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ON FRACTAL DIMENSIONS OF FRACTAL FUNCTIONS USING FUNCTION SPACES

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Abstract

Based on the work of Mauldin and Williams ['On the Hausdorff dimension of some graphs', *Trans. Amer. Math. Soc.* **298**(2) (1986), 793–803] on convex Lipschitz functions, we prove that fractal interpolation functions belong to the space of convex Lipschitz functions under certain conditions. Using this, we obtain some dimension results for fractal functions. We also give some bounds on the fractal dimension of fractal functions with the help of oscillation spaces.

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1. Introduction and preliminaries

The fractal dimension is one of the major themes in Fractal Geometry. Estimation of the fractal dimension of sets and graphs has received much attention (see [5, 6, 10]). The study of dimensions of graphs began with the Hausdorff dimension of Weierstrass-type functions (see [12, 17]). In [17], Mauldin and Williams considered such a function,

$$W_b(x) = \sum_{n=-\infty}^{\infty} b^{-\alpha n}[\Phi(b^n x + \theta_n) - \Phi(\theta_n)],$$

where $b > 1$, $0 < \alpha < 1$, Φ has period one and θ_n is an arbitrary number, and established results on the Hausdorff dimension when the function satisfies a convex-Lipschitz condition. This is the major motivation for our work. By using the definition of a convex Lipschitz function, we introduce the convex-Lipschitz space and estimate the Hausdorff dimension and box dimension of a general fractal interpolation function (FIF).

The concept of FIF was introduced by Barnsley [5] using the notion of an iterated function system (IFS). Recent related work on dimension theory can be seen in

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[3, 4]. In [3], Bárány *et al.* applied a result of Hochman [11] on self-similar sets with overlaps, to compute the Hausdorff dimension of self-affine sets. They also studied the dimension theory of diagonally homogeneous triangular planar self-affine sets in [4].

1.1. Fractal interpolation functions. We outline the construction of FIF and refer to [5, 6] for the details.

Assume that $\{(x_n, y_n) : n = 1, 2, \ldots, N\}$ is a set of interpolation points. We write $I = [x_1, x_N]$ and $J = \{1, 2, \ldots, N-1\}$, and let $I_j = [x_j, x_{j+1}]$ for $j \in J$. Let $L_j : I \to I_j$, $j \in J$, be contractive homeomorphisms with

$$L_j(x_1) = x_j, \quad L_j(x_N) = x_{j+1}, \quad j \in J.$$

Let $F_j : I \times \mathbb{R} \to \mathbb{R}$ be a mapping satisfying, for $j \in J$, $r_j \in [0, 1)$:

(i) $F_j(x_1, y_1) = y_j, F_j(x_N, y_N) = y_{j+1}$;
(ii) $|F_j(x, y) - F_j(x, y')| \leq r_j|y - y'|$ for all $x \in I$ and $y, y' \in \mathbb{R}$.

We consider

$$L_j(x) = a_j x + b_j, \quad F_j(x, y) = \alpha_j y + q_j(x).$$

Here, a_j and b_j can be determined by using the conditions $L_j(x_1) = x_j, L_j(x_N) = x_{j+1}$. The scaling factor α_j satisfies $-1 < \alpha_j < 1$ and we set $|\alpha_j|_\infty = \max_j\{\alpha_j\}$. The 'join-up conditions', which are imposed on the maps F_j, are given by $q_j(x_1) = y_j - \alpha_j y_1$ and $q_j(x_N) = y_{j+1} - \alpha_j y_N$ for all $j \in J$ for suitable continuous functions $q_j : I \to \mathbb{R}$. Let us define $W_j : I \times \mathbb{R} \to I \times \mathbb{R}$ for $j \in J$ by

$$W_j(x, y) = (L_j(x), F_j(x, y)).$$

Then $\mathcal{I} := \{I \times \mathbb{R}; W_1, W_2, \ldots, W_{N-1}\}$ is the IFS. Barnsley [5] proved that $\mathcal{I}$ has a unique invariant set which is the graph of a continuous function $f : I \to \mathbb{R}$, referred to as a FIF, and that it satisfies the self-referential equation

$$f(x) = \alpha_j f(L_j^{-1}(x)) + q_j(L_j^{-1}(x)), \quad x \in I_j, j \in J.$$

There are various approaches to fractal dimensions of fractal functions. These include the use of the mass-distribution principle, potential theory, Fourier transforms and positive operators to compute or estimate the Hausdorff dimension of a set [10, 18, 24]. Using the potential theoretic approach, Barnsley [5] gave results on the Hausdorff dimension of an affine FIF. Falconer [10] also gave estimates for the Hausdorff dimension of an affine FIF. Results on the Hausdorff dimension using the positive operators approach are given in [18, 24]. This approach is used to discuss the continuity of the Hausdorff dimension of the invariant set in [20].

Pandey *et al.* [19] considered the fractal dimension for set valued mappings using the δ-covering method. Jha and Verma [13] gave results for the fractal dimensions of fractal functions and some invariant sets. They estimated fractal dimensions for a class of FIFs, known as α-fractal functions. Agrawal and Som [1, 2] gave results for α-fractal functions on Sierpiński gaskets. Sahu and Priyadarshi [23] estimated the box dimension of the graph of harmonic functions on Sierpiński gaskets.

Ruan *et al.* [21] estimated the box dimension of a new class of linear FIFs by using the δ-covering method. Additionally, they established a relationship between the order of a fractional integral and box dimensions of two linear FIFs. Estimates of the box dimension of bilinear fractal interpolation surfaces are given in [14]. A recurrent FIF is the generalisation of a linear FIF and the graph of a recurrent FIF is the invariant set of a recurrent IFS. Ruan *et al.* [22] gave the construction of a recurrent FIF under certain assumptions and estimated the box dimension of the self-affine recurrent FIFs.

Work on the fractal dimension of fractional integrals can be seen as a connection between fractal geometry and fractional calculus. The bounded variation property of a continuous function plays a significant role in estimating the box dimension. Using this approach, Liang [15] gave interesting results on the box dimension of the Riemann–Liouville fractional integral. He proved that if a function f is continuous and of bounded variation on $[0, 1]$, then $\dim_B(f) = 1$ and the box dimension of the Riemann–Liouville fractional integral corresponding to f is also $= 1$ [16]. Liang estimated the exact box dimension of the Riemann–Liouville fractional integral of one-dimensional continuous functions. We gave the fractal dimension of the mixed Riemann–Liouville fractional integral on a rectangular region in [8] and estimated fractal dimensions for various choices of continuous functions such as Hölder continuous function, functions having box dimension two and unbounded variational continuous functions.

In Sections 2 and 3, we give dimension results on convex-Lipschitz space and oscillation space, respectively.

1.2. Definitions. We complete Section 1 with some definitions and terminologies. For further definitions related to the fractal dimension, we refer to [10].

Let $F \neq \emptyset$ be a subset of $\mathbb{R}^n$. The diameter of F is given by

$$|F| = \sup\{\|x - y\|_2 : x, y \in F\}.$$

If $\{F_i\}$ is a countable (or finite) collection of sets having diameter at most δ which cover the set $E \subseteq \mathbb{R}^n$, then we say that $\{F_i\}$ is a δ-cover of E. For $\delta > 0$ and a nonnegative real number s, we define

$$H^s_\delta(E) = \inf\bigg\{\sum_{i=1}^{\infty} |F_i|^s : \{F_i\} \text{ is a } \delta\text{-cover of } E\bigg\}. \tag{1.1}$$

DEFINITION 1.1. The s-dimensional Hausdorff measure of E is $H^s(E) = \lim_{\delta\to 0} H^s_\delta(E)$.

DEFINITION 1.2 (Hausdorff dimension). Let $s \geq 0$ and $E \subseteq \mathbb{R}^n$. The Hausdorff dimension of E is defined as

$$\dim_H(E) = \inf\{s : H^s(E) = 0\} = \sup\{s : H^s(E) = \infty\}.$$

DEFINITION 1.3 (Box dimension). Let $E \subseteq \mathbb{R}^n$ be bounded and nonempty and let $N_\delta(E)$ be the smallest number of sets of diameter at most δ which cover E. The lower box dimension of E is

$$\underline{\dim}_B(E) = \liminf_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta}$$

and the upper box dimension of E is

$$\overline{\dim}_B(E) = \limsup_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta}.$$

If both, lower and upper box dimensions are the same, then that quantity is called the box dimension of E and it is given by

$$\dim_B(E) = \lim_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta}.$$

2. Convex-Lipschitz space

In this section, first we show that fractal functions associated with some IFS belong to the class of convex Lipschitz functions. Then we estimate the Hausdorff dimension and the box dimension of fractal functions in this class.

DEFINITION 2.1 [17]. Let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$. A function f is called convex Lipschitz of order θ on an interval $[a, b]$ provided there exists a constant M such that

$$|\Delta(x, y, \delta)| := |f(x + \delta y) - (\delta f(x + y) + (1 - \delta) f(x))| \leq M\theta(y),$$

for $a \leq x < x + y \leq b$ and $0 \leq \delta \leq 1$. The convex-Lipschitz space of order θ is

$$V^\theta(I) = \{f : I \to \mathbb{R} : f \text{ is convex Lipschitz of order } \theta\}.$$

It can be seen that V^θ is a vector space over the field $\mathbb{R}$. For $f \in V^\theta(I)$, we define $\|f\|_{V^\theta} = \|f\|_\infty + [f]^*$, where

$$[f]^* = \sup_{a\leq x<x+y\leq b} \frac{|f(x + \delta y) - (\delta f(x + y) + (1 - \delta) f(x))|}{\theta(y)}.$$

It is easy to check that $\|\cdot\|_{V^\theta}$ defines a norm on $V^\theta(I)$.

LEMMA 2.2. *If $f : I \to \mathbb{R}$ and (f_k) is a sequence of continuous functions which converges uniformly to f, then $[f_n - f]^* \leq \liminf_{k\to\infty} [f_n - f_k]^*$.*

PROOF. By using the triangle inequality,

$$\begin{aligned}
&\frac{|f_n(x + \delta y) - f(x + \delta y) - [\delta f_n(x + y) - \delta f(x + y) + (1 - \delta) f_n(x) - (1 - \delta) f(x)]|}{\theta(y)} \\
&\quad = \lim_{k\to\infty} \frac{1}{\theta(y)} \{|f_n(x + \delta y) - f_k(x + \delta y) \\
&\qquad\qquad - [\delta f_n(x + y) - \delta f_k(x + y) + (1 - \delta) f_n(x) - (1 - \delta) f_k(x)]|\} \\
&\quad \leq \liminf_{k\to\infty} \sup_{a\leq x<x+y\leq b} \frac{1}{\theta(y)} \{|f_n(x + \delta y) - f_k(x + \delta y) \\
&\qquad\qquad - [\delta f_n(x + y) - \delta f_k(x + y) + (1 - \delta) f_n(x) - (1 - \delta) f_k(x)]|\}.
\end{aligned}$$

This completes the proof. □

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THEOREM 2.3. *The space* $(V^{\theta}(I), \|\cdot\|_{V^{\theta}})$ *is a Banach space.*

PROOF. Let (f_n) be a Cauchy sequence with respect to $\|\cdot\|_{V^{\theta}}$ in $V^{\theta}(I)$. This means that for any $\epsilon > 0$, there exists a natural number n_0 such that $\|f_n - f_k\|_{V^{\theta}} < \epsilon$ for all $n, k \geq n_0$.

From the definition of the norm $\|\cdot\|_{V^{\theta}}$, it follows that $\|f_n - f_k\|_{\infty} < \epsilon$ for all $n, k \geq n_0$. Because $(C(I), \|\cdot\|_{\infty})$ is a Banach space, there is a continuous function f such that $\|f_n - f\|_{\infty} \to 0$ as $n \to \infty$. We claim that $f \in V^{\theta}(I)$ and $\|f_n - f\|_{V^{\theta}} \to 0$ as $n \to \infty$.

Let $n \geq n_0$. In view of Lemma 2.2,

$$\begin{aligned}\|f_n - f\|_{V^{\theta}} = \|f_n - f\|_{\infty} + [f_n - f]^* &\leq \liminf_{k\to\infty}\{\|f_n - f_k\|_{\infty} + [f_n - f_k]^*\}\\ &\leq \sup_{k\geq n_0} \|f_n - f_k\|_{V^{\theta}} \leq \epsilon.\end{aligned}$$

Hence, we obtain $f - f_{n_0} \in V^{\theta}(I)$. Consequently, $f = f - f_{n_0} + f_{n_0} \in V^{\theta}(I)$ and we have $\|f_n - f\|_{V^{\theta}} \leq \epsilon$ for all $n \geq n_0$. This completes the proof. □

DEFINITION 2.4 [10, Section 2.5]. Let $E \subset \mathbb{R}^n$ and suppose that the dimension function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing and continuous. Analogously to (1.1), we define

$$H^{\theta}_{\delta}(E) = \inf\Big\{\sum \theta(|F_i|) : \{F_i\} \text{ is a } \delta\text{-cover of } E\Big\}.$$

This leads to a measure, by taking $H^{\theta}(E) = \lim_{\delta\to 0} H^{\theta}_{\delta}(E)$. If $\theta(t) = t^s$, it is the usual definition of an s-dimensional Hausdorff measure.

THEOREM 2.5 [17]. *Let* $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ *be a continuous map such that:*

(i) $\theta(t) > 0$ *for* $t > 0$*;*
(ii) $\limsup_{t\to 0} t/\theta(t) < \infty$ *and*
(iii) *there is a* $\beta \geq 0$ *such that* $\lim_{t\to 0} \theta(ct)/\theta(t) = c^{\beta}$ *for all* $c > 0$.

If f is a continuous map on $[0, 1]$ *and also convex Lipschitz of order* θ*, then f has* σ*-finite h measure, where* $h(y) = y^2/\theta(y)$.

THEOREM 2.6. *Under the hypotheses of the above theorem:*

- *if* $\theta(y) = y^{\alpha}$ *then* $\dim_H(\mathrm{Graph}(f)) \leq \overline{\dim}_B(\mathrm{Graph}(f)) \leq 2 - \alpha$*;*
- *if* $\theta(y) = y\ln(1/y)$ *then* $\dim_H(\mathrm{Graph}(f)) = \dim_B(\mathrm{Graph}(f)) = 1$.

PROOF. The results follow from Theorem 2.5 and the definitions of the Hausdorff measure and Hausdorff dimension. □

THEOREM 2.7. *Let* $q_j \in V^{\theta}(I)$ *and* $\alpha_j \in (-1, 1)$. *Then the associated fractal interpolation function f is in* $V^{\theta}(I)$ *provided that* $\max\{|\alpha_j|_{\infty}, \max_j |\alpha_j|\theta(Y)/\theta(a_jY)\} < 1$.

PROOF. We first define $V^{\theta}_*(I) := \{f \in V^{\theta} : f(x_1) = y_1,\ f(x_N) = y_N\}$. Since $V^{\theta}_*(I)$ is a closed subset of $V^{\theta}(I)$, it follows that $V^{\theta}_*(I)$ is a complete metric space with respect to the metric induced by the norm $\|\cdot\|_{V^{\theta}}$. Let us define a map $T : V^{\theta}_*(I) \to V^{\theta}_*(I)$ by

$$(Tf)(x) = \alpha_j f(L_j^{-1}(x)) + q_j(L_j^{-1}(x)), \quad x \in I_j, j \in J.$$

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Here, $L_j(x) = a_j x + b_j$ and $L_j^{-1}(x) = x/a_j - b_j$. Set $X = x/a_j - b_j$ and $Y = y/a_j$. The mapping T is well defined and, for $f, g \in V_*^\theta(I)$,

$$\begin{aligned}
\|Tf - Tg\|_{V^\theta} &= \|Tf - Tg\|_\infty + [Tf - Tg]^* \\
&\leq |\alpha_j|_\infty \|f(X) - g(X)\|_\infty + \max_j \sup_{a \leq a_j(X+b_j) < a_j(X+Y+b_j) \leq b} \\
&\qquad \frac{|\alpha_j||(f-g)(X+\delta Y) - (\delta(f-g)(X+Y) + (1-\delta)(f-g)(X)|}{\theta(Y)} \times \frac{\theta(Y)}{\theta(a_j Y)} \\
&\leq |\alpha_j|_\infty \|f - g\|_\infty + \max_j |\alpha_j| \frac{\theta(Y)}{\theta(a_j Y)} [f-g]^* \\
&\leq \max\left\{ |\alpha_j|_\infty, \max_j |\alpha_j| \frac{\theta(Y)}{\theta(a_j Y)} \right\} \|f - g\|_{V^\theta}.
\end{aligned}$$

Since $\max\{|\alpha_j|_\infty, \max_j |\alpha_j| \theta(Y)/\theta(a_j Y)\} < 1$, the mapping T is a contraction on $V_*^\theta(I)$. From the Banach fixed point theorem, T has a unique fixed point $f \in V_*^\theta(I)$. From $T(f) = f$, we can write

$$f(L_j(x)) = \alpha_j f(L_j(x)) + q_j(x) \quad \text{for } x \in I, j \in J. \tag{2.1}$$

For $j \in J := \{1, 2, 3, \ldots, N-1\}$, let us define $W_j : I \times \mathbb{R} \to I \times \mathbb{R}$ by

$$W_j(x, y) = (L_j(x), \alpha_j y + q_j(x)). \tag{2.2}$$

We have shown above that the graph of f is an attractor of the IFS $\{I \times \mathbb{R}; W_j, j \in J\}$. By using the proof of Theorem 1 in [5], we can show that the attractor associated with this IFS is the graph of f. In fact, it is the graph of the fractal perturbation of f. To see this, we take the functional equation (2.1), the definition of W_j from (2.2) and $I = \bigcup_{j \in J} L_j(I)$, and get

$$\begin{aligned}
\bigcup_{j \in J} W_j(\text{Graph}(f)) &= \bigcup_{j \in J} \{(L_j(x), f(L_j(x))) : x \in I\} \\
&= \bigcup_{j \in J} \{(x, f(x)) : x \in L_j(I)\} = \text{Graph}(f),
\end{aligned}$$

completing the proof. □

By combining Theorems 2.6 and 2.7, we can estimate the fractal dimension of certain fractal interpolation functions.

THEOREM 2.8. *Let $q_j \in V^\theta(I)$ and $\alpha_j \in (-1, 1)$ be such that*

$$\max\{|\alpha_j|_\infty, \max_j |\alpha_j| \theta(Y)/\theta(a_j Y)\} < 1.$$

Then we have the following bounds for the fractal dimension of the graph of the associated fractal interpolation function f.

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- *If* $\theta(y) = y^{\alpha}$, *then* $\dim_H(\mathrm{Graph}(f)) \leq \overline{\dim}_B(\mathrm{Graph}(f)) \leq 2 - \alpha$.
- *If* $\theta(y) = y\ln(1/y)$, *then* $\dim_H(\mathrm{Graph}(f)) = \dim_B(\mathrm{Graph}(f)) = 1$.

EXAMPLE 2.9 (Weierstrass-type function). For more details on this example, we refer to [17]. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a bounded function which is convex Lipschitz of order 1. For $b > 1,\ 0 < \alpha < 1$, define

$$f(x) = \sum_{n=0}^{\infty} b^{-\alpha n}\Phi(b^n x + \theta_n).$$

Then f is convex Lipschitz of order α. Consequently, $\dim_H(\mathrm{Graph}(f)) \leq 2 - \alpha$. If $\alpha = 1$, then $\dim_H(\mathrm{Graph}(f)) = 1$.

REMARK 2.10. Note that any continuous function $f : [a, b] \to \mathbb{R}$ satisfies the convex-Lipschitz condition with $\theta(y) =$ constant. For any continuous function f, we have $1 \leq \dim(\mathrm{Graph}(f)) \leq 2$. So, for constant θ, we cannot conclude any nontrivial dimension estimates.

3. Oscillation spaces

We refer to [7, 9] for more details on oscillation spaces. Let $Q \subset [0, 1]$ be a p-adic subinterval, that is, $Q = [jp^{-m}, (j + 1)p^{-m}]$ for some integers j and m with $m \geq 0$ and $0 \leq j < p^{-m}$. The oscillation of a continuous function $g : [0, 1] \to \mathbb{R}$ over Q is given by

$$\mathcal{R}_g(Q) = \sup_{x_1,x_2 \in Q} |g(x_1) - g(x_2)| = \sup_{x_1 \in Q} g(x_1) - \inf_{x_2 \in Q} g(x_2),$$

and the total oscillation of order m is given by

$$\mathrm{Osc}(m, g) = \sum_{|Q|=p^{-m}} \mathcal{R}_g(Q),$$

where the sum is taken over all p-adic intervals $Q \subset [0, 1]$ having length $|Q| = p^{-m}$.

The oscillation space $V^{\beta}(I)$, $\beta \in \mathbb{R}$, is defined by

$$V^{\beta}(I) = \Big\{g \in C(I) : \sup_{m \in \mathbb{N}} \frac{\mathrm{Osc}(m, g)}{p^{m(1-\beta)}} < \infty\Big\}.$$

We also define

$$V^{\beta-}(I) = \{g \in C(I) : g \in V^{\beta-\epsilon}(I) \text{ for all } \epsilon > 0\}$$

and

$$V^{\beta+}(I) = \{g \in C(I) : g \notin V^{\beta+\epsilon}(I) \text{ for all } \epsilon > 0\}.$$

THEOREM 3.1 [7, Theorem 4.1]; see also [9]. *For a real-valued continuous function g which is defined on I and* $0 < \beta \leq 1$,

$$\overline{\dim}_B(\mathrm{Graph}(g)) \leq 2 - \beta \quad \textit{if and only if } g \in V^{\beta-}(I)$$

and

$$\overline{\dim}_B(\mathrm{Graph}(g)) \geq 2 - \beta \quad \textit{if and only if } g \in V^{\beta+}(I).$$

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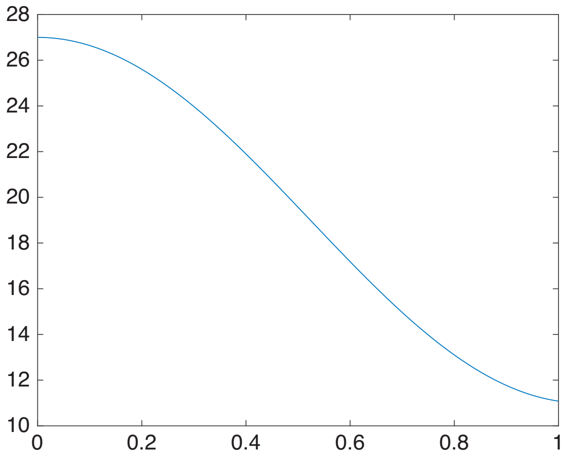

FIGURE 1. Plot for $\alpha = 0.0$.

THEOREM 3.2. *Let* $q_j \in V^\beta(I)$, $\alpha_j \in (-1, 1)$ *and* $\max\{|\alpha_j|_\infty, \sum_{j\in J} |\alpha_j|_\infty\} < 1$. *Then the fractal interpolation function* $f \in V^\beta(I)$. *Moreover,* $\overline{\dim}_B(\mathrm{Graph}(f)) \le 2 - \beta$.

PROOF. Let $V_*^\beta(I) = \{f \in V^\beta(I) : f(x_1) = y_1,\ f(x_N) = y_N\}$. It can be seen that the space V_*^β is a closed subset of $V^\beta(I)$. It follows that $V_*^\beta(I)$ is a complete metric space with respect to the metric induced by the norm $\|\cdot\|_{V^\beta}$. Let us define a map $T : V_*^\beta(I) \to V_*^\beta(I)$ by

$$(Tf)(x) = \alpha_j f(L_j^{-1}(x)) + q_j(L_j^{-1}(x)), \quad x \in I_j, j \in J.$$

Set $X = x/a_j - b_j$, so that $L_j^{-1}(x) = X$. Then T is well defined and, for $g, h \in V_*^\beta(I)$,

$$\begin{aligned}\|Tf - Tg\|_{V^\beta} &= \|Tf - Tg\|_\infty + \sup_{m\in\mathbb{N}} \frac{\mathrm{Osc}(m, Tf - Tg)}{p^{m(1-\beta)}} \\ &\le |\alpha_j|_\infty \|f(X) - g(X)\|_\infty + \sum_{j\in J} |\alpha_j|_\infty \sup_{m\in\mathbb{N}} \frac{\mathrm{Osc}(m, f(X) - g(X))}{p^{m(1-\beta)}} \\ &\le \max\Big\{|\alpha_j|_\infty, \sum_{j\in J} |\alpha_j|_\infty\Big\} \|f - g\|_{V^\beta}.\end{aligned}$$

Since $\max\{|\alpha_j|_\infty, \sum_{j\in J} |\alpha_j|_\infty\} < 1$, the mapping T is a contraction on $V_*^\beta(I)$. From the Banach fixed point principle, T has a unique fixed point $f \in V_*^\beta(I)$, completing the proof. □

3.1. Graphs of fractal interpolation functions. Figures 1–4 give approximate graphs of some fractal interpolation functions.

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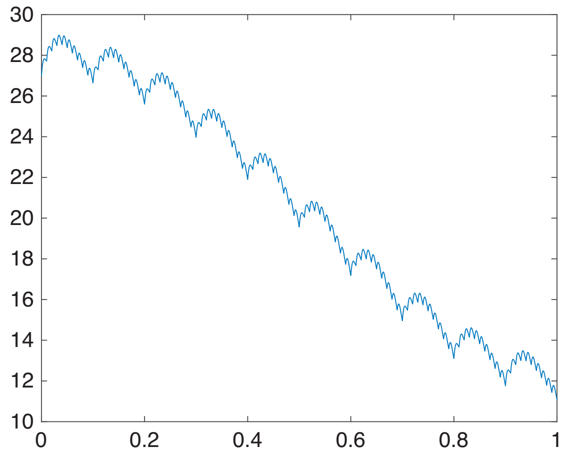

FIGURE 2. Plot for $\alpha = 0.3$.

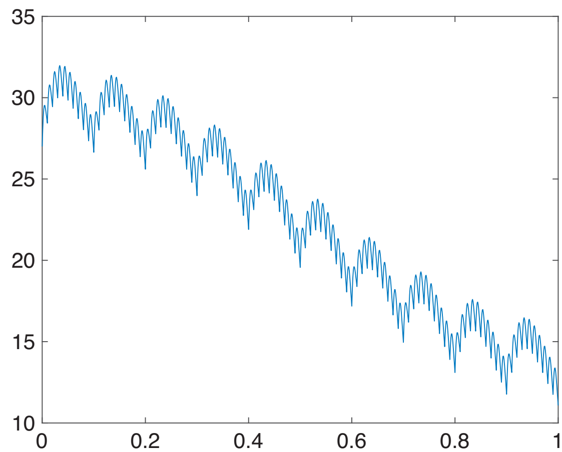

FIGURE 3. Plot for $\alpha = 0.6$.

For this example, let $g = 19 + 8\cos(3x)$ and $q(x) = (1 - \alpha(1 + x^2 - x)) \cdot g(x)$, $x \in [0, 1]$. We show the graphs of the fractal interpolation function f for scaling factors $\alpha = 0.0,\ 0.3, 0.6,\ 0.9$ in Figures 1–4 respectively.

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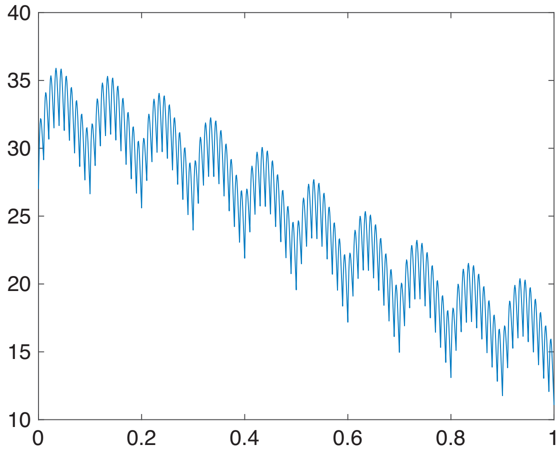

FIGURE 4. Plot for $\alpha = 0.9$.

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