that

$$\chi(S^3 \setminus K) = b_0 - b_1 + b_2 = 1 - d + (d-1) = 0,$$

we obtain that

$$\mathrm{rank}_{\mathbb{F}[t^{\pm 1}]}(H_1^{\varepsilon,\rho}(S^3 \setminus K, \mathbb{F}[t^{\pm 1}])) = -\chi(S^3 \setminus K) = 0.$$

Hence, the first twisted Alexander module $H_1^{\varepsilon,\rho}(S^3 \setminus K, \mathbb{F}[t^{\pm 1}])$ is also torsion over $\mathbb{F}[t^{\pm 1}]$. Finally, by [12, theorem 4.1], we get that

$$\Delta_1^{\varepsilon,\rho}(t)/\Delta_0^{\varepsilon,\rho}(t) = (\det(t^{\varepsilon(x_0)}\rho(x_0) - \mathrm{Id}))^{d-2}.$$

□

2.2.2. *Links of A_{odd}-singularities*

Let $C = \{x^2 - y^{2n} = 0\} \subset \mathbb{C}^2$, $n > 1$, and fix (ε, ρ) as before, with ε non-trivial. The germ $(C, 0)$ of C at the origin of $\mathbb{C}^2$ is known as the A_{2n-1}-singularity. The curve C is the union of two smooth curves that intersect non-transversely at the origin. Let $K \subset S^3$ be the link of $(C, 0)$. Since the defining polynomial of $(C, 0)$

is weighted homogeneous, it follows that $S^3 \setminus K \simeq \mathbb{C}^2 \setminus C$ fibres over $S^1 \simeq \mathbb{C}^*$, with the fibre homotopy equivalent to a wedge of circles. In particular, $S^3 \setminus K$ is aspherical, so its twisted Alexander invariants can be computed by Fox calculus from a presentation of the fundamental group. By [22], we have that

$$\pi_1(S^3 \setminus K) \cong \pi_1(\mathbb{C}^2 \setminus C) \cong G(2,2n) = \langle a_i, \beta \mid \beta = a_1 a_0, R_1, R_2 \rangle,$$

where

$$R_1 \colon a_{i+2n} = a_i, \quad R_2 \colon a_{i+2} = \beta^{-1} a_i \beta, \quad i = 0, \dots, 2n-1.$$

So, explicitly,

$$\pi_1(S^3 \setminus K) \cong \langle a_0, a_1, \dots, a_{2n-1}, \beta \mid a_1 a_0 \beta^{-1}, \beta a_2 \beta^{-1} a_0^{-1}, \beta a_4 \beta^{-1} a_2^{-1}, \dots, \\ \beta a_0 \beta^{-1} a_{2n-2}^{-1}, \beta a_3 \beta^{-1} a_1^{-1}, \beta a_5 \beta^{-1} a_3^{-1}, \dots, \beta a_1 \beta^{-1} a_{2n-1}^{-1} \rangle.$$

By direct computation, it can be seen that in the corresponding twisted chain complex one has $\ker(\partial_2) = 0$, so $H_2^{\varepsilon,\rho}(S^3 \setminus K, \mathbb{F}[t^{\pm 1}]) = 0$. Also, since ε is non-trivial, we get by proposition 2.6 that $H_0^{\varepsilon,\rho}(S^3 \setminus K, \mathbb{F}[t^{\pm 1}])$ is a torsion $\mathbb{F}[t^{\pm 1}]$-module. An Euler characteristic argument similar to that of the previous example then yields that $H_1^{\varepsilon,\rho}(S^3 \setminus K, \mathbb{F}[t^{\pm 1}])$ is a torsion $\mathbb{F}[t^{\pm 1}]$-module.

3. Twisted Alexander invariants of plane curve complements

Twisted Alexander invariants were adapted to the study of plane algebraic curves by Cogolludo and Florens [2], who showed that these twisted invariants can detect Zariski pairs that share the same (classical) Alexander polynomial. In this section, we study torsion properties of the twisted Alexander modules of plane curve complements and study splitting fields containing the roots of the corresponding twisted Alexander polynomials. We focus here on homological invariants, while similar statements about their cohomological counterparts can be obtained via (2.1).

Let C be a reduced curve in $\mathbb{CP}^2$ of degree d with r irreducible components, and let L be a line in $\mathbb{CP}^2$. Set

$$\mathcal{U} := \mathbb{CP}^2 \setminus (C \cup L) = \mathbb{C}^2 \setminus (C \setminus (C \cap L)),$$

where we use the natural identification of $\mathbb{C}^2$ with $\mathbb{CP}^2 \setminus L$. The line L will usually be referred to as the *line at infinity*. Alternatively, if $f(x,y) \colon \mathbb{C}^2 \to \mathbb{C}$ is a square-free polynomial of degree d defining an affine plane curve $C^a := \{f = 0\}$, we let C be the zero locus in $\mathbb{CP}^2$ (with homogeneous coordinates x, y, z) of the homogenization f^h of f, with L given by $z = 0$. Then $\mathcal{U} = \mathbb{C}^2 \setminus C^a$.

Recall that $H_1(\mathcal{U}, \mathbb{Z}) \cong \mathbb{Z}^r$, generated by homology classes ν_i of meridian loops γ_i bounding transversal discs at a smooth point in each irreducible component of C^a. Let $n_1, \dots, n_r$ be positive integers with $\gcd(n_1, \dots, n_r) = 1$. Let $ab \colon \pi_1(\mathcal{U}) \to H_1(\mathcal{U}, \mathbb{Z})$ denote the abelianization map, sending $[\gamma_i]$ to ν_i. Then the composition

$$\varepsilon \colon \pi_1(\mathcal{U}) \xrightarrow{ab} H_1(\mathcal{U}, \mathbb{Z}) \xrightarrow{\psi \colon \nu_i \to n_i} \mathbb{Z}$$

defines a *positive* epimorphism. If all $n_i = 1$, then ε can be identified with the total linking number homomorphism

$$lk \colon \pi_1(\mathcal{U}) \xrightarrow{[\alpha] \mapsto lk(\alpha, C \cup -dL)} \mathbb{Z},$$

which is just the homomorphism $f_{\#}\colon \pi_1(\mathcal{U}) \to \pi_1(\mathbb{C}^*) \cong \mathbb{Z}$ induced by the restriction of f to $\mathcal{U}$ (see, for example, [3, p. 77]).

Fix, as before, a field $\mathbb{F}$ and a finite ℓ-dimensional $\mathbb{F}$-vector space $\mathbb{V}$ endowed with a linear representation $\rho\colon \pi_1(\mathcal{U}) \to \mathrm{GL}(\mathbb{V})$. As in § 2.1, the $\mathbb{F}[t^{\pm 1}]$-module $H_i^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ is defined for any $i \geqslant 0$, and is called the *ith (homological) twisted Alexander module of* C *with respect to* L. The twisted Alexander modules $H_i^{lk,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ associated with the total linking number homomorphism lk are denoted by $H_i^{\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ and we let $\Delta^{\rho}_{i,\mathcal{U}}(t)$ be the corresponding Alexander polynomials. In the case of the trivial representation, these further reduce to the classical Alexander invariants, as originally studied in [13].

Note that, since $\mathcal{U}$ is the complement of a plane affine curve, it is a complex two-dimensional affine manifold. Therefore, $\mathcal{U}$ has the homotopy type of a real two-dimensional finite CW-complex (see, for example, [11,20]). Hence, $H_i^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}]) = 0$ for $i \geqslant 3$, and $H_2^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ is a free $\mathbb{F}[t^{\pm 1}]$-module. For $i = 0, 1$, the $\mathbb{F}[t^{\pm 1}]$-modules $H_i^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ are of finite type, and we investigate their torsion properties below.

3.1. Torsion properties

In this section, we prove the following result.

THEOREM 3.1. *Let* C *be a reduced complex projective plane curve and* $\varepsilon\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ *be a positive epimorphism. If* C *is irreducible and* ρ *is abelian (i.e. the image of* ρ *is abelian), or if* C *is in general position at infinity (i.e.* C *is transversal to the line at infinity* L*), then the twisted Alexander modules* $H_i^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ *are torsion* $\mathbb{F}[t^{\pm 1}]$*-modules, for* $i = 0, 1$.

Proof. The claim about $H_0^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ follows from proposition 2.6 since ε is non-trivial.

If C is irreducible and ρ is abelian, it follows from [15] that the classical Alexander modules of an irreducible curve complement determine the twisted ones. So the claim follows in this case from [13].

Assume now that the line at infinity L is transversal to the curve C, and let $d = \deg(C)$. Let $S^3_\infty \subset \mathbb{C}^2$ be a sphere of sufficiently large radius. Then the link of C at infinity, $K_\infty = S^3_\infty \cap C$, is the Hopf link on d components, as described in § 2.2.1. (Indeed, there exists a deformation of C to a union of d lines passing through the origin of $\mathbb{C}^2$, so the transversality at infinity assumption holds for all curves appearing during the deformation.) Let $i\colon S^3_\infty \backslash K_\infty \hookrightarrow \mathcal{U}$ denote the inclusion map. Then by [13, lemma 5.2], the induced homomorphism

$$\pi_1(S^3_\infty \setminus K_\infty) \cong \langle x_0, x_1, \ldots, x_d \mid x_d x_{d-1} \cdots x_1 x_0^{-1}, x_0 x_i x_0^{-1} x_i^{-1}, i = 1, \ldots, d\rangle \xrightarrow{i_\#} \pi_1(\mathcal{U})$$

is surjective. Moreover, as in [13, § 7], the groups $\pi_1(\mathcal{U})$ and $\pi_1(S^3_\infty \setminus K_\infty)$ have the same generators, while the relations in $\pi_1(\mathcal{U})$ are those of $\pi_1(S^3_\infty \backslash K_\infty)$ together with relations describing the monodromy about exceptional lines by using the Zariski–van Kampen method. Therefore, $\varepsilon \circ i_\# = \varepsilon$ and $\rho \circ i_\# = \rho$ (as this can be checked on generators).

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Up to homotopy, $\mathcal{U}$ is obtained from $S^3_\infty \setminus K_\infty$ by attaching cells of dimension $\geqslant 2$. So the homomorphism

$$H_k^{\varepsilon,\rho}(S^3_\infty \setminus K_\infty, \mathbb{F}[t^{\pm 1}]) \to H_k^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$$

induced by the inclusion map i is an isomorphism for $k = 0$, and an epimorphism for $k = 1$. Here, $H_k^{\varepsilon,\rho}(S^3_\infty \setminus K_\infty, \mathbb{F}[t^{\pm 1}])$ is defined with respect to the pair $(\varepsilon \circ i_\# = \varepsilon, \rho \circ i_\# = \rho)$ induced by the inclusion map i. As a consequence, in order to conclude that $H_1^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ is an $\mathbb{F}[t^{\pm 1}]$-torsion module, it suffices to prove the torsion property for the $\mathbb{F}[t^{\pm 1}]$-module $H_1^{\varepsilon,\rho}(S^3_\infty \setminus K_\infty, \mathbb{F}[t^{\pm 1}])$. Hence, by proposition 2.9, it suffices to show that $\varepsilon \circ i_\#(x_0) = \varepsilon(x_0) \neq 0$.

We have the following commutative diagram:

$$\begin{array}{ccccc}
\pi_1(S^3_\infty \setminus K_\infty) & \xrightarrow{i_\#} & \pi_1(\mathcal{U}) & \xrightarrow{\varepsilon} & \mathbb{Z} \\
\downarrow{\scriptstyle ab} & & \downarrow{\scriptstyle ab} & \nearrow{\scriptstyle \psi} & \\
H_1(S^3_\infty \setminus K_\infty, \mathbb{Z}) & \xrightarrow{i_*} & H_1(\mathcal{U}, \mathbb{Z}) & &
\end{array}$$

So, $\varepsilon \circ i_\# = \psi \circ i_* \circ ab$, and thus it is enough to understand the maps ab and i_*. Recall that the Hopf link complement $S^3_\infty \setminus K_\infty$ is homotopy equivalent to the complement $\mathbb{C}^2 \setminus \mathcal{A}$ of a central line arrangement $\mathcal{A}$ of d lines in $\mathbb{C}^2$. So

$$H_1(S^3_\infty \setminus K_\infty, \mathbb{Z}) \cong \mathbb{Z}^d = \langle \mu_1, \ldots, \mu_d \rangle,$$

where μ_k is the homology class of the meridian about the line $l_k \subset \mathcal{A}$. Moreover, $ab(x_k) = \mu_k$ for $k = 1, \ldots, d$, and hence

$$ab(x_0) = \mu_1 + \cdots + \mu_d.$$

On the other hand, $H_1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}^r$, generated by the homology classes ν_l of the meridians about each irreducible component of C^a. Since $\mathcal{A}$ is defined by the homogeneous part of the defining equation of C^a, it is clear that i_* takes each μ_k to one of the ν_l. In fact, exactly d_l of the μ_k are mapped by i_* to ν_l, where d_l is the degree of the component C_l of C. Finally, since $\psi(\nu_l) = n_l$, for all $k \geqslant 1$ we have that $\varepsilon \circ i_\#(x_k) = n_{l_j}$ for some l_j, and

$$\varepsilon \circ i_\#(x_0) = \psi \circ i_*(\mu_1 + \cdots + \mu_d) = \sum_{l=1}^{r} d_l n_l > 0.$$

This concludes the proof that $H_1^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ is a finitely generated $\mathbb{F}[t^{\pm 1}]$-torsion module. □

Remark 3.2. The above result will be generalized to arbitrary hypersurfaces in theorem 4.1. The reason for stating it in this section is our study of splitting fields containing the roots of the associated twisted Alexander polynomials (see theorem 3.5).

As a consequence of theorem 3.1 and proposition 2.9, we obtain the following.

Corollary 3.3. *If C is a reduced curve of degree d in general position at infinity, then the first twisted Alexander polynomial $\Delta^{\varepsilon,\rho}_{1,\mathcal{U}}(t)$ of $\mathcal{U}$ divides the product*

$$(\det(t^{\sum_{l=1}^{r} d_l n_l}\rho(x_0) - \mathrm{Id}))^{d-2} \cdot \gcd(\det(t^{\sum_{l=1}^{r} d_l n_l}\rho(x_0) - \mathrm{Id}), \det(t^{n_{l_1}}\rho(x_1) - \mathrm{Id}), \ldots, \det(t^{n_{l_{d-1}}}\rho(x_{d-1}) - \mathrm{Id})).$$

In particular, if $\varepsilon = lk$, then $\Delta^{\rho}_{1,\mathcal{U}}(t)$ divides

$$(\det(t^d\rho(x_0)-\mathrm{Id}))^{d-2}\cdot\gcd(\det(t^d\rho(x_0)-\mathrm{Id}), \det(t\rho(x_1)-\mathrm{Id}), \ldots, \det(t\rho(x_{d-1})-\mathrm{Id})).$$

Remark 3.4. For curves in general position at infinity, corollary 3.3 generalizes Libgober's divisibility result [13, theorem 2], which states that the Alexander polynomial $\Delta_{1,\mathcal{U}}(t) := \Delta^{lk,\mathrm{triv}}_{1,\mathcal{U}}(t)$ of C divides the Alexander polynomial of the link at infinity, which is given by $(t-1)(t^d-1)^{d-2}$.

3.2. Roots of twisted Alexander polynomials

In [15, theorem 5.4], Libgober used Hodge theory to show that for an irreducible plane curve C, and for ρ a unitary representation, the roots of the first twisted Alexander polynomial of C are in a cyclotomic extension of the field generated by the rationals and the eigenvalues of $\rho(\gamma)$, where γ is a meridian about C at a non-singular point. Libgober's result does not touch upon the extension degree.

In this section, we give a topological proof of Libgober's result, and identify such a cyclotomic extension explicitly.

Theorem 3.5. *Let C be a reduced projective plane curve of degree d and in general position at infinity, and assume that $\varepsilon = lk\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ is the linking number homomorphism. Suppose $\mathbb{F} = \mathbb{C}$, and let $\rho\colon \pi_1(\mathcal{U}) \to \mathrm{GL}_\ell(\mathbb{C})$ be an arbitrary representation. Denote by x_0 the (homotopy class of the) meridian about the line H at infinity, and let $\lambda_1, \ldots, \lambda_\ell$ be the eigenvalues of $\rho(x_0)^{-1}$. Then the roots of $\Delta^{\rho}_{1,\mathcal{U}}(t)$ lie in the splitting field $\mathbb{S}$ of $\prod_{i=1}^{\ell}(t^d - \lambda_i)$ over $\mathbb{Q}$, which is cyclotomic over $\mathbb{K} = \mathbb{Q}(\lambda_1, \ldots, \lambda_\ell)$.*

Proof. Using the notation from the proof of theorem 3.1, we denote by $x_1, \ldots, x_d$ the (homotopy classes of) meridians about the components of the link of C at infinity (see also remark 2.8).

If there is no common eigenvalue for all of $\rho(x_1), \ldots, \rho(x_d)$, then corollary 3.3 yields that $\Delta^{\rho}_{1,\mathcal{U}}(t)$ divides $(\det(t^d\rho(x_0) - \mathrm{Id}))^{d-2}$. In particular, the prime factors of $\Delta^{\rho}_{1,\mathcal{U}}(t)$ are among the prime factors of $\det(t^d\rho(x_0) - \mathrm{Id})$. Let $p(t)$ be the characteristic polynomial of $\rho(x_0)^{-1}$. Then

$$\det(t^d\rho(x_0)-\mathrm{Id}) = (-1)^r\det(\rho(x_0))\cdot p(t^d) = (-1)^r\det(\rho(x_0))\cdot(t^d-\lambda_1)\cdots(t^d-\lambda_\ell).$$

Therefore, the roots of $\Delta^{\rho}_{1,\mathcal{U}}(t)$ are contained in the splitting field $\mathbb{S}$ of $\prod_{i=1}^{\ell}(t^d - \lambda_i)$ over $\mathbb{Q}$.

If α is a common eigenvalue of all matrices $\rho(x_1), \ldots, \rho(x_d)$, then one of the eigenvalues of $\rho(x_0) = \rho(x_d)\rho(x_{d-1})\cdots\rho(x_1)$ is α^d. Without loss of generality, assume that $\alpha^d = \lambda_1^{-1}$. Then $\alpha \in \mathbb{S}$. □

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4. Twisted Alexander invariants of complex hypersurface complements

In this section, we generalize the above results to the context of complex hypersurfaces with arbitrary singularities. We study the torsion properties of the associated twisted Alexander modules, and estimate their corresponding twisted Alexander polynomials in terms of local topological data encoded by the singularities.

4.1. Definitions

Let V be a (globally defined) degree-d hypersurface in $\mathbb{CP}^{n+1}$ ($n \geqslant 1$) and let H be a hyperplane in $\mathbb{CP}^{n+1}$, called the 'hyperplane at infinity'. Let

$$\mathcal{U} := \mathbb{CP}^{n+1} \setminus (V \cup H) = \mathbb{C}^{n+1} \setminus V^a,$$

where $V^a \subset \mathbb{C}^{n+1} = \mathbb{CP}^{n+1} \setminus H$ denotes the affine part of V. Alternatively, we can start with a degree-d polynomial $f(z_1, \dots, z_{n+1}) \colon \mathbb{C}^{n+1} \to \mathbb{C}$, and take $V^a = \{f = 0\}$, with $V \subset \mathbb{CP}^{n+1}$ the projectivization of V^a, and H given by $z_0 = 0$. (Here, $z_0, z_1, \dots, z_{n+1}$ denote the homogeneous coordinates on $\mathbb{CP}^{n+1}$.)

Assume that the underlying reduced hypersurface V_{red} of V has r irreducible components $V_1, \dots, V_r$, with $d_i = \deg(V_i)$ for $i = 1, \dots, r$. Then

$$H_1(\mathcal{U}, \mathbb{Z}) \cong \mathbb{Z}^r,$$

generated by the homology classes ν_i of meridians γ_i about the irreducible components V_i of V_{red} (see, for example, [3, (4.1.3), (4.1.4)]). Moreover, if γ_∞ denotes the meridian loop in $\mathcal{U}$ about the hyperplane H at infinity, with homology class ν_∞, then the following relation holds in $H_1(\mathcal{U}, \mathbb{Z})$:

$$\nu_\infty + \sum_{i=1}^{r} d_i \nu_i = 0. \tag{4.1}$$

Let n_i be r positive integers with $\gcd(n_1, \dots, n_r) = 1$, and define the *positive* epimorphism $\varepsilon \colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ by the composition

$$\varepsilon \colon \pi_1(\mathcal{U}) \xrightarrow{ab} H_1(\mathcal{U}, \mathbb{Z}) \xrightarrow{\nu_i \mapsto n_i} \mathbb{Z}.$$

Note that if the defining equation f of the affine hypersurface V^a has an irreducible decomposition given by $f = f_1^{n_1} \cdots f_r^{n_r}$, then ε coincides with the homomorphism $f_\# \colon \pi_1(\mathcal{U}) \to \pi_1(\mathbb{C}^*) \cong \mathbb{Z}$ induced by the restriction of f to $\mathcal{U}$, or, equivalently, with the *total linking number homomorphism* (see [3, pp. 76–77])

$$lk \colon \pi_1(\mathcal{U}) \xrightarrow{[\alpha] \to lk(\alpha, V \cup -dH)} \mathbb{Z}.$$

Fix a finite ℓ-dimensional $\mathbb{F}$-vector space $\mathbb{V}$ endowed with a linear representation $\rho \colon \pi_1(\mathcal{U}) \to \mathrm{GL}(\mathbb{V})$. As in § 2.1, the $\mathbb{F}[t^{\pm 1}]$-modules $H_i^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ and $H^i_{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ are defined for any $i \geqslant 0$, and are called the *ith (co)homological twisted Alexander modules of V with respect to the hyperplane at infinity, H*. The twisted Alexander modules $H_i^{lk,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ associated with the total linking number homomorphism lk are denoted by

$$H_i^{\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}]),$$

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and similarly for their cohomology counterparts $H^i_\rho(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$. In the case of the trivial representation, these further reduce to the classical Alexander modules, as studied, for example, in [4, 17, 19].

Note that, since $\mathcal{U}$ is the complement of a complex n-dimensional affine hypersurface, it is an $(n + 1)$-dimensional affine variety, and hence has the homotopy type of a finite CW-complex of real dimension $n + 1$ (see, for example, [11, 20], or [3, (1.6.7), (1.6.8)]). Therefore, $H^{\varepsilon,\rho}_i(\mathcal{U}, \mathbb{F}[t^{\pm 1}]) = 0$ for $i \geqslant n+1$, $H^{\varepsilon,\rho}_{n+1}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ is a free $\mathbb{F}[t^{\pm 1}]$-module, and the $\mathbb{F}[t^{\pm 1}]$-modules $H^{\varepsilon,\rho}_i(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ are of finite type for $0 \leqslant i \leqslant n$. In the following subsections, we investigate the torsion properties of the latter.

4.2. Torsion properties

In the notation of the previous subsection, we say that the hypersurface $V \subset \mathbb{CP}^{n+1}$ is *in general position (with respect to the hyperplane H) at infinity* if the reduced hypersurface V_{red} underlying V is transversal to H in the stratified sense.

The main result of this section is the following high-dimensional generalization of theorem 3.1.

Theorem 4.1. *Let $V \subset \mathbb{CP}^{n+1}$ be a hypersurface in general position at infinity, let $\varepsilon\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ be a positive epimorphism and let $\rho\colon \pi_1(\mathcal{U}) \to \mathrm{GL}(\mathbb{V})$ be an arbitrary representation. Then the twisted Alexander modules $H^{\varepsilon,\rho}_i(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ are torsion $\mathbb{F}[t^{\pm 1}]$-modules for any $0 \leqslant i \leqslant n$.*

In order to prove theorem 4.1, we introduce the following notation and develop some prerequisites.

Let S^{2n+1}_∞ be a $(2n + 1)$-sphere in $\mathbb{C}^{n+1}$ of sufficiently large radius (that is, the boundary of a small tubular neighbourhood in $\mathbb{CP}^{n+1}$ of the hyperplane H at infinity). Denote by

$$K_\infty = S^{2n+1}_\infty \cap V^a$$

the *link of V^a at infinity*, and by

$$\mathcal{U}^\infty = S^{2n+1}_\infty \setminus K_\infty$$

its complement in S^{2n+1}_∞. Note that $\mathcal{U}^\infty$ is homotopy equivalent to $T(H) \setminus (V \cup H)$, where $T(H)$ is the tubular neighbourhood of H in $\mathbb{CP}^{n+1}$ for which S^{2n+1}_∞ is the boundary. Then a classical argument based on the Lefschetz hyperplane theorem yields that the homomorphism

$$\pi_i(\mathcal{U}^\infty) \to \pi_i(\mathcal{U})$$

induced by inclusion is an isomorphism for $i < n$ and it is surjective for $i = n$ (see [4, § 4.1] for more details). It follows that

$$\pi_i(\mathcal{U}, \mathcal{U}^\infty) = 0 \quad \text{for all } i \leqslant n, \tag{4.2}$$

and hence $\mathcal{U}$ has the homotopy type of a CW complex obtained from $\mathcal{U}^\infty$ by adding cells of dimension $\geqslant n + 1$.

We denote by $(\varepsilon_\infty, \rho_\infty)$ the epimorphism and representation on $\pi_1(\mathcal{U}^\infty)$ induced by composing (ε, ρ) with the homomorphism $\pi_1(\mathcal{U}^\infty) \to \pi_1(\mathcal{U})$. Hence, the *twisted*

Alexander modules of V at infinity, $H_i^{\varepsilon_\infty,\rho_\infty}(\mathcal{U}^\infty, \mathbb{F}[t^{\pm 1}])$, can be defined (and similarly for the corresponding cohomology modules). Then (4.2) and the fact that twisted Alexander modules are homotopy invariants yield the following.

Proposition 4.2. *The inclusion map $\mathcal{U}^\infty \hookrightarrow \mathcal{U}$ induces the $\mathbb{F}[t^{\pm 1}]$-module isomorphisms*

$$H_i^{\varepsilon_\infty,\rho_\infty}(\mathcal{U}^\infty, \mathbb{F}[t^{\pm 1}]) \xrightarrow{\cong} H_i^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$$

for any $i < n$, and an epimorphism of the $\mathbb{F}[t^{\pm 1}]$-modules

$$H_n^{\varepsilon_\infty,\rho_\infty}(\mathcal{U}^\infty, \mathbb{F}[t^{\pm 1}]) \twoheadrightarrow H_n^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}]).$$

Corollary 4.3. *For any $0 \leqslant i \leqslant n$, if $H_i^{\varepsilon_\infty,\rho_\infty}(\mathcal{U}^\infty, \mathbb{F}[t^{\pm 1}])$ is a torsion $\mathbb{F}[t^{\pm 1}]$-module, then so is $H_i^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$.*

Let us now assume that the complex projective hypersurface V is in general position at infinity, i.e. V_{red} is transversal in the stratified sense to the hyperplane at infinity, H. Then the complement of the link at infinity $\mathcal{U}^\infty$ is a circle fibration over $H \setminus (V \cap H)$, which is homotopy equivalent to the complement in $\mathbb{C}^{n+1}$ to the affine cone over the projective hypersurface $V \cap H \subset H = \mathbb{CP}^n$ (for a similar argument see [4, § 4.1]). Hence, by the Milnor fibration theorem (see, for example, [3, (3.1.9), (3.1.11)]), $\mathcal{U}^\infty$ fibres over $\mathbb{C}^* \simeq S^1$, with fibre homotopy equivalent to a finite n-dimensional CW-complex. Moreover, it is known that this fibre is also homotopy equivalent to the infinite cyclic cover of $\mathcal{U}^\infty$ defined by the kernel of the total linking number homomorphism defined with respect to V^a.

We can now complete the proof of theorem 4.1.

Proof. By corollary 4.3, it suffices to prove that, for any $0 \leqslant i \leqslant n$, the $\mathbb{F}[t^{\pm 1}]$-module $H_i^{\varepsilon_\infty,\rho_\infty}(\mathcal{U}^\infty, \mathbb{F}[t^{\pm 1}])$ is torsion. The idea is to replace V^a by another affine hypersurface X with the same underlying reduced structure, and hence also the same complement $\mathcal{U}$, so that ε becomes the homomorphism defined by the total linking number with X.

Let $f_1 \cdots f_r = 0$ be a square-free polynomial equation defining V^a_{red}, the reduced affine hypersurface underlying $V^a = V \setminus H$. Recall that if γ_i is the meridian about the irreducible component $f_i = 0$, then by definition we have that $\varepsilon([\gamma_i]) = n_i$. Let us now consider the polynomial $g = f_1^{n_1} \cdots f_r^{n_r}$ on $\mathbb{C}^{n+1}$ defining an affine hypersurface

$$X = \{g = 0\},$$

and replace V by the projective hypersurface $\bar{X}$ defined by the homogenization of g. Clearly, the underlying reduced hypersurface X_{red} coincides with V^a_{red}, so X and V^a have the same complement:

$$\mathcal{U} := \mathbb{C}^n \setminus V^a = \mathbb{C}^n \setminus X.$$

Moreover, the given homomorphism $\varepsilon\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ (hence also $\varepsilon_\infty\colon \pi_1(\mathcal{U}^\infty) \to \mathbb{Z}$) coincides with the total linking number homomorphism defined with respect to X (see [3, pp. 76–77]). Finally, since V is in general position at infinity, so is $\bar{X}$, and the corresponding complements of the links at infinity coincide. Therefore (as

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explained in the paragraph before the proof of theorem 4.1), the complement $\mathcal{U}^\infty$ of the link at infinity admits a locally trivial topological fibration

$$F \hookrightarrow \mathcal{U}^\infty \to \mathbb{C}^*$$

whose fibre F has the homotopy type of a finite n-dimensional CW-complex, and is also homotopy equivalent to the infinite cyclic cover of $\mathcal{U}^\infty$ defined by the kernel of the linking number with respect to X (i.e. by $\ker(\varepsilon_\infty)$).

Altogether, for any $0 \leqslant i \leqslant n$, we have

$$H_i^{\varepsilon_\infty,\rho_\infty}(\mathcal{U}^\infty, \mathbb{F}[t^{\pm 1}]) \cong H_i(F, \mathbb{V}_{\rho_\infty}),$$

which is a finite-dimensional $\mathbb{F}$-vector space, hence a torsion $\mathbb{F}[t^{\pm 1}]$-module. □

As an immediate consequence of theorem 4.1, we have the following.

Corollary 4.4. *Using the notation and assumptions of theorem 4.1, we have*

$$\operatorname{rank}_{\mathbb{F}[t^{\pm 1}]} H_{n+1}^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}]) = (-1)^{n+1} \cdot \ell \cdot \chi(\mathcal{U}),$$

with ℓ the rank of the representation ρ.

By applying theorem 4.1 to the dual representation ρ^*, we deduce from (2.1) the following.

Corollary 4.5. *Let $V \subset \mathbb{CP}^{n+1}$ be a hypersurface in general position at infinity, let $\varepsilon\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ be a positive epimorphism and let $\rho\colon \pi_1(\mathcal{U}) \to \mathrm{GL}(\mathbb{V})$ be an arbitrary representation. Then the cohomological twisted Alexander modules $H^i_{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ are torsion $\mathbb{F}[t^{\pm 1}]$-modules for any $0 \leqslant i \leqslant n$.*

Remark 4.6. If V is in general position at infinity, and $\dim_{\mathbb{C}} \mathrm{Sing}(V) \leqslant n-2$ (in which case V is already irreducible), then $\pi_1(\mathcal{U}) \cong \mathbb{Z}$ (see, for example, [14, lemma 1.5]). So, in this case, the representation ρ is abelian, and the twisted Alexander invariants of $\mathcal{U}$ are determined by the classical ones (studied in [4, 17, 19]). The results of this paper are particularly interesting for hypersurfaces with singularities in codimension 1 (e.g. hyperplane arrangements) and non-abelian representations.

4.3. Local twisted Alexander invariants

For each point $x \in V$, consider the local complement

$$\mathcal{U}_x := \mathcal{U} \cap B_x,$$

for B_x a small open ball about x in $\mathbb{CP}^{n+1}$ chosen so that (V, x) has a conic structure in $\bar{B}_x$. Let

$$\varepsilon_x\colon \pi_1(\mathcal{U}_x) \xrightarrow{(i_x)_\#} \pi_1(\mathcal{U}) \xrightarrow{\varepsilon} \mathbb{Z}$$

and

$$\rho_x\colon \pi_1(\mathcal{U}_x) \xrightarrow{(i_x)_\#} \pi_1(\mathcal{U}) \xrightarrow{\rho} \mathrm{GL}(\mathbb{V}) = \mathrm{GL}_\ell(\mathbb{F})$$

be induced by the inclusion $i_x\colon \mathcal{U}_x \hookrightarrow \mathcal{U}$. Then we can consider the *local* (co)homological twisted Alexander modules $H_k^{\varepsilon_x,\rho_x}(\mathcal{U}_x, \mathbb{F}[t^{\pm 1}])$ and $H^k_{\varepsilon_x,\rho_x}(\mathcal{U}_x, \mathbb{F}[t^{\pm 1}])$ for $k \in \mathbb{Z}$.

Remark 4.7. Note that ε_x is not necessarily onto, so the infinite cyclic cover of $\mathcal{U}_x$ defined by $\ker(\varepsilon_x)$ may be disconnected.

Definition 4.8. We say that (ε, ρ) is *acyclic at* $x \in V$ if (ε_x, ρ_x) is acyclic in the sense of definition 2.3, i.e. if $H_k^{\varepsilon_x,\rho_x}(\mathcal{U}_x, \mathbb{F}[t^{\pm 1}])$ are torsion $\mathbb{F}[t^{\pm 1}]$-modules for all $k \in \mathbb{Z}$. We say that (ε, ρ) is *locally acyclic* along a subset $Y \subseteq V$ if (ε, ρ) is acyclic at any point $x \in Y$.

The next result provides one important geometric example of local acyclicity.

Proposition 4.9. *Let $V \subset \mathbb{CP}^{n+1}$ be a degree-d projective hypersurface in general position at infinity. Then (ε, ρ) is locally acyclic along V for any positive epimorphism $\varepsilon\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ and any representation $\rho\colon \pi_1(\mathcal{U}) \to \mathrm{GL}(\mathbb{V})$.*

Proof. As in the proof of theorem 4.1, after changing V^a (respectively, V) by an affine hypersurface X (respectively, by its projectivization $\bar{X}$) with the same underlying reduced structure, and hence also preserving the (local) complements, we can assume without loss of generality (and without changing the notation) that ε is the total linking number homomorphism lk. Therefore, for any $x \in V$, the local homomorphism ε_x becomes $lk_x := lk \circ (i_x)_\#$. Denote by $\mathcal{U}_{x,\infty}$ the infinite cyclic cover of $\mathcal{U}_x$ defined by $\ker(lk_x)$.

Let $\mathcal{U}' = \mathbb{CP}^{n+1} \setminus V$, and for any point $x \in V$ let $\mathcal{U}'_x := \mathcal{U}' \cap B_x$, for B_x denoting, as before, a small open ball about x in $\mathbb{CP}^{n+1}$ for which (V, x) has a conic structure in $\bar{B}_x$. Let $S_x := \partial \bar{B}_x$, with $K_x := V \cap S_x$ denoting the corresponding link of (V, x). Note that $\mathcal{U}'_x$ is homotopy equivalent to the link complement $S_x \setminus K_x$. Moreover, since K_x is an algebraic link, the Milnor fibration theorem (see, for example, [3, ch. 3] and the references therein) implies that the complement $S_x \setminus K_x$ fibres over a circle, with (Milnor) fibre F_x homotopy equivalent to a finite CW-complex. It is also known that F_x is homotopy equivalent to the infinite cyclic cover of $S_x \setminus K_x$ defined by the linking number with respect to K_x. For future reference, let us denote by lk'_x the epimorphism on $\pi_1(S_x \setminus K_x) \cong \pi_1(\mathcal{U}'_x)$ defined by the total linking number with K_x.

If $x \in V \setminus H$, then $\mathcal{U}_x = \mathcal{U}'_x \simeq S_x \setminus K_x$, so in this case

$$H_k^{\varepsilon_x,\rho_x}(\mathcal{U}_x, \mathbb{F}[t^{\pm 1}]) = H_k^{lk_x,\rho_x}(\mathcal{U}_x, \mathbb{F}[t^{\pm 1}]) \cong H_k(\mathcal{U}_{x,\infty}, \mathbb{V}_{\rho_x}) \cong H_k(F_x, \mathbb{V}_{\rho_x})$$

is a finite-dimensional $\mathbb{F}$-vector space, and hence a torsion $\mathbb{F}[t^{\pm 1}]$-module for any $k \in \mathbb{Z}$.

If $x \in V \cap H$, then by the transversality assumption we have that $\mathcal{U}_x \simeq \mathcal{U}'_x \times S^1$, with the restrictions of lk_x to the factors of this product described as follows: on $\pi_1(\mathcal{U}'_x)$, lk_x restricts to the homomorphism lk'_x defined by the linking number with K_x (this is, of course, the same as $lk_{x'}$ at a nearby point $x' \in V \setminus H$ in the same stratum as x), while on $\pi_1(S^1)$ it can be seen from (4.1) that lk_x acts by sending the generator (which coincides with the homotopy class of the meridian loop γ_∞ about H) to $-d$. The acyclicity at $x \in V \cap H$ then follows by the Künneth formula, since the homotopy factors of $\mathcal{U}_x$, endowed with the corresponding homomorphisms and representations induced from the pair (lk_x, ρ_x), are acyclic. ☐

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By applying proposition 4.9 to the dual representation ρ^*, we deduce from (2.1) the following (which shall be referred to below as the *local cohomological acyclicity* along V).

Corollary 4.10. *Let $V \subset \mathbb{CP}^{n+1}$ be a hypersurface in general position at infinity, let $\varepsilon\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ be a positive epimorphism and let $\rho\colon \pi_1(\mathcal{U}) \to \mathrm{GL}(\mathbb{V})$ be an arbitrary representation. Then, for any $x \in V$, the local cohomological twisted Alexander modules $H^k_{\varepsilon_x,\rho_x}(\mathcal{U}_x, \mathbb{F}[t^{\pm 1}])$ are torsion $\mathbb{F}[t^{\pm 1}]$-modules for any $k \in \mathbb{Z}$. (Here (ε_x, ρ_x) is induced as above from (ε, ρ) via the inclusion $i_x\colon \mathcal{U}_x \hookrightarrow \mathcal{U}$.)*

4.4. Sheaf (co)homology interpretation of twisted Alexander modules

In the remainder of the paper, we employ the language of perverse sheaves and homological algebra techniques to relate the local and global properties of twisted Alexander invariants. For this purpose, we first rephrase the definition of twisted Alexander modules as the (co)homology of a certain local system defined on the complement $\mathcal{U}$.

Let $\mathcal{L}$ be the local system of $\mathbb{F}[t^{\pm 1}]$-modules on $\mathcal{U}$, with stalk $\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V}$, and action of the fundamental group corresponding to the right $\mathbb{F}[\pi]$-module structure of the stalk, i.e.

$$\begin{aligned} \pi_1(\mathcal{U}) &\to \mathrm{Aut}(\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V}) \cong \mathrm{GL}_\ell(\mathbb{F}[t^{\pm 1}]), \\ [\alpha] &\mapsto (p \otimes v \mapsto (p \otimes v) \cdot \alpha = pt^{\varepsilon(\alpha)} \otimes v\rho(\alpha)). \end{aligned}$$

(Here ℓ denotes, as before, the rank of the representation ρ, and we regard the elements of $\mathbb{V} \cong \mathbb{F}^\ell$ as row vectors.) Then it is clear from the definition of the (co)homological twisted Alexander modules that we have the following isomorphisms of $\mathbb{F}[t^{\pm 1}]$-modules (see, for example, [1, p. 355]):

$$H_i^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}]) \cong H_i(\mathcal{U}, \mathcal{L}) \quad \text{and} \quad H^i_{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}]) \cong H^i(\mathcal{U}, \mathcal{L}). \tag{4.3}$$

If $x \in V$, let $i_x\colon \mathcal{U}_x := \mathcal{U} \cap B_x \hookrightarrow \mathcal{U}$ denote the inclusion of the local complement at x, with corresponding induced local pair (ε_x, ρ_x) as in § 4.3. Let

$$\mathcal{L}_x := i_x^* \mathcal{L} = \mathcal{L}|_{\mathcal{U}_x}$$

be the restriction of the local system $\mathcal{L}$ to $\mathcal{U}_x$, i.e. $\mathcal{L}_x$ is defined via the action of (ε_x, ρ_x). Then, for any $k \in \mathbb{Z}$, it follows, as above, that the local kth (co)homological twisted Alexander modules at x can be described by

$$H_k^{\varepsilon_x,\rho_x}(\mathcal{U}_x, \mathbb{F}[t^{\pm 1}]) \cong H_k(\mathcal{U}_x, \mathcal{L}_x) \quad \text{and} \quad H^k_{\varepsilon_x,\rho_x}(\mathcal{U}_x, \mathbb{F}[t^{\pm 1}]) \cong H^k(\mathcal{U}_x, \mathcal{L}_x).$$

4.5. Local-to-global analysis: divisibility results

In this section, we assume that the projective hypersurface V is in general position at infinity. By theorem 4.1 and corollary 4.5, for $\varepsilon\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ a positive epimorphism and $\rho\colon \pi_1(\mathcal{U}) \to \mathrm{GL}(\mathbb{V})$ an arbitrary representation, the (co)homological twisted Alexander modules $H_i^{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$ and $H^i_{\varepsilon,\rho}(\mathcal{U}, \mathbb{F}[t^{\pm 1}])$, respectively, are torsion $\mathbb{F}[t^{\pm 1}]$-modules for any $0 \leqslant i \leqslant n$. Following definition 2.5, we denote the corresponding twisted Alexander polynomials by $\Delta^{\varepsilon,\rho}_{i,\mathcal{U}}(t)$ and $\Delta^i_{\varepsilon,\rho,\mathcal{U}}(t)$, respectively, with $0 \leqslant i \leqslant n$.

The sheaf theoretic realization of twisted Alexander modules in § 4.4 allows us to use the language of perverse sheaves (or intersection homology), which, when coupled with homological algebra techniques, gives a concise relationship between the global twisted Alexander invariants of complex hypersurface complements and the corresponding local ones at singular points (respectively, at infinity). For simplicity, we formulate our results in this section in cohomological terms (but see also remark 4.15). Our approach is similar to [5, § 3].

We work with sheaves of $\mathbb{F}[t^{\pm 1}]$-modules. For a topological space Y, we denote by $D^b_c(Y; \mathbb{F}[t^{\pm 1}])$ the bounded derived category of complexes of sheaves of $\mathbb{F}[t^{\pm 1}]$-modules on Y with constructible cohomology, and we let $\mathrm{Perv}(Y)$ be the abelian category of perverse sheaves of $\mathbb{F}[t^{\pm 1}]$-modules on Y.

The first result in this section singles out the contribution of the meridian 'at infinity', γ_∞, to the global twisted Alexander invariants, and it can be regarded as a high-dimensional generalization (for arbitrary singularities) of corollary 3.3, where γ_∞ plays the role of x_0.

Theorem 4.11. *Let $V \subset \mathbb{CP}^{n+1}$ be a projective hypersurface in general position (with respect to the hyperplane H) at infinity, with complement $\mathcal{U} = \mathbb{CP}^{n+1} \setminus (V \cup H)$. Fix a positive epimorphism $\varepsilon\colon \pi_1(\mathcal{U}) \to \mathbb{Z}$ and a rank-ℓ representation $\rho\colon \pi_1(\mathcal{U}) \to \mathrm{GL}(\mathbb{V})$. Then, for any $0 \leqslant i \leqslant n$, the zeros of the global cohomological Alexander polynomial $\Delta^i_{\varepsilon,\rho,\mathcal{U}}(t)$ are among those of the order of the cokernel of the endomorphism $t^{-\varepsilon(\gamma_\infty)} \otimes \rho(\gamma_\infty)^{-1} - \mathrm{Id} \in \mathrm{End}(\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V})$. (Here we use the left $\mathbb{Z}[\pi]$-module structure on $\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V}$, as dictated by the use of cohomological invariants, as in definition 2.1.)*

Proof. Let $\mathbb{C}^{n+1} = \mathbb{CP}^{n+1} \setminus H$, and denote by $u\colon \mathcal{U} \hookrightarrow \mathbb{C}^{n+1}$ and $v\colon \mathbb{C}^{n+1} \hookrightarrow \mathbb{CP}^{n+1}$ the two inclusions. Since $\mathcal{U}$ is smooth and $(n+1)$ dimensional, and $\mathcal{L}$ is a local system on $\mathcal{U}$, it follows that $\mathcal{L}[n+1] \in \mathrm{Perv}(\mathcal{U})$. Moreover, since u is a quasi-finite affine morphism, we also have that

$$\mathcal{F}^\bullet := Ru_*(\mathcal{L}[n+1]) \in \mathrm{Perv}(\mathbb{C}^{n+1})$$

(see, for example, [24, theorem 6.0.4]). But $\mathbb{C}^{n+1}$ is an affine $(n+1)$-dimensional variety, so by Artin's vanishing theorem for perverse sheaves (see, for example, [24, corollary 6.0.4]) we obtain that

$$\mathbb{H}^k(\mathbb{C}^{n+1}, \mathcal{F}^\bullet) = 0 \quad \text{for all } k > 0 \tag{4.4}$$

and

$$\mathbb{H}^k_c(\mathbb{C}^{n+1}, \mathcal{F}^\bullet) = 0 \quad \text{for all } k < 0. \tag{4.5}$$

Let $a\colon \mathbb{CP}^{n+1} \to \text{point}$ be the constant map. Then

$$\mathbb{H}^k(\mathbb{C}^{n+1}, \mathcal{F}^\bullet) \cong H^{k+n+1}(\mathcal{U}, \mathcal{L}) \cong H^k(Ra_* Rv_* \mathcal{F}^\bullet). \tag{4.6}$$

Similarly,

$$\mathbb{H}^k_c(\mathbb{C}^{n+1}, \mathcal{F}^\bullet) \cong H^k(Ra_! Rv_! \mathcal{F}^\bullet), \tag{4.7}$$

where the last equality follows since a is a proper map; hence, $Ra_! = Ra_*$.

Consider the canonical morphism $Rv_!\mathcal{F}^\bullet \to Rv_*\mathcal{F}^\bullet$, and extend it to the distinguished triangle:

$$Rv_!\mathcal{F}^\bullet \to Rv_*\mathcal{F}^\bullet \to \mathcal{G}^\bullet \xrightarrow{[1]} \tag{4.8}$$

in $D^b_c(\mathbb{CP}^{n+1};\mathbb{F}[t^{\pm 1}])$. Since $v^*Rv_! \cong \mathrm{id} \cong v^*Rv_*$, after applying v^* to the above triangle we get that $v^*\mathcal{G} \cong 0$ or, equivalently, $\mathcal{G}$ is supported on H. Next, we apply $Ra_! = Ra_*$ to the distinguished triangle (4.8) to obtain a new triangle in $D^b_c(\mathrm{point};\mathbb{F}[t^{\pm 1}])$:

$$Ra_!Rv_!\mathcal{F}^\bullet \to Ra_*Rv_*\mathcal{F}^\bullet \to Ra_*\mathcal{G}^\bullet \xrightarrow{[1]} . \tag{4.9}$$

Upon applying the cohomology functor to the distinguished triangle (4.9), and using the vanishing from (4.4) and (4.5) together with the identifications (4.6) and (4.7), we obtain that

$$H^{k+n+1}(\mathcal{U},\mathcal{L}) \cong \mathbb{H}^k(\mathbb{CP}^{n+1},\mathcal{G}^\bullet) \cong \mathbb{H}^k(H,\mathcal{G}^\bullet) \quad \text{for } k < -1,$$

and $H^n(\mathcal{U},\mathcal{L})$ is a submodule of the $\mathbb{F}[t^{\pm 1}]$-module $\mathbb{H}^{-1}(H,\mathcal{G}^\bullet)$. So in order to prove the theorem, it remains to show that the $\mathbb{F}[t^{\pm 1}]$-modules $\mathbb{H}^k(H,\mathcal{G}^\bullet)$ are torsion for $k \leqslant -1$, and the zeros of their corresponding orders are among those of the order of the cokernel of $t^{-\varepsilon(\gamma_\infty)} \otimes \rho(\gamma_\infty)^{-1} - \mathrm{Id} \in \mathrm{End}(\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V})$.

Note that $\mathbb{H}^k(H,\mathcal{G}^\bullet)$ is the abutment of a hypercohomology spectral sequence with the E_2-term defined by

$$E_2^{p,q} = H^p(H,\mathcal{H}^q(\mathcal{G}^\bullet)). \tag{4.10}$$

This prompts us to investigate the stalk cohomology of $\mathcal{G}^\bullet$ at points along H.

For $x \in H$, let us, as before, denote by $\mathcal{U}_x = \mathcal{U} \cap B_x$ the local complement at x for B_x a small ball in $\mathbb{CP}^{n+1}$ centred at x. Then we have the following identification:

$$\mathcal{H}^q(\mathcal{G}^\bullet)_x \cong H^{q+n+1}(\mathcal{U}_x,\mathcal{L}_x), \tag{4.11}$$

where $\mathcal{L}_x$ is the restriction of $\mathcal{L}$ to $\mathcal{U}_x$. Indeed, the following isomorphisms of $\mathbb{F}[t^{\pm 1}]$-modules hold:

$$\begin{aligned}\mathcal{H}^q(\mathcal{G}^\bullet)_x &\cong \mathcal{H}^q(Rv_*\mathcal{F}^\bullet)_x \\ &\cong \mathcal{H}^{q+n+1}(Rv_*Ru_*\mathcal{L})_x \\ &\cong \mathbb{H}^{q+n+1}(B_x, R(v\circ u)_*\mathcal{L}) \\ &\cong H^{q+n+1}(\mathcal{U}_x,\mathcal{L}_x).\end{aligned}$$

If $x \in H \setminus V$, then $\mathcal{U}_x$ is homotopy equivalent to S^1, and the corresponding local system $\mathcal{L}_x$ is defined by the action of γ_∞, i.e. by the *right* multiplication by $t^{\varepsilon(\gamma_\infty)} \otimes \rho(\gamma_\infty)$ on $\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V}$. In particular, $H^*(U_x,\mathcal{L}_x)$ is the cohomology of the co-chain complex of $\mathbb{F}[t^{\pm 1}]$-modules:

$$0 \longleftarrow \mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V} \xleftarrow{t^{-\varepsilon(\gamma_\infty)}\otimes\rho(\gamma_\infty)^{-1}-\mathrm{Id}} \mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V} \longleftarrow 0,$$

i.e.

$$H^k(U_x,\mathcal{L}_x) = \begin{cases}\mathrm{Coker}(t^{-\varepsilon(\gamma_\infty)} \otimes \rho(\gamma_\infty)^{-1}), & k = 1,\\ 0, & k \neq 1.\end{cases} \tag{4.12}$$

If $x \in H \cap V$, then we know by corollary 4.10 that the local cohomological twisted Alexander modules $H^k(\mathcal{U}_x, \mathcal{L}_x)$ are $\mathbb{F}[t^{\pm 1}]$-torsion modules for all $k \in \mathbb{Z}$. Moreover, in the notation of proposition 4.9, we have that $\mathcal{U}_x \simeq \mathcal{U}'_x \times S^1$, and the local system $\mathcal{L}_x$ is an external tensor product, the second factor being defined by the action of γ_∞ as in the previous case. So, it follows from the Künneth formula that the zeros of the local cohomological twisted Alexander polynomials at points in $H \cap V$ are among those of the order of the cokernel of $t^{-\varepsilon(\gamma_\infty)} \otimes \rho(\gamma_\infty)^{-1} \in \mathrm{End}(\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V})$.

By (4.11) and the above calculations, it then follows that the $\mathbb{F}[t^{\pm 1}]$-modules $\mathcal{H}^q(\mathcal{G}^\bullet)_{x \in H}$ are torsion, and the zeros of their associated orders are among those of the order of the cokernel of $t^{-\varepsilon(\gamma_\infty)} \otimes \rho(\gamma_\infty)^{-1} - \mathrm{Id} \in \mathrm{End}(\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V})$. Hence, using the spectral sequence (4.10), each hypercohomology group $\mathbb{H}^k(H, \mathcal{G}^\bullet)$ is a torsion $\mathbb{F}[t^{\pm 1}]$-module, and the zeros of its associated order are among those of the order of the cokernel of $t^{-\varepsilon(\gamma_\infty)} \otimes \rho(\gamma_\infty)^{-1} - \mathrm{Id} \in \mathrm{End}(\mathbb{F}[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{V})$. This ends the proof of our theorem. □

REMARK 4.12. If $\mathbb{F} = \mathbb{C}$ and $\varepsilon = lk$ is the total linking number homomorphism, theorem 4.11 implies that any root λ of $\Delta^i_{\rho,\mathcal{U}}(t)$, $i \leqslant n$, must satisfy the condition that λ^d is an eigenvalue of $\rho(\gamma_\infty)$, where $d = \sum_{i=1}^r n_i d_i$ is the degree of V. If, in addition, $\rho = \mathrm{triv}$ is the trivial representation, the statement of theorem 4.11 reduces to the fact that the zeros of the classical cohomological Alexander polynomials $\Delta^i_{\mathcal{U}}(t)$, $i \leqslant n$, are roots of unity of order $d = \deg(V)$, a fact also shown in [4, 17, 19] in the reduced case.

In the next theorem, we assume for simplicity of exposition that V is a reduced hypersurface. Recall from §§ 4.3 and 4.4 that for any point x in V with local complement $\mathcal{U}_x = \mathcal{U} \cap B_x$ we get from (ε, ρ) an induced pair (ε_x, ρ_x) via the inclusion map $i_x : \mathcal{U}_x \hookrightarrow \mathcal{U}$. Moreover, the local twisted Alexander modules have a sheaf description in terms of the local system $\mathcal{L}_x := i_x^* \mathcal{L}$, namely $H_k^{\varepsilon_x, \rho_x}(\mathcal{U}_x, \mathbb{F}[t^{\pm 1}]) \cong H_k(\mathcal{U}_x, \mathcal{L}_x)$ and $H^k_{\varepsilon_x, \rho_x}(\mathcal{U}_x, \mathbb{F}[t^{\pm 1}]) \cong H^k(\mathcal{U}_x, \mathcal{L}_x)$ for all $k \in \mathbb{Z}$. We define by

$$\Delta_{k,x}(t) := \Delta^{\varepsilon_x, \rho_x}_{k, \mathcal{U}_x}(t) \quad \text{and} \quad \Delta^k_x(t) := \Delta^k_{\varepsilon_x, \rho_x, \mathcal{U}_x}(t)$$

the *local* (co)homological twisted Alexander polynomials at x.

Let us now assume also that V is in general position at infinity. Then if $x \in V \cap H$, in the notation of proposition 4.9 there is a homotopy equivalence $\mathcal{U}_x \simeq \mathcal{U}'_x \times S^1$, where $\mathcal{U}'_x = B_x \setminus V$ and with the S^1-factor corresponding to the meridian loop about the hyperplane at infinity, H. On the other hand, $\mathcal{U}'_x$ is homeomorphic to any local complement $\mathcal{U}_{x'}$ at a point $x' \in V \setminus H$ in the same stratum with x. So, by the Künneth formula, the zeros of the local twisted Alexander polynomials $\Delta^k_x(t)$ of $(\mathcal{U}_x, \varepsilon_x, \rho_x)$ are among those associated with $(\mathcal{U}_{x'}, \varepsilon_{x'}, \rho_{x'})$, for $x' \in V \setminus H$ a nearby point in the same stratum of V as x. For brevity, points of $V^a = V \setminus H$ will be referred to as *affine points of* V.

The next result shows that the zeros of the global twisted Alexander polynomials can be estimated from those of the local twisted Alexander polynomials at (affine) points along some irreducible component of V.

THEOREM 4.13. *Let $V \subset \mathbb{CP}^{n+1}$ be a reduced hypersurface in general position at infinity, with complement $\mathcal{U} = \mathbb{CP}^{n+1} \setminus (V \cup H)$, and let V_1 be a fixed irreducible component of V. Fix a positive epimorphism $\varepsilon \colon \pi_1(\mathcal{U}) \to \mathbb{Z}$, a rank ℓ representation*

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$\rho\colon \pi_1(\mathcal{U}) \to \mathrm{GL}(\mathbb{V})$ and a non-negative integer σ. If $\lambda \in \mathbb{F}$ is not a root of the ith local twisted Alexander polynomial $\Delta_x^i(t)$ for any $i < n+1-\sigma$ and any (affine) point $x \in V_1 \setminus H$, then λ is not a root of the global twisted Alexander polynomial $\Delta^i_{\varepsilon,\rho,\mathcal{U}}(t)$ for any $i < n+1-\sigma$.

Proof. First, note that by the transversality assumption and the Künneth formula it follows by the above considerations that the hypothesis on local twisted Alexander polynomials implies that λ is not a root of the ith local twisted Alexander polynomial $\Delta_x^i(t)$ for any $i < n+1-\sigma$ and any point $x \in V_1$ (including points in $V_1 \cap H$).

As in the proof of theorem 4.11, after replacing $\mathbb{C}^{n+1}$ by $\mathcal{U}_1 = \mathbb{CP}^{n+1} \setminus V_1$, it follows that, for $k \leqslant -1$, $H^{k+n+1}(\mathcal{U}, \mathcal{L})$ is a submodule of $\mathbb{H}^k(\mathbb{CP}^{n+1}, \mathcal{G}^\bullet)$, where $\mathcal{G}^\bullet$ is now a complex of sheaves of $\mathbb{F}[t^{\pm 1}]$-modules supported on V_1. It thus suffices to show that $\mathbb{H}^k(\mathbb{CP}^{n+1}, \mathcal{G}^\bullet)$, $k < -\sigma$, is a torsion $\mathbb{F}[t^{\pm 1}]$-module whose order does not vanish at λ.

As in (4.11), the cohomology stalks of $\mathcal{G}^\bullet$ at any $x \in V_1$ are given by

$$\mathcal{H}^q(\mathcal{G}^\bullet)_x \cong H^{q+n+1}(\mathcal{U}_x, \mathcal{L}_x),$$

and these are all torsion $\mathbb{F}[t^{\pm 1}]$-modules by corollary 4.10. Therefore, for a fixed $x \in V_1$ the fact that λ is not a root of $\Delta_x^i(t)$ for any $i < n+1-\sigma$ is equivalent to the assertion that the order of $\mathcal{H}^q(\mathcal{G}^\bullet)_x$ does not vanish at λ for all $i < -\sigma$. The desired claim now follows by using the hypercohomology spectral sequence with the E_2-term defined by $E_2^{p,q} = H^p(V_1, \mathcal{H}^q(\mathcal{G}^\bullet))$, which computes the groups $\mathbb{H}^k(V_1, \mathcal{G}^\bullet) \cong \mathbb{H}^k(\mathbb{CP}^{n+1}, \mathcal{G}^\bullet)$. □

Remark 4.14. Note that the proofs of theorems 4.11 and 4.13 indicate that we can give a more general condition than transversality with respect to H in order to conclude that the global cohomological twisted Alexander modules $H^i_{\varepsilon,\rho}(\mathcal{U}; \mathbb{F}[t^{\pm 1}])$ are torsion for all $i \leqslant n$. Indeed, it suffices to assume that the pair (ε, ρ) is locally cohomologically acyclic along $V \cap H$ (or even $V_1 \cap H$, in the context of theorem 4.13), i.e. the corresponding local cohomological twisted Alexander modules are torsion at points in $V \cap H$ (or $V_1 \cap H$). Of course this assumption is satisfied if V is in general position at infinity, as proposition 4.9 and corollary 4.10 show. But there are other instances when it is satisfied, such as in the examples discussed in § 2.2.

Remark 4.15. Let us conclude with a few observations about other possible approaches for studying twisted Alexander-type invariants of hypersurface complements.

If $\mathbb{F} = \mathbb{C}$, one can argue as in [4] if similar divisibility results are desired for the homological twisted Alexander polynomials. In more detail, the study of such twisted homological invariants is reduced via a twisted version of the Milnor sequence to studying the vanishing (except in the middle degree) of the homology groups $H_k(\mathcal{U}, \mathcal{L}_\lambda \otimes \mathbb{V}_\rho)$ (or equivalently, of cohomology groups $H^k(\mathcal{U}, \mathcal{L}_\lambda \otimes \mathbb{V}_\rho)$), where $\mathcal{L}_\lambda$ is the rank-1 $\mathbb{C}$-local system on $\mathcal{U}$ defined by the character

$$\pi_1(\mathcal{U}) \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{1 \mapsto \lambda} \mathbb{C}^*.$$

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The language of $\mathbb{C}$-perverse sheaves can then be employed as in the proofs of theorems 4.11 and 4.13 to get the desired vanishing, thus providing a twisted generalization of results from [4, 19].

Alternatively, one can use the approach from [18,19] to study the (co)homological twisted Alexander invariants by using the associated *residue complex* $\mathcal{R}^\bullet$ of $\mathcal{U}$, which is defined as the cone of the natural morphism $Rj_!\mathcal{L} \to Rj_*\mathcal{L}$ for $j\colon \mathcal{U} \hookrightarrow \mathbb{CP}^{n+1}$ the inclusion map.

Lastly, such results can also be derived by using more elementary techniques, as follows:

- by transversality and a Lefschetz-type argument one can reduce, as in [14], the study of the twisted Alexander modules of $\mathcal{U}$ to those of a regular neighbourhood $\mathcal{N}$ in $\mathbb{C}^{n+1}$ of the affine part V^a of V;

- Alexander-type invariants of $\mathcal{N}$ can be computed via the Mayer–Vietoris spectral sequence for the induced stratification of such a neighbourhood.

We leave the details and precise formulations as an exercise for the interested reader.

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