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A VARIANT OF HARISH-CHANDRA FUNCTORS

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Abstract Harish-Chandra induction and restriction functors play a key role in the representation theory of reductive groups over finite fields. In this paper, extending earlier work of Dat, we introduce and study generalisations of these functors which apply to a wide range of finite and profinite groups, typical examples being compact open subgroups of reductive groups over non-archimedean local fields. We prove that these generalisations are compatible with two of the tools commonly used to study the (smooth, complex) representations of such groups, namely Clifford theory and the orbit method. As a test case, we examine in detail the induction and restriction of representations from and to the Siegel Levi subgroup of the symplectic group Sp_4 over a finite local principal ideal ring of length two. We obtain in this case a Mackey-type formula for the composition of these induction and restriction functors which is a perfect analogue of the well-known formula for the composition of Harish-Chandra functors. In a different direction, we study representations of the Iwahori subgroup I_n of $\mathrm{GL}_n(\mathsf{F})$, where F is a non-archimedean local field. We establish a bijection between the set of irreducible representations of I_n and tuples of primitive irreducible representations of smaller Iwahori subgroups, where primitivity is defined by the vanishing of suitable restriction functors.

Keywords: Harish-Chandra induction; parabolic induction; compact $\mathfrak{p}$-adic groups; representations of profinite groups

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Contents

1. Introduction

1.1. Overview

Harish-Chandra (or parabolic) induction and restriction are fundamental operations in the representation theory of reductive groups over finite fields, allowing efficient transport of representations between such groups and establishing a close connection to the representation theory of finite Coxeter groups; see [32, 46] for a particularly elegant development of this connection for finite classical groups. Let us briefly recall the definition of these functors. Let G be the group of rational points of a connected reductive group defined over a finite field, and let L and U be the respective groups of rational points of a Levi factor of and the unipotent radical of a rational parabolic subgroup P of G. Then the Harish-Chandra induction functor $\mathrm{i}_{\mathsf{L}}^{\mathsf{G}}$ is the functor from the complex representations of L to the complex representations of G given by tensor product with the $\mathcal{H}(\mathsf{G})$-$\mathcal{H}(\mathsf{L})$ bimodule $\mathcal{H}(\mathsf{G})e_{\mathsf{U}}$, where $\mathcal{H}$ denotes the complex group algebra and e_{U} is the idempotent associated with the trivial representation of U. Dually, tensoring with the bimodule $e_{\mathsf{U}}\mathcal{H}(\mathsf{G})$ gives the Harish-Chandra restriction functor $\mathrm{r}_{\mathsf{L}}^{\mathsf{G}}$ that is adjoint to $\mathrm{i}_{\mathsf{L}}^{\mathsf{G}}$. The two functors are related by a variant of Mackey's double-coset formula:

$$\mathrm{r}_{\mathsf{L}}^{\mathsf{G}}\,\mathrm{i}_{\mathsf{L}}^{\mathsf{G}} \cong \bigoplus_{g\in W_{\mathsf{L}}\backslash W_{\mathsf{G}}/W_{\mathsf{L}}} \mathrm{i}_{\mathsf{L}\cap g\mathsf{L}g^{-1}}^{\mathsf{L}}\ \mathrm{Ad}_g\ \mathrm{r}_{g^{-1}\mathsf{L}g\cap\mathsf{L}}^{\mathsf{L}} \tag{1.1}$$

where W denotes the Weyl group. See [13] for the precise general formulation and proof, and for a sampling of the applications of this formula; and see [17, 41] for the original work of Harish-Chandra.

In this paper we study induction and restriction functors which generalise the Harish-Chandra functors to a rich family of profinite groups, to which the family of reductive groups over finite fields is only a partial first approximation. Our motivating examples are classical groups over compact discrete valuation rings, but our framework covers many other cases, including arbitrary open compact subgroups of reductive groups over local fields. Certain representations of such open compact subgroups play an important role in the construction and classification of smooth representations of the reductive groups via the theory of types. However, the representation theory of these compact subgroups per se is not so well understood.

Before we introduce the functors that are at the heart of the present paper we remark that the most obvious generalisation of the Harish-Chandra functors to the setting

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considered here tends to produce representations whose decomposition into irreducibles is rather complicated, and in this sense lacks the efficiency of the 'classical' Harish-Chandra functors, for which the problem of decomposing induced representations is tractable through Weyl group calculations, thanks to (1.1) (see [24] for a comprehensive account).

For a concrete example of this inefficiency, let $\mathfrak{o}$ be the ring of integers in a non-archimedean local field F (so F is either the field of Laurent series over a finite field, or a finite extension of the p-adic numbers). Let $\mathfrak{p}$ denote the maximal ideal of $\mathfrak{o}$, and for every $\ell \in \mathbb{N}$ set $\mathfrak{o}_\ell = \mathfrak{o}/\mathfrak{p}^\ell$. Let $\mathrm{T}_n \subset \mathrm{B}_n \subset \mathrm{GL}_n$ denote the standard diagonal torus and the standard upper-triangular Borel subgroup in the general linear group, and let U_n denote the unipotent radical of B_n. The classical Harish-Chandra induction functor from representations of $\mathrm{T}_n(\mathfrak{o}_1)$ to representations of $\mathrm{GL}_n(\mathfrak{o}_1)$ is given by tensor product with the bimodule $\mathcal{H}(\mathrm{GL}_n(\mathfrak{o}_1))e_{\mathrm{U}_n(\mathfrak{o}_1)}$. There is an obvious way to generalise this definition to $\mathrm{GL}_n(\mathfrak{o}_\ell)$ for $\ell > 1$, by taking the tensor product with the bimodule $\mathcal{H}(\mathrm{GL}_n(\mathfrak{o}_\ell))e_{\mathrm{U}_n(\mathfrak{o}_\ell)}$. In particular, this functor sends the trivial representation of $\mathrm{T}_n(\mathfrak{o}_\ell)$ to the permutation representation of $\mathrm{GL}_n(\mathfrak{o}_\ell)$ given by

$$\mathcal{H}(\mathrm{GL}_n(\mathfrak{o}_\ell))e_{\mathrm{U}_n(\mathfrak{o}_\ell)} \otimes_{\mathcal{H}(\mathrm{T}_n(\mathfrak{o}_\ell))} 1 \cong \mathcal{H}(\mathrm{GL}_n(\mathfrak{o}_\ell)/\mathrm{B}_n(\mathfrak{o}_\ell)). \tag{1.2}$$

When $\ell = 1$ the Mackey formula (1.1) gives a decomposition of (1.2) according to the regular representation of the symmetric group on n letters. For $\ell > 1$ the decomposition of (1.2) into irreducibles gets very quickly out of control, owing to the complicated nature of the double-coset space $\mathrm{B}_n(\mathfrak{o}_\ell)\backslash \mathrm{GL}_n(\mathfrak{o}_\ell)/\mathrm{B}_n(\mathfrak{o}_\ell)$. Misleadingly simple is the case $n = 2$, where the induced representation has $\ell + 1$ irreducible components (see [5]); already for $n = 3$ the decomposition of the induced representation is rather complicated and, in particular, depends on the degree of the residue field $\mathfrak{o}_1 = \mathfrak{o}/\mathfrak{p}$, see [37].

Our proposed variant of Harish-Chandra induction, in this GL_n example, sends the trivial representation of $\mathrm{T}_n(\mathfrak{o}_\ell)$ to the image of the intertwining operator

$$\mathcal{H}(\mathrm{GL}_n(\mathfrak{o}_\ell)/\mathrm{B}_n(\mathfrak{o}_\ell)) \to \mathcal{H}(\mathrm{GL}_n(\mathfrak{o}_\ell)/\mathrm{B}_n^t(\mathfrak{o}_\ell))$$

which averages right $\mathrm{B}_n(\mathfrak{o}_\ell)$-invariant functions on $\mathrm{GL}_n(\mathfrak{o}_\ell)$ by the right action of $\mathrm{U}_n^t(\mathfrak{o}_\ell)$, where t means transpose, to obtain right $\mathrm{B}_n^t(\mathfrak{o}_\ell)$-invariant functions. This image is isomorphic—regardless of ℓ—to the module $\mathcal{H}(\mathrm{GL}_n(\mathfrak{o}_1)/\mathrm{B}_n(\mathfrak{o}_1))$, on which $\mathrm{GL}_n(\mathfrak{o}_\ell)$ acts through the quotient map $\mathrm{GL}_n(\mathfrak{o}_\ell) \to \mathrm{GL}_n(\mathfrak{o}_1)$.

This process of passing to the image of a canonical intertwining operator between two induced representations fits into a rather general setting, which we shall now describe. In the main body of the paper we study representations of profinite groups, such as groups of matrices over compact discrete valuation rings; but our results also apply to (and are interesting for) finite groups, such as matrix groups over the finite rings $\mathfrak{o}_\ell$, and in order to minimise the technicalities in this introduction we shall restrict our attention here to the finite case.

Let G be a finite group, and suppose that U, L and V are subgroups of G such that L normalises U and V, and such that the map

$$U \times L \times V \hookrightarrow G$$

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given by multiplication in G is injective. We let e_U and e_V denote the idempotents in the complex group algebra $\mathcal{H}(G)$ associated to the trivial representations of U and V, and we consider the $\mathcal{H}(G)$-$\mathcal{H}(L)$ bimodule $\mathcal{H}(G)e_U e_V$. Let $i_{U,V}$ be the functor from the category $\mathcal{R}(L)$ of complex representations of L to the category of complex representations of G defined by tensoring with this bimodule:

$$i_{U,V} : \mathcal{R}(L) \to \mathcal{R}(G), \quad M \mapsto \mathcal{H}(G)e_U e_V \otimes_{\mathcal{H}(L)} M.$$

Similarly, define

$$r_{U,V} : \mathcal{R}(G) \to \mathcal{R}(L), \quad N \mapsto e_U e_V \mathcal{H}(G) \otimes_{\mathcal{H}(G)} N.$$

This definition, suitably extended to profinite groups, generalises the parahoric functors defined by Dat in [10] for the purpose of studying representations of $\mathfrak{p}$-adic reductive groups. (Note, though, that we consider only complex representations, whereas Dat studied representations over more general commutative rings.) In the situations studied by Dat, which we shall recall in detail in Example 2.4, G is the isotropy group of a point in the Bruhat–Tits building of a reductive group $\mathbf{G}$, and the product ULV is a parahoric subgroup of G. The main novelty of the above definition relative to Dat's is that we do not require the product ULV to be a group; for instance, ULV may be the intersection with G of a Bruhat cell of $\mathbf{G}$. One source of motivation for considering this more general situation is a question raised by Dat [10, Question 2.15], to which we shall provide a negative answer in Corollary 5.21. The precise relationship between our definition and Dat's is discussed in more detail in Examples 2.4 and 2.12, Remark 2.8, and in § 5.5.

1.2. Description of the main results

Basic properties of the functors $i_{U,V}$ and $r_{U,V}$, in the abstract setting for profinite groups, are presented in § 2. For instance, these functors are adjoints on both sides; they do not depend on the order of U and V, up to natural isomorphism; they preserve finite dimensionality; and they satisfy a version of 'induction in stages'.

The analysis of the functors $i_{U,V}$ and $r_{U,V}$ becomes considerably less complicated in cases where the product map $U \times L \times V \to G$ is a bijection. In many examples, such as the GL_n example considered above, this is not the case, but there is a normal subgroup $G_0 \lhd G$ such that the product map $U_0 \times L_0 \times V_0 \to G_0$ is a bijection, where H_0 means $H \cap G_0$. In the GL_n example we can take G_0 to be the principal congruence subgroup $G_0 = \{g \in \mathrm{GL}_n(\mathfrak{o}_\ell) \mid g \equiv 1 \text{ modulo } \mathfrak{p}\}$.

Suppose that G admits such a normal subgroup G_0. The representation categories $\mathcal{R}(L)$ and $\mathcal{R}(G)$ decompose according to L_0- and G_0-isotypic components, and the individual components can be described using Clifford theory. In § 3 we prove that the Clifford analysis is compatible with the induction and restriction functors $i_{U,V}$ and $r_{U,V}$.

More precisely, let ψ be an irreducible representation of L_0, and let $\varphi = i_{U_0,V_0}(\psi)$ be the corresponding (irreducible) induced representation of G_0. Let $L(\psi)$ and $G(\varphi)$ denote the inertia groups of ψ and φ. We prove in Theorems 3.4, 3.6 and 3.14 that there is a

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commutative diagram for induction (and a similar diagram for restriction) :

$$\begin{array}{ccc}
\mathcal{R}(L)_{\psi} & \xrightarrow{\;i_{U,V}\;} & \mathcal{R}(G)_{\varphi} \\
\uparrow \cong & & \uparrow \cong \\
\mathcal{R}\left(L(\psi)\right)_{\psi} & \xrightarrow{\;i_{U(\varphi),V(\varphi)}\;} & \mathcal{R}(G(\varphi))_{\varphi} \\
\uparrow \cong & & \uparrow \cong \\
\mathcal{R}^{\gamma}(L(\psi)/L_0) & \xrightarrow{\;i_{U(\varphi)/U_0,V(\varphi)/V_0}\;} & \mathcal{R}^{\gamma}(G(\varphi)/G_0)
\end{array}$$

where $\mathcal{R}(H)_{\theta}$ stands for the representations of H whose restriction to $H_0 \lhd H$ is a sum of conjugates of the irreducible representation θ; and $\mathcal{R}^{\gamma}(H(\theta)/H_0)$ stands for projective representations (for a certain cocycle γ) of the quotient $H(\theta)/H_0$.

As for the groups L_0 and G_0, in many of our motivating examples they are amenable to the orbit method: their irreducible representations correspond bijectively to coadjoint orbits in the Pontryagin duals of certain Lie algebras $\mathfrak{l}_0$ and $\mathfrak{g}_0$. This situation is studied in § 4, where we show that under appropriate assumptions the induction functor $i_{U_0,V_0} : \mathcal{R}(L_0) \to \mathcal{R}(G_0)$ corresponds to a natural inclusion of coadjoint orbits $\Lambda^* : L_0\backslash\widehat{\mathfrak{l}_0} \hookrightarrow G_0\backslash\widehat{\mathfrak{g}_0}$. That is, the diagram

$$\begin{array}{ccc}
\mathrm{Irr}\,(L_0) & \xrightarrow{\;i_{U_0,V_0}\;} & \mathrm{Irr}\,(G_0) \\
\cong \downarrow & & \downarrow \cong \\
L_0\backslash\widehat{\mathfrak{l}_0} & \xrightarrow{\;\Lambda^*\;} & G_0\backslash\widehat{\mathfrak{g}_0}
\end{array}$$

commutes.

Returning from the abstract setting to our motivating examples, the functors $i_{U,V}$ and $r_{U,V}$ provide a new approach to the representation theory of classical groups over compact discrete valuation rings, and the results of §§ 3 and 4 provide tools to analyse these functors. In § 5 we illustrate the method for the symplectic group $\mathrm{Sp}_4(\mathfrak{o}_2)$. The main result (Theorem 5.2) is a Mackey-type formula for the composition of restriction and induction to/from the Siegel Levi subgroup. The formula is the same as the usual formula (1.1) for the composition of Harish-Chandra induction and restriction for the corresponding group $\mathrm{Sp}_4(\mathfrak{o}_1)$ over the residue field of $\mathfrak{o}$, which lends some support to the analogy between our functors and the Harish-Chandra functors. This analogy is further supported by an analysis for the groups GL_n, which is presented in [9] and in the forthcoming paper [8].

The general methods developed in §§ 3 and 4 and used in § 5 apply equally well to Dat's parahoric induction and restriction functors. In § 5.5 we prove that, for the Siegel Levi subgroup of Sp_4, Dat's parahoric induction and restriction functors are not isomorphic to the functors appearing in our Mackey formula; this gives a negative answer to Dat's question [10, Question 2.15]. We also prove that the parahoric induction and restriction functors do not satisfy the analogue of (1.1) in this example.

While our primary motivation for studying the functors $i_{U,V}$ and $r_{U,V}$ is their application to classical groups, these functors are defined in much broader generality, and we believe that they have a useful role to play in the representation theory of more general matrix groups. In § 6 we use these functors to study one such example, the representation theory of the Iwahori subgroup I_n of $GL_n(\mathfrak{o})$. The Iwahori in SL_2 was previously studied from a similar point of view in [10] and [7]. The main result of this section, Theorem 6.11, states that the functors $i_{U,V}$ and $r_{U,V}$ in this context give a bijection

$$\operatorname{Irr}(I_n) \longleftrightarrow \bigsqcup_{n_1+\cdots+n_k=n} \operatorname{Prim}(I_{n_1}) \times \cdots \times \operatorname{Prim}(I_{n_k}) \tag{1.3}$$

between the irreducible representations of I_n, and tuples of *primitive* irreducible representations of smaller Iwahori subgroups (where primitivity is defined by the vanishing of the functors $r_{U,V}$). The problem of classifying the irreducible representations of I_n remains a very difficult one—it contains the problem of counting the conjugacy classes in the group of upper-triangular matrices over the residue field $\mathfrak{o}_1$—but the bijection (1.3) shows that part of this classification is very simple and combinatorial in nature.

1.3. Related constructions

Representations of open compact subgroups of reductive groups over local fields have received much attention in the past two decades. One approach, taken by Lusztig and Stasinski, is to generalise Deligne–Lusztig theory [12] (which is itself a generalisation of the Harish-Chandra theory) to such groups; see [34, 44], and also [6] and [35]. Another approach, taken by Hill [18–21], consists of a direct Clifford-theoretic analysis of representations according to their restrictions to congruence kernels. In particular, in [18] Hill establishes a Jordan decomposition for characters of general linear groups over rings of integers in p-adic fields, analogous to the Jordan decomposition of irreducible characters of finite reductive groups established by Lusztig, cf. [33]. Hill's work relies on an analysis of certain Hecke algebras building on the work of Howe and Moy [23]. Another approach was proposed by the third author in [36] using a different variant of Harish-Chandra induction that allows one to import representations from automorphism groups of finite modules over discrete valuation rings, yielding a complete and characteristic-independent treatment in rank two. The work of Dat [10], in which representations of parahoric subgroups of p-adic reductive groups are studied using methods closely related to those of the present paper, has already been mentioned above.

It would be of great interest to understand how all these approaches align with the one taken in this paper. The relationship between our work and that of Dat is addressed in Examples 2.4 and 2.12, Remark 2.8, and § 5.5. As for the other works cited above, let us make a couple of general observations.

The first point to note is that the natural filtration on the valuation ring $\mathfrak{o}$ does not enter a priori into the definition of our induction/restriction functors, and in this sense our approach is more elementary than those of the above-cited works. For instance, induction functors of the form $i_{U,V}$ are used in [9] to give a simple construction of a 'principal series' of representations of $GL_n(R)$, where R is any finite commutative ring.

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A second difference is one of scope. Our focus here is on producing induced representations with tractable intertwining properties, rather than on producing all or most of the irreducible representations of a given group. To give an example, we note that in the case of $G = \mathrm{GL}_2(\mathfrak{o})$, $L = \mathrm{T}_2(\mathfrak{o})$, $U = \mathrm{U}_2(\mathfrak{o})$ and $V = \mathrm{U}_2^{t}(\mathfrak{o})$ as in (1.2), all of the representations of G which are of types (3) and (4) in the classification of Stasinski [43, § 2.1] (or equivalently types (iii) and (iv) in [36, § 5.2]) are annihilated by the restriction functor $\mathrm{r}_{\mathrm{U}_2(\mathfrak{o}),\mathrm{U}_2^{t}(\mathfrak{o})} : \mathcal{R}(\mathrm{GL}_2(\mathfrak{o})) \to \mathcal{R}(\mathrm{T}_2(\mathfrak{o}))$; hence these representations do not appear as subrepresentations of $\mathrm{i}_{\mathrm{U}_2(\mathfrak{o}),\mathrm{U}_2^{t}(\mathfrak{o})}(M)$ for any representation M of $\mathrm{T}_2(\mathfrak{o})$. Our goal here is to develop an analogue of Harish-Chandra theory which mirrors as closely as possible the theory for reductive groups over a finite field, yielding a description of arbitrary representations in terms of 'primitive' ones, and of Weyl group combinatorics. We leave untouched for now the problem of constructing or classifying the primitive representations.

2. Notation, definitions, and basic properties

In this section we define and develop basic properties of the functors $\mathrm{i}_{U,V}$ and $\mathrm{r}_{U,V}$ in an abstract setting. The pivotal point in this section is Proposition 2.16, which allows us to generalise many of Dat's results from [10, § 2] to the situation considered in this paper. We begin by setting up the notation that will be used throughout the paper.

2.1. Notation

For a profinite group G we let $\mathcal{R}(G)$ denote the category of smooth, complex representations of G, that is, linear representations $\varphi : G \to \mathrm{GL}_{\mathbb{C}}(M)$ in which each vector in M is fixed by some open subgroup of G. We denote such a representation either by the map φ or by the space M, as convenient. If M is any representation of G (not necessarily smooth), we let M^{∞} denote the G-subspace of vectors fixed by some open subgroup of G.

Let $\mathcal{H}(G)$ denote the algebra of locally constant, complex-valued functions on G, with product given by convolution with respect to some Haar measure on G. Different choices of Haar measure give isomorphic algebras, the isomorphism being multiplication by the ratio $\mathrm{vol}_1(G)/\mathrm{vol}_2(G)$ of the total volumes of the two measures. The category $\mathcal{R}(G)$ is equivalent to the category of nondegenerate left $\mathcal{H}(G)$-modules, i.e. those modules M which satisfy $M = \mathcal{H}(G)M$. If G is finite then we usually use counting measure as the Haar measure on G, in which case the map sending $g \in G$ to the δ-function δ_g at g extends to an isomorphism from the complex group ring $\mathbb{C}(G)$ to $\mathcal{H}(G)$.

Let $\mathrm{Irr}(G)$ denote the set of isomorphism classes of irreducible smooth representations of G. When chances for confusion are slim we also write $\rho \in \mathrm{Irr}(G)$ for an actual irreducible representation. For each $\rho \in \mathrm{Irr}(G)$ let $\mathrm{ch}_\rho \in \mathcal{H}(G)$ be the character $g \mapsto \mathrm{tr}(\rho(g))$ of ρ, and let $e_\rho \in \mathcal{H}(G)$ be the idempotent defined by

$$e_\rho : g \mapsto \frac{\dim_{\mathbb{C}}(\rho)}{\mathrm{vol}(G)} \mathrm{ch}_\rho(g^{-1}).$$

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If M is a smooth representation of G then e_ρ acts on M by projecting M onto its ρ-isotypical submodule. For the special case of the trivial representation we write e_G for the corresponding idempotent, namely, the function on G with constant value $1/\operatorname{vol}(G)$. The element e_G acts on each smooth representation M by projecting onto the submodule M^G of G-fixed vectors.

If M is a nondegenerate left $\mathcal{H}(G)$-module, and N is a nondegenerate right $\mathcal{H}(G)$-module, then by definition

$$N \otimes_{\mathcal{H}(G)} M = N \otimes_{\mathbb{C}} M \big/ \operatorname{span}\{nf \otimes m - n \otimes fm \mid n \in N,\ m \in M,\ f \in \mathcal{H}(G)\}.$$

Equivalently, viewing N and M as smooth representations of G, $N \otimes_{\mathcal{H}(G)} M$ is the space of coinvariants for the action $g : n \otimes m \to ng^{-1} \otimes gm$ of G on $N \otimes_{\mathbb{C}} M$.

If H is a closed subgroup of G, then $\mathcal{H}(G)$ is a smooth representation of H under both left and right translation, and consequently $\mathcal{H}(G)$ is an $\mathcal{H}(H)$-bimodule. Given a smooth representation M of H we write

$$\operatorname{ind}_H^G M = \left\{ f : G \xrightarrow[\text{constant}]{\text{locally}} M \;\middle|\; f(hg) = h \cdot f(g), \forall h \in H, g \in G \right\}$$

for the induced representation, on which G acts by right translation. This is isomorphic to the tensor product $\mathcal{H}(G) \otimes_{\mathcal{H}(H)} M$. If the subgroup H is a semidirect product $U \rtimes L$, then representations may be induced from L to G by first inflating to H (i.e. pulling back along the quotient map $H \to L$), and then applying the functor ind_H^G. The resulting functor from $\mathcal{R}(L)$ to $\mathcal{R}(G)$ is isomorphic to the functor of tensor product with the $\mathcal{H}(G)$-$\mathcal{H}(L)$ bimodule $\mathcal{H}(G)e_U \cong \mathcal{H}(G/U)$, where $\mathcal{H}(G/U)$ denotes the space of locally constant functions on G/U.

Whenever a group G acts on a set X we write $G(x)$ for the stabiliser in G of $x \in X$.

The first three chapters of [38] are a convenient reference for all of the above. Many of the examples considered here will be groups of matrices over compact subrings of non-archimedean local fields; see [38, Chapter V], for instance, for more background on these.

2.2. Virtual Iwahori decompositions

Let us begin by describing the kind of groups that we shall be interested in, and giving several examples.

Definition 2.1. Let G be a profinite group. A *virtual Iwahori decomposition* of G is a triple of closed subgroups (U, L, V) of G, where L normalises U and V, such that

(1) The multiplication map $U \times L \times V \to G$ is an open embedding (and therefore a homeomorphism onto its image).

(2) G contains arbitrarily small open, normal subgroups K for which the multiplication map

$$(U \cap K) \times (L \cap K) \times (V \cap K) \to K$$

is a homeomorphism.

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An *Iwahori decomposition* of G is a virtual Iwahori decomposition for which the multiplication map in (1) is surjective (and therefore, a homeomorphism).

The following immediate observation shows that the notion of virtual Iwahori decomposition is inherited by subgroups and quotients.

Observation 2.2. Let $(\mathsf{U}, \mathsf{L}, \mathsf{V})$ be a virtual Iwahori decomposition of G.

(1) If J is a closed subgroup of G, then $(\mathsf{U} \cap \mathsf{J}, \mathsf{L} \cap \mathsf{J}, \mathsf{V} \cap \mathsf{J})$ is a virtual Iwahori decomposition of J.

(2) If $(\mathsf{X}, \mathsf{H}, \mathsf{Y})$ is a virtual Iwahori decomposition of L, then $(\mathsf{U} \rtimes \mathsf{X}, \mathsf{H}, \mathsf{Y} \ltimes \mathsf{V})$ is a virtual Iwahori decomposition of G.

(3) If K is an open normal subgroup of G with an Iwahori decomposition as in part (2) of Definition 2.1, then $(\mathsf{U}/(\mathsf{U} \cap \mathsf{K}), \mathsf{L}/(\mathsf{L} \cap \mathsf{K}), \mathsf{V}/(\mathsf{V} \cap \mathsf{K}))$ is a virtual Iwahori decomposition of G/K.

The concept of Iwahori decomposition first appeared in the work of Iwahori and Matsumoto on $\mathfrak{p}$-adic Chevalley groups [26]. The 'virtual' version defined above is likewise motivated by examples occurring naturally in the study of reductive groups:

Example 2.3. Let $\mathbf{G}$ be a connected reductive group over a non-archimedean local field F, and let G be any compact open subgroup of $\mathbf{G}(\mathsf{F})$. There is a maximal F-split torus $\mathbf{T} \subset \mathbf{G}$ (depending on G) with the property that if $\mathbf{L}$ is an F-rational Levi subgroup of $\mathbf{G}$ containing $\mathbf{T}$, and $\mathbf{U}$ and $\mathbf{V}$ are the unipotent radicals of an opposite pair of F-rational parabolic subgroups of $\mathbf{G}$ with common Levi factor $\mathbf{L}$, then the triple of subgroups $(\mathsf{G} \cap \mathbf{U}(\mathsf{F}), \mathsf{G} \cap \mathbf{L}(\mathsf{F}), \mathsf{G} \cap \mathbf{V}(\mathsf{F}))$ is a virtual Iwahori decomposition of G. This follows from the Bruhat–Tits theory: one can take $\mathbf{T}$ to be any torus whose associated apartment in the affine building of $\mathbf{G}(\mathsf{F})$ contains a point fixed by G. An explicit filtration of G by open normal subgroups admitting Iwahori decompositions is constructed in [39, § 1.2]; cf. [10, 2.11].

Example 2.4. Keeping the notation of the previous example, let us recall the Iwahori decompositions used by Dat in his construction of parahoric induction and restriction; see [10, 2.11] for more details. Let x be a point in the Bruhat–Tits building of $\mathbf{G}(\mathsf{F})$, lying in the apartment associated to some maximal split torus $\mathbf{T}$. Let $\mathsf{G} = \mathbf{G}(\mathsf{F})(x)$ be the stabiliser of x in $\mathbf{G}(\mathsf{F})$; this is a compact open subgroup of $\mathbf{G}(\mathsf{F})$. The Bruhat–Tits theory distinguishes an open subgroup G^+ of G, the *pro-*$\mathfrak{p}$ *radical*. Let $\mathbf{P} = \mathbf{LU}$ and $\mathbf{Q} = \mathbf{LV}$ be an opposite pair of rational parabolic subgroups of $\mathbf{G}$ whose common Levi factor $\mathbf{L}$ contains $\mathbf{T}$. In addition to the groups $\mathsf{U} = \mathsf{G} \cap \mathbf{U}(\mathsf{F})$, $\mathsf{L} = \mathsf{G} \cap \mathbf{L}(\mathsf{F})$, and $\mathsf{V} = \mathsf{G} \cap \mathbf{V}(\mathsf{F})$ considered in the previous example, we consider $\mathsf{U}^+ = \mathsf{U} \cap \mathsf{G}^+$ and $\mathsf{V}^+ = \mathsf{V} \cap \mathsf{G}^+$. Then $(\mathsf{U}, \mathsf{L}, \mathsf{V}^+)$ and $(\mathsf{U}^+, \mathsf{L}, \mathsf{V})$ are virtual Iwahori decompositions of G, with the property that the products $\mathsf{G}_{\mathbf{P}} = \mathsf{ULV}^+$ and $\mathsf{G}_{\mathbf{Q}} = \mathsf{U}^+\mathsf{LV}$ are subgroups of G (and so, $(\mathsf{U}, \mathsf{L}, \mathsf{V}^+)$ and $(\mathsf{U}^+, \mathsf{L}, \mathsf{V})$ are Iwahori decompositions of the groups $\mathsf{G}_{\mathbf{P}}$ and $\mathsf{G}_{\mathbf{Q}}$). By contrast, for the virtual Iwahori decomposition $(\mathsf{U}, \mathsf{L}, \mathsf{V})$ considered in Example 2.3, the product ULV is generally not a subgroup of G.

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Example 2.5. For a specific instance of the previous two examples, let $\mathbf{G} = \mathrm{GL}_n$, and let F be a non-archimedean local field with ring of integers $\mathfrak{o}$ and maximal ideal $\mathfrak{p}$. For each positive integer ℓ we let $\mathfrak{o}_\ell = \mathfrak{o}/\mathfrak{p}^\ell$.

Given an ordered partition $n = n_1 + \cdots + n_m$ of n as a sum of positive integers, let $\mathbf{L} \cong \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_m}$ be the corresponding block-diagonal Levi subgroup of $\mathbf{G}$; let $\mathbf{U}$ be the group of upper-triangular matrices in G with diagonal blocks $1_{n_1 \times n_1} \times \cdots \times 1_{n_m \times n_m}$; let $\mathbf{P} = \mathbf{LU}$ be the parabolic subgroup of block-upper-triangular matrices; and let $\mathbf{Q} = \mathbf{LV}$ be the opposite parabolic subgroup of block-lower-triangular matrices.

The group $G = \mathbf{G}(\mathfrak{o}) = \mathrm{GL}_n(\mathfrak{o})$ is the stabiliser of a vertex in the building of $\mathbf{G}(F)$, lying in the apartment associated to the diagonal torus $\mathbf{T}$. The group $L = \mathbf{L}(\mathfrak{o})$ is the group of block-diagonal matrices in G; similarly $U = \mathbf{U}(\mathfrak{o})$ and $V = \mathbf{V}(\mathfrak{o})$. The triple (U, L, V) is a virtual Iwahori decomposition of G: the principal congruence subgroups

$$K_\ell := \ker\left(\mathrm{GL}_n(\mathfrak{o}) \to \mathrm{GL}_n(\mathfrak{o}_\ell)\right)$$

all admit Iwahori decompositions. Passing to quotients by the K_ℓ yields virtual Iwahori decompositions of the finite groups $\mathrm{GL}_n(\mathfrak{o}_\ell)$.

The pro-p radical G^+ is the first principal congruence subgroup K_1. The subgroups $G_{\mathbf{P}} = ULV^+$ and $G_{\mathbf{Q}} = U^+LV$ are the preimages of the parabolic subgroups $\mathbf{P}(\mathfrak{o}_1)$ and $\mathbf{Q}(\mathfrak{o}_1)$ (respectively) under the reduction-mod-$\mathfrak{p}$ map $G \to \mathbf{G}(\mathfrak{o}_1)$. For instance, if the partition is $n = 1 + \cdots + 1$, then $G_{\mathbf{P}}$ is the standard *Iwahori subgroup* of $\mathrm{GL}_n(\mathfrak{o})$, comprising those matrices which are upper-triangular modulo $\mathfrak{p}$. By contrast, ULV is in this example the set of $\mathfrak{o}$-points of the open Bruhat cell $\mathbf{PQ} \subset \mathbf{G}$; this is clearly not a subgroup of G.

Let us return now to the general setting of Definition 2.1. If G is finite then the condition (2) of that definition is always satisfied, e.g. by the trivial subgroup $K = \{1\}$. Since the smooth representation theory of a profinite group G is determined in a very simple way by the representations of the finite quotients of G, the condition (2) is therefore not essential to much of the sequel. On the other hand, this condition is convenient in places for shortening some proofs, and it is satisfied by all of our motivating examples. Nevertheless, let us note the following quite general construction of examples which satisfy condition (1) without—at least a priori—satisfying (2).

Example 2.6. Let $\mathcal{G}$ be a totally disconnected locally compact group and let $\alpha : \mathcal{G} \to \mathcal{G}$ be a topological group automorphism. Suppose that the contraction subgroups

$$\mathcal{U}_\alpha = \{g \in \mathcal{G} \mid \alpha^n(g) \to 1 \text{ as } n \to \infty\} \quad \text{and} \quad \mathcal{V}_\alpha = \mathcal{U}_{\alpha^{-1}}$$

are closed in $\mathcal{G}$. This is always the case, for example, if $\mathcal{G}$ is a p-adic Lie group.

These contraction subgroups are both normalised by the closed subgroup

$$\mathcal{L}_\alpha = \{g \in \mathcal{G} \mid \{\alpha^n(g) \mid n \in \mathbb{Z}\} \text{ is precompact in } \mathcal{G}\},$$

and the multiplication map

$$\mathcal{U}_\alpha \times \mathcal{L}_\alpha \times \mathcal{V}_\alpha \to \mathcal{G}$$

is an open embedding. So if G is any compact open subgroup of $\mathcal{G}$, then the triple $(\mathcal{U}_\alpha \cap G, \mathcal{L}_\alpha \cap G, \mathcal{V}_\alpha \cap G)$ satisfies condition (1) of Definition 2.1. Moreover, G contains arbitrarily small open subgroups K for which the multiplication map

$$(\mathcal{U}_\alpha \cap K) \times (\mathcal{L}_\alpha \cap K) \times (\mathcal{V}_\alpha \cap K) \to K$$

is a homeomorphism (the so-called *tidy* subgroups for α). It is not clear to us whether G contains arbitrarily small open *normal* subgroups K with this property. If $\mathcal{G}$ is an analytic Lie group over a local field and the automorphism α is analytic (keeping the assumption that the contraction groups are closed), then it is at least true that $\mathcal{G}$ contains arbitrarily small open subgroups K with Iwahori decomposition $(\mathcal{U}_\alpha \cap K, \mathcal{L}_\alpha \cap K, \mathcal{V}_\alpha \cap K)$; cf. Example 4.8 for the characteristic 0 case. We thank George Willis and Helge Glöckner for a discussion of this example. See [1] and [16] for details.

2.3. Definition and basic properties of the functors i and r

We now come to the main definition of the paper. Whenever H is a closed subgroup of a profinite group G, the space $\mathcal{H}(G)$ is a bimodule over $\mathcal{H}(H)$. If L, U and V are closed subgroups of G, and L normalises U and V, then the action of $\mathcal{H}(L)$ on $\mathcal{H}(G)$ commutes with the idempotents $e_U \in \mathcal{H}(U)$ and $e_V \in \mathcal{H}(V)$. Thus $\mathcal{H}(G)e_U e_V$ is an $\mathcal{H}(G)$-$\mathcal{H}(L)$ bimodule, and $e_U e_V \mathcal{H}(G)$ is an $\mathcal{H}(L)$-$\mathcal{H}(G)$ bimodule.

Definition 2.7. Let (U, L, V) be a virtual Iwahori decomposition of a profinite group G. Define the following functors:

$$\mathrm{i}_{U,V} : \mathcal{R}(L) \to \mathcal{R}(G), \quad \mathrm{i}_{U,V} : M \mapsto \mathcal{H}(G)e_U e_V \otimes_{\mathcal{H}(L)} M$$
$$\mathrm{r}_{U,V} : \mathcal{R}(G) \to \mathcal{R}(L), \quad \mathrm{r}_{U,V} : N \mapsto e_U e_V \mathcal{H}(G) \otimes_{\mathcal{H}(G)} N.$$

Remark 2.8. The definition in the case where ULV is a subgroup of G is due to Dat, who considered situations like Example 2.4; see [10, 2.6, 2.11]. The novelty of Definition 2.7 is that we relax the requirement that ULV be a group, so as to cover cases like Example 2.3. See Example 2.12 and § 5.5 for a discussion of the relationship between this more general definition and Dat's definition of *parahoric induction*. Note that Dat makes a further assumption in [10], namely that the group L should contain an open normal subgroup $L^\dagger$ such that the set $UL^\dagger V$ is a pro-p subgroup of G. This assumption, which is needed to ensure the integrality of certain constructions in [10], plays no role here, where all representations are over $\mathbb{C}$.

Let us make a few further remarks on Definition 2.7. Firstly, since $\mathcal{H}(G)e_U e_V$ is the image of the bimodule map $f \mapsto fe_V$ from $\mathcal{H}(G)e_U$ to $\mathcal{H}(G)e_V$, and since every $M \in \mathcal{R}(L)$ is a direct sum of representations of finite quotients of L, and hence flat as a module over $\mathcal{H}(L)$, the module $\mathrm{i}_{U,V}(M)$ is isomorphic to the image of the map

$$J_V : \mathcal{H}(G)e_U \otimes_{\mathcal{H}(L)} M \xrightarrow{f\otimes m \mapsto fe_V \otimes m} \mathcal{H}(G)e_V \otimes_{\mathcal{H}(L)} M. \tag{2.9}$$

The module $\mathcal{H}(G)e_U \otimes_{\mathcal{H}(L)} M$ is isomorphic as a representation of G to the induced representation $\mathrm{ind}_{LU}^{G}(M)$, where M is inflated to a representation of LU by letting U act

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trivially (cf. § 2.1). We similarly have $\mathcal{H}(G)e_V \otimes_{\mathcal{H}(L)} M \cong \mathrm{ind}_{LV}^G(M)$, and the map J_V corresponds in this picture to the 'standard intertwining operator'

$$J_V : \mathrm{ind}_{LU}^G(M) \to \mathrm{ind}_{LV}^G(M), \quad J_V(f) : g \mapsto \int_V f(vg)\, dv.$$

Similarly, $r_{U,V}(N)$ is isomorphic to the image of the canonical projection

$$e_U : N^V \to N^U, \quad n \mapsto \int_U un\, du$$

from the V-invariants to the U-invariants of N.

As a final remark on Definition 2.7, we note that the definition makes sense if we assume only that L, U and V are closed subgroups of G such that L normalises U and V. Some of the properties of the functors $i_{U,V}$ and $r_{U,V}$ that we shall establish below remain valid in this degree of generality: e.g., parts (1), (2), (4) and (7) of Theorem 2.18. For the applications we have in mind, the assumption that (U, L, V) is a virtual Iwahori decomposition is both a natural and a useful one.

Example 2.10. Let G be a reductive group over a finite field, and let LU and LV be an opposite pair of parabolic subgroups of G. A theorem of Howlett and Lehrer (see [25, Theorem 2.4]) asserts that in this case the map (2.9) is an isomorphism for every $M \in \mathcal{R}(L)$, and this implies that the functor $i_{U,V}$ is equal to the Harish-Chandra induction functor $M \mapsto \mathcal{H}(G)e_V \otimes_{\mathcal{H}(L)} M$ (and isomorphic to the analogous functor with U in place of V). Similarly, $r_{U,V}$ is isomorphic to the functor of Harish-Chandra restriction. See [13, Chapter 4] for background on Harish-Chandra functors for finite reductive groups.

Example 2.11. Example 2.10 notwithstanding, the map (2.9) is usually far from being an isomorphism. For instance, if G is a compact open subgroup of a reductive group $\mathbf{G}(F)$ as in Example 2.3, and (U, L, V) is the virtual Iwahori decomposition of G corresponding to an opposite pair of proper parabolic subgroups of $\mathbf{G}$, then the subgroups LU and LV have infinite index in G, and hence the representations $\mathrm{ind}_{LU}^G(M)$ and $\mathrm{ind}_{LV}^G(M)$ are infinite-dimensional for every $M \in \mathcal{R}(L)$. By contrast, the representation $i_{U,V}(M)$ is finite-dimensional whenever M is: see Theorem 2.18(6).

Example 2.12 (Parahoric induction [10]). We return to the setting of Example 2.4: G is the stabiliser of a point in the Bruhat–Tits building of the reductive group $\mathbf{G}(F)$; $\mathbf{P} = \mathbf{LU}$ and $\mathbf{Q} = \mathbf{LV}$ are a suitably chosen pair of opposite parabolic subgroups; U, L and V denote the intersections with G of $\mathbf{U}(F)$, $\mathbf{L}(F)$ and $\mathbf{V}(F)$; and U^+ denotes the intersection of U with the pro-p-radical of G. In [10], Dat considers the *parahoric induction* functor $i_{U^+,V} : \mathcal{R}(L) \to \mathcal{R}(G)$ associated to the virtual Iwahori decomposition (U^+, L, V) of G.

The product $G_{\mathbf{Q}} = U^+LV$ is a group, and it is straightforward to see that the functor $i_{U^+,V}$ factors into the composition

$$\mathcal{R}(L) \xrightarrow{i_{U^+,V}} \mathcal{R}(G_{\mathbf{Q}}) \xrightarrow{\mathrm{ind}_{G_{\mathbf{Q}}}^G} \mathcal{R}(G). \tag{2.13}$$

Dat observed [10, p.275] that the functor $i_{U^+,V} : \mathcal{R}(L) \to \mathcal{R}(G_{\mathbf{Q}})$ can be viewed as a kind of generalised 'inflation' from L to the parahoric subgroup $G_{\mathbf{Q}}$, and that the composition

(2.13) bears a resemblance to the functor of parabolic induction, which inflates smooth representations of $\mathbf{L}(\mathsf{F})$ to $\mathbf{Q}(\mathsf{F})$, then induces to $\mathbf{G}(\mathsf{F})$.

As we observed in Example 2.3, the triple $(\mathsf{U}, \mathsf{L}, \mathsf{V})$ gives a second virtual Iwahori decomposition of G, and thus a second induction functor $\mathrm{i}_{\mathsf{U},\mathsf{V}} : \mathcal{R}(\mathsf{L}) \to \mathcal{R}(\mathsf{G})$. Since ULV is not necessarily a subgroup of G, the functor $\mathrm{i}_{\mathsf{U},\mathsf{V}}$ does not a priori admit a decomposition as in (2.13). Now, U^+ is a subgroup of U, and so we have $e_{\mathsf{U}} = e_{\mathsf{U}} e_{\mathsf{U}^+}$, and consequently

$$\mathcal{H}(\mathsf{G}) e_{\mathsf{U}} e_{\mathsf{V}} \subseteq \mathcal{H}(\mathsf{G}) e_{\mathsf{U}^+} e_{\mathsf{V}}. \tag{2.14}$$

Thus $\mathrm{i}_{\mathsf{U},\mathsf{V}}$ is a subfunctor of $\mathrm{i}_{\mathsf{U}^+,\mathsf{V}}$.

In [10, Question 2.15], Dat asks whether there exists a distribution D on G such that $e_{\mathsf{U}^+} e_{\mathsf{V}} = \mathsf{D} e_{\mathsf{U}} e_{\mathsf{V}}$. The existence of such a D would imply that the inclusion (2.14) is in fact an equality, so that the functor $\mathrm{i}_{\mathsf{U},\mathsf{V}}$ is equal to the parahoric induction functor $\mathrm{i}_{\mathsf{U}^+,\mathsf{V}}$. Dat proves that this is indeed the case when $\mathbf{L}$ is a *minimal* Levi subgroup of $\mathbf{G}$. In § 5.5 we shall produce an example where the inclusion (2.14) is proper, showing in particular that there does not always exist a distribution D as above, and the functor $\mathrm{i}_{\mathsf{U},\mathsf{V}}$ is generally a proper subfunctor of the parahoric induction functor $\mathrm{i}_{\mathsf{U}^+,\mathsf{V}}$.

Example 2.15. Suppose that $(\mathsf{U}, \mathsf{L}, \mathsf{V})$ is a virtual Iwahori decomposition of G such that the subgroups U and V commute with one another. Then the product $\mathsf{H} := \mathsf{ULV}$ is an open subgroup of G, isomorphic to $(\mathsf{U} \times \mathsf{V}) \rtimes \mathsf{L}$. We have $e_{\mathsf{U}} e_{\mathsf{V}} = e_{\mathsf{U}\times\mathsf{V}}$, and there are isomorphisms of $\mathcal{H}(\mathsf{G})$-$\mathcal{H}(\mathsf{L})$ bimodules

$$\mathcal{H}(\mathsf{G}) e_{\mathsf{U}} e_{\mathsf{V}} = \mathcal{H}(\mathsf{G}) e_{\mathsf{U}\times\mathsf{V}} \cong \mathcal{H}(\mathsf{G}/(\mathsf{U} \times \mathsf{V})).$$

Consequently the functor $\mathrm{i}_{\mathsf{U},\mathsf{V}}$ is of the form $\mathcal{R}(\mathsf{L}) \xrightarrow{\text{inf}} \mathcal{R}(\mathsf{H}) \xrightarrow{\text{ind}} \mathcal{R}(\mathsf{G})$ discussed in § 2.1.

We shall now establish some basic properties of the functors $\mathrm{i}_{\mathsf{U},\mathsf{V}}$ and $\mathrm{r}_{\mathsf{U},\mathsf{V}}$. Many of these properties were established in [10] for the case where $(\mathsf{U}, \mathsf{L}, \mathsf{V})$ is an actual, as opposed to a virtual, Iwahori decomposition of G. The proofs in [10] mostly carry over with only minor changes to the case of a virtual Iwahori decomposition, thanks to the following analogue of [10, Proposition 2.2]. The proofs of these propositions, though, are quite different.

Proposition 2.16. *Let G be a profinite group and let L, U and V be closed subgroups of G such that L normalises U and V. For every $\mathsf{M} \in \mathcal{R}(\mathsf{G})$ there is a linear automorphism $z_{\mathsf{M}} \in \mathrm{GL}(\mathsf{M})$, commuting with the actions of L, e_{U} and e_{V}, such that $z_{\mathsf{M}}^{-1} e_{\mathsf{U}} e_{\mathsf{V}}$ is an idempotent in $\mathrm{End}(\mathsf{M})$.*

Proof. Each smooth representation $\mathsf{M} \in \mathcal{R}(\mathsf{G})$ may be regarded as a representation of the infinite dihedral group $\Gamma = \langle s, t \mid s^2 = t^2 = 1 \rangle$, by sending $s \mapsto 2e_{\mathsf{U}} - 1$ and $t \mapsto 2e_{\mathsf{V}} - 1$. Since G is profinite, every $\mathsf{M} \in \mathcal{R}(\mathsf{G})$ is isomorphic to a direct sum of finite-dimensional unitary representations of G, which restrict to finite-dimensional unitary representations of Γ (unitary because the idempotents e_{U} and e_{V} are self-adjoint in $\mathcal{H}(\mathsf{G})$). It follows that every $\mathsf{M} \in \mathcal{R}(\mathsf{G})$ is semisimple as a representation of Γ, and so M decomposes (uniquely) as the direct sum of its Γ-isotypic components.

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We claim that in each irreducible representation W of Γ there is a nonzero $z_W \in \mathbb{C}$ such that $z_W^{-1}pq$ is an idempotent in $\mathrm{GL}(W)$, where $p = \frac{1}{2}(s+1)$ and $q = \frac{1}{2}(t+1)$. Indeed, since the dihedral group has an abelian normal subgroup of index two, every irreducible representation of Γ is either one- or two-dimensional. In the one-dimensional case p and q commute and so we may take $z_W = 1$. In the two-dimensional case, pq and $(pq)^2$ are two nonzero maps between the one-dimensional subspaces qW and pW, and so there is a (unique) nonzero scalar z_W such that $pq = z_W^{-1}(pq)^2$.

Having established the claim, we let $z_M \in \mathrm{GL}(M)$ be the automorphism of M which acts as the scalar z_W on the W-isotypical component of M. It is clear from the construction that z_M commutes with e_U and e_V, and that $z_M^{-1}e_Ue_V$ is an idempotent. If $T \in \mathrm{End}(M)$ commutes with e_U and e_V then T preserves the Γ-isotypic components, and so commutes with z_M. In particular, z_M commutes with the L-action on M. □

Remark 2.17. If W is a two-dimensional irreducible unitary representation of the infinite dihedral group, then $z_W = \cos^2(\alpha_W)$, where α_W is the angle between the images of p and q in the Hilbert space W. Thus the eigenvalues of z_M all lie in the interval $(0, 1]$. If the multiplication map $U \times L \times V \to G$ is a homeomorphism, and M is an irreducible representation of G, then z_M is the scalar operator

$$z_M = \begin{cases} \dim r_{U,V}(M)/\dim M & \text{if } r_{U,V}(M) \neq 0, \\ 1 & \text{if } r_{U,V}(M) = 0; \end{cases}$$

see [7, Proposition 1.11]. Moreover, Dat has shown that if L contains an open normal subgroup $L^\dagger$ such that $UL^\dagger V$ is a pro-p subgroup of G, then the eigenvalues lie in $\mathbb{Z}[1/p]$; see [10, Proposition 2.2].

With the automorphisms z_M in hand, many of the arguments from [10, § 2] carry over to our setting, and establish the following properties of the functors $i_{U,V}$ and $r_{U,V}$.

Theorem 2.18. *Let* (U, L, V) *be a virtual Iwahori decomposition of a profinite group* G, *and consider the functors* $i_{U,V}$ *and* $r_{U,V}$. *Then:*

(1) *There are natural isomorphisms* $i_{U,V} \cong i_{V,U}$ *and* $r_{U,V} \cong r_{V,U}$.

(2) $i_{U,V}$ *is naturally isomorphic to the functor*

$$i'_{U,V} : M \mapsto \mathrm{Hom}_{\mathcal{H}(L)}(e_Ve_U\mathcal{H}(G), M)^\infty,$$

and is therefore right-adjoint to $r_{U,V}$.

(3) $r_{U,V}$ *is naturally isomorphic to the functor*

$$r'_{U,V} : N \mapsto \mathrm{Hom}_{\mathcal{H}(G)}(\mathcal{H}(G)e_Ve_U, N)^\infty,$$

and is therefore right-adjoint to $i_{U,V}$.

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(4) *Let* (U', L, V') *be a second virtual Iwahori decomposition of* G*, such that*

$$U = (U \cap U')(U \cap V'), \quad V = (V \cap U')(V \cap V'),$$
$$U' = (U' \cap U)(U' \cap V), \quad \text{and} \quad V' = (V' \cap U)(V' \cap V).$$

Then $i_{U,V} \cong i_{U',V'}$ *and* $r_{U,V} \cong r_{U',V'}$.

(5) *Let* K *be an open normal subgroup of* G *with an Iwahori decomposition* $(U_K, L_K, V_K) := (U \cap K, L \cap K, V \cap K)$. *The diagrams*

$$\begin{array}{ccc} \mathcal{R}(L) & \xrightarrow{i_{U,V}} & \mathcal{R}(G) \\ \uparrow{\scriptstyle \inf} & & \uparrow{\scriptstyle \inf} \\ \mathcal{R}(L/L_K) & \xrightarrow{i_{U/U_K, V/V_K}} & \mathcal{R}(G/K) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{R}(G) & \xrightarrow{r_{U,V}} & \mathcal{R}(L) \\ \uparrow{\scriptstyle \inf} & & \uparrow{\scriptstyle \inf} \\ \mathcal{R}(G/K) & \xrightarrow{r_{U/U_K, V/V_K}} & \mathcal{R}(L/L_K) \end{array}$$

commute up to natural isomorphism. (Here $\inf$ *denotes inflation.)*

(6) $i_{U,V}(M)$ *is nonzero whenever* M *is nonzero, and* $i_{U,V}(M)$ *is finite-dimensional whenever* M *is finite-dimensional.*

(7) *If* (X, H, Y) *is a virtual Iwahori decomposition of* L*, then*

$$i_{U,V} \circ i_{X,Y} \cong i_{U \rtimes X, Y \ltimes V}$$

as functors $\mathcal{R}(H) \to \mathcal{R}(G)$.

Parts (2) and (3) are instances of the following general fact, whose proof generalises the argument of [10, Corollaire 2.7]:

Lemma 2.19. *Let* H *and* K *be closed subgroups of a profinite group* G. *Let* $X \subseteq \mathcal{H}(G)$ *be an* $\mathcal{H}(H)$*-*$\mathcal{H}(K)$ *subbimodule, and denote by* X^* *the image of* X *under the involution* $f^*(g) = \overline{f(g^{-1})}$ *on* $\mathcal{H}(G)$*; note that* X^* *is an* $\mathcal{H}(K)$*-*$\mathcal{H}(H)$ *bimodule. Suppose that for every open normal subgroup* $H_1 \subseteq H$ *there is an open normal subgroup* $G_1 \subseteq G$ *satisfying*

$$e_{H_1} X \subseteq e_{G_1} \mathcal{H}(G). \tag{2.20}$$

Then the functors $\mathcal{R}(K) \to \mathcal{R}(H)$ *defined by*

$$M \mapsto X \otimes_{\mathcal{H}(K)} M \quad \text{and} \quad M \mapsto \mathrm{Hom}_{\mathcal{H}(K)}(X^*, M)^\infty$$

are naturally isomorphic.

Proof. Consider the natural transformation

$$\Phi : X \otimes_{\mathcal{H}(K)} M \to \mathrm{Hom}_{\mathcal{H}(K)}(X^*, M), \quad \Phi(x_1 \otimes m) : x_2^* \mapsto (x_2^* x_1)|_K \cdot m$$

where $(x_2^* x_1)|_K$ means the restriction of the convolution product $x_2^* x_1 \in \mathcal{H}(G)$ to the subgroup K. Fix an open normal subgroup $H_1 \subseteq H$. We show that the map Φ restricts to an isomorphism between the respective subspaces of H_1-fixed vectors.

Let G_1 be an open normal subgroup of G satisfying condition (2.20), so that the subspace $e_{H_1} X \subset \mathcal{H}(G)$ consists exclusively of G_1-invariant functions. We may then

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replace G by G/G_1, and assume for the rest of the proof that G is a finite group. Furthermore, the module M decomposes as a direct sum of finite-dimensional modules, and the natural map Φ commutes with direct sums, so we may assume that M is finite-dimensional.

Now, the pairing

$$X^* \times X \to \mathbb{C} \quad (x_2^*, x_1) \mapsto (x_2^* x_1)(1)$$

is nondegenerate, since it is the restriction to X of the natural L^2-inner product on $\mathcal{H}(G)$. It follows from this, and from the standard duality theory of finite-dimensional vector spaces, that the map

$$\Psi : X \otimes_{\mathbb{C}} M \to \mathrm{Hom}_{\mathbb{C}}(X^*, M) \quad \Psi(x_1 \otimes m) : x_2^* \mapsto (x_2^* x_1)(1) \cdot m$$

is an isomorphism. The map Ψ descends to an isomorphism of K-coinvariants

$$X \otimes_{\mathcal{H}(K)} M \xrightarrow{\cong} (\mathrm{Hom}_{\mathbb{C}}(X^*, M))_K, \tag{2.21}$$

where the K-action on $\mathrm{Hom}_{\mathbb{C}}(X^*, M)$ is by conjugation. Averaging over K gives an isomorphism of K-coinvariants with K-invariants:

$$(\mathrm{Hom}_{\mathbb{C}}(X^*, M))_K \xrightarrow[\cong]{T \mapsto \int_K kTk^{-1}\, dk} \mathrm{Hom}_{\mathcal{H}(K)}(X^*, M), \tag{2.22}$$

and the map Φ is the composition of the isomorphisms (2.21) and (2.22). □

Proof of Theorem 2.18. To prove part (1), let $z_{\mathcal{H}(G)}$ be the automorphism of $\mathcal{H}(G)$ obtained by applying Proposition 2.16 to the left-translation action of G. Then the maps

$$e_U e_V \mathcal{H}(G) \xrightarrow{f \mapsto e_V f} e_V e_U \mathcal{H}(G) \quad \text{and} \quad e_V e_U \mathcal{H}(G) \xrightarrow{f \mapsto z_{\mathcal{H}(G)}^{-1} e_U f} e_U e_V \mathcal{H}(G)$$

are mutually inverse isomorphisms of $\mathcal{H}(L)$-$\mathcal{H}(G)$ bimodules, giving rise to a natural isomorphism of functors $r_{U,V} \cong r_{V,U}$. A similar argument, using the right action of G on $\mathcal{H}(G)$, gives $i_{U,V} \cong i_{V,U}$.

To prove part (2) we apply Lemma 2.19 with $H = G$, $K = L$, and $X = \mathcal{H}(G) e_U e_V$. The hypothesis (2.20) is trivially satisfied and we conclude that the functor $i_{U,V}$ is naturally isomorphic to $i'_{U,V}$. The standard $\otimes$-Hom adjunction implies that $i'_{U,V}$ is right-adjoint to $r_{V,U}$, and we have $r_{V,U} \cong r_{U,V}$ by part (1); see [38, I.2.2 (Corollaire)] for a formulation and proof of the adjunction in the present context.

To prove part (3) we apply Lemma 2.19 again, this time with $H = L$, $K = G$ and $X = e_U e_V \mathcal{H}(G)$. To verify the hypothesis (2.20), fix an open normal subgroup $H_1 \subseteq L$. Then there is an open normal subgroup $G_0 \subseteq G$ having an Iwahori decomposition (U_0, L_0, V_0), where L_0 is contained in H_1. Here Y_0 means $Y \cap G_0$ for every subset $Y \subset G$. We then have

$$e_{H_1} e_U e_V \mathcal{H}(G) \subseteq e_{L_0} e_U e_V \mathcal{H}(G) = e_U (e_{U_0} e_{L_0} e_{V_0}) e_V \mathcal{H}(G) \subseteq e_{G_0} \mathcal{H}(G),$$

so (2.20) is satisfied by $G_1 = G_0$. Now Lemma 2.19 implies that $r_{U,V}$ is isomorphic to $r'_{U,V}$, which is right-adjoint to $i_{U,V}$ by the argument of part (2) .

Part (4) follows from Proposition 2.16, as in [10, Lemme 2.9].

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Part (5) follows from the equality $e_K = e_{U_K} e_{L_K} e_{V_K}$, as was remarked in [10, p. 272]. It is also a consequence of Theorem 3.4, below.

The finite-dimensionality assertion in part (6) follows from part (5) : every finite-dimensional smooth representation of L is inflated from a representation of some finite quotient L/L_K, and the functor $i_{U/U_K, V/V_K}$ obviously preserves finite dimensionality. To prove that $i_{U,V}(M) \neq 0$ as long as $M \neq 0$, fix a nonzero $m \in M$ and let $f \in \mathrm{ind}_{LU}^{G}(M)$ be the function supported on the open set $ULV \subset G$, and given there by $f(ulv) = l \cdot m$. The image of f under the intertwiner $J_V : \mathrm{ind}_{LU}^{G}(M) \to \mathrm{ind}_{LV}^{G}(M)$ (see (2.9)) is nonzero, because

$$(J_V f)(1) = \int_V f(v)\, dv = m,$$

and so $i_{U,V}(M) \cong \mathrm{Im}(J_V)$ is nonzero.

For part (7), convolution over L gives an isomorphism of $\mathcal{H}(G)$-$\mathcal{H}(H)$ bimodules

$$\mathcal{H}(G) e_U e_V \otimes_{\mathcal{H}(L)} \mathcal{H}(L) e_X e_Y \xrightarrow{\cong} \mathcal{H}(G) e_U e_V e_X e_Y.$$

Now e_X commutes with e_V since X normalises V, and we have $e_U e_X = e_{U \rtimes X}$ and $e_V e_Y = e_{Y \ltimes V}$, and so we have produced an isomorphism between the bimodules representing the functors $i_{U,V} \circ i_{X,Y}$ and $i_{U \rtimes X, Y \ltimes V}$. □

If the triple (U, L, V) is an actual Iwahori decomposition of G, then the functors $i_{U,V}$ and $r_{U,V}$ enjoy the following additional properties, which we recall from [10] and [7] for the reader's convenience:

Theorem 2.23. *Suppose that* (U, L, V) *is an Iwahori decomposition of a profinite group* G*. Then:*

(1) $i_{U,V}$ *sends* $\mathrm{Irr}(L)$ *to* $\mathrm{Irr}(G)$*, while* $r_{U,V}$ *sends* $\mathrm{Irr}(G)$ *to* $\mathrm{Irr}(L) \sqcup \{0\}$.

(2) $r_{U,V} \circ i_{U,V} \cong \mathrm{id}_{\mathcal{R}(L)}$.

(3) *If* $M \in \mathrm{Irr}(G)$ *and* $r_{U,V}(M) \neq 0$*, then* $i_{U,V} r_{U,V}(M) \cong M$.

(4) *If* $M \in \mathrm{Irr}(G)$*, then either* $\mathrm{Hom}_L(M^U, M^V)$ *is zero, in which case* $r_{U,V}(M) = 0$*; or* $\mathrm{Hom}_L(M^U, M^V)$ *is one-dimensional, in which case it is spanned by the operator* $e_V e_U$*, which is an isomorphism, and* $r_{U,V}(M) \cong M^U \cong M^V$.

(5) *Given* $M \in \mathrm{Irr}(G)$ *and* $N \in \mathrm{Irr}(L)$*, one has* $M \cong i_{U,V}(N)$ *if and only if* N *is a common subrepresentation of* M^U *and* M^V.

(6) *For each* $(\varphi, M) \in \mathrm{Irr}(G)$ *there is a nonzero scalar* c *such that*

$$e_U e_\varphi e_V = c e_U e_{r_{U,V}(\varphi)} e_V$$

as operators on $\mathcal{H}(G)$.

Proof. Parts (1) and (2) are proved in [10, Corollaire 2.10]. Part (4) is proved in [7, Lemma 1.10], and part (5) follows from parts (4) and (3).

To prove part (3) : if $r_{U,V}(M) \neq 0$ then the adjunction in part (2) of Theorem 2.18 gives a nonzero intertwiner $M \to i_{U,V} r_{U,V}(M)$. Both of these representations are irreducible (by part (1)), and so they are isomorphic.

Part (6) follows from the character formula for $r_{U,V}$ proved in [7, Proposition 1.11]. That formula implies that there is a nonzero $s \in \mathbb{C}$ such that

$$\mathrm{ch}_{r_{U,V}(\varphi)}(l) = s \int_U \int_V \mathrm{ch}_\varphi(vlu)\, du\, dv$$

for all $l \in L$. Writing $\sim$ to indicate equality up to a nonzero scalar multiple, the operator $e_U e_{r_{U,V}(\varphi)} e_V$ is thus given by

$$\begin{aligned} e_U e_{r_{U,V}(\varphi)} e_V &\sim \int_U \int_L \int_V \mathrm{ch}_{r_{U,V}(\varphi)}(l^{-1}) ulv\, du\, dl\, dv \\ &\sim \int_U \int_L \int_V \left(\int_U \int_V \mathrm{ch}_\varphi(v_1^{-1} l^{-1} u_1^{-1})\, du_1 \;\; dv_1 \right) ulv\, du\, dl\, dv \\ &\sim \int_U \int_L \int_V \left(\int_U \int_V \mathrm{ch}_\varphi(v_1^{-1} l^{-1} u_1^{-1})\, du_1 \;\; dv_1 \right) u(u_1 l v_1)v\, du\, dl\, dv \\ &\sim \int_U \int_G \int_V \mathrm{ch}_\varphi(g^{-1}) ugv\, du\, dg\, dv \\ &\sim e_U e_\varphi e_V \end{aligned}$$

where in the third step we used invariance of the Haar measures, and in fourth we used the fact that the product of the Haar measures on U, L and V is a Haar measure on $G = ULV$. □

3. Relations of the functors i and r with Clifford theory

The functors $i_{U,V}$ and $r_{U,V}$ can be difficult to work with, since the bimodule $\mathcal{H}(G)e_U e_V$ is not obviously the space of functions on any nice $G \times L$-space. The situation where (U, L, V) is an actual, rather than a virtual, Iwahori decomposition of G is significantly easier to deal with; see §§ 4 and 6, for instance. If G admits only a virtual Iwahori decomposition, then G contains an open normal subgroup G_0 which admits an actual Iwahori decomposition (this is part of the definition). In this section we firstly recall how Clifford theory reduces the study of representations of G to that of projective representations of certain subgroups of G/G_0; and then we show how the induction and restriction functors i and r are compatible with this reduction.

3.1. Review of Clifford theory

Let us recall the basic assertions of Clifford theory. Details can be found in [28], for example.

Let G_0 be an open normal subgroup of a profinite group G. Then G acts by conjugation on the set $\mathrm{Irr}(G_0)$ of isomorphism classes of irreducible representations of G_0. For each $\varphi \in \mathrm{Irr}(G_0)$ we let $\mathcal{R}(G)_\varphi$ denote the category of smooth representations of G whose restriction to G_0 contains only representations in the G-orbit $G \cdot \varphi$. The first assertion of Clifford theory is:

$$\mathcal{R}(G) \text{ is equivalent to the product } \prod_{G\cdot\varphi \in G \backslash \mathrm{Irr}(G_0)} \mathcal{R}(G)_\varphi. \tag{C1}$$

Fix an irreducible representation $\varphi : G_0 \to \mathrm{GL}(W)$ of G_0, and let $G(\varphi)$ denote the stabiliser of φ in G. Since φ is smooth, it is trivial on some open normal subgroup G_{00} of G, and replacing G by G/G_{00} we might as well assume—as we shall, for the rest of § 3.1—that G is finite. We use the counting measure on G to define the convolution on $\mathcal{H}(G)$, so that the δ-functions δ_g satisfy $\delta_g\delta_h = \delta_{gh}$.

Representations may be induced from $G(\varphi)$ to G in the usual way (see § 2.1). The second assertion of Clifford theory is:

$$\text{The functor } \operatorname{ind} : \mathcal{R}(G(\varphi))_\varphi \to \mathcal{R}(G)_\varphi \text{ is an equivalence of categories.} \tag{C2}$$

An inverse is given by the functor which sends a representation $M \in \mathcal{R}(G)_\varphi$ to its $G(\varphi)$-subspace $e_\varphi M$, where $e_\varphi \in \mathcal{H}(G_0)$ is the central idempotent associated to φ. Note that the category $\mathcal{R}(G(\varphi))_\varphi$ is equivalent, in an obvious way, to the category of modules over the direct-summand $e_\varphi\mathcal{H}(G(\varphi))$ of the algebra $\mathcal{H}(G(\varphi))$.

Let $\overline{G}$ denote the quotient G/G_0, and let $\theta : G \to \overline{G}$ be the quotient map. Schur's lemma implies that φ admits a *projective extension* to $G(\varphi)$, i.e. a map $\varphi' : G(\varphi) \to \mathrm{GL}(W)$ which becomes a group homomorphism upon passing to the quotient $\mathrm{PGL}(W)$, and which satisfies

$$\varphi'(g_0 g) = \varphi(g_0)\varphi'(g) \quad \text{and} \quad \varphi'(g g_0) = \varphi'(g)\varphi(g_0)$$

for all $g \in G(\varphi)$ and all $g_0 \in G_0$. These two properties imply that there is a two-cocycle $\gamma : \overline{G}(\varphi) \times \overline{G}(\varphi) \to \mathbb{C}^\times$ whose inflation to a cocycle on $G(\varphi)$, which we also denote by γ, satisfies

$$\varphi'(g_1)\varphi'(g_2) = \gamma(g_1, g_2)^{-1}\varphi'(g_1 g_2). \tag{3.1}$$

We call γ^{-1} the two-cocycle associated to φ'; the cocycles associated to different choices of projective extensions of φ are cohomologous. The projective representation φ' may be regarded as a module over the twisted group algebra $\mathcal{H}^{\gamma^{-1}}(G(\varphi))$, a construction that we now recall. To a two-cocycle α on a finite group Γ, one may associate the *twisted group algebra* $\mathcal{H}^\alpha(\Gamma)$, that is, the algebra of complex-valued functions on G, with twisted convolution multiplication $\cdot_\alpha$ defined on the basis $\{\delta_g \mid g \in \Gamma\}$ of δ-functions by

$$\delta_{g_1} \cdot_\alpha \delta_{g_2} := \alpha(g_1, g_2)\delta_{g_1 g_2}.$$

We let $\mathcal{R}^\alpha(\Gamma)$ denote the category of $\mathcal{H}^\alpha(\Gamma)$-modules. An immediate consequence of the definition is that if $\alpha, \beta : \Gamma \times \Gamma \to \mathbb{C}^\times$ are two-cocycles, and M and N are modules over $\mathcal{H}^\alpha(\Gamma)$ and $\mathcal{H}^\beta(\Gamma)$ respectively, then $M \otimes N$ is naturally an $\mathcal{H}^{\alpha\beta}(\Gamma)$-module with respect to the diagonal action.

Returning to our setup, if M is an $\mathcal{H}^\gamma(\overline{G}(\varphi))$-module, then by inflation M is also an $\mathcal{H}^\gamma(G(\varphi))$-module. As $W \in \mathcal{R}^{\gamma^{-1}}(G(\varphi))$ we get that $M \otimes W$ is an $\mathcal{H}^{\gamma\cdot\gamma^{-1}}(\overline{G}(\varphi))$-module, i.e. an ordinary (as opposed to a projective) representation of $G(\varphi)$, and the third assertion of Clifford theory is:

$$\text{The functor } \otimes\varphi' : \mathcal{R}^\gamma(\overline{G}(\varphi)) \xrightarrow{M \mapsto M\otimes W} \mathcal{R}(G(\varphi))_\varphi \text{ is an equivalence of categories.} \tag{C3}$$

An equivalent formulation of (C3), which we shall use below, is that the map

$$\theta \otimes \varphi' : \mathcal{H}(G(\varphi))e_\varphi \to \mathcal{H}^\gamma(\overline{G}(\varphi)) \otimes_{\mathbb{C}} \mathrm{End}(W) \quad \delta_g e_\varphi \mapsto \delta_{\theta(g)} \otimes \varphi'(g)$$

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is an isomorphism of algebras. Then the equivalence (C3) decomposes as

$$\mathcal{R}^{\gamma}(\overline{G}(\varphi)) \xrightarrow[\cong]{M \mapsto M\otimes W} \operatorname{Mod}(\mathcal{H}^{\gamma}(\overline{G}(\varphi)) \otimes \operatorname{End}(W)) \xrightarrow[\cong]{(\theta\otimes\varphi')^*} \mathcal{R}(G(\varphi))_{\varphi}. \tag{3.2}$$

Here Mod (R) denotes the category of left R-modules.

This ends our review of Clifford theory. We shall now explain the compatibility with the functors i and r.

3.2. Induction and Clifford theory

In this section we let G be a profinite group, with a virtual Iwahori decomposition (U, L, V) as in Definition 2.1. We also fix one open normal subgroup $G_0 \subset G$ for which the product mapping

$$U_0 \times L_0 \times V_0 \to G_0$$

is a homeomorphism, where $H_0 := H \cap G_0$ for every subgroup $H \subseteq G$. We consider the induction functors

$$i = i_{U,V} : \mathcal{R}(L) \to \mathcal{R}(G) \quad \text{and} \quad i_0 = i_{U_0,V_0} : \mathcal{R}(L_0) \to \mathcal{R}(G_0),$$

along with their adjoint restriction functors r and r_0, as in Definition 2.7.

It follows from part (1) of Theorem 2.23 that the functor i_0 sends irreducible representations of L_0 to irreducible representations of G_0, and thus produces a map from Irr (L_0) to Irr (G_0).

Lemma 3.3. *The map* $i_0 : \operatorname{Irr}(L_0) \to \operatorname{Irr}(G_0)$ *is* L*-equivariant and injective.*

Proof. L normalises U_0 and V_0, and so commutes with e_{U_0} and e_{V_0}. The injectivity follows from part (2) of Theorem 2.23. □

The functors i and r are compatible with the decomposition (C1), in the following sense.

Theorem 3.4. *With the above notation one has* $i(\mathcal{R}(L)_{\psi}) \subseteq \mathcal{R}(G)_{i_0(\psi)}$, *for every* $\psi \in \operatorname{Irr}(L_0)$.

Proof. We first claim that $\mathcal{H}(G)e_U e_V$ is isomorphic, as an $\mathcal{H}(G)$-$\mathcal{H}(L)$ bimodule, to some submodule of $\mathcal{H}(G)e_{U_0}e_{V_0}$. This is because

$$\mathcal{H}(G)e_U e_V \subseteq \mathcal{H}(G)e_{U_0}e_V \cong \mathcal{H}(G)e_V e_{U_0} \subseteq \mathcal{H}(G)e_{V_0}e_{U_0} \cong \mathcal{H}(G)e_{U_0}e_{V_0},$$

where the inclusions hold because U_0 and V_0 are subgroups of U and V, respectively, and the isomorphisms hold by part (1) of Theorem 2.18.

For each $N \in \mathcal{R}(L)_{\psi}$ we now have (up to G-equivariant isomorphism)

$$i(N) \subseteq \mathcal{H}(G)e_{U_0}e_{V_0} \otimes_{\mathcal{H}(L)} N \subseteq \mathcal{H}(G)e_{U_0}e_{V_0} \otimes_{\mathcal{H}(L_0)} \operatorname{res}^L_{L_0}(N) \tag{3.5}$$

where the first inclusion holds because of the inclusion of bimodules established above, and the second inclusion holds because the tensor product over $\mathcal{H}(L)$ is a quotient—and therefore also a submodule—of the tensor product over the subalgebra $\mathcal{H}(L_0)$. The

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restriction to G_0 of the right-hand side in (3.5) is a direct sum of G-conjugates of $i_0(\operatorname{res}^L_{L_0}(N))$, where $\operatorname{res}^L_{L_0}(N)$ is a direct sum of L-conjugates of ψ. Since i_0 is L-equivariant this implies that the restriction of $i(N)$ to G_0 is a direct sum of G-conjugates of $i_0(\psi)$, as claimed. □

For the rest of this section we shall fix an irreducible representation $\psi : L_0 \to GL(W)$ of L_0, and study the functor i on the subcategory $\mathcal{R}(L)_\psi$. There is an open normal subgroup $G_{00} \subset G$ with an Iwahori decomposition $G_{00} = U_{00}L_{00}V_{00}$, such that ψ is trivial on L_{00}. Part (5) of Theorem 2.18 allows us to replace G by the quotient G/G_{00}, and so we may assume without loss of generality for the rest of this section that **G is a finite group**. We consequently take all Haar measures to be counting measures, so that the δ functions on elements of G satisfy $\delta_g\delta_h = \delta_{gh}$ inside $\mathcal{H}(G)$.

To simplify the notation let us write φ for $i_0(\psi)$. Lemma 3.3 implies that $G(\varphi) \cap L = L(\psi)$. Let $U(\varphi)$ and $V(\varphi)$ denote the inertia groups of φ in U and V, respectively, and consider the functor

$$i_\psi := i_{U(\varphi),V(\varphi)} : \mathcal{R}(L(\psi))_\psi \to \mathcal{R}(G(\varphi))_\varphi$$

given by tensor product with the $e_\varphi\mathcal{H}(G(\varphi))$-$e_\psi\mathcal{H}(L(\psi))$ bimodule $e_\varphi\mathcal{H}(G(\varphi))e_{U(\varphi)}e_{V(\varphi)}e_\psi$.

The functors i and r are compatible with the equivalence (C2) as follows:

Theorem 3.6. *The diagram*

$$\begin{array}{ccc} \mathcal{R}(L)_\psi & \xrightarrow{\quad i \quad} & \mathcal{R}(G)_\varphi \\ {\scriptstyle \text{ind}}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \text{ind}} \\ \mathcal{R}(L(\psi))_\psi & \xrightarrow{\quad i_\psi \quad} & \mathcal{R}(G(\varphi))_\varphi \end{array}$$

commutes up to a natural isomorphism.

Proof. We replace the right-hand vertical arrow in the diagram by its inverse, and prove that the diagram

$$\begin{array}{ccc} \mathcal{R}(L)_\psi & \xrightarrow{\quad i \quad} & \mathcal{R}(G)_\varphi \\ {\scriptstyle \text{ind}}\uparrow & & \downarrow{\scriptstyle e_\varphi} \\ \mathcal{R}(L(\psi))_\psi & \xrightarrow{\quad i_\psi \quad} & \mathcal{R}(G(\varphi))_\varphi \end{array} \tag{3.7}$$

commutes up to natural isomorphism. This amounts to producing an isomorphism of $G(\varphi)$-$L(\psi)$ bimodules

$$e_\varphi\mathcal{H}(G)e_U e_V e_\psi \cong e_\varphi\mathcal{H}(G(\varphi))e_{U(\varphi)}e_{V(\varphi)}e_\psi. \tag{3.8}$$

We first claim that

$$e_\varphi\mathcal{H}(G)e_U e_V e_\psi = e_\varphi\mathcal{H}(G(\varphi))e_{U(\varphi)}e_V e_\psi, \tag{3.9}$$

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the equality holding inside $\mathcal{H}(G)$. For vectors x and y we write $x \sim y$ if they differ by a nonzero scalar multiple. To prove (3.9) we compute for every $g \in G$:

$$\begin{aligned}\delta_g e_U e_V e_\psi &= \delta_g e_U (e_{U_0} e_\psi e_{V_0}) e_V e_\psi \sim \delta_g e_U (e_{U_0} e_\varphi e_{V_0}) e_V e_\psi \\ &= \delta_g e_U e_\varphi e_V e_\psi \sim \sum_{u \in U/U(\varphi)} \delta_{gu} e_\varphi e_{U(\varphi)} e_V e_\psi,\end{aligned}$$

where in the first step we have used that e_ψ is an idempotent which commutes with e_V, and in the second step we have used part (6) of Theorem 2.23. Orthogonality of characters then implies

$$\begin{aligned}e_\varphi \delta_g e_U e_V e_\psi &\sim \sum_{u \in U/U(\varphi)} e_\varphi e_{(gu)\cdot\varphi} \delta_{gu} e_{U(\varphi)} e_V e_\psi \\ &\sim \begin{cases} e_\varphi \delta_{gu} e_{U(\varphi)} e_V e_\psi & \text{if } \exists u \in U \text{ with } gu \in G(\varphi), \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

This shows that every element of the left-hand side of (3.9) can be written as an element of the right-hand side, and vice versa.

Now, the $G(\varphi) \times L(\psi)$-equivariant map

$$e_\varphi \mathcal{H}(G(\varphi)) e_{U(\varphi)} e_{V(\varphi)} e_\psi \xrightarrow{f \mapsto f e_V} e_\varphi \mathcal{H}(G(\varphi)) e_{U(\varphi)} e_V e_\psi \tag{3.10}$$

is obviously surjective. It is injective as well, for if $f \in \mathcal{H}(G(\varphi)) e_{V(\varphi)}$ then

$$f e_V \sim \sum_{v \in V(\varphi)\backslash V} f e_{V(\varphi)} \delta_v = \sum_{v \in V(\varphi)\backslash V} f \delta_v,$$

where the functions $f\delta_v$ (as v varies over $V(\varphi)\backslash V$) are supported on the disjoint cosets $G(\varphi)v$ and are therefore linearly independent. Thus the map (3.10) is an isomorphism. Composing with the equality (3.9) gives the desired isomorphism (3.8). □

We are still fixing an irreducible representation $\psi : L_0 \to \mathrm{GL}(W)$ and letting φ denote the induced representation $i_0(\psi) : G_0 \to \mathrm{GL}(i_0(W))$. Consider the quotients

$$\overline{G}(\varphi) = G(\varphi)/G_0, \quad \overline{L}(\psi) = L(\psi)/L_0, \quad \overline{U}(\varphi) = U(\varphi)/U_0, \quad \overline{V}(\varphi) = V(\varphi)/V_0.$$

Choose a projective extension φ' of φ to $G(\varphi)$, and let γ^{-1} be the associated two-cocycle as in (3.1). Part (2) of Theorem 2.23 implies that there is an L_0-equivariant isomorphism

$$\Theta : W \xrightarrow{\cong} \varphi(e_{U_0})\varphi(e_{V_0})\, i_0(W),$$

unique up to a nonzero scalar multiple. Since the subgroup $L(\psi)$ normalises the subgroups U_0 and V_0 it follows that $\varphi'(l)$ commutes with $\varphi(e_{U_0})\varphi(e_{V_0})$ for every $l \in L(\psi)$, and therefore $\varphi'(l)$ stabilises the subspace $\varphi(e_{U_0})\varphi(e_{V_0})\, i_0(W) \subset i_0(W)$. The map $\psi' : L(\psi) \to \mathrm{GL}(W)$ defined by

$$\psi'(l) = \Theta^{-1} \varphi'(l) \Theta \tag{3.11}$$

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is then a projective extension of ψ to $L(\psi)$, independent of the choice of Θ. (However, it does depend on the choice of φ'.) An easy argument shows that the resulting two-cocycle on $\overline{L}(\psi)$ is just the restriction of the two-cocycle γ to $\overline{L}(\psi)$, and we shall therefore denote both two-cocycles by the same letter.

Lemma 3.12. *Given a projective extension φ' of φ as above, there are unique scalars $a_x, b_y \in \mathbb{C}^\times$ for $x \in \overline{U}(\varphi)$ and $y \in \overline{V}(\varphi)$ such that the elements*

$$e^{\varphi'}_{\overline{U}(\varphi)} := \frac{1}{|\overline{U}(\varphi)|} \sum_{x \in \overline{U}(\varphi)} a_x \delta_x \quad \textit{and} \quad e^{\varphi'}_{\overline{V}(\varphi)} := \frac{1}{|\overline{V}(\varphi)|} \sum_{y \in \overline{V}(\varphi)} b_y \delta_y$$

are idempotents in $\mathcal{H}^\gamma(\overline{G}(\varphi))$, commute with the subalgebra $\mathcal{H}^\gamma(\overline{L}(\psi))$, and such that the image of the elements $e_{U(\varphi)}e_\varphi$ and $e_{V(\varphi)}e_\varphi$ under the isomorphism of algebras

$$\theta \otimes \varphi' : \mathcal{H}(G(\varphi))e_\varphi \to \mathcal{H}^\gamma(\overline{G}(\varphi)) \otimes \operatorname{End}(i_0(W))$$

are $e^{\varphi'}_{\overline{U}(\varphi)} \otimes \varphi(e_{U_0})$ and $e^{\varphi'}_{\overline{V}(\varphi)} \otimes \varphi(e_{V_0})$, respectively.

Proof. We prove the lemma for the U-subgroups. The proof for the V-subgroups is identical.

We know, by Parts (2) and (4) of Theorem 2.23, that $\varphi(e_{U_0})\, i_0(W) \cong W$ as representations of L_0, and in particular that $\varphi(e_{U_0})\, i_0(W)$ is irreducible over L_0.

Let $u \in U(\varphi)$. Since U_0 is a normal subgroup of $U(\varphi)$, we know that u commutes with e_{U_0} in $\mathcal{H}(G(\varphi))$, and so $\varphi'(u)$ induces a linear automorphism of the subspace $\varphi(e_{U_0})\, i_0(W)$. Moreover, for $l \in L_0$ we have that $ulu^{-1}l^{-1} \in U \cap G_0 = U_0$. Writing $ul = luu_0$ with $u_0 \in U_0$, we observe that on $\varphi(e_{U_0})\, i_0(W)$ the operator $\varphi'(u)$ commutes with $\varphi(l)$ for every $l \in L_0$. Hence, by Schur's lemma, the operator $\varphi'(u)$ acts on $\varphi(e_{U_0})\, i_0(W)$ as a nonzero scalar $a_u \in \mathbb{C}^\times$. The scalar a_u depends only on the class of u in the quotient $\overline{U}(\varphi) = U(\varphi)/U_0$, and so for each $x \in \overline{U}(\varphi)$ we may define $a_x := a_{\tilde{x}}$, where $\tilde{x} \in U(\varphi)$ is any lift of x.

The image of the idempotent $e_{U(\varphi)}e_\varphi$ under $\theta \otimes \varphi'$ is therefore

$$\begin{aligned}
\theta \otimes \varphi'(e_{U(\varphi)}e_\varphi) &= \frac{1}{|U(\varphi)|} \sum_{u \in U(\varphi)} \theta(\delta_u) \otimes \varphi'(u) \\
&= \frac{1}{|\overline{U}(\varphi)| \cdot |U_0|} \sum_{\substack{x \in \overline{U}(\varphi) \\ u_0 \in U_0}} \delta_x \otimes \varphi'(\tilde{x})\varphi(u_0) \\
&= \frac{1}{|\overline{U}(\varphi)|} \sum_{x \in \overline{U}(\varphi)} \delta_x \otimes \varphi'(\tilde{x})\varphi(e_{U_0}) = e^{\varphi'}_{\overline{U}(\varphi)} \otimes \varphi(e_{U_0}).
\end{aligned}$$

From the fact that the element $e_{U(\varphi)}e_\varphi$ is an idempotent which commutes with the elements of $L(\psi)$ in $\mathcal{H}(G(\varphi))$, and the fact that $\theta \otimes \varphi'$ is an algebra homomorphism, it follows immediately that $e^{\varphi'}_{\overline{U}(\varphi)}$ is an idempotent which commutes with the subalgebra $\mathcal{H}^\gamma(\overline{L}(\psi))$.

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Finally, the uniqueness of the scalars a_x follows from the linear independence of the elements $\delta_x \otimes \varphi(e_{U_0})$ in $\mathcal{H}^\gamma(\overline{G}(\varphi)) \otimes \operatorname{End}(\mathrm{i}_0(W))$. $\square$

Remark 3.13. From the proof of Lemma 3.12 it follows that the restriction of γ to $\overline{U}(\varphi)$ and to $\overline{V}(\varphi)$ is cohomologous to the trivial two-cocycle. Indeed, following the proof of the lemma, we see that for $x_1, x_2 \in \overline{U}(\varphi)$ one has $a_{x_1} a_{x_2} \gamma(x_1, x_2) = a_{x_1 x_2}$. In other words, $a : \overline{U}(\varphi) \to \mathbb{C}^\times$ is a coboundary which provides a trivialisation of the restriction of γ to $\overline{U}(\varphi)$. The same is true for $\overline{V}(\varphi)$ and b. By changing the choice of φ' by a suitable coboundary, one can therefore arrange that all the numbers a_x and b_y are 1. We shall continue to work with an arbitrary choice of φ' in what follows.

Lemma 3.12 implies that $\mathcal{H}^\gamma(\overline{G}(\varphi)) e^{\varphi'}_{\overline{U}(\varphi)} e^{\varphi'}_{\overline{V}(\varphi)}$ is an $\mathcal{H}^\gamma(\overline{G}(\varphi))$-$\mathcal{H}^\gamma(\overline{L}(\psi))$ bimodule. We denote by $\mathrm{i}^{\varphi'}_{\overline{U}(\varphi),\overline{V}(\varphi)} : \mathcal{R}^\gamma(\overline{L}(\psi)) \to \mathcal{R}^\gamma(\overline{G}(\varphi))$ the functor of tensor product with this bimodule. Likewise, we denote by $\mathrm{r}^{\varphi'}_{\overline{U}(\varphi),\overline{V}(\varphi)}$ the functor of tensor product with the $\mathcal{H}^\gamma(\overline{L}(\psi))$-$\mathcal{H}^\gamma(\overline{G}(\varphi))$ bimodule $e^{\varphi'}_{\overline{U}(\varphi)} e^{\varphi'}_{\overline{V}(\varphi)} \mathcal{H}^\gamma(\overline{G}(\varphi))$. The arguments of Theorem 2.18 carry over to this twisted setting, and show that the functors $\mathrm{i}^{\varphi'}_{\overline{U}(\varphi),\overline{V}(\varphi)}$ and $\mathrm{r}^{\varphi'}_{\overline{U}(\varphi),\overline{V}(\varphi)}$ are two-sided adjoints, and that up to natural isomorphism they do not depend on the order of $\overline{U}(\varphi)$ and $\overline{V}(\varphi)$.

Our induction functors are compatible with the final assertion (C3) of Clifford theory, as follows:

Theorem 3.14. *Let φ' be a projective extension of φ, with corresponding cocycle γ^{-1}, and let ψ' be the projective extension of ψ defined by (3.11). The diagram*

$$\begin{array}{ccc}
\mathcal{R}(L(\psi))_\psi & \xrightarrow{\ \mathrm{i}_\psi\ } & \mathcal{R}(G(\varphi))_\varphi \\
{\scriptstyle \otimes\psi'} \uparrow {\scriptstyle \cong} & & {\scriptstyle \cong} \uparrow {\scriptstyle \otimes\varphi'} \\
\mathcal{R}^\gamma(\overline{L}(\psi)) & \xrightarrow{\ \mathrm{i}^{\varphi'}_{\overline{U}(\varphi),\overline{V}(\varphi)}\ } & \mathcal{R}^\gamma(\overline{G}(\varphi))
\end{array}$$

is commutative up to natural isomorphism.

Example 3.15. Suppose that the irreducible representation ψ of L_0 satisfies $G(\mathrm{i}_0\,\psi) = L(\psi)G_0$. We then have $\overline{G}(\varphi) = \overline{L}(\psi)$ in Theorem 3.14, and $\mathrm{i}^{\varphi'}_{\overline{U}(\varphi),\overline{V}(\varphi)}$ is the identity functor, and so we conclude from Theorems 3.6 and 3.14 that in this case

$$\mathrm{i}_{U,V} : \mathcal{R}(L)_\psi \to \mathcal{R}(G)_{\mathrm{i}_0(\psi)}$$

is an equivalence of categories. Specific examples of this kind arise in § 5.

The proof of Theorem 3.14 uses the following lemma, whose proof is a matter of straightforward linear algebra:

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Lemma 3.16. *Let* E *be a finite-dimensional vector space over* $\mathbb{C}$, *and let* $S : E \to E$ *be a linear endomorphism. Let* $A \subseteq \mathrm{End}_{\mathbb{C}}(E)$ *be the centraliser of* S *in* $\mathrm{End}_{\mathbb{C}}(E)$. *Then we have isomorphisms*

$$E \otimes (\mathrm{Im}(S))^* \xrightarrow{\ e\otimes f \longmapsto [e' \mapsto e f(e')]\ } \mathrm{Hom}(\mathrm{Im}(S), E) \xrightarrow{\ T \mapsto T\circ S\ } \mathrm{End}(E)S$$

of $\mathrm{End}(E)$-A-*bimodules.* □

In our application the space E will be $i_0(W)$, and the endomorphism S will be the action of $e_{U_0} e_{V_0} e_{\psi}$.

Proof of Theorem 3.14. Consider the functor

$$T : \mathrm{Mod}(\mathcal{H}^{\gamma}(\overline{L}(\psi)) \otimes \mathrm{End}(W))) \to \mathrm{Mod}(\mathcal{H}^{\gamma}(\overline{G}(\varphi)) \otimes \mathrm{End}(i_0(W))$$

of tensor product with the bimodule $\mathcal{H}^{\gamma}(\overline{G}(\varphi)) e^{\varphi'}_{\overline{U}(\varphi)} e^{\varphi'}_{\overline{V}(\varphi)} \otimes_{\mathbb{C}} (i_0(W) \otimes_{\mathbb{C}} W^*)$, where $i_0(W)$ is viewed as a left $\mathrm{End}(i_0(W))$ module and W^* is viewed as a right $\mathrm{End}(W)$-module in the obvious way.

We shall decompose the equivalences $\otimes\psi'$ and $\otimes\varphi'$ into compositions of two equivalences, as in (3.2), and show both squares in the diagram

$$\begin{array}{ccc}
\mathcal{R}(L(\psi))_{\psi} & \xrightarrow{\quad i_{\psi}\quad} & \mathcal{R}(G(\varphi))_{\varphi} \\
{\scriptstyle (\theta\otimes\psi')^*}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle (\theta\otimes\varphi')^*} \\
\mathrm{Mod}(\mathcal{H}^{\beta}(\overline{L}(\psi)) \otimes \mathrm{End}(W)) & \xrightarrow{\quad T\quad} & \mathrm{Mod}(\mathcal{H}^{\gamma}(\overline{G}(\varphi)) \otimes \mathrm{End}(i_0(W))) \\
{\scriptstyle \otimes W}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \otimes i_0(W)} \\
\mathcal{R}^{\beta}(\overline{L}(\psi)) & \xrightarrow{\quad i^{\varphi'}_{\overline{U}(\varphi),\overline{V}(\varphi)}\quad} & \mathcal{R}^{\gamma}(\overline{G}(\varphi))
\end{array} \tag{3.17}$$

commute.

To show that the bottom square of (3.17) commutes, let M be an $\mathcal{H}^{\gamma}(\overline{L}(\psi))$ module. We have natural isomorphisms of $\mathcal{H}^{\gamma}(\overline{G}(\varphi)) \otimes \mathrm{End}(i_0(W))$ modules

$$\begin{aligned}
T(M \otimes W) &= \left(\mathcal{H}^{\gamma}(\overline{G}(\varphi)) e^{\varphi'}_{\overline{U}(\varphi)} e^{\varphi'}_{\overline{V}(\varphi)} \otimes (i_0(W) \otimes W^*)\right) \otimes_{\mathcal{H}^{\gamma}(\overline{L}(\psi)) \otimes \mathrm{End}(W)} (M \otimes W) \\
&\cong \left(\mathcal{H}^{\gamma}(\overline{G}(\varphi)) e^{\varphi'}_{\overline{U}(\varphi)} e^{\varphi'}_{\overline{V}(\varphi)} \otimes_{\mathcal{H}^{\gamma}(\overline{L}(\psi))} M\right) \otimes (i_0(W) \otimes (W^* \otimes_{\mathrm{End}(W)} W)) \\
&\cong i^{\varphi'}_{\overline{U}(\varphi),\overline{V}(\varphi)}(M) \otimes i_0(W),
\end{aligned}$$

because $W^* \otimes_{\mathrm{End}(W)} W \cong \mathbb{C}$, as W is finite-dimensional. Thus the bottom square of the diagram commutes.

To show that the top square of (3.17) commutes, it is enough to construct a linear isomorphism

$$F : e_{\varphi} \mathcal{H}(G(\varphi)) e_{U(\varphi)} e_{V(\varphi)} e_{\psi} \longrightarrow \mathcal{H}^{\gamma}(\overline{G}(\varphi)) e^{\varphi'}_{\overline{U}(\varphi)} e^{\varphi'}_{\overline{V}(\varphi)} \otimes_{\mathbb{C}} (i_0(W) \otimes_{\mathbb{C}} W^*)$$

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between the bimodules associated to the functors i_ψ and T, satisfying

$$F(f\cdot h\cdot k) = (\theta\otimes\varphi')(f)\cdot F(h)\cdot(\theta\otimes\psi')(k) \tag{3.18}$$

for all $f\in e_\varphi\mathcal{H}(G(\varphi))$, $h\in e_\varphi\mathcal{H}(G(\varphi))e_{U(\varphi)}e_{V(\varphi)}e_\psi$ and $k\in e_\psi\mathcal{H}(L(\psi))$.

We shall construct F as a composition $F=F_3F_2F_1$:

$$\begin{aligned} e_\varphi\mathcal{H}(G(\varphi))e_{U(\varphi)}e_{V(\varphi)}e_\psi &\xrightarrow[\cong]{F_1} \mathcal{H}^\gamma(\overline{G}(\varphi))e^{\varphi'}_{U(\varphi)}e^{\varphi'}_{V(\varphi)}\otimes \mathrm{End}\,(\,i_0\,(W))\varphi(e_{U_0}e_{V_0}e_\psi)\\ &\xrightarrow[\cong]{F_2} \mathcal{H}^\gamma(\overline{G}(\varphi))e^{\varphi'}_{U(\varphi)}e^{\varphi'}_{V(\varphi)}\otimes i_0\,(W)\otimes(\varphi(e_\psi e_{U_0}e_{V_0})\,i_0\,(W))^*\\ &\xrightarrow[\cong]{F_3} \mathcal{H}^\gamma(\overline{G}(\varphi))e^{\varphi'}_{U(\varphi)}e^{\varphi'}_{V(\varphi)}\otimes i_0\,(W)\otimes W^*, \end{aligned}$$

where the isomorphisms F_1, F_2 and F_3 are defined below.

The map F_1 is the restriction of the algebra isomorphism $\theta\otimes\varphi'$ to the bimodule $e_\varphi\mathcal{H}(G(\varphi))e_{U(\varphi)}e_{V(\varphi)}e_\psi$. The image is as claimed because of Lemma 3.12. By definition, F_1 satisfies

$$F_1(f\cdot h\cdot k) = (\theta\otimes\varphi')(f)\cdot F_1(h)\cdot(\theta\otimes\varphi')(k)$$

(where f, h and k are as in (3.18)).

The map F_2 is the identity on the first tensor factor, while on the second factor it is the isomorphism given by Lemma 3.16, with $E=i_0\,(W)$ and $S=\varphi(e_{U_0}e_{V_0}e_\psi)$. Clearly F_2 is a map of left $\mathcal{H}^\gamma(\overline{G}(\varphi))\otimes\mathrm{End}\,(\,i_0\,(W))$ modules. Turning to the right module structure, fix $l\in L(\psi)$. The operator $\varphi'(e_\psi\delta_l)\in\mathrm{End}\,(\,i_0\,(W))$ commutes with $\varphi(e_\psi e_{U_0}e_{V_0})$, because $L(\psi)$ centralises ψ and normalises U_0 and V_0. Therefore, $\varphi'(e_\psi\delta_l)$ lies in the algebra A of Lemma 3.16, and so the isomorphism F_2 satisfies

$$F_2(F_1(h)\cdot\theta(l)\otimes\varphi'(e_\psi\delta_l)) = F_2F_1(h)\cdot(\theta(l)\otimes\varphi'(e_\psi\delta_l)).$$

The map F_3 is the identity on the first two tensor factors, while on the third factor it is the linear dual Θ^* of the isomorphism $\Theta: W\to\varphi(e_{U_0}e_{V_0})\,i_0\,(W)$. (Note that e_ψ acts as the identity on W.) Clearly F_3 is a map of left $\mathcal{H}^\gamma(\overline{G}(\varphi))\otimes\mathrm{End}\,(\,i_0\,(W))$ modules. The isomorphism Θ satisfies $\varphi'(e_\psi\delta_l)\circ\Theta=\Theta\circ\psi'(e_\psi\delta_l)$, by the definition (3.11) of ψ', and we therefore have

$$F_3[F_2F_1(h)\cdot(\theta(l)\otimes\varphi'(e_\psi\delta_l))] = F(h)\cdot(\theta(l)\otimes\psi'(e_\psi\delta_l)).$$

We have now shown that the isomorphism F satisfies (3.18), and this completes the proof of Theorem 3.14. □

4. The functor i and the orbit method

In this section we examine the induction functor $i_{U,V}$ in situations to which the orbit method applies, and show that it corresponds to a natural inclusion map on coadjoint orbits. We begin with an abstract formulation and then discuss a natural family of groups to which it applies, namely uniform pro-p-groups and finite p-groups of nilpotency class less than p. In particular, this family includes many compact open subgroups in reductive groups over p-adic fields.

4.1. An abstract formulation

The orbit method in the context of profinite groups goes back to the work of Howe [22]. An abstract formulation was given by Boyarchenko and Sabitova in [4], and it is this latter point of view that we shall adopt here.

Let G be a profinite group, let $\mathfrak{g}$ be an abelian profinite group, and let $\exp : \mathfrak{g} \to \mathsf{G}$ be a homeomorphism satisfying

(A) The formula $\mathrm{Ad}_g(x) := \log(g \exp(x) g^{-1})$ for $g \in \mathsf{G}$, $x \in \mathfrak{g}$, and $\log = \exp^{-1}$ defines an action of G on $\mathfrak{g}$ by group automorphisms.

(B) The pullback map $\exp^* : \mathcal{H}(\mathsf{G})^{\mathsf{G}} \to \mathcal{H}(\mathfrak{g})^{\mathsf{G}}$, from the Ad_{G}-invariant locally constant functions on G to those on $\mathfrak{g}$, is an isomorphism of convolution algebras.

The adjoint action of G on $\mathfrak{g}$ induces a coadjoint action on the Pontryagin dual group $\widehat{\mathfrak{g}}$. It is shown in [4, Theorem 1.1] that for each irreducible smooth representation τ of G, with character $\mathrm{ch}_\tau \in \mathcal{H}(\mathsf{G})^{\mathsf{G}}$, there is an Ad_{G}-orbit $\Omega \subset \widehat{\mathfrak{g}}$ such that

$$\exp^*(\mathrm{ch}_\tau) = |\Omega|^{-1/2} \sum_{\psi \in \Omega} \psi, \tag{4.1}$$

and the map $\mathrm{ch}_\tau \mapsto \Omega$ sets up a bijection $\mathcal{O}_{\mathsf{G}} : \mathrm{Irr}(\mathsf{G}) \to \mathsf{G}\backslash\widehat{\mathfrak{g}}$ from the set of isomorphism classes of irreducible representations of G to the set of coadjoint orbits in $\widehat{\mathfrak{g}}$.

Theorem 4.2. *Let* G, $\mathfrak{g}$ *and* $\exp$ *be as above. Let* U, L *and* V *be closed subgroups of* G *such that:*

(1) $(\mathsf{U}, \mathsf{L}, \mathsf{V})$ *is an Iwahori decomposition of* G.

(2) *The preimages* $\mathfrak{u}, \mathfrak{l}, \mathfrak{v}$ *of* $\mathsf{U}, \mathsf{L}, \mathsf{V}$ *under* $\exp$ *are subgroups of* $\mathfrak{g}$, *and* $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{l} \oplus \mathfrak{v}$ *as abelian groups.*

(3) *The map* $\exp : \mathfrak{g} \to \mathsf{G}$ *restricts to homeomorphisms*

$$\mathfrak{l} \to \mathsf{L}, \quad \mathfrak{l} \oplus \mathfrak{u} \to \mathsf{LU} \quad \textit{and} \quad \mathfrak{l} \oplus \mathfrak{v} \to \mathsf{LV},$$

each of which satisfies the conditions (A) and (B).

Then the projection Λ *of* $\mathfrak{g}$ *onto its summand* $\mathfrak{l}$ *induces an injective map*

$$\Lambda^* : \mathsf{L}\backslash\widehat{\mathfrak{l}} \to \mathsf{G}\backslash\widehat{\mathfrak{g}}, \quad \Omega \mapsto \mathrm{Ad}^*_{\mathsf{G}}(\Omega \circ \Lambda)$$

which makes the diagram

$$\begin{array}{ccc} \mathrm{Irr}(\mathsf{L}) & \xrightarrow{i_{\mathsf{U},\mathsf{V}}} & \mathrm{Irr}(\mathsf{G}) \\ {\scriptstyle \mathcal{O}_{\mathsf{L}}}\downarrow & & \downarrow{\scriptstyle \mathcal{O}_{\mathsf{G}}} \\ \mathsf{L}\backslash\widehat{\mathfrak{l}} & \xrightarrow{\Lambda^*} & \mathsf{G}\backslash\widehat{\mathfrak{g}} \end{array}$$

commutative.

We require the following lemma:

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Lemma 4.3. *Let* Ω *be an orbit in* $LU\backslash\widehat{\mathfrak{lu}}$*, corresponding via the orbit method to an irreducible representation* τ *of* LU*. Then* τ *is trivial on* U *if and only if every* $\psi \in \Omega$ *is trivial on* $\mathfrak{u}$.

Proof. If every $\psi \in \Omega$ is trivial on $\mathfrak{u}$, then the character formula (4.1) ensures that the character of τ is constant on U, and therefore that τ is trivial on U. Conversely, if τ is trivial on U then its character is constant on U, with constant value $\dim(\tau) = |\Omega|^{1/2}$, and so (4.1) implies that $\sum_{\psi\in\Omega} \psi(y) = |\Omega|$ for every $y \in \mathfrak{u}$. Since each $\psi(y)$ is a complex number of modulus one, this equality forces $\psi(y) = 1$ for every ψ and every y. □

Note that by Theorem 2.23(1) the functor $i = i_{U,V} : \mathcal{R}(L) \to \mathcal{R}(G)$ preserves irreducibility, and therefore induces a map $i : \mathrm{Irr}(L) \to \mathrm{Irr}(G)$. We recall the following characterisation of the map i from Theorem 2.23(4) : given irreducible representations $\tau \in \mathcal{R}(G)$ and $\sigma \in \mathcal{R}(L)$, one has $\tau \cong i(\sigma)$ if and only if σ is a common subrepresentation of τ^U and τ^V.

Proof of Theorem 4.2. Fix an orbit $\Omega \in L\backslash\widehat{\mathfrak{l}}$ and let $\sigma = \mathcal{O}_L^{-1}(\Omega) \in \mathrm{Irr}(L)$ be the corresponding irreducible representation of L. Let $\tau = \mathcal{O}_G^{-1}(\Lambda^*\Omega) \in \mathrm{Irr}(G)$ be the corresponding irreducible representation of G. We show that σ is isomorphic to a subrepresentation of τ^U. The same argument shows that σ is isomorphic to a subrepresentation of τ^V, and then Theorem 2.23(4) gives $\tau \cong i(\sigma)$ as required.

The characters ch_ρ, where ρ ranges over $\mathrm{Irr}(LU)$, constitute a linear basis for the space $\mathcal{H}(LU)^{LU}$. We let $P : \mathcal{H}(LU)^{LU} \to \mathcal{H}(LU)^{LU}$ be the projection

$$P(\mathrm{ch}_\rho) = \begin{cases} \mathrm{ch}_\rho & \text{if } \rho \text{ is trivial on } U \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, the functions

$$\chi_\Psi := |\Psi|^{-1/2} \sum_{\psi\in\Psi} \psi,$$

as Ψ ranges over $LU\backslash\widehat{\mathfrak{lu}}$, constitute a basis for $\mathcal{H}(\mathfrak{lu})^{LU}$, and we let Q be the idempotent operator on $\mathcal{H}(\mathfrak{lu})^{LU}$ defined by

$$Q(\chi_\Psi) = \begin{cases} \chi_\Psi & \text{if every } \psi \in \Psi \text{ is trivial on } \mathfrak{u} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.3 implies the commutativity of the middle square in the diagram

$$\begin{array}{ccccccc} \mathcal{H}(G)^G & \xrightarrow{\text{restrict}} & \mathcal{H}(LU)^{LU} & \xrightarrow{P} & \mathcal{H}(LU)^{LU} & \xrightarrow{\text{restrict}} & \mathcal{H}(L)^L \\ \downarrow{\scriptstyle \exp^*} & & \downarrow{\scriptstyle \exp^*} & & \downarrow{\scriptstyle \exp^*} & & \downarrow{\scriptstyle \exp^*} \\ \mathcal{H}(\mathfrak{g})^G & \xrightarrow{\text{restrict}} & \mathcal{H}(\mathfrak{lu})^{LU} & \xrightarrow{Q} & \mathcal{H}(\mathfrak{lu})^{LU} & \xrightarrow{\text{restrict}} & \mathcal{H}(\mathfrak{l})^L \end{array} \tag{4.4}$$

where 'restrict' means restriction of functions. The two outer squares in the diagram obviously commute. For each irreducible representation ρ of G, the composition along the top row of (4.4) sends the character of ρ to the character of ρ^U.

Choose a point ψ in the orbit $\Omega \subset \widehat{\mathfrak{l}}$, and write $\Lambda^*\Omega = \{\psi \circ \Lambda, \varphi_1, \ldots, \varphi_n\}$. Since the character $\psi \circ \Lambda \in \widehat{\mathfrak{g}}$ is trivial on $\mathfrak{u}$, and has $\psi \circ \Lambda|_{\mathfrak{l}} = \psi$, we find that the composition along the bottom row of (4.4) sends $\chi_{\Lambda^*\Omega} = \exp^*(\mathrm{ch}_\tau)$ to the function

$$|\Lambda^*\Omega|^{-1/2}\Big(\psi + \sum_{\varphi_i \equiv 1 \text{ on } \mathfrak{u}} \varphi_i|_{\mathfrak{l}}\Big).$$

Since this sum contains ψ—and hence $\chi_\Omega = \exp^*(\mathrm{ch}_\sigma)$—with a positive coefficient, we conclude from the commutativity of (4.4) that τ^U contains a copy of σ. $\square$

4.2. Application to (pro-) p-groups

The results of the previous subsection apply to a rich and well-behaved family of (pro-) p-groups which we now discuss. Roughly speaking these groups admit good linearisations, that is, to each such group one may associate a Lie algebra that carries complete information on the group.

Uniform pro-p-groups. A finite p-group is called *powerful* if $[G, G] \subset G^p$ when p is odd (and $[G, G] \subset G^4$ when $p = 2$). Here G^m is the group generated by m-powers. A pro-p group is called powerful if it is the inverse limit of finite powerful groups. A pro-p group is called *uniform* if it is powerful, finitely generated (as a pro-p group), and torsion-free. To each uniform pro-p group G one may associate a *uniform* $\mathbb{Z}_p$-Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, that is, a $\mathbb{Z}_p$-Lie algebra which is free of finite rank as a $\mathbb{Z}_p$-module, and which satisfies $[\mathfrak{g}, \mathfrak{g}]_{\mathrm{Lie}} \subset p\mathfrak{g}$ for p odd (and $[\mathfrak{g}, \mathfrak{g}]_{\mathrm{Lie}} \subset 4\mathfrak{g}$ for $p = 2$); see [14] for a comprehensive treatment. This association defines an equivalence of categories between the category of uniform pro-p-groups and uniform $\mathbb{Z}_p$-Lie algebras. Starting with a uniform Lie algebra $\mathfrak{g}$ this association is made concrete using the Campbell–Hausdorff series

$$H(u, v) = \log(\exp(u)\exp(v)) = u + v + (\text{Lie brackets}) \in \mathbb{Q}\langle\langle u, v\rangle\rangle,$$

which is expressible in terms of $u, v \in \mathfrak{g}$ by means of the Lie bracket, and which allows one to define a uniform pro-p group G having the same underlying set as $\mathfrak{g}$ and group operation $u \cdot v = H(u, v)$. Let exp denote the identity map on $\mathfrak{g}$, thought of as a map from the Lie algebra $\mathfrak{g}$ to the group G. This map is well-behaved with respect to passage to subgroups or quotients: Lie subalgebras of $\mathfrak{g}$ correspond bijectively to closed uniform subgroups of G, and ideals in $\mathfrak{g}$ correspond to normal subgroups in G; see [14, § 4.5]. Moreover, it is shown in [4, Theorem 2.6] that for $p \neq 2$ this map exp satisfies the conditions (A) and (B) from § 4.1, meaning that the orbit method applies and gives a bijection $\mathcal{O}_G : \mathrm{Irr}(G) \to G\backslash\widehat{\mathfrak{g}}$. This generalises an earlier result of Howe [22, Theorem 1.1].

Finite p-groups of nilpotency class less than p. There is a similar Lie-type correspondence for finite p-groups of nilpotency class less than p. To each group G of this type one may associate a finite $\mathbb{Z}$-Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ which is nilpotent of class less than p, and whose additive group is a p-group, such that G is isomorphic to the group $\exp(\mathfrak{g})$ whose underlying set is $\mathfrak{g}$ and whose multiplication is given by the Campbell–Hausdorff series (which is finite, in this case); see [29, § 10.2]. If p is odd then the orbit method applies to the map $\exp : \mathfrak{g} \to G$; see [4, Theorem 2.6].

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Application of Theorem 4.2. For the rest of this section let $G = \exp(\mathfrak{g})$ be either a uniform pro-p group or a finite p-group of nilpotency class less than p, with corresponding Lie algebra $\mathfrak{g}$. For each subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ we write H for the corresponding subgroup $\exp(\mathfrak{h})$ of G.

Definition 4.5. An *Iwahori decomposition* of $\mathfrak{g}$ is a triple of Lie subalgebras $(\mathfrak{u}, \mathfrak{l}, \mathfrak{v})$ of $\mathfrak{g}$ such that $[\mathfrak{l}, \mathfrak{u}] \subseteq \mathfrak{u}$, $[\mathfrak{l}, \mathfrak{v}] \subseteq \mathfrak{v}$, and such that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{l} \oplus \nu$ as $\mathbb{Z}_p$-modules (in the uniform pro-p case) or as $\mathbb{Z}$-modules (in the finite p-group case).

Lemma 4.6. *If $(\mathfrak{u}, \mathfrak{l}, \mathfrak{v})$ is an Iwahori decomposition of $\mathfrak{g}$, then (U, L, V) is an Iwahori decomposition of G.*

Proof. The Lie correspondence ensures that U, L and V are closed subgroups of G such that L normalises U and V. The subgroups V and $B := UL$ have trivial intersection in G, because the subalgebras $\mathfrak{v}$ and $\mathfrak{u} \oplus \mathfrak{l}$ have trivial intersection in $\mathfrak{g}$, and so the product map $U \times L \times V \to G$ is injective. We shall now show that this map is surjective.

We must show that for each $x \in \mathfrak{b} := \mathfrak{u} \oplus \mathfrak{l}$ and each $y \in \mathfrak{v}$ one has $\exp(x + y) \in BV$. The Campbell–Hausdorff formula implies that $\exp(x + y) = \exp(x) \exp(z_1) \exp(y)$ for some $z_1 \in \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$. Writing $z_1 = x_1 + y_1$, where $x_1 \in \mathfrak{b}$ and $y_1 \in \mathfrak{v}$, another application of Campbell–Hausdorff gives $\exp(z_1) = \exp(x_1) \exp(z_2) \exp(y_1)$ for some $z_2 \in \mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}_1]$. Continuing in this way we find $z_n \in \mathfrak{g}_n = [\mathfrak{g}, \mathfrak{g}_{n-1}]$, $x_{n-1} \in \mathfrak{b}$ and $y_{n-1} \in \mathfrak{v}$, for every $n \in \mathbb{N}$, such that $\exp(z_{n-1}) = \exp(x_{n-1}) \exp(z_n) \exp(y_{n-1})$, and we deduce that

$$\exp(x + y) \in \bigcap_{n \geqslant 0} B \exp(\mathfrak{g}_n) V = B \left(\bigcap_{n \geqslant 0} \exp(\mathfrak{g}_n) \right) V = BV,$$

where the first equality holds because the groups $\exp(\mathfrak{g}_n)$ form a descending chain and G is compact, and the second holds because $\mathfrak{g}$ is either uniform or nilpotent.

We are left to verify condition (2) of Definition 2.1. If G is finite this condition is trivially satisfied, so suppose that G is a uniform pro-p group. For each $n \geqslant 0$ the triple $(p^n\mathfrak{u}, p^n\mathfrak{l}, p^n\mathfrak{v})$ is an Iwahori decomposition of the ideal $p^n\mathfrak{g}$ of $\mathfrak{g}$, and so the above argument shows that the open normal subgroups $K_n = \exp(p^n\mathfrak{g})$ of G satisfy condition (2). □

We now have the following corollary of Theorem 4.2:

Corollary 4.7. *Let p be an odd prime. Let G be either a uniform pro-p group, or a finite p-group of nilpotency class less than p. Let $(\mathfrak{u}, \mathfrak{l}, \mathfrak{v})$ be an Iwahori decomposition of the Lie algebra $\mathfrak{g}$ of G, and let (U, L, V) be the corresponding Iwahori decomposition of G. The diagram*

$$\begin{array}{ccc} \mathrm{Irr}(L) & \xrightarrow{\ i_{U,V}\ } & \mathrm{Irr}(G) \\ \downarrow{\scriptstyle \mathcal{O}_L} & & \downarrow{\scriptstyle \mathcal{O}_G} \\ L\backslash\widehat{\mathfrak{l}} & \xrightarrow{\ \Lambda^*\ } & G\backslash\widehat{\mathfrak{g}} \end{array}$$

is commutative.

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Proof. This follows from Theorem 4.2. The hypothesis (1) of that theorem is satisfied because of Lemma 4.6; hypothesis (2) is satisfied by assumption; and the hypothesis (3) is satisfied because of [4, Theorem 2.6]. □

We remark that for uniform pro-2-groups the orbit method does not fully apply, though one has weaker versions; see [4, 27].

Example 4.8. In 'real life' one may find a rich supply of groups to which the corollary may be applied. Let $\mathcal{G}$ be a p-adic Lie group, let α be an automorphism of $\mathcal{G}$, and denote by α_* the derived automorphism of the Lie algebra $\mathfrak{g}$ of G. Then

$$\mathfrak{u}_\alpha := \{x \in \mathfrak{g} \mid \alpha_*^n(x) \to 0 \text{ as } n \to \infty\} \quad \text{and} \quad \mathfrak{v}_\alpha := \mathfrak{u}_{\alpha^{-1}}$$

are nilpotent Lie subalgebras of $\mathfrak{g}$, normalised by the subalgebra

$$\mathfrak{l}_\alpha := \{x \in \mathfrak{g} \mid \{\alpha_*^n(x) \mid n \in \mathbb{Z}\} \text{ is precompact in } \mathfrak{g}\},$$

and we have $\mathfrak{g} = \mathfrak{u}_\alpha \oplus \mathfrak{l}_\alpha \oplus \mathfrak{v}_\alpha$ as $\mathbb{Q}_p$-vector spaces. Moreover, $\mathfrak{u}_\alpha$, $\mathfrak{l}_\alpha$ and $\mathfrak{v}_\alpha$ are the respective Lie algebras of the closed subgroups $\mathcal{U}_\alpha$, $\mathcal{L}_\alpha$ and $\mathcal{V}_\alpha$ of $\mathcal{G}$, where we are using the notation of Example 2.6. These assertions are proved in [45, Theorem 3.5]. It is shown in [15, Lemma 3.3] that $\mathfrak{g}$ contains arbitrary small open uniform $\mathbb{Z}_p$-Lie subalgebras $\mathfrak{k}$ having $\mathfrak{k} = (\mathfrak{u}_\alpha \cap \mathfrak{k}) \oplus (\mathfrak{l}_\alpha \cap \mathfrak{k}) \oplus (\mathfrak{v}_\alpha \cap \mathfrak{k})$. The compact open subgroups $K = \exp(\mathfrak{k})$ of $\mathcal{G}$ then have Iwahori decompositions $(\mathcal{U}_\alpha \cap K, \mathcal{L}_\alpha \cap K, \mathcal{V}_\alpha \cap K)$. If p is odd, Corollary 4.7 describes the induction functor $i_{\mathcal{U}_\alpha \cap K, \mathcal{V}_\alpha \cap K} : \mathcal{R}(\mathcal{L}_\alpha \cap K) \to \mathcal{R}(K)$ in terms of the orbit method and of the projection $\mathfrak{k} \to \mathfrak{l}_\alpha \cap \mathfrak{k}$.

Example 4.9. For a finite example, let $\mathfrak{o}$ be a compact discrete valuation ring with maximal ideal $\mathfrak{p}$ and residue characteristic p. Let $K = K_1$ be the first principal congruence subgroup in $\mathrm{GL}_n(\mathfrak{o}/\mathfrak{p}^\ell)$, for $\ell > 1$. Then K is a finite p-group of nilpotency class $\ell - 1$, with Lie algebra $\mathfrak{k} = M_n(\mathfrak{p}/\mathfrak{p}^\ell)$. As explained in Example 2.5, each partition $n = n_1 + \cdots + n_m$ gives an Iwahori decomposition $(U \cap K, L \cap K, V \cap K)$ of K, corresponding to the decomposition of $\mathfrak{k}$ into block-upper-triangular, block-diagonal and block-lower-triangular matrices. If $p > \ell - 1$ and odd, Corollary 4.7 gives a description of the resulting induction functor $i_{U \cap K, V \cap K} : \mathcal{R}(L \cap K) \to \mathcal{R}(K)$ in terms of the orbit method and the projection of $\mathfrak{k}$ onto its subalgebra of block-diagonal matrices.

Remark 4.10. We have taken the point of view of the theory of uniform groups due to its fairly concrete and algebraic formulation. Historically, the Lie correspondence for (pro-)p-groups goes back to the seminal work of Lazard [30], [31]. The technique of obtaining Iwahori decompositions of groups from decompositions of Lie algebras is well known in the setting of p-adic reductive groups: see [2] and [11], for example.

5. Case study: Siegel Levi subgroup in $\mathrm{Sp}_4(\mathfrak{o}_2)$

Let $\mathfrak{o}$ be a compact discrete valuation ring with maximal ideal $\mathfrak{p}$, a fixed uniformiser π and finite residue field $\Bbbk$ of odd characteristic. Let $\mathfrak{o}_\ell := \mathfrak{o}/\mathfrak{p}^\ell$. In this section we illustrate how the results of the previous sections may be applied to study the representations

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of the symplectic group $\mathrm{Sp}_4(\mathfrak{o}_2)$ that are induced, in the sense of Definition 2.7, from the Siegel Levi subgroup of 2×2 block-diagonal matrices. Note that this is equivalent to studying those induced representations of $\mathrm{Sp}_4(\mathfrak{o})$ which factor through $\mathrm{Sp}_4(\mathfrak{o}_2)$: see Theorem 2.18(5). The main results in this section are a double-coset formula, à la Mackey, for the composition of induction and restriction for these groups (Theorem 5.2); and an answer to a question of Dat regarding parahoric induction (Corollary 5.21).

Let us introduce the notation used to state the Mackey formula. Let

$$G = \mathrm{Sp}_4(\mathfrak{o}_2) = \{g \in \mathrm{GL}_4(\mathfrak{o}_2) \mid g^t j g = j\}, \quad \text{where } j = \begin{bmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{bmatrix}. \tag{5.1}$$

This group admits a virtual Iwahori decomposition (U, L, V), with

$$\begin{aligned} L &= \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-t} \end{bmatrix} \,\middle|\, a \in \mathrm{GL}_2(\mathfrak{o}_2) \right\}, \\ U &= \left\{ \begin{bmatrix} 1 & m \\ & 1 \end{bmatrix} \,\middle|\, m \in M_2(\mathfrak{o}_2),\ m = m^t \right\}, \quad \text{and} \quad V = U^t, \end{aligned}$$

where $(\cdot)^t$ means transpose and $(\cdot)^{-t}$ means transpose inverse. We consider the associated functors

$$i_L^G := i_{U,V} : \mathcal{R}(L) \to \mathcal{R}(G) \quad \text{and} \quad r_L^G := r_{U,V} : \mathcal{R}(G) \to \mathcal{R}(L).$$

The subgroup $L \cong \mathrm{GL}_2(\mathfrak{o}_2)$ has a virtual Iwahori decomposition (U', D, V'), where

$$\begin{aligned} D &= \{\mathrm{diag}(\alpha, \delta, \alpha^{-1}, \delta^{-1}) \mid \alpha, \delta \in \mathfrak{o}_2^\times\}, \\ U' &= \left\{ \mathrm{diag}\left(\begin{bmatrix} 1 & \beta \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ -\beta & 1 \end{bmatrix} \right) \,\middle|\, \beta \in \mathfrak{o}_2 \right\} \quad \text{and} \quad V' = (U')^t. \end{aligned}$$

We consider the associated functors

$$i_D^L := i_{U',V'} : \mathcal{R}(D) \to \mathcal{R}(L) \quad \text{and} \quad r_D^L := r_{U',V'} : \mathcal{R}(L) \to \mathcal{R}(D).$$

We let

$$W_G := N_G(D)/D \quad \text{and} \quad W_L := N_L(D)/D$$

denote the Weyl groups of D in G and in L, respectively. We write Ad_g for the conjugation action of a group on itself and subsets thereof, and with a slight abuse of notation also for the corresponding action on representations.

Theorem 5.2. *There is a natural isomorphism of functors $\mathcal{R}(L) \to \mathcal{R}(L)$,*

$$r_L^G\, i_L^G \cong \bigoplus_{g \in W_L \backslash W_G / W_L} i^L_{gLg^{-1}\cap L}\, \mathrm{Ad}_g\, r^L_{L\cap g^{-1}Lg}.$$

Remarks 5.3. Let us unpack Theorem 5.2 a little.

(1) The right-hand side of the formula in Theorem 5.2 is a sum over a set of representatives $g \in N_G(D)$ for the double cosets of W_L in W_G; the resulting functor does not depend on the choices made, up to natural isomorphism.

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(2) For each $g \in N_G(D)$, the intersection $gLg^{-1} \cap L$ is either L or D. The functors i_D^L and r_D^L were defined above; the functors i_L^L and r_L^L are, by definition, the identity functors on $\mathcal{R}(L)$.

(3) The group W_L is the two-element group generated (modulo D) by the matrix

$$t = \operatorname{diag}(\sigma, \sigma) \in L, \quad \text{where } \sigma = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \in \mathrm{GL}_2(\mathfrak{o}_2).$$

The eight-element dihedral group W_G is generated (modulo D) by W_L together with the matrix

$$w := \begin{bmatrix} 0 & & -1 & \\ & 1 & & \\ 1 & & 0 & \\ & & & 1 \end{bmatrix}.$$

Defining

$$s := \begin{bmatrix} & \sigma \\ \sigma^{-1} & \end{bmatrix} \in G,$$

we have the double-coset decomposition

$$W_G = W_L \sqcup W_L s W_L \sqcup W_L w W_L.$$

The element s normalises L, while $wLw^{-1} \cap L = D$. Putting all of this together, the formula in Theorem 5.2 takes the following more explicit form:

$$r_L^G \, i_L^G \cong \mathrm{id} \oplus \mathrm{Ad}_s \oplus i_D^L \, \mathrm{Ad}_w \, r_D^L .$$

(4) Note that the definition of the functors i_L^G and r_L^G, and the statement of Theorem 5.2, continue to make sense when $\mathfrak{o}_2$ is replaced by $\mathfrak{o}_\ell$, or indeed by any finite (or profinite) commutative ring. Over $\mathfrak{o}_1$, the formula is valid: as explained in Example 2.10, the functors i_L^G, r_L^G, i_D^L and r_D^L are isomorphic in that case to Harish-Chandra functors, and the formula in Theorem 5.2 is an instance of the well-known formula (1.1) for the composition of these functors (cf. [13, Theorem 5.1]). We do not know whether the formula in Theorem 5.2 is valid for Sp_4 over more general rings; the proof presented below relies on some very special features of $\mathfrak{o}_2$.

Our strategy for proving Theorem 5.2 is as follows. Reduction modulo π gives rise to a surjective group homomorphism $G = \mathrm{Sp}_4(\mathfrak{o}_2) \to \mathrm{Sp}_4(\mathbb{k})$, whose kernel is an abelian group isomorphic to the Lie algebra $\mathfrak{sp}_4(\mathbb{k})$. In §§ 5.1 and 5.2 we apply the orbit method and Clifford theory to reduce Theorem 5.2 to a statement about orbits and representations of stabilisers for the adjoint action of $\mathrm{Sp}_4(\mathbb{k})$ on $\mathfrak{sp}_4(\mathbb{k})$. In §§ 5.3 and 5.4 we verify the theorem through a case-by-case analysis of the orbits (with some details postponed to Appendix A).

For the semisimple orbits our induction and restriction functors correspond to Harish-Chandra induction and restriction for (reductive) subgroups of $\mathrm{Sp}_4(\mathbb{k})$, and our Mackey formula follows from the well-known Mackey formula (1.1) for Harish-Chandra functors. The computation for the non-semisimple orbits—and in particular, for the one nilpotent orbit that is relevant here—is more subtle. In Corollary 5.17 we shall see that it is precisely this nilpotent orbit that witnesses the difference between our induction functor and Dat's parahoric induction.

5.1. The congruence subgroup

Let G_0 denote the kernel of the reduction map $\mathrm{Sp}_4(\mathfrak{o}_2) \to \mathrm{Sp}_4(\mathbb{k})$, and let

$$\mathfrak{g} = \mathfrak{sp}_4(\mathbb{k}) = \{y \in M_4(\mathbb{k}) \mid jy + y^t j = 0\},$$

viewed as an additive abelian group on which G acts via the adjoint action of its quotient $\mathrm{Sp}_4(\mathbb{k})$. To reduce the notational load we shall write

$$g \cdot y = \mathrm{Ad}_g(y) = gyg^{-1} \pmod{\pi}, \quad g \in G, y \in \mathfrak{g}.$$

Lemma 5.4. *The map* $\exp : \mathfrak{g} \to G_0$ *defined as the composition*

$$\mathfrak{g} \xrightarrow{y \mapsto \pi y} \pi\mathfrak{sp}_4(\mathfrak{o}_2) \xrightarrow{z \mapsto 1+z} G_0$$

is a G*-equivariant group isomorphism.*

Proof. Clear. □

For every subgroup H of G we set

$$H_0 := H \cap G_0, \quad \overline{H} := HG_0/G_0 \cong H/H_0, \quad \text{and} \quad \mathfrak{h} := \log(H_0),$$

where $\log : G_0 \to \mathfrak{g}$ denotes the inverse to exp. In particular, $\mathfrak{l}$ is the additive subgroup of $M_4(\mathbb{k})$ consisting of the block-diagonal matrices $\mathrm{diag}(x, -x^t)$, for $x \in M_2(\mathbb{k})$.

It is easily checked that the triple of subgroups $(\mathfrak{u}, \mathfrak{l}, \mathfrak{v})$ forms an Iwahori decomposition of $\mathfrak{g}$, and it follows that the triple (U_0, L_0, V_0) is an Iwahori decomposition of G_0. Similarly, $(\mathfrak{u}', \mathfrak{d}, \mathfrak{v}')$ is an Iwahori decomposition of $\mathfrak{l}$, and so (U_0', D_0, V_0') is an Iwahori decomposition of L_0.

Lemma 5.5. *Choose and fix a nontrivial character* $\zeta : \mathbb{k} \to \mathbb{C}^\times$*. For each* $y \in \mathfrak{g}$*, denote by* $\varphi_y : G_0 \to \mathbb{C}^\times$ *the character*

$$\varphi_y : g \mapsto \zeta \circ \mathrm{tr}(\log(g) y).$$

The mapping $y \mapsto \varphi_y$ *is a* G*-equivariant bijection* $\mathfrak{g} \xrightarrow{\cong} \mathrm{Irr}(G_0)$*, which restricts to an* L*-equivariant bijection* $\mathfrak{l} \xrightarrow{\cong} \mathrm{Irr}(L_0)$*, and to a* D*-equivariant bijection* $\mathfrak{d} \xrightarrow{\cong} \mathrm{Irr}(D_0)$*.*

Proof. Let $\langle z, y \rangle := \zeta \circ \mathrm{tr}(zy)$ for $z, y \in M_4(\mathbb{k})$. It is well known that the map $M_4(\mathbb{k}) \to \widehat{M_4(\mathbb{k})}$ sending y to $\langle \cdot, y \rangle$ is an isomorphism. By Pontryagin duality this map restricts to an isomorphism between $\mathfrak{g}$ and the dual of $M_4(\mathbb{k})/\mathfrak{g}^\perp$, where $\mathfrak{g}^\perp = \{z \in M_4(\mathbb{k}) \mid \langle z, \mathfrak{g} \rangle = 1\}$. Let $\mathfrak{g}' = \{z \in M_4(\mathbb{k}) \mid jz - z^t j = 0\}$. For each $z \in \mathfrak{g}'$ and $y \in \mathfrak{g}$ we have $\mathrm{tr}(zy) = \mathrm{tr}(\mathrm{Ad}_j(z)\,\mathrm{Ad}_j(y)) = -\mathrm{tr}(zy)$, showing that $\mathfrak{g}' \subseteq \mathfrak{g}^\perp$. We also have $M_4(\mathbb{k}) = \mathfrak{g} \oplus \mathfrak{g}'$ (this is the eigenspace decomposition for the involution $y \mapsto \mathrm{Ad}_j(y^t)$), and since $\mathfrak{g}$ and its dual $M_4(\mathbb{k})/\mathfrak{g}^\perp$ have the same cardinality we must have $\mathfrak{g}' = \mathfrak{g}^\perp$. Thus the pairing $\langle \cdot, \cdot \rangle$ restricts to an isomorphism $\mathfrak{g} \to \widehat{\mathfrak{g}}$. Composing with the isomorphism $\widehat{\log} : \widehat{\mathfrak{g}} \to \widehat{G_0} = \mathrm{Irr}(G_0)$ shows that $y \mapsto \varphi_y$ is an isomorphism $\mathfrak{g} \to \mathrm{Irr}(G_0)$. The G-equivariance of this map follows from the invariance of the trace. Similar arguments apply to $\mathfrak{l}$ and $\mathfrak{d}$. □

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Theorem 4.2, applied to this particularly simple setting, gives the following identification of the induction maps

$$i_0 := i_{U_0, V_0} : \mathrm{Irr}\,(L_0) \to \mathrm{Irr}\,(G_0) \quad \text{and} \quad i'_0 := i_{U'_0, V'_0} : \mathrm{Irr}\,(D_0) \to \mathrm{Irr}\,(L_0).$$

Lemma 5.6. *The diagram*

$$\begin{array}{ccccc} \mathrm{Irr}\,(D_0) & \xrightarrow{\;i'_0\;} & \mathrm{Irr}\,(L_0) & \xrightarrow{\;i_0\;} & \mathrm{Irr}\,(G_0) \\ {\scriptstyle \mathfrak{y}\mapsto\varphi_{\mathfrak{y}}}\uparrow{\scriptstyle\cong} & & {\scriptstyle \mathfrak{y}\mapsto\varphi_{\mathfrak{y}}}\uparrow{\scriptstyle\cong} & & {\scriptstyle \mathfrak{y}\mapsto\varphi_{\mathfrak{y}}}\uparrow{\scriptstyle\cong} \\ \mathfrak{d} & \xrightarrow{\text{inclusion}} & \mathfrak{l} & \xrightarrow{\text{inclusion}} & \mathfrak{g} \end{array}$$

is commutative.

Proof. In view of Theorem 4.2 and Lemma 5.4, it is enough to observe that the diagram

$$\begin{array}{ccccc} \widehat{\mathfrak{d}} & \xrightarrow{\;(\Lambda')^*\;} & \widehat{\mathfrak{l}} & \xrightarrow{\;\Lambda^*\;} & \widehat{\mathfrak{g}} \\ {\scriptstyle \mathfrak{y}\mapsto\langle\cdot,\mathfrak{y}\rangle}\uparrow{\scriptstyle\cong} & & {\scriptstyle \mathfrak{y}\mapsto\langle\cdot,\mathfrak{y}\rangle}\uparrow{\scriptstyle\cong} & & {\scriptstyle \mathfrak{y}\mapsto\langle\cdot,\mathfrak{y}\rangle}\uparrow{\scriptstyle\cong} \\ \mathfrak{d} & \xrightarrow{\text{inclusion}} & \mathfrak{l} & \xrightarrow{\text{inclusion}} & \mathfrak{g} \end{array}$$

commutes, where Λ is the projection of $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{l} \oplus \mathfrak{v}$ onto its summand $\mathfrak{l}$, and Λ' is the projection of $\mathfrak{l}$ onto its summand $\mathfrak{d}$. □

5.2. Application of Clifford theory

We shall use Theorems 3.4, 3.6 and 3.14 to transport the functors i_L^G and r_L^G to the setting of (projective) representations of the centralisers $\overline{L}(\mathfrak{y}) = \overline{L}(\varphi_{\mathfrak{y}}) \subseteq GL_2\,(\Bbbk)$ and $\overline{G}(\mathfrak{y}) = \overline{G}(\varphi_{\mathfrak{y}}) \subseteq Sp_4\,(\Bbbk)$ associated to the characters $\varphi_{\mathfrak{y}}$.

The first assertion (C1) of Clifford theory decomposes the categories $\mathcal{R}(D)$, $\mathcal{R}(L)$ and $\mathcal{R}(G)$ as products over the sets $D\backslash\,\mathrm{Irr}\,(D_0)$, $L\backslash\,\mathrm{Irr}\,(L_0)$ and $G\backslash\,\mathrm{Irr}\,(G_0)$, respectively. For each $\mathfrak{y} \in \mathfrak{g}$, let $\varphi_{\mathfrak{y}}$ be the character in $\mathrm{Irr}\,(G_0)$ defined in Lemma 5.5. We denote by

$$E_{\mathfrak{y}}^G : \mathcal{R}(G) \to \mathcal{R}(G)_{\varphi_{\mathfrak{y}}}$$

the projection onto the subcategory associated to (the G-orbit of) the character $\varphi_{\mathfrak{y}}$. We similarly define $E_{\mathfrak{y}}^L$ and $E_{\mathfrak{y}}^D$, for $\mathfrak{y} \in \mathfrak{l}$ and $\mathfrak{y} \in \mathfrak{d}$ respectively.

For each $\mathfrak{y} \in \mathfrak{l}$ we write

$$G(\mathfrak{y}, \mathfrak{l}) := \{g \in G \mid g \cdot \mathfrak{y} \in \mathfrak{l}\}$$

for the set of elements in G which conjugate $\mathfrak{y}$ back into $\mathfrak{l}$. Notice that $G(\mathfrak{y}, \mathfrak{l})$ is stable under left multiplication by L, and under right multiplication by $G(\mathfrak{y})$.

The first step is to show that we may deal with ordinary, as opposed to projective, representations of the centralisers.

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Lemma 5.7. *There is a family of maps* $(\varphi'_y)_{y\in\mathfrak{l}}$ *with the following properties:*

(1) φ'_y *is a one-dimensional (ordinary) representation of the centraliser* $G(y)$ *that extends* φ_y.

(2) *For each* $g \in G(y, \mathfrak{l})$ *one has* $\mathrm{Ad}_g(\varphi'_y) = \varphi'_{g\cdot y}$.

(3) $\varphi'_y(g) = 1$ *for all* $g \in U(y) \cup V(y)$.

(4) *If* $y \in \mathfrak{d}$ *then* $\varphi'_y(g) = 1$ *for all* $g \in U'(y) \cup V'(y)$.

Proof. For each $y \in \mathfrak{l} \subset M_4(\Bbbk)$, let $H(y)$ denote the centraliser of y in the group $\mathrm{GL}_4(\mathfrak{o}_2)$ (which acts on $M_4(\Bbbk)$ through the adjoint action of its quotient $\mathrm{GL}_4(\Bbbk)$). Singla showed in [40, Proposition 2.2] that the character φ_y extends to a linear character of $H(y)$. If φ'_y is such an extension, then for each $g \in G(y, \mathfrak{l})$ the character $\mathrm{Ad}_g(\varphi'(y))$ is an extension of $\varphi_{g\cdot y}$ to $H(g\cdot y)$. Moreover, if $g \in G(y)$ then $\mathrm{Ad}_g(\varphi'_y) = \varphi'_y$. We may thus choose a family of characters φ'_y satisfying (1) and (2) by fixing one y in each G-orbit, choosing an extension φ'_y as above, and then defining $\varphi'_{g\cdot y} := \mathrm{Ad}_g(\varphi'_y)$ for each $g \in G(y, \mathfrak{l})$.

We prove that the characters φ'_y constructed above are trivial on $U(y)$ and $V(y)$ by showing that these two groups belong to the commutator subgroup of $H(y)$. Indeed, let $m \in M_2(\mathfrak{o}_2)$ be any matrix such that the 4×4 matrix $u = \left[\begin{smallmatrix}1 & m\\ & 1\end{smallmatrix}\right]$ lies in $H(y)$. Then the matrices $u' = \left[\begin{smallmatrix}1 & m/2\\ & 1\end{smallmatrix}\right]$ and $z = \left[\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right]$ also lie in $H(y)$, and we have $u = [u', z]$. This shows that $U(y)$ lies in $[H(y), H(y)]$, and a similar argument applies to $V(y)$. Thus the family φ'_y constructed above satisfies condition (3).

Finally, if $y \in \mathfrak{d}$, then a similar argument to the above shows that $U'(y)$ and $V'(y)$ belong to the commutator subgroup of the centraliser of y inside the block-diagonal subgroup $\mathrm{diag}(\mathrm{GL}_2(\mathfrak{o}_2), \mathrm{GL}_2(\mathfrak{o}_2)) \subset \mathrm{GL}_4(\mathfrak{o}_2)$, and so property (4) is also satisfied. □

For the rest of § 5 we fix a family of characters φ'_y as in Lemma 5.7. As explained in § 3.1, Clifford theory gives equivalences of categories

$$F^L_y : \mathcal{R}(\overline{L}(y)) \xrightarrow{\otimes\varphi'_y} \mathcal{R}(L(y))_{\varphi_y} \xrightarrow{\mathrm{ind}^L_{L(y)}} \mathcal{R}(L)_{\varphi_y}$$

and

$$F^G_y : \mathcal{R}(\overline{G}(y)) \xrightarrow{\otimes\varphi'_y} \mathcal{R}(G(y))_{\varphi_y} \xrightarrow{\mathrm{ind}^G_{G(y)}} \mathcal{R}(G)_{\varphi_y}.$$

Lemma 5.8. *For each* $y \in \mathfrak{l}$, *each* $g \in G(y, \mathfrak{l})$ *and each* $h \in N_G(L)$, *the diagrams*

$$\begin{array}{ccc} \mathcal{R}(G)_{\varphi_y} & \xrightarrow{\mathrm{id}} & \mathcal{R}(G)_{\varphi_{g\cdot y}} \\ \uparrow{\scriptstyle F^G_y} & & \uparrow{\scriptstyle F^G_{g\cdot y}} \\ \mathcal{R}(\overline{G}(y)) & \xrightarrow{\mathrm{Ad}_g} & \mathcal{R}(\overline{G}(g\cdot y)) \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathcal{R}(L)_{\varphi_y} & \xrightarrow{\mathrm{Ad}_h} & \mathcal{R}(L)_{\varphi_{h\cdot y}} \\ \uparrow{\scriptstyle F^L_y} & & \uparrow{\scriptstyle F^L_{h\cdot y}} \\ \mathcal{R}(\overline{L}(y)) & \xrightarrow{\mathrm{Ad}_h} & \mathcal{R}(\overline{L}(h\cdot y)) \end{array}$$

commute up to natural isomorphism.

Proof. The commutativity of the second diagram follows from property (2) of Lemma 5.7, and from the well-known fact that $\mathrm{Ad}_h \circ \mathrm{ind}^L_{L(y)} \cong \mathrm{ind}^L_{L(h\cdot y)} \circ \mathrm{Ad}_h$. The commutativity of the first diagram follows from the same argument, plus the fact that Ad_g is isomorphic to the identity functor on $\mathcal{R}(G)$ for every $g \in G$. □

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For each $\mathfrak{y} \in \mathfrak{l}$ we consider the functors

$$i_{\overline{L}(\mathfrak{y})}^{\overline{G}(\mathfrak{y})} := i_{\overline{U}(\mathfrak{y}),\overline{V}(\mathfrak{y})} : \mathcal{R}(\overline{L}(\mathfrak{y})) \to \mathcal{R}(\overline{G}(\mathfrak{y})) \quad \text{and} \quad r_{\overline{L}(\mathfrak{y})}^{\overline{G}(\mathfrak{y})} : \mathcal{R}(\overline{G}(\mathfrak{y})) \to \mathcal{R}(\overline{L}(\mathfrak{y})).$$

Lemma 5.9. *For each* $\mathfrak{y} \in \mathfrak{l}$ *the diagrams*

$$\begin{array}{ccc} \mathcal{R}(L)_{\varphi_{\mathfrak{y}}} & \xrightarrow{\;i_L^G\, E_{\mathfrak{y}}^L\;} & \mathcal{R}(G)_{\varphi_{\mathfrak{y}}} \\ {\scriptstyle F_{\mathfrak{y}}^L}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle F_{\mathfrak{y}}^G} \\ \mathcal{R}(\overline{L}(\mathfrak{y})) & \xrightarrow{\;i_{\overline{L}(\mathfrak{y})}^{\overline{G}(\mathfrak{y})}\;} & \mathcal{R}(\overline{G}(\mathfrak{y})) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{R}(G)_{\varphi_{\mathfrak{y}}} & \xrightarrow{\;E_{\mathfrak{y}}^L\, r_L^G\;} & \mathcal{R}(L)_{\varphi_{\mathfrak{y}}} \\ {\scriptstyle F_{\mathfrak{y}}^G}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle F_{\mathfrak{y}}^L} \\ \mathcal{R}(\overline{G}(\mathfrak{y})) & \xrightarrow{\;r_{\overline{L}(\mathfrak{y})}^{\overline{G}(\mathfrak{y})}\;} & \mathcal{R}(\overline{L}(\mathfrak{y})) \end{array}$$

commute up to natural isomorphism.

Proof. The fact that $\varphi'_{\mathfrak{y}}$ is trivial on the subgroups $U(\mathfrak{y})$ and $V(\mathfrak{y})$ ensures that the functions $\mathfrak{a}$ and $\mathfrak{b}$ of Lemma 3.12 are identically equal to 1, and thus that the functor $i_{\overline{U}(\mathfrak{y}),\overline{V}(\mathfrak{y})}^{\varphi'_{\mathfrak{y}}}$ appearing in Theorem 3.14 is equal to $i_{\overline{L}(\mathfrak{y})}^{\overline{G}(\mathfrak{y})}$. This proves the commutativity of the i-diagram; taking adjoints proves the commutativity of the r-diagram. □

Combining Lemmas 5.8 and 5.9 gives immediately:

Lemma 5.10. *For each* $\mathfrak{y} \in \mathfrak{l}$ *and each* $g \in G(\mathfrak{y}, \mathfrak{l})$ *the diagram*

$$\begin{array}{ccccccc} \mathcal{R}(L)_{\varphi_{\mathfrak{y}}} & \xrightarrow{\;i_L^G\, E_{\mathfrak{y}}^L\;} & \mathcal{R}(G)_{\varphi_{\mathfrak{y}}} & \xrightarrow{\;\mathrm{id}\;} & \mathcal{R}(G)_{\varphi_{g\cdot\mathfrak{y}}} & \xrightarrow{\;E_{g\cdot\mathfrak{y}}^L\, r_L^G\;} & \mathcal{R}(L)_{\varphi_{g\cdot\mathfrak{y}}} \\ {\scriptstyle F_{\mathfrak{y}}^L}\uparrow & & {\scriptstyle F_{\mathfrak{y}}^G}\uparrow & & {\scriptstyle F_{g\cdot\mathfrak{y}}^G}\uparrow & & {\scriptstyle F_{g\cdot\mathfrak{y}}^L}\uparrow \\ \mathcal{R}(\overline{L}(\mathfrak{y})) & \xrightarrow{\;i_{\overline{L}(\mathfrak{y})}^{\overline{G}(\mathfrak{y})}\;} & \mathcal{R}(\overline{G}(\mathfrak{y})) & \xrightarrow{\;\mathrm{Ad}_g\;} & \mathcal{R}(\overline{G}(g\cdot\mathfrak{y})) & \xrightarrow{\;r_{\overline{L}(g\cdot\mathfrak{y})}^{\overline{G}(g\cdot\mathfrak{y})}\;} & \mathcal{R}(\overline{L}(g\cdot\mathfrak{y})) \end{array}$$

commutes up to natural isomorphism. □

Now we use Clifford theory to analyse the right-hand side $\mathrm{id} \oplus \mathrm{Ad}_s \oplus i_D^L\, \mathrm{Ad}_w\, r_D^L$ of the Mackey formula (cf. Remarks 5.3(3)). For each pair of elements $\mathfrak{y}, \mathfrak{z} \in \mathfrak{l}$, define a functor

$$\Delta(\mathfrak{z}, \mathfrak{y}) : \mathcal{R}(\overline{L}(\mathfrak{y})) \to \mathcal{R}(\overline{L}(\mathfrak{z})), \quad \Delta(\mathfrak{z}, \mathfrak{y}) = \begin{cases} \mathrm{Ad}_l & \text{if } \mathfrak{z} = l\cdot\mathfrak{y} \\ 0 & \text{if } \mathfrak{z} \notin L\cdot\mathfrak{y}. \end{cases}$$

Note that $\Delta(\mathfrak{z}, l)$ is well-defined up to natural isomorphism, because $\mathrm{Ad}_l \cong \mathrm{id}$ on $\mathcal{R}(\overline{L}(\mathfrak{y}))$ for every $l \in L(\mathfrak{y})$.

Lemma 5.11. *For each* $\mathfrak{y}, \mathfrak{z} \in \mathfrak{l}$ *the diagrams*

$$\begin{array}{ccc} \mathcal{R}(L)_{\varphi_{\mathfrak{y}}} & \xrightarrow{\;E_{\mathfrak{z}}^L E_{\mathfrak{y}}^L\;} & \mathcal{R}(L)_{\varphi_{\mathfrak{z}}} \\ {\scriptstyle F_{\mathfrak{y}}^L}\uparrow & & \uparrow{\scriptstyle F_{\mathfrak{z}}^L} \\ \mathcal{R}(\overline{L}(\mathfrak{y})) & \xrightarrow{\;\Delta(\mathfrak{z},\mathfrak{y})\;} & \mathcal{R}(\overline{L}(\mathfrak{z})) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{R}(L)_{\varphi_{\mathfrak{y}}} & \xrightarrow{\;E_{\mathfrak{z}}^L\, \mathrm{Ad}_s\, E_{\mathfrak{y}}^L\;} & \mathcal{R}(L)_{\varphi_{\mathfrak{z}}} \\ {\scriptstyle F_{\mathfrak{y}}^L}\uparrow & & \uparrow{\scriptstyle F_{\mathfrak{z}}^L} \\ \mathcal{R}(\overline{L}(\mathfrak{y})) & \xrightarrow{\;\Delta(\mathfrak{z},s\cdot\mathfrak{y})\,\mathrm{Ad}_s\;} & \mathcal{R}(\overline{L}(\mathfrak{z})) \end{array}$$

commute up to natural isomorphism.

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Proof. The commutativity of the first diagram follows from Lemma 5.8, and from the fact that $\mathsf{E}^{\mathsf{L}}_z\mathsf{E}^{\mathsf{L}}_y = 0$ unless y and z are L-conjugate. The commutativity of the second diagram follows from a similar argument, plus the equality $\mathrm{Ad}_s\,\mathsf{E}^{\mathsf{L}}_y = \mathsf{E}^{\mathsf{L}}_{s\cdot y}\,\mathrm{Ad}_s$. □

The analysis of the functors $\mathrm{i}^{\mathsf{L}}_{\mathsf{D}}$ and $\mathrm{r}^{\mathsf{L}}_{\mathsf{D}}$ follows the above analysis of $\mathrm{i}^{\mathsf{G}}_{\mathsf{L}}$ and $\mathrm{r}^{\mathsf{G}}_{\mathsf{L}}$. Because D is abelian we have $\mathsf{D}(y) = \mathsf{D}$ for each $y \in \mathfrak{d}$. Clifford theory gives an equivalence of categories

$$\mathsf{F}^{\mathsf{D}}_y : \mathcal{R}(\overline{\mathsf{D}}) \xrightarrow{\otimes\varphi'_y} \mathcal{R}(\mathsf{D})_{\varphi_y},$$

such that for each $h \in \mathsf{N}_{\mathsf{G}}(\mathsf{D})$ the diagram

$$\begin{array}{ccc} \mathcal{R}(\mathsf{D})_{\varphi_y} & \xrightarrow{\mathrm{Ad}_h} & \mathcal{R}(\mathsf{D})_{\varphi_{h\cdot y}} \\ \uparrow{\scriptstyle \mathsf{F}^{\mathsf{D}}_y} & & \uparrow{\scriptstyle \mathsf{F}^{\mathsf{D}}_{h\cdot y}} \\ \mathcal{R}(\overline{\mathsf{D}}) & \xrightarrow{\mathrm{Ad}_h} & \mathcal{R}(\overline{\mathsf{D}}) \end{array}$$

commutes up to natural isomorphism.

We consider the functors

$$\mathrm{i}^{\overline{\mathsf{L}}(y)}_{\overline{\mathsf{D}}} := \mathrm{i}_{\overline{\mathsf{U}'_0(y)},\overline{\mathsf{V}'_0(y)}} : \mathcal{R}(\overline{\mathsf{D}}) \to \mathcal{R}(\overline{\mathsf{L}}(y)) \quad \text{and} \quad \mathrm{r}^{\overline{\mathsf{L}}(y)}_{\overline{\mathsf{D}}} := \mathrm{r}_{\overline{\mathsf{U}'_0(y)},\overline{\mathsf{V}'_0(y)}} : \mathcal{R}(\overline{\mathsf{L}}(y)) \to \mathcal{R}(\overline{\mathsf{D}}).$$

For each $y, z \in \mathfrak{l}$, we define the functor

$$\Xi(z,y) : \mathcal{R}(\overline{\mathsf{L}}(y)) \to \mathcal{R}(\overline{\mathsf{L}}(z))$$

as the direct sum, over $d \in \mathfrak{d}$, of the compositions

$$\mathcal{R}(\overline{\mathsf{L}}(y)) \xrightarrow{\Delta(d,y)} \mathcal{R}(\overline{\mathsf{L}}(d)) \xrightarrow{\mathrm{r}^{\overline{\mathsf{L}}(d)}_{\overline{\mathsf{D}}}} \mathcal{R}(\overline{\mathsf{D}}) \xrightarrow{\mathrm{Ad}_w} \mathcal{R}(\overline{\mathsf{D}}) \xrightarrow{\mathrm{i}^{\overline{\mathsf{L}}(w\cdot d)}_{\overline{\mathsf{D}}}} \mathcal{R}(\overline{\mathsf{L}}(w\cdot d)) \xrightarrow{\Delta(z,w\cdot d)} \mathcal{R}(\overline{\mathsf{L}}(z)).$$

Lemma 5.12. *For each $y, z \in \mathfrak{l}$ the diagram*

$$\begin{array}{ccc} \mathcal{R}(\mathsf{L})_{\varphi_y} & \xrightarrow{\mathsf{E}^{\mathsf{L}}_z(\mathrm{i}^{\mathsf{L}}_{\mathsf{D}}\,\mathrm{Ad}_w\,\mathrm{r}^{\mathsf{L}}_{\mathsf{D}})\mathsf{E}^{\mathsf{L}}_y} & \mathcal{R}(\mathsf{L})_{\varphi_z} \\ \uparrow{\scriptstyle \mathsf{F}^{\mathsf{L}}_y} & & \uparrow{\scriptstyle \mathsf{F}^{\mathsf{L}}_z} \\ \mathcal{R}(\overline{\mathsf{L}}(y)) & \xrightarrow{\Xi(z,y)} & \mathcal{R}(\overline{\mathsf{L}}(z)) \end{array}$$

commutes up to natural isomorphism.

Proof. Decomposing the category $\mathcal{R}(\mathsf{D})$ over $\mathrm{Irr}\,(\mathsf{D}_0) \cong \mathfrak{d}$, we have

$$\begin{aligned} \mathrm{i}^{\mathsf{L}}_{\mathsf{D}}\,\mathrm{Ad}_w\,\mathrm{r}^{\mathsf{L}}_{\mathsf{D}} &= \bigoplus_{d\in\mathfrak{d}} \mathrm{i}^{\mathsf{L}}_{\mathsf{D}}\,\mathrm{Ad}_w\,\mathsf{E}^{\mathsf{D}}_d\,\mathrm{r}^{\mathsf{L}}_{\mathsf{D}} = \bigoplus_{d\in\mathfrak{d}} \mathrm{i}^{\mathsf{L}}_{\mathsf{D}}\,\mathsf{E}^{\mathsf{D}}_{w\cdot d}\,\mathrm{Ad}_w\,\mathsf{E}^{\mathsf{D}}_d\,\mathrm{r}^{\mathsf{L}}_{\mathsf{D}} \\ &= \bigoplus_{d\in\mathfrak{d}} \mathsf{E}^{\mathsf{L}}_{w\cdot d}\,\mathrm{i}^{\mathsf{L}}_{\mathsf{D}}\,\mathsf{E}^{\mathsf{D}}_{w\cdot d}\,\mathrm{Ad}_w\,\mathsf{E}^{\mathsf{D}}_d\,\mathrm{r}^{\mathsf{L}}_{\mathsf{D}}\,\mathsf{E}^{\mathsf{L}}_d \end{aligned}$$

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where in the last equality we have used Theorem 3.4. After applying Lemma 5.11 to the functors $E^L_y E^L_{w\cdot d}$ and $E^L_d E^L_z$, we are left to prove the commutativity of

$$\begin{CD}
\mathcal{R}(L)_{\varphi_d} @>{E^D_d\, r^L_D}>> \mathcal{R}(D)_{\varphi_d} @>{\mathrm{Ad}_w}>> \mathcal{R}(D)_{\varphi_{w\cdot d}} @>{i^L_D}>> \mathcal{R}(L)_{\varphi_{w\cdot d}}\\
@A{F^L_d}AA @A{F^D_d}AA @A{F^D_{w\cdot d}}AA @A{F^L_{w\cdot d}}AA\\
\mathcal{R}(\overline{L}(d)) @>{r^{\overline{L}(d)}_{\overline{D}}}>> \mathcal{R}(\overline{D}) @>{\mathrm{Ad}_w}>> \mathcal{R}(\overline{D}) @>{i^{\overline{L}(w\cdot d)}_{\overline{D}}}>> \mathcal{R}(\overline{L}(w\cdot d))
\end{CD}$$

for each $d \in \mathfrak{d}$. This follows from Theorem 3.14, just as in Lemma 5.10. □

The end result of our Clifford analysis is as follows:

Corollary 5.13. *Theorem 5.2 is equivalent to the assertion that for every $y \in \mathfrak{l}$ and every $g \in G(y, \mathfrak{l})$, there is a natural isomorphism*

$$r^{\overline{G}(g\cdot y)}_{\overline{L}(g\cdot y)}\ \mathrm{Ad}_g\ i^{\overline{G}(y)}_{\overline{L}(y)} \cong \Delta(g\cdot y, y) \bigoplus \Delta(g\cdot y, s\cdot y)\,\mathrm{Ad}_s \bigoplus \Xi(g\cdot y, y)$$

of functors $\mathcal{R}(\overline{L}(y)) \to \mathcal{R}(\overline{L}(g\cdot y))$.

Proof. By Lemma 5.5 and (C1) we have

$$r^G_L\, i^G_L = \bigoplus_{L\cdot y,\ L\cdot z \in L\backslash \mathfrak{l}} E^L_z\, r^G_L\, i^G_L\, E^L_y.$$

Theorem 3.4 implies that

$$E^L_z\, r^G_L\, i^G_L\, E^L_y = E^L_z\, r^G_L\, E^G_z E^G_y\, i^G_L\, E^L_y,$$

and $E^G_z E^G_y = 0$ unless z and y lie in the same G-orbit. This proves that

$$r^G_L\, i^G_L = \bigoplus_{\substack{L\cdot y \in L\backslash \mathfrak{l},\\ g\in L\backslash G(y,\mathfrak{l})/G(y)}} E^L_{g\cdot y}\, r^G_L\, i^G_L\, E^L_y. \tag{5.14}$$

Clifford theory likewise gives a decomposition

$$\mathrm{id} \oplus \mathrm{Ad}_s \oplus i^L_D\, \mathrm{Ad}_w\, r^L_D = \bigoplus_{L\cdot y,\ L\cdot z \in L\backslash \mathfrak{l}} (E^L_z E^L_y \oplus E^L_z\, \mathrm{Ad}_s\, E^L_y \oplus E^L_z\, i^L_D\, \mathrm{Ad}_w\, r^L_D\, E^L_y),$$

and Lemmas 5.11 and 5.12 imply that each term in the sum vanishes if z and y are not G-conjugate. This proves that

$$\mathrm{id} \oplus \mathrm{Ad}_s \oplus i^L_D\, \mathrm{Ad}_w\, r^L_D = \bigoplus_{\substack{L\cdot y \in L\backslash \mathfrak{l},\\ g\in L\backslash G(y,\mathfrak{l})/G(y)}} (E^L_{g\cdot y} E^L_y \oplus E^L_{g\cdot y}\, \mathrm{Ad}_s\, E^L_y \oplus E^L_{g\cdot y}\, i^L_D\, \mathrm{Ad}_w\, r^L_D\, E^L_y). \tag{5.15}$$

Thus the reformulation of Theorem 5.2 given in Remarks 5.3(3) is equivalent to the existence of a natural isomorphism, for each $y \in \mathfrak{l}$ and each $g \in G(y, \mathfrak{l})$, between the (y, g) terms on the right-hand sides of (5.14) and (5.15). Conjugating each of these terms by the equivalences F^L_y and $F^L_{g\cdot y}$, and applying Lemmas 5.10, 5.11 and 5.12, we bring Theorem 5.2 into the asserted form. □

5.3. Centralisers

We now present the facts about the centralisers $\overline{G}(y)$ and $\overline{L}(y)$ and about the orbit spaces $L\backslash G(y,\mathfrak{l})/G(y)$ that will be needed for the proof of Theorem 5.2. More details, and the proofs of the assertions made here, are given in Appendix A.

Fix $x \in M_2(\Bbbk)$, and let $y = \operatorname{diag}(x, -x^t)$ be the corresponding element of $\mathfrak{l}$. We divide our analysis according to the Jordan normal form of x. Up to conjugacy by $\overline{L} \cong \mathrm{GL}_2(\Bbbk)$, the following nine cases exhaust all of the possibilities. In the following s, t, w and σ are as in Remarks 5.3. We shall write '$L\backslash G(y,\mathfrak{l})/G(y) = \{g, h, k\}$' to mean that $G(y,\mathfrak{l}) = LgG(y) \sqcup LhG(y) \sqcup LkG(y)$.

Case 1. $x = \operatorname{diag}(\mu,\mu)$, $\mu \in \Bbbk$. In this case $\overline{L}(y) = \overline{L} \cong \mathrm{GL}_2(\Bbbk)$. There are two subcases:

1A: $\mu \neq 0$. Here $\overline{G}(y) = \overline{L}(y)$, and $L\backslash G(y,\mathfrak{l})/G(y) = \{1, s, w\}$.

1B: $\mu = 0$. Here $\overline{G}(y) = \overline{G}$ and $L\backslash G(y,\mathfrak{l})/G(y) = \{1\}$.

Case 2. $x = \operatorname{diag}(\mu,\nu)$, $\mu \neq \nu$. In this case $\overline{L}(y) = \overline{D}$. There are three subcases:

2A: $\mu \neq \pm\nu$, $\mu \neq 0 \neq \nu$. Here $\overline{G}(y) = \overline{L}(y)$, and $L\backslash G(y,\mathfrak{l})/G(y) = \{1, s, w, wt\}$.

2A*: $\nu = 0$. Here $\overline{G}(y)$ is a reductive group over $\Bbbk$; $\overline{L}(y) = \overline{D}$ is a rational maximal torus, whose Weyl group in $\overline{G}(y)$ is generated (modulo $\overline{D}$) by the involution $t^{-1}wt$; and $\overline{U}(y)$ and $\overline{V}(y)$ are the unipotent radicals of an opposite pair of rational Borel subgroups of $\overline{G}(y)$ containing $\overline{L}(y)$. We have $L\backslash \overline{G}(y)/G(y) = \{1, w\}$.

2B: $\mu = -\nu$. In this case we have $\overline{G}(y) = \mathrm{Ad}_w(\overline{L})$, $\overline{U}(y) = \mathrm{Ad}_w(\overline{V'})$ and $\overline{V}(y) = \mathrm{Ad}_w(\overline{U'})$, while $L\backslash G(y,\mathfrak{l})/G(y) = \{1, w, wt\}$.

Case 3. $x = \left[\begin{smallmatrix} \alpha & \beta \\ \mu\beta & \alpha \end{smallmatrix}\right]$, $\mu \in \Bbbk$ non-square, $\alpha \in \Bbbk$, $\beta \in \Bbbk^\times$. In this case $\Bbbk_2 := M_2(\Bbbk)(x)$ is a quadratic field extension of $\Bbbk$, and $\overline{L}(y) = \{\operatorname{diag}(a, a^{-t}) \mid a \in \Bbbk_2^\times\} \cong \Bbbk_2^\times$. There are two subcases.

3A: $\alpha \neq 0$. We have $\overline{G}(y) = \overline{L}(y)$ and $L\backslash G(y,\mathfrak{l})/G(y) = \{1, s\}$.

3B: $\alpha = 0$. Here $\overline{G}(y)$ is a reductive group over $\Bbbk$; $\overline{L}(y)$ is a rational maximal torus of $\overline{G}(y)$ whose Weyl group is generated by s; and $\overline{U}(y)$ and $\overline{V}(y)$ are the unipotent radicals of an opposite pair of rational Borel subgroups of $\overline{G}(y)$ containing $\overline{L}(y)$. In this case $L\backslash G(y,\mathfrak{l})/G(y) = \{1\}$.

Case 4. $x = \left[\begin{smallmatrix} \mu & 1 \\ & \mu \end{smallmatrix}\right]$. In this case $\overline{L}(y) = \{\operatorname{diag}(a, a^{-t}) \mid a \in \Bbbk[x]\} \cong \mathrm{GL}_1(\Bbbk[\varepsilon]/(\varepsilon^2))$. There are two subcases.

4A: $\mu \neq 0$. Here $\overline{G}(y) = \overline{L}(y)$ and $L\backslash G(y,\mathfrak{l})/G(y) = \{1, s\}$.

4B: $\mu = 0$. The subgroups $\overline{U}(y)$ and $\overline{V}(y)$ commute with one another in $\overline{G}(y)$, and we have

$$\overline{G}(y) = (\overline{U}(y) \times \overline{V}(y)) \rtimes (\overline{L}(y) \rtimes S)$$

where S is the two-element group generated by s. We have $L\backslash G(y,\mathfrak{l})/G(y) = \{1\}$.

5.4. Proof of Theorem 5.2

In this section we shall use the results of the previous section to prove that for each $y \in \mathfrak{l}$ and each $g \in G(y,\mathfrak{l})$, there is a natural isomorphism

$$\mathrm{r}^{\overline{G}(g\cdot y)}_{\overline{L}(g\cdot y)} \mathrm{Ad}_g\, \mathrm{i}^{\overline{G}(y)}_{\overline{L}(y)} \cong \Delta(g\cdot y, y) \bigoplus \Delta(g\cdot y, s\cdot y)\,\mathrm{Ad}_s \bigoplus \Xi(g\cdot y, y) \tag{5.16}$$

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of functors $\mathcal{R}(\overline{\mathrm{L}}(y)) \to \mathcal{R}(\overline{\mathrm{L}}(g\cdot y))$. By Corollary 5.13, this constitutes a proof of Theorem 5.2. We recall from § 5.2 that

$$\Delta(z,y) = \begin{cases} \mathrm{Ad}_{l} & \text{if } z = l\cdot y \\ 0 & \text{if } z \notin \mathrm{L}\cdot y \end{cases} \quad \text{and} \quad \Xi(z,y) = \bigoplus_{d\in\mathfrak{d}} \left(\Delta(z, w\cdot d)\, \mathrm{i}_{\overline{\mathrm{D}}}^{\overline{\mathrm{L}}(w\cdot d)}\, \mathrm{Ad}_{w}\, \mathrm{r}_{\overline{\mathrm{D}}}^{\overline{\mathrm{L}}(d)}\, \Delta(d,y)\right)$$

for all $y, z \in \mathfrak{l}$.

The proof of (5.16) goes through a case-by-case analysis of the various possibilities for $y = \mathrm{diag}\,(x, -x^{t})$. The cases are labelled as in § 5.3. The reader who is more interested in ideas than in details might like to focus on cases 1A , 2A* and 4B, which together contain all of the techniques used in the other cases.

Case 1A. Take $y = \mathrm{diag}\,(\mu,\mu,-\mu,-\mu)$, $\mu \neq 0$. We must consider $g = 1$, $g = s$ and $g = w$.

For $g = 1$, the left-hand side of (5.16) is the identity on $\mathcal{R}(\overline{\mathrm{L}})$, because all of the centralisers are equal to $\overline{\mathrm{L}}$. We have $\Delta(y, s\cdot y) = 0$ because y and $s\cdot y = -y$ are not L-conjugate. The only diagonal matrix $d \in \mathfrak{d}$ that is L-conjugate to y is $d = y$ itself, and we have $\Delta(y, w\cdot y) = 0$, and so $\Xi(y, y) = 0$. Thus the only nonzero term on the right-hand side of (5.16) is $\Delta(y, y)$, which is the identity on $\mathcal{R}(\overline{\mathrm{L}})$. Thus the two sides of (5.16) are isomorphic.

For $g = s$ all of the centralisers are again equal to $\overline{\mathrm{L}}$, and so the left-hand side of (5.16) equals Ad_s. One finds as above that the only nonzero term on the right-hand side is $\Delta(s\cdot y, s\cdot y)\,\mathrm{Ad}_s$, which equals Ad_s.

For $g = w$ we have $w\cdot y = \mathrm{diag}\,(-\mu,\mu,\mu,-\mu)$, and so the centralisers of $g\cdot y$ are as in case 2B. The left-hand side of (5.16) is thus equal to $\mathrm{r}_{\overline{\mathrm{D}}}^{\overline{\mathrm{G}}(w\cdot y)}\,\mathrm{Ad}_w$. Since $\Delta(w\cdot y, y)$ and $\Delta(w\cdot y, s\cdot y)$ are both zero, the only potentially nonzero term on the right-hand side of (5.16) is $\Xi(w\cdot y, y)$. Since the only diagonal matrix that is L-conjugate to y is y itself, we have

$$\Xi(w\cdot y, y) = \Delta(w\cdot y, w\cdot y)\, \mathrm{i}_{\overline{\mathrm{D}}}^{\overline{\mathrm{L}}(w\cdot y)}\, \mathrm{Ad}_w\, \mathrm{r}_{\overline{\mathrm{D}}}^{\overline{\mathrm{L}}(y)}\, \Delta(y,y) = \mathrm{Ad}_w\, \mathrm{r}_{\overline{\mathrm{D}}}^{\overline{\mathrm{L}}}.$$

Now, we have $\overline{\mathrm{G}}(w\cdot y) = \mathrm{Ad}_w\,(\overline{\mathrm{L}})$, $\overline{\mathrm{U}}(w\cdot y) = \mathrm{Ad}_w\,(\overline{\mathrm{V}'})$, and $\overline{\mathrm{V}}(w\cdot y) = \mathrm{Ad}_w\,(\overline{\mathrm{U}'})$ (see case 2B in § 5.3), and therefore

$$\mathrm{r}_{\overline{\mathrm{D}}}^{\overline{\mathrm{G}}(w\cdot y)}\,\mathrm{Ad}_w = \mathrm{r}_{\overline{\mathrm{U}}(w\cdot y),\overline{\mathrm{V}}(w\cdot y)}\,\mathrm{Ad}_w \cong \mathrm{Ad}_w\,\mathrm{r}_{\overline{\mathrm{V}'},\overline{\mathrm{U}'}} \cong \mathrm{Ad}_w\,\mathrm{r}_{\overline{\mathrm{U}'},\overline{\mathrm{V}'}} = \mathrm{Ad}_w\,\mathrm{r}_{\overline{\mathrm{D}}}^{\overline{\mathrm{L}}}$$

where we used Theorem 2.18(1) to switch $\overline{\mathrm{U}'}$ and $\overline{\mathrm{V}'}$. This completes the proof of (5.16) in case 1A.

Case 2A. Take $y = \mathrm{diag}\,(\mu,\nu,-\mu,-\nu)$, where μ and ν are nonzero and $\mu \neq \pm\nu$. We must consider $g = 1$, $g = s$, $g = w$ and $g = wt$. For each of these g the matrix $g\cdot y$ is again of the form 2A, and so all of the centralisers appearing in (5.16) are equal to $\overline{\mathrm{D}}$, and the left-hand side of (5.16) is equal to the functor Ad_g on $\mathcal{R}(\overline{\mathrm{D}})$.

For $g = 1$ the functor $\Delta(y, y)$ equals the identity, while $\Delta(y, s\cdot y) = 0$ (because y and $s\cdot y$ are not L-conjugate) and $\Xi(y, y) = 0$ (because the only diagonal matrices that are

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L-conjugate to y are y and $t\cdot y$, and neither of these is L-conjugate to $w\cdot y$). So both sides of (5.16) equal the identity.

For $g=s$ the functor $\Delta(s\cdot y, s\cdot y)$ is the identity, while $\Delta(s\cdot y, y)$ and $\Xi(s\cdot y, s\cdot y)$ are both zero. So both sides of (5.16) equal Ad_s.

For $g=w$, the only potentially nonzero term on the right-hand side of (5.16) is $\Xi(w\cdot y, y)$. There are two diagonal matrices $d\in\mathfrak{d}$ that are L-conjugate to y, namely y itself and $t\cdot y$. Since $w\cdot y=\mathrm{diag}(-\mu,\nu,\mu,\nu)$ and $wt\cdot y=\mathrm{diag}(-\nu,\mu,\nu,-\mu)$ are not L-conjugate, we have $\Delta(w\cdot y, wt\cdot y)=0$, and so the summand in $\Xi(w\cdot y, w\cdot y)$ corresponding to $d=t\cdot y$ is equal to zero. Therefore,

$$\Xi(w\cdot y, y)=\Delta(w\cdot y, w\cdot y)\,\mathrm{i}_{\overline{D}}^{\overline{L}(w\cdot y)}\,\mathrm{Ad}_w\,\mathrm{r}_{\overline{D}}^{\overline{L}(y)}\,\Delta(y,y)=\mathrm{Ad}_w$$

as required.

For $g=wt$ the argument of the previous paragraph shows that the right-hand side of (5.16) is equal to $\Xi(wt\cdot y, y)$, and that only the $d=t\cdot y$ summand in the latter is nonzero. We have

$$\Xi(wt\cdot y, y)=\Delta(wt\cdot y, wt\cdot y)\,\mathrm{i}_{\overline{D}}^{\overline{L}(wt\cdot y)}\,\mathrm{Ad}_w\,\mathrm{r}_{\overline{D}}^{\overline{L}(t\cdot y)}\,\Delta(t\cdot y, y)=\mathrm{Ad}_w\,\mathrm{Ad}_t$$

because $\Delta(t\cdot y, y)=\mathrm{Ad}_t$ and all of the centralisers equal $\overline{D}$. This completes the proof of (5.16) in case 2A.

Case 2A*. Let $y=\mathrm{diag}(\mu,0,-\mu,0)$ where $\mu\neq 0$. We must consider $g=1$ and $g=w$.

For $g=1$, the left-hand side of (5.16) equals $\mathrm{r}_{\overline{D}}^{\overline{G}(y)}\,\mathrm{i}_{\overline{D}}^{\overline{G}(y)}$. We are in the situation of Example 2.10, and so $\mathrm{r}_{\overline{D}}^{\overline{G}(y)}$ and $\mathrm{i}_{\overline{D}}^{\overline{G}(y)}$ are isomorphic to the functors of Harish-Chandra restriction and induction (respectively) for the maximal torus $\overline{D}\subset\overline{G}(y)$. Since the Weyl group of $\overline{D}$ in $\overline{G}(y)$ is equal to $\{1, t^{-1}wt\}$, the usual Mackey formula (1.1) (cf. [13, Theorem 5.1]) for the composition of Harish-Chandra functors gives

$$\mathrm{r}_{\overline{D}}^{\overline{G}(y)}\,\mathrm{i}_{\overline{D}}^{\overline{G}(y)}\cong\mathrm{id}\oplus\mathrm{Ad}_{t^{-1}wt}\,.$$

Still taking $g=1$, we have $\Delta(g\cdot y, s\cdot y)=0$, and so the right-hand side of (5.16) equals $\mathrm{id}\oplus\Xi(y,y)$. The only diagonal matrices that are L-conjugate to y are $d=y$ and $d=t\cdot y$. For $d=y$ we have $\Delta(y, w\cdot y)=0$, and so the only potentially nonzero summand in $\Xi(y,y)$ is the one corresponding to $d=t\cdot y$. Computing this summand, we find

$$\Xi(y,y)=\Delta(y, wt\cdot y)\,\mathrm{i}_{\overline{D}}^{\overline{L}(wt\cdot y)}\,\mathrm{Ad}_w\,\mathrm{r}_{\overline{D}}^{\overline{L}(t\cdot y)}\,\Delta(t\cdot y, y)=\mathrm{Ad}_{t^{-1}}\,\mathrm{Ad}_w\,\mathrm{Ad}_t,$$

because $\overline{L}(wt\cdot y)=\overline{L}(t\cdot y)=\overline{D}$. Thus the right-hand side of (5.16) is, like the left-hand side, isomorphic to $\mathrm{id}\oplus\mathrm{Ad}_{t^{-1}wt}$.

Now take $g=w$. Notice that $w\cdot y=-y$. The left-hand side of (5.16) is

$$\mathrm{r}_{\overline{D}}^{\overline{G}(y)}\,\mathrm{Ad}_w\,\mathrm{i}_{\overline{D}}^{\overline{G}(y)}\cong\mathrm{Ad}_w\,\mathrm{r}_{\overline{D}}^{\overline{G}(y)}\,\mathrm{i}_{\overline{D}}^{\overline{G}(y)}\cong\mathrm{Ad}_w\oplus\mathrm{Ad}_{ts},$$

where for the first isomorphism we have used Theorem 2.18(1), and for the second we have used the Mackey formula (1.1) for Harish-Chandra induction together with the equality $wt^{-1}wt=ts$ in G.

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Keeping $g = w$ and turning to the right-hand side of (5.16), the term $\Delta(w \cdot y, y)$ vanishes, while the fact that $w \cdot y = ts \cdot y$ implies that $\Delta(w \cdot y, s \cdot y)\,\mathrm{Ad}_s = \mathrm{Ad}_{ts}$. So we are left to show that $\Xi(w \cdot y, y) = \mathrm{Ad}_w$. The $d = y$ term in $\Xi(w \cdot y, y)$ is equal to

$$\Delta(w \cdot y, w \cdot y)\, \mathrm{i}_{\mathrm{D}}^{\overline{\mathrm{D}}}\, \mathrm{Ad}_w\, \mathrm{r}_{\mathrm{D}}^{\overline{\mathrm{D}}}\, \Delta(y, y) = \mathrm{Ad}_w,$$

while the $d = t \cdot y$ term vanishes because $\Delta(w \cdot y, t \cdot y) = 0$. Thus both sides of (5.16) are isomorphic to $\mathrm{Ad}_w \oplus \mathrm{Ad}_{ts}$ in this case.

Case 3A. Take $x = \left[\begin{smallmatrix} \alpha & \beta \\ \beta\mu & \alpha \end{smallmatrix}\right]$, where $\mu \in \Bbbk$ is a non-square and $\alpha, \beta \in \Bbbk^\times$, and let $y = \operatorname{diag}(x, -x^t)$. We must consider $g = 1$ and $g = s$. We have $\overline{\mathrm{G}}(y) = \overline{\mathrm{G}}(s \cdot y) = \overline{\mathrm{L}}(s \cdot y) = \overline{\mathrm{L}}(y)$, so that the left-hand side of (5.16) is equal to Ad_g for each g. Note that since y is not L-conjugate to a diagonal matrix we have $\Xi(z, y) = 0$ for every z.

For $g = 1$ we have $\Delta(y, y) = \mathrm{id}$ while $\Delta(y, s \cdot y) = 0$, so both sides of (5.16) equal the identity.

For $g = s$ we have $\Delta(s \cdot y, y) = 0$ while $\Delta(s \cdot y, s \cdot y) = \mathrm{id}$ and so both sides of (5.16) equal Ad_s. So (5.16) holds in case 3A.

Case 4A. Take $x = \left[\begin{smallmatrix} \mu & 1 \\ & \mu \end{smallmatrix}\right]$, where $\mu \neq 0$, and let $y = \operatorname{diag}(x, -x^t)$. The argument is the same as in case 3A.

Case 1B. Take $y = 0$. We need only consider $g = 1$. Then (5.16) becomes the assertion that

$$\mathrm{r}_{\mathrm{L}}^{\overline{\mathrm{G}}}\, \mathrm{i}_{\mathrm{L}}^{\overline{\mathrm{G}}} \cong \mathrm{id} \oplus \mathrm{Ad}_s \oplus \mathrm{i}_{\mathrm{D}}^{\overline{\mathrm{L}}}\, \mathrm{Ad}_w\, \mathrm{r}_{\mathrm{D}}^{\overline{\mathrm{L}}}.$$

This is true: the functors $\mathrm{i}_{\mathrm{L}}^{\overline{\mathrm{G}}}$ and $\mathrm{r}_{\mathrm{L}}^{\overline{\mathrm{G}}}$ identify, as in Example 2.10, with the functors of Harish-Chandra induction and restriction for the Siegel Levi subgroup in $\overline{\mathrm{G}} = \mathrm{Sp}_4(\Bbbk)$, and the above formula is just the standard Mackey formula (1.1) for the composition of these functors.

Case 2B. Let $y = \operatorname{diag}(\mu, -\mu, -\mu, \mu)$, $\mu \neq 0$. We must consider $g = 1$, $g = w$ and $g = wt$.

For $g = 1$ the left-hand side of (5.16) is equal to

$$\mathrm{r}_{\mathrm{D}}^{\mathrm{Ad}_w(\overline{\mathrm{L}})}\, \mathrm{i}_{\mathrm{D}}^{\mathrm{Ad}_w(\overline{\mathrm{L}})} = \mathrm{Ad}_w\, \mathrm{r}_{\mathrm{D}}^{\overline{\mathrm{L}}}\, \mathrm{i}_{\mathrm{D}}^{\overline{\mathrm{L}}}\, \mathrm{Ad}_{w^{-1}} \cong \mathrm{Ad}_w\, (\mathrm{id} \oplus \mathrm{Ad}_t)\, \mathrm{Ad}_{w^{-1}} \cong \mathrm{id} \oplus \mathrm{Ad}_s$$

where we have identified $\mathrm{i}_{\mathrm{D}}^{\overline{\mathrm{L}}}$ and $\mathrm{r}_{\mathrm{D}}^{\overline{\mathrm{L}}}$ with Harish-Chandra functors and applied the usual Mackey formula (1.1) for the group $\overline{\mathrm{L}} \cong \mathrm{GL}_2(\Bbbk)$ and its diagonal torus $\overline{\mathrm{D}}$. On the right-hand side of (5.16) we have $\Delta(y, y) = \mathrm{id}$ and $\Delta(y, s \cdot y) = \mathrm{id}$, so we are left to show that $\Xi(y, y) = 0$. The only $d \in \mathfrak{d}$ with $\Delta(d, y) \neq 0$ are $d = y$ and $d = t \cdot y$. In both of these cases we have $\Delta(y, w \cdot d) = 0$, and so $\Xi(y, y) = 0$ as required.

The $g = w$ and $g = wt$ cases follow the argument for the '$g = w$ component' of case 1A. The left-hand side of (5.16) is isomorphic to $\mathrm{Ad}_g\, \mathrm{i}_{\mathrm{D}}^{\mathrm{Ad}_g(\overline{\mathrm{L}})}$, while the right-hand side is isomorphic to $\mathrm{i}_{\mathrm{D}}^{\overline{\mathrm{L}}}\, \mathrm{Ad}_g$, and the two sides are isomorphic to each other by Theorem 2.18(1).

Case 3B. Let $x = \left[\begin{smallmatrix} & \beta \\ \beta\mu & \end{smallmatrix}\right]$, $\mu \in \Bbbk$ a non-square, $\beta \in \Bbbk^\times$, and take $y = \operatorname{diag}(x, -x^t)$. We need consider only $g = 1$. We have on the one hand $\Delta(y, s \cdot y) = \mathrm{id}$, while on the other

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hand $\Xi(y, y) = 0$ (since y is not L-conjugate to any $d \in \mathfrak{d}$), and so (5.16) reads

$$r^{\overline{G}(y)}_{\overline{L}(y)} \, i^{\overline{G}(y)}_{\overline{L}(y)} \cong \mathrm{id} \oplus \mathrm{Ad}_s \, .$$

This is true: the functors on the right-hand side are isomorphic to Harish-Chandra functors as in Example 2.10, and the above formula is the usual Mackey formula for these functors.

Case 4B. Take $x = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $y = \operatorname{diag}(x, -x^t)$. We need only consider $g = 1$. As in case 3B, the right-hand side of (5.16) is $\mathrm{id} \oplus \mathrm{Ad}_s$, while the left-hand side is $r^{\overline{G}(y)}_{\overline{L}(y)} \, i^{\overline{G}(y)}_{\overline{L}(y)}$. Since $\overline{U}(y)$ and $\overline{V}(y)$ commute, the latter functor is isomorphic as in Example 2.15 to the tensor product with the $\mathcal{H}(\overline{L}(y))$-bimodule

$$\begin{aligned} e_{\overline{U}(y)} e_{\overline{V}(y)} \mathcal{H}(\overline{G}(y)) e_{\overline{U}(y)} e_{\overline{V}(y)} &\cong \mathcal{H}((\overline{U}(y) \times \overline{V}(y)) \backslash \overline{G}(y) / (\overline{U}(y) \times \overline{V}(y)) \\ &= \mathcal{H}((\overline{U}(y) \times \overline{V}(y)) \backslash \overline{G}(y)), \end{aligned}$$

with the last equality holding because $\overline{U}(y) \times \overline{V}(y)$ is normal in $\overline{G}(y)$. The semidirect product decomposition of $\overline{G}(y)$ given in § 5.3 for this case implies that

$$\mathcal{H}((\overline{U}(y) \times \overline{V}(y)) \backslash \overline{G}(y)) \cong \mathcal{H}(\overline{L}(y) \rtimes S) \cong \mathcal{H}(\overline{L}(y)) \oplus \mathcal{H}(\overline{L}(y))s$$

as $\mathcal{H}(\overline{L}(y))$-bimodules, and so the corresponding tensor product functor $r^{\overline{G}(y)}_{\overline{L}(y)} \, i^{\overline{G}(y)}_{\overline{L}(y)}$ is isomorphic to $\mathrm{id} \oplus \mathrm{Ad}_s$ as required.

This completes the proof of (5.16) and hence, by Corollary 5.13, of Theorem 5.2. □

5.5. Comparison with parahoric induction

We now come to the second corollary of the analysis of §§ 5.1–5.3. In addition to the virtual Iwahori decomposition (U, L, V) of G that we have been considering until now, we shall also consider the triple (U_0, L, V), which is a virtual Iwahori decomposition of G because the subgroup $U_0 \subset U$ is normalised by L. This second virtual Iwahori decomposition gives rise to a second induction functor $i_{U_0,V} : \mathcal{R}(L) \to \mathcal{R}(G)$ which, as we shall see below, is an example of Dat's parahoric induction ([10], cf. Example 2.12). It follows immediately from the definitions that we have a natural inclusion $i_{U,V} \subseteq i_{U_0,V}$. We shall show that this inclusion is proper, and then we shall explain why this gives a negative answer to [10, Question 2.15].

Corollary 5.17. *Let $x \in M_2(\mathbb{k})$ and consider $y = \operatorname{diag}(x, -x^t) \in \mathfrak{l}$. The restrictions of the functors*

$$i_{U,V}, \; i_{U_0,V} : \mathcal{R}(L) \to \mathcal{R}(G)$$

to the subcategory $\mathcal{R}(L)_{\varphi_y}$ are mutually nonisomorphic if x is nonzero and nilpotent; and these restrictions are mutually isomorphic if x is zero or non-nilpotent.

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Proof. The computations of § 5.2, in particular Lemma 5.9, show that there are commutative (up to natural isomorphism) diagrams

$$\begin{array}{ccc} \mathcal{R}(L)_{\varphi_y} & \xrightarrow{i_{U,V}} & \mathcal{R}(G)_{\varphi_y} \\ {\scriptstyle F_y^L}\uparrow\cong & & \cong\uparrow{\scriptstyle F_y^G} \\ \mathcal{R}(\overline{L}(y)) & \xrightarrow{i_{\overline{U}(y),\overline{V}(y)}} & \mathcal{R}(\overline{G}(y)) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{R}(L)_{\varphi_y} & \xrightarrow{i_{U_0,V}} & \mathcal{R}(G)_{\varphi_y} \\ {\scriptstyle F_y^L}\uparrow\cong & & \cong\uparrow{\scriptstyle F_y^G} \\ \mathcal{R}(\overline{L}(y)) & \xrightarrow{i_{\overline{V}(y)}} & \mathcal{R}(\overline{G}(y)) \end{array}$$

(In the second diagram we have used the fact that the group $\overline{U_0}$ is trivial, so that $i_{\overline{U_0}(y),\overline{V}(y)} = i_{\overline{V}(y)}$.)

If x is semisimple then $\overline{G}(y)$ is a finite reductive group, and $\overline{U}(y)$ and $\overline{V}(y)$ are the unipotent radicals of an opposite pair of rational parabolic subgroups with common Levi subgroup $\overline{L}(y)$. The functor $i_{\overline{V}(y)}$is the Harish-Chandra induction functor associated to the parabolic subgroup $\overline{L}(y)\overline{V}(y)$ of $\overline{G}(y)$, and as in Example 2.10 the natural inclusion $i_{\overline{U}(y),\overline{V}(y)} \subseteq i_{\overline{V}(y)}$ is an isomorphism.

If x is neither semisimple nor nilpotent, as in Case 4A, then the groups $\overline{U}(y)$ and $\overline{V}(y)$ are both trivial, $\overline{G}(y) = \overline{L}(y)$, and the functors $i_{\overline{U}(y),\overline{V}(y)}$ and $i_{\overline{V}(y)}$ are both isomorphic to the identity.

We are left to consider the case where x is nilpotent; say $x = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. In this case we have

$$\overline{G}(y) = (\overline{U}(y) \times \overline{V}(y)) \rtimes (\overline{L}(y) \rtimes S),$$

from which it follows (cf. Example 2.15) that the functors $i_{\overline{U}(y),\overline{V}(y)}$ and $i_{\overline{V}(y)}$ are isomorphic, respectively, to the compositions

$$\begin{aligned} i_{\overline{U}(y),\overline{V}(y)} &: \mathcal{R}(\overline{L}(y)) \xrightarrow{\text{inf}} \mathcal{R}((\overline{U}(y) \times \overline{V}(y)) \rtimes \overline{L}(y)) \xrightarrow{\text{ind}} \mathcal{R}(\overline{G}(y)), \\ i_{\overline{V}(y)} &: \mathcal{R}(\overline{L}(y)) \xrightarrow{\text{inf}} \mathcal{R}(\overline{V}(y) \rtimes \overline{L}(y)) \xrightarrow{\text{ind}} \mathcal{R}(\overline{G}(y)). \end{aligned}$$

The functor $i_{\overline{U}(y),\overline{V}(y)}$ thus scales the $\mathbb{C}$-dimension of representations by a factor of

$$[\overline{G}(y) : (\overline{U}(y) \times \overline{V}(y)) \rtimes \overline{L}(y)] = |S| = 2,$$

while $i_{\overline{V}(y)}$ scales the dimension by

$$[\overline{G}(y) : \overline{V}(y) \rtimes \overline{L}(y)] = |S| \cdot |\overline{U}(y)| = 2|\Bbbk|.$$

Thus $i_{U,V}$ is not isomorphic to $i_{U_0,V}$ as functors on $\mathcal{R}(L)_{\varphi_y}$. □

The above proof also shows that the parahoric induction and restriction functors do not satisfy the analogue of Theorem 5.2:

Corollary 5.18. *Let* $x \in M_2(\Bbbk)$ *be nonzero and nilpotent, and let* $y = \operatorname{diag}(x, -x^t)$. *The restriction of the functor*

$$r_{U_0,V}\, i_{U_0,V} : \mathcal{R}(L) \to \mathcal{R}(L)$$

to the subcategory $\mathcal{R}(L)_{\varphi_y}$ *is not isomorphic to* $\mathrm{id} \oplus \mathrm{Ad}_s$.

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Proof. The proof of Corollary 5.17 showed that for each nonzero $M \in \mathcal{R}(L)_{\varphi_y}$ there is a proper inclusion $i_{U,V}(M) \subsetneq i_{U_0,V}(M)$, and hence a proper inclusion

$$\begin{aligned} \mathrm{Hom}_L(M, M \oplus \mathrm{Ad}_s(M)) &\cong \mathrm{End}_G(i_{U,V}(M)) \subsetneq \mathrm{End}_G(i_{U_0,V}(M)) \\ &\cong \mathrm{Hom}_L(M, r_{U_0,V}\, i_{U_0,V}(M)). \end{aligned}$$

Thus $r_{U_0,V}\, i_{U_0,V}(M)$ is not isomorphic to $M \oplus \mathrm{Ad}_s(M)$. □

Remarks 5.19. (1) A straightforward computation with the functors $i_{\overline{V}(y)}$ and $r_{\overline{V}(y)}$, using the semidirect product decomposition of $\overline{G}(y)$, shows that for each irreducible $M \in \mathcal{R}(L)_{\varphi_y}$ one has

$$\dim_{\mathbb{C}} \mathrm{End}_G(i_{U_0,V}(M)) = \begin{cases} |\Bbbk| + 1 & \text{if } M \cong \mathrm{Ad}_s(M), \\ |\Bbbk| & \text{if } M \ncong \mathrm{Ad}_s(M). \end{cases}$$

(2) The nilpotent orbit $L \cdot y$ is the only one on which the Mackey formula fails to hold for the functors $i_{U_0,V}$ and $r_{U_0,V}$: on all of the other orbits our proof of Theorem 5.2 carries over to the parahoric functors, thanks to Corollary 5.17.

Let us now explain the connection to parahoric induction. Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$ and residue field $\Bbbk$ of odd characteristic. As usual, we let $\mathfrak{o}_\ell = \mathfrak{o}/\mathfrak{p}^\ell$. Let $\mathbf{G}$ be the group Sp_4, realised as a subgroup of GL_4 as in (5.1). Let $\mathbf{P} = \mathbf{LU}$ be the Siegel parabolic subgroup of block-upper-triangular matrices, and let $\mathbf{Q} = \mathbf{LV}$ be the opposite parabolic subgroup of block-lower-triangular matrices, as above.

The group $\mathbf{G}(\mathfrak{o}) = \mathrm{Sp}_4(\mathfrak{o})$ is the stabiliser of a point in the Bruhat–Tits building of $\mathbf{G}(F)$ lying in the apartment associated to the diagonal torus $\mathbf{D}$. The pro-p radical of $\mathbf{G}(\mathfrak{o})$ is equal to the congruence subgroup

$$K_1 = \ker(\mathbf{G}(\mathfrak{o}) \to \mathbf{G}(\Bbbk)).$$

We have $\mathbf{U}(F) \cap \mathbf{G}(\mathfrak{o}) = \mathbf{U}(\mathfrak{o})$, and likewise for $\mathbf{L}$ and $\mathbf{V}$. Let $\mathbf{U}(\mathfrak{o})^+ = \mathbf{U}(\mathfrak{o}) \cap K_1$.

We consider two induction functors $\mathcal{R}(\mathbf{L}(\mathfrak{o})) \to \mathcal{R}(\mathbf{G}(\mathfrak{o}))$: the parahoric induction functor $i_{\mathbf{U}(\mathfrak{o})^+,\mathbf{V}(\mathfrak{o})}$ defined by Dat, and the functor $i_{\mathbf{U}(\mathfrak{o}),\mathbf{V}(\mathfrak{o})}$. There is a natural inclusion

$$i_{\mathbf{U}(\mathfrak{o}),\mathbf{V}(\mathfrak{o})} \subseteq i_{\mathbf{U}(\mathfrak{o})^+,\mathbf{V}(\mathfrak{o})}. \tag{5.20}$$

As we explained in Example 2.12, an affirmative answer to [10, Question 2.15] would imply that this inclusion is in fact an equality.

Corollary 5.21. *The inclusion (5.20) is proper, and so [10, Question 2.15] has a negative answer in this case.*

Proof. Reduction modulo $\mathfrak{p}^2$ gives a surjective homomorphism $\mathbf{G}(\mathfrak{o}) \to \mathbf{G}(\mathfrak{o}_2)$, which restricts to surjective homomorphisms on the subgroups $\mathbf{U}$, $\mathbf{L}$ and $\mathbf{V}$. Let $K_2 \subset \mathbf{G}(\mathfrak{o})$ denote the kernel of this homomorphism. The triple $(\mathbf{U}(\mathfrak{o}) \cap K_2, \mathbf{L}(\mathfrak{o}) \cap K_2, \mathbf{V}(\mathfrak{o}) \cap K_2)$ is an Iwahori decomposition of K_2: this follows easily from the corresponding decomposition of the second congruence subgroup of $\mathrm{GL}_4(\mathfrak{o})$ (cf. Example 2.5). Recalling the notation G, L, U, V, etc. from previously in this section, and noting that U_0 is the image of $\mathbf{U}(\mathfrak{o})^+$ under the reduction map $\mathbf{U}(\mathfrak{o}) \to \mathbf{U}(\mathfrak{o}_2)$, we conclude from Theorem 2.18(5) that the diagrams

$$\begin{array}{ccc} \mathcal{R}(\mathbf{L}(\mathfrak{o})) & \xrightarrow{\ i_{\mathbf{U}(\mathfrak{o}),\mathbf{V}(\mathfrak{o})}\ } & \mathcal{R}(\mathbf{G}(\mathfrak{o})) \\ \uparrow \scriptstyle{\inf} & & \uparrow \scriptstyle{\inf} \\ \mathcal{R}(L) & \xrightarrow{\ i_{U,V}\ } & \mathcal{R}(G) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{R}(\mathbf{L}(\mathfrak{o})) & \xrightarrow{\ i_{\mathbf{U}(\mathfrak{o})^+,\mathbf{V}(\mathfrak{o})}\ } & \mathcal{R}(\mathbf{G}(\mathfrak{o})) \\ \uparrow \scriptstyle{\inf} & & \uparrow \scriptstyle{\inf} \\ \mathcal{R}(L) & \xrightarrow{\ i_{U_0,V}\ } & \mathcal{R}(G) \end{array}$$

commute. Corollary 5.17 shows that $i_{U,V}$ is a proper subfunctor of $i_{U_0,V}$, and so the inclusion (5.20) is proper. □

6. Representations of the Iwahori subgroup of the general linear group

Let $\mathfrak{o}$ be a compact discrete valuation ring with maximal ideal $\mathfrak{p}$. In this section we shall present a simple application of the functors $i_{U,V}$ and $r_{U,V}$ to the representation theory of the *Iwahori subgroups*

$$I_n = I_n(\mathfrak{o}) = \{g \in \mathrm{GL}_n(\mathfrak{o}) \mid g \text{ is upper-triangular modulo } \mathfrak{p}\}.$$

We shall relate the representations of I_n to representations of its block-diagonal subgroups. Before stating the main result let us establish some notation (borrowed from [3]) for these subgroups.

Let $\mathcal{P}_n$ denote the set of compositions (also called ordered partitions) of n: an element $\alpha \in \mathcal{P}_n$ is thus an ordered tuple of positive integers $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ having $\sum \alpha_i = n$. The *blocks* of α are the subsets

$$b_1(\alpha) = \{1, \ldots, \alpha_1\}, \quad b_2(\alpha) = \{\alpha_1 + 1, \ldots, \alpha_1 + \alpha_2\}, \quad \text{etc.}$$

of $\{1, \ldots, n\}$. We shall usually write n, instead of (n), for the composition with one block.

The set $\mathcal{P}_n$ is partially ordered by refinement: $\alpha \leqslant \beta$ if each block of β is a union of blocks of α. This partial order makes $\mathcal{P}_n$ into a lattice, the greatest lower bound $\alpha \wedge \beta$ of two compositions being the composition whose blocks are the nonempty intersections $b_i(\alpha) \cap b_j(\beta)$ of the blocks of α and β. We also have an associative order-preserving product

$$\mathcal{P}_n \times \mathcal{P}_m \to \mathcal{P}_{n+m}, \quad (\alpha, \beta) \mapsto \alpha \cdot \beta$$

given by concatenation.

Given a composition $\alpha \in \mathcal{P}_n$ we denote by

$$I_\alpha = \{g \in I_n \mid g_{ij} = 0 \text{ unless } i \text{ and } j \text{ lie in the same block of } \alpha\}$$

the closed subgroup of α-block-diagonal matrices in I_n. These groups are compatible with the concatenation product:

$$I_{\alpha \cdot \beta} \cong I_\alpha \times I_\beta \tag{6.1}$$

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in an obvious way, and this gives an equivalence on smooth representations,

$$\mathcal{R}(I_\alpha) \times \mathcal{R}(I_\beta) \xrightarrow[\cong]{(M_\alpha, M_\beta) \mapsto M_\alpha \otimes M_\beta} \mathcal{R}(I_{\alpha\cdot\beta}).$$

We also consider the groups

$$U_\alpha = \left\{ g \in I_n \;\middle|\; \begin{array}{l} g \text{ is upper triangular; } g_{ii} = 1 \text{ for every } i; \text{ and} \\ g_{ij} = 0 \text{ if } i \neq j \text{ and } i \text{ and } j \text{ lie in the same block of } \alpha \end{array} \right\}, \quad \text{and}$$

$$V_\alpha = U_\alpha^t \cap I_n.$$

If β is a second composition with $\alpha \leqslant \beta$, we define

$$U_\alpha^\beta = U_\alpha \cap I_\beta \quad \text{and} \quad V_\alpha^\beta = V_\alpha \cap I_\beta.$$

If $\alpha \leqslant \beta \in \mathcal{P}_n$ and $\gamma \leqslant \delta \in \mathcal{P}_m$, then the isomorphism $I_{\beta\cdot\delta} \cong I_\beta \times I_\delta$ of (6.1) restricts to isomorphisms

$$U_{\alpha\cdot\gamma}^{\beta\cdot\delta} \cong U_\alpha^\beta \times U_\gamma^\delta \quad \text{and} \quad V_{\alpha\cdot\gamma}^{\beta\cdot\delta} \cong V_\alpha^\beta \times V_\gamma^\delta. \tag{6.2}$$

Example 6.3. If $\alpha = (2, 1)$ and $\beta = (3)$, then

$$I_\alpha = \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} & \\ \mathfrak{p} & \mathfrak{o}^\times & \\ & & \mathfrak{o}^\times \end{bmatrix}, \quad U_\alpha^\beta = \begin{bmatrix} 1 & & \mathfrak{o} \\ & 1 & \mathfrak{o} \\ & & 1 \end{bmatrix}, \quad V_\alpha^\beta = \begin{bmatrix} 1 & & \\ & 1 & \\ \mathfrak{p} & \mathfrak{p} & 1 \end{bmatrix}$$

where the blanks indicate zeros.

Lemma 6.4.

(1) *For each pair of compositions $\alpha \leqslant \beta$ in $\mathcal{P}_n$, the triple $(U_\alpha^\beta, I_\alpha, V_\alpha^\beta)$ is an Iwahori decomposition of I_β.*

(2) *For each triple of compositions $\alpha \leqslant \beta \leqslant \gamma$ one has*

$$U_\alpha^\gamma = U_\alpha^\beta \ltimes U_\beta^\gamma \quad \textit{and} \quad V_\alpha^\gamma = V_\alpha^\beta \ltimes V_\beta^\gamma.$$

(3) *For each pair of compositions $\alpha, \beta \in \mathcal{P}_n$ one has*

$$U_{\alpha\wedge\beta}^\alpha = U_\beta \cap I_\alpha \quad \textit{and} \quad V_{\alpha\wedge\beta}^\alpha = V_\beta \cap I_\alpha.$$

Proof. Part (1) is well known, and can be established by elementary linear algebra as in [3, 3.11]. Part (2) follows immediately from the Iwahori decompositions. Part (3) boils down to the (manifestly true) assertion that for integers i and j lying in the same block of α, i and j lie in the same block of $\alpha \wedge \beta$ if and only if they lie in the same block of β. □

Definition 6.5. For each pair of compositions $\alpha \leqslant \beta$ in $\mathcal{P}_n$, consider the functors

$$i_\alpha^\beta = i_{U_\alpha^\beta, V_\alpha^\beta} : \mathcal{R}(I_\alpha) \to \mathcal{R}(I_\beta) \quad \text{and} \quad r_\alpha^\beta = r_{U_\alpha^\beta, V_\alpha^\beta} : \mathcal{R}(I_\beta) \to \mathcal{R}(I_\alpha).$$

The functors i_α^β and r_α^β are examples of parahoric induction as defined in [10]. Theorems 2.18 and 2.23 give some basic properties of these functors. Let us mention two that will be used below:

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Lemma 6.6. (1) *If $\alpha \leqslant \beta \leqslant \gamma$ are compositions of n, then*

$$\mathrm{i}_{\alpha}^{\gamma} \cong \mathrm{i}_{\beta}^{\gamma}\,\mathrm{i}_{\alpha}^{\beta} \quad \textit{and} \quad \mathrm{r}_{\alpha}^{\gamma} \cong \mathrm{r}_{\alpha}^{\beta}\,\mathrm{r}_{\beta}^{\gamma}.$$

(2) *If $\alpha \leqslant \beta \in \mathcal{P}_n$ and $\gamma \leqslant \delta \in \mathcal{P}_m$, then the diagram*

$$\begin{array}{ccc} \mathcal{R}(I_\alpha) \times \mathcal{R}(I_\gamma) & \xrightarrow{\ \mathrm{i}_{\alpha}^{\beta} \times \mathrm{i}_{\gamma}^{\delta}\ } & \mathcal{R}(I_\beta) \times \mathcal{R}(I_\delta) \\ \otimes \downarrow & & \downarrow \otimes \\ \mathcal{R}(I_{\alpha\cdot\gamma}) & \xrightarrow{\ \mathrm{i}_{\alpha\cdot\gamma}^{\beta\cdot\delta}\ } & \mathcal{R}(I_{\beta\cdot\delta}) \end{array}$$

commutes up to natural isomorphism, as does the corresponding diagram of adjoint functors r.

Proof. Part (1) follows from part (2) of Lemma 6.4 and part (7) of Theorem 2.18. Part (2) follows from the compatibility of the decompositions (6.1) and (6.2). □

Definition 6.7. An irreducible representation M of I_n will be called *primitive* if $\mathrm{r}_{\alpha}^{n}(M) = 0$ for every composition $\alpha \in \mathcal{P}_n$ except for $\alpha = n$. We denote the set of isomorphism classes of primitive irreducible representations by $\mathrm{Prim}\,(I_n)$.

The following lemma is key to our analysis of the functors i and r.

Lemma 6.8. *Let $\alpha, \beta \in \mathcal{P}_n$ be compositions of n, and let M be an irreducible representation of I_n. If $\mathrm{r}_{\alpha}^{n}(M)$ and $\mathrm{r}_{\beta}^{n}(M)$ are both nonzero, then so is $\mathrm{r}_{\alpha\wedge\beta}^{n}(M)$.*

Proof. Since the representation M is irreducible and smooth, it factors through the quotient map $I_n(\mathfrak{o}) \to I_n(\mathfrak{o}/\mathfrak{p}^{\ell})$ for some ℓ. The functors i and r commute with inflation (Theorem 2.18(5)), and so we may replace $\mathfrak{o}$ by $\mathfrak{o}/\mathfrak{p}^{\ell}$ and assume throughout the proof that I_n is a finite group.

We know that $N := \mathrm{r}_{\alpha}^{n}(M)$ is nonzero. Therefore, up to isomorphism, we can write

$$M = \mathrm{i}_{\alpha}^{n}(N) = \mathcal{H}(I_n)e_{U_\alpha}e_{V_\alpha} \otimes_{\mathcal{H}(I_\alpha)} N = \mathcal{H}(I_n)e_{U_\alpha}e_{V_\alpha}e_{U_\alpha}e_{V_\alpha} \otimes_{\mathcal{H}(I_\alpha)} N.$$

(In the last equality we used Proposition 2.16.)

We know that the subspace

$$\mathrm{r}_{\beta}^{n}(M) = e_{U_\beta}e_{V_\beta}\mathcal{H}(I_n)e_{U_\alpha}e_{V_\alpha}e_{U_\alpha}e_{V_\alpha} \otimes_{\mathcal{H}(I_\alpha)} N \tag{6.9}$$

of M is nonzero. By part (2) of Lemma 6.4 we know that each element of $V_\alpha \subseteq V_{\alpha\wedge\beta}$ can be written as the product of an element of V_β with an element of $V_{\alpha\wedge\beta}^{\beta} = V_\alpha \cap I_\beta$. Therefore, using the Iwahori decomposition of I_n with respect to α, we get that $I_n = V_\alpha I_\alpha U_\alpha = V_\beta V_{\alpha\wedge\beta}^{\beta} I_\alpha U_\alpha$, which allows us to replace $\mathcal{H}(I_n)$ by $\mathcal{H}(V_{\alpha\wedge\beta}^{\beta})$ in (6.9) and write

$$\begin{aligned} \mathrm{r}_{\beta}^{n}(M) &= e_{U_\beta}e_{V_\beta}\mathcal{H}(V_{\alpha\wedge\beta}^{\beta})e_{U_\alpha}e_{V_\alpha}e_{U_\alpha}e_{V_\alpha} \otimes_{\mathcal{H}(I_\alpha)} N \\ &= \mathcal{H}(V_{\alpha\wedge\beta}^{\beta})e_{U_\beta}e_{V_\beta}e_{U_\alpha}e_{V_\alpha}e_{U_\alpha}e_{V_\alpha} \otimes_{\mathcal{H}(I_\alpha)} N, \end{aligned}$$

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where the second equality holds because the elements of $\mathcal{H}(V^{\beta}_{\alpha\wedge\beta}) \subset \mathcal{H}(I_\beta)$ commute with e_{U_β} and e_{V_β}. So we see that $r^n_\beta(M)$ is generated as a representation of I_β by its subspace

$$e_{U_\beta} e_{V_\beta} e_{U_\alpha} e_{V_\alpha} e_{U_\alpha} e_{V_\alpha} \otimes_{\mathcal{H}(I_\alpha)} N \tag{6.10}$$

and hence that this subspace is nonzero.

We now write $e_{V_\beta} = e_{V_\beta} e_{V^{\alpha}_{\alpha\wedge\beta}}$. We use the fact that elements of $\mathcal{H}(V^{\alpha}_{\alpha\wedge\beta}) \subset \mathcal{H}(I_\alpha)$ commute with e_{U_α} and that $e_{V^{\alpha}_{\alpha\wedge\beta}} e_{V_\alpha} = e_{V_{\alpha\wedge\beta}}$ (by part (2) of Lemma 6.4), to obtain

$$e_{V_\beta} e_{U_\alpha} e_{V_\alpha} = e_{V_\beta} e_{V^{\alpha}_{\alpha\wedge\beta}} e_{U_\alpha} e_{V_\alpha} = e_{V_\beta} e_{U_\alpha} e_{V^{\alpha}_{\alpha\wedge\beta}} e_{V_\alpha} = e_{V_\beta} e_{U_\alpha} e_{V_{\alpha\wedge\beta}}.$$

A similar argument shows that $e_{U_\beta} e_{V_\beta} e_{U_\alpha} = e_{U_{\alpha\wedge\beta}} e_{V_\beta} e_{U_\alpha}$, and so the subspace (6.10) is equal to

$$e_{U_{\alpha\wedge\beta}} e_{V_\beta} e_{U_\alpha} e_{V_{\alpha\wedge\beta}} e_{U_\alpha} e_{V_\alpha} \otimes_{\mathcal{H}(I_\alpha)} N.$$

This nonzero subspace of M is contained in the subspace

$$\begin{aligned} e_{U_{\alpha\wedge\beta}} \mathcal{H}(I_n) e_{V_{\alpha\wedge\beta}} e_{U_\alpha} e_{V_\alpha} \otimes_{\mathcal{H}(I_\alpha)} N &= e_{U_{\alpha\wedge\beta}} \mathcal{H}(I_{\alpha\wedge\beta}) e_{V_{\alpha\wedge\beta}} e_{U_\alpha} e_{V_\alpha} \otimes_{\mathcal{H}(I_\alpha)} N \\ &= e_{U_{\alpha\wedge\beta}} e_{V_{\alpha\wedge\beta}} e_{U_\alpha} e_{V_\alpha} \otimes_{\mathcal{H}(I_\alpha)} N, \end{aligned}$$

where we have used the Iwahori decomposition of I_n with respect to $\alpha\wedge\beta$, and the inclusion $I_{\alpha\wedge\beta} \subseteq I_\alpha$. But this last nonzero subspace of M is exactly $r^n_{\alpha\wedge\beta}(M)$, so we are done. □

Let us now present the main results of this section:

Theorem 6.11. *Let M be an irreducible representation of the Iwahori subgroup $I_n \subset GL_n(\mathfrak{o})$. There is a unique composition $\alpha = (\alpha_1, \dots, \alpha_m)$ of n, and unique primitive irreducible representations $M_i \in \mathrm{Prim}(I_{\alpha_i})$, such that*

$$M \cong i^n_\alpha(M_1 \otimes \cdots \otimes M_m).$$

Proof. First note the following consequence of part (2) of Lemma 6.6: if $M_1, \dots, M_m$ are irreducible representations of $I_{\alpha_1}, \dots, I_{\alpha_m}$, then

$$M_i \text{ is primitive for all } i \quad \Longleftrightarrow \quad r^\alpha_\gamma(M_1 \otimes \cdots \otimes M_m) = 0 \text{ for all } \gamma \lneq \alpha. \tag{6.12}$$

Consider the set

$$\mathcal{Q} = \{\alpha \in \mathcal{P}_n \mid r^n_\alpha(M) \neq 0\},$$

which is nonempty since it contains the composition n. Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be the greatest lower bound of $\mathcal{Q}$ in the lattice $\mathcal{P}_n$; Lemma 6.8 implies that $\alpha \in \mathcal{Q}$. The (nonzero) irreducible representation $r^n_\alpha(M)$ of the group I_α decomposes uniquely as a tensor product

$$r^n_\alpha(M) \cong \bigotimes_{i=1}^{m} M_i$$

of irreducible representations of the factors I_{α_i} of I_α (cf. (6.1)). If $\gamma \lneq \alpha$ then

$$r^\alpha_\gamma\Big(\bigotimes M_i\Big) \cong r^n_\gamma(M) = 0$$

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by Lemma 6.6 part (1) and the minimality of α, and so all of the M_i's are primitive by (6.12). Since by part (3) of Theorem 2.23 we have $M \cong i^n_\alpha \, r^n_\alpha \,(M)$, we are done with the 'existence' part of the proof.

The uniqueness follows from (6.12): if $r^n_\beta \,(M) \cong N_1 \otimes \cdots \otimes N_\ell$, where the N_i are all primitive, then we must have $\beta = \alpha$ by minimality, and then $N_i \cong M_i$ for each i by the uniqueness of the tensor product decomposition. □

Lemma 6.8 also implies the following simple formula for the composition of induction and restriction:

Proposition 6.13. *For all* $\alpha, \beta \in \mathcal{P}_n$ *and all* $M \in \mathrm{Irr}\,(I_\alpha)$ *one has*

$$r^n_\beta \, i^n_\alpha \,(M) \cong i^\beta_{\alpha\wedge\beta} \, r^\alpha_{\alpha\wedge\beta} \,(M).$$

Proof. If $r^n_\beta \,(\, i^n_\alpha \,(M))$ is nonzero, then—since $r^n_\alpha \,(\, i^n_\alpha \, M) \cong M$ is also nonzero—Lemma 6.8 implies that

$$r^\alpha_{\alpha\wedge\beta} \,(M) \cong r^n_{\alpha\wedge\beta} \,(\, i^n_\alpha \,(M)) \neq 0.$$

In other words, if $r^\alpha_{\alpha\wedge\beta} \,(M) = 0$, then $r^n_\beta \, i^n_\alpha \,(M) = 0$ too.

If $r^\alpha_{\alpha\wedge\beta} \,(M) \neq 0$, then we can use Theorem 2.23 and Lemma 6.6(1) to compute

$$\begin{aligned} r^n_\beta \, i^n_\alpha \,(M) &\cong r^n_\beta \, i^n_\alpha \,(\, i^\alpha_{\alpha\wedge\beta} \, r^\alpha_{\alpha\wedge\beta} \,(M)) \cong r^n_\beta \, i^n_{\alpha\wedge\beta} \, r^\alpha_{\alpha\wedge\beta} \,(M) \cong r^n_\beta \, i^n_\beta \,(\, i^\beta_{\alpha\wedge\beta} \, r^\alpha_{\alpha\wedge\beta} \,(M)) \\ &\cong i^\beta_{\alpha\wedge\beta} \, r^\alpha_{\alpha\wedge\beta} \,(M) \end{aligned}$$

as claimed. □

Theorem 6.11 has the following corollary, which gives a neat description of the way the representations of all the groups I_n (for $n \geqslant 0$) fit in together. Namely, let $\mathcal{K} := \bigoplus_{n\geqslant 0} K_0(\mathcal{R}_f(I_n))$ denote the direct sum of the Grothendieck groups of the categories of finite-dimensional smooth representations of the groups I_n, with the convention that I_0 is the trivial group. The maps induced on Grothendieck groups by the functors

$$\mathcal{R}(I_n) \times \mathcal{R}(I_m) \to \mathcal{R}(I_{n+m}), \quad (M_1, M_2) \mapsto i^{n+m}_{(n,m)} \,(M_1 \otimes M_2)$$

equip $\mathcal{K}$ with a graded multiplication structure. It follows from Lemma 6.6 that this multiplication is associative. Since the irreducible representations of I_n constitute a $\mathbb{Z}$-basis for $K_0(\mathcal{R}_f(I_n))$, Theorem 6.11 implies the following result:

Corollary 6.14. *The ring* $\mathcal{K}$ *is isomorphic to* $\mathbb{Z}\langle \bigsqcup_{n\geqslant 0} \mathrm{Prim}\,(I_n)\rangle$, *the non-commutative polynomial algebra with indeterminates the primitive irreducible representations.* □

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Appendix A. Centralisers for the adjoint action of $\mathrm{Sp}_4(\Bbbk)$

In this section we give proofs of the assertions in § 5.3 regarding the centralisers $\overline{\mathsf{G}}(\mathsf{y})$ and $\overline{\mathsf{L}}(\mathsf{y})$ and the spaces $\mathsf{L}\backslash\mathsf{G}(\mathsf{y},\mathfrak{l})/\mathsf{G}(\mathsf{y})$. Part of the computations here can be deduced from [42], where the cardinalities of the centralisers of elements of $\mathrm{Sp}_4(\Bbbk)$ are computed, by using the Cayley map. As we require the precise structure of the centralisers we give a detailed computation below.

Fix $x \in M_2(\Bbbk)$, and let $y = \mathrm{diag}(x, -x^t)$ be the corresponding element of $\mathfrak{l}$. Clearly we have

$$\overline{\mathsf{L}}(y) = \{\mathrm{diag}(a, a^{-t}) \mid a \in M_2(\Bbbk)(x)\},$$

where $M_2(\Bbbk)(x)$ denotes the centraliser of x in the algebra $M_2(\Bbbk)$. Elements of $M_2(\Bbbk)$ are either scalar or regular (in the sense of admitting a cyclic vector in $\Bbbk^2$). We therefore have

$$M_2(\Bbbk)(x) = \begin{cases} M_2(\Bbbk) & \text{if } x \text{ is a scalar matrix,} \\ \Bbbk[x] & \text{if } x \text{ is non-scalar.} \end{cases}$$

Turning to the centralisers in $\overline{\mathsf{G}} = \mathrm{Sp}_4(\Bbbk)$, let us first note that the matrices x and $-x^t$ give rise to two $\Bbbk[\mathsf{T}]$-module structures on $\Bbbk^2$, and that the centraliser of y in $\mathrm{GL}_4(\Bbbk)$ is isomorphic, in an obvious way, to the automorphism group of the direct sum $\Bbbk^2_x \oplus \Bbbk^2_{-x^t}$ of these modules.

Lemma A.1. *For each $x \in M_2(\Bbbk)$ with $\mathrm{tr}(x) = 0$, the centraliser $\mathrm{GL}_4(\Bbbk)(y)$ of $y = \mathrm{diag}(x, -x^t) \in M_4(\Bbbk)$ inside $\mathrm{GL}_4(\Bbbk)$ is given by*

$$\mathrm{GL}_4(\Bbbk)(y) = \Sigma \cdot \mathrm{GL}_2(M_2(\Bbbk)(x)) \cdot \Sigma^{-1}$$

where $\sigma = \left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right] \in \mathrm{GL}_2(\Bbbk)$ and $\Sigma = \left[\begin{smallmatrix} 1 & \\ & \sigma \end{smallmatrix}\right] \in \mathrm{GL}_4(\Bbbk)$.

Proof. If $\mathrm{tr}(x) = 0$ then $\sigma x \sigma^{-1} = -x^t$, and so $\mathrm{id} \oplus \sigma : \Bbbk^2_x \oplus \Bbbk^2_x \to \Bbbk^2_x \oplus \Bbbk^2_{-x^t}$ is a $\Bbbk[\mathsf{T}]$-module isomorphism. Conjugating $\mathrm{GL}_2(M_2(\Bbbk)(x)) = \mathrm{Aut}(\Bbbk^2_x \oplus \Bbbk^2_x)$ by this isomorphism gives the asserted description of $\mathrm{GL}_4(\Bbbk)(y)$. □

We now proceed to the computation of $\overline{\mathsf{L}}(y)$, $\overline{\mathsf{G}}(y)$ and $\mathsf{L}\backslash\mathsf{G}(y,\mathfrak{l})/\mathsf{G}(y)$ in each of the cases listed in § 5.3. Note that $\mathrm{tr}(x) \neq 0$ in the 'A' cases, while $\mathrm{tr}(x) = 0$ in the 'B' cases.

Case 1. $x = \mathrm{diag}(\mu, \mu)$.

We have $M_2(\Bbbk)(x) = M_2(\Bbbk)$, so $\overline{\mathsf{L}}(y) = \overline{\mathsf{L}}$. For $\overline{\mathsf{G}}(y)$ and $\mathsf{L}\backslash\mathsf{G}(y,\mathfrak{l})/\mathsf{G}(y)$ there are two subcases to consider:

1A: $\mu \neq 0$. Since x and $-x^t$ share no eigenvalue, there are no nonzero morphisms between the $\mathsf{k}[\mathsf{T}]$ modules $\Bbbk^2_x$ and $\Bbbk^2_{-x^t}$, and consequently we have $\overline{\mathsf{G}}(y) = \overline{\mathsf{L}}(y) = \overline{\mathsf{L}}$.

We claim that $\mathsf{L}\backslash\mathsf{G}(y,\mathfrak{l})/\mathsf{G}(y) = \{1, s, w\}$. This is equivalent to the claim that there are, up to conjugacy by L, three G-conjugates of y lying in $\mathfrak{l}$: namely y itself, $s \cdot y$, and $w \cdot y$. Indeed, any G-conjugate of y in $\mathfrak{l}$ must be split and semisimple, and must therefore be L-conjugate to a diagonal matrix z whose entries form a permutation of the entries of y. Since z lies in $\mathfrak{l}$, and hence is of the form $\mathrm{diag}(z_1, z_2, -z_1, -z_2)$, the only possibilities for z are

$$\mathrm{diag}(\mu, \mu, -\mu, -\mu), \ \mathrm{diag}(-\mu, -\mu, \mu, \mu), \quad \mathrm{diag}(-\mu, \mu, \mu, -\mu), \ \text{or} \ \mathrm{diag}(\mu, -\mu, -\mu, \mu).$$

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The first three are equal to y, $s \cdot y$ and $w \cdot y$ respectively, while the last is L-conjugate to $w \cdot y$.

1B: $\mu = 0$. Obviously $\overline{H}(y) = \overline{H}$ for all $H \subseteq G$, and $G(y, \mathfrak{l}) = G(y)$.

Case 2. $x = \mathrm{diag}\,(\mu, \nu)$, $\mu \neq \nu$.

We have $M_2(\mathbb{k})(x) = \{[{}^{\alpha}\ {}_{\beta}] \mid \alpha, \beta \in \mathbb{k}\} \cong \mathbb{k} \oplus \mathbb{k}$, and so $\overline{L}(y) = \overline{D}$ is the group of diagonal matrices in $\overline{G}$. There are three subcases to consider:

2A. $\nu \neq \pm\mu$, $\mu \neq 0 \neq \nu$. Similar arguments to those of Case 1A show that $\overline{G}(y) = \overline{L}(y)$, and that $L\backslash G(y, \mathfrak{l})/G(y) = \{1, s, w, wt\}$.

2A*. $\nu = 0$. The space $\mathrm{Hom}_{\mathbb{k}[T]}\,(\mathbb{k}^2_x, \mathbb{k}^2_{-x^t})$ is one-dimensional, spanned by $p = [{}^{0}\ {}_{1}]$, and so we have

$$\mathrm{GL}_4\,(\mathbb{k})(y) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle|\ a, d \in \overline{D},\ b, c \in \mathbb{k}p \right\}.$$

Applying the condition $j^{-1} g^t j = g^{-1}$ defining $\mathrm{Sp}_4\,(\mathbb{k})$ to a matrix of the above form, we find that

$$\overline{G}(y) = \left\{ \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \beta \\ & & \delta_1 & \\ & \gamma & & \delta_2 \end{bmatrix} \in \mathrm{GL}_4\,(\mathbb{k}) \ \middle|\ \alpha_1\delta_1 = 1 = \alpha_2\delta_2 - \beta\gamma \right\} \cong \mathrm{GL}_1\,(\mathbb{k}) \times \mathrm{SL}_2\,(\mathbb{k}).$$

The Weyl group of $\mathrm{SL}_2\,(\mathbb{k})$ with respect to its diagonal torus is generated by the matrix σ, and so the Weyl group of $\overline{G}(y)$ with respect to $\overline{D}$ is generated by the matrix

$$\begin{bmatrix} 1 & & & \\ & 0 & & -1 \\ & & 1 & \\ & 1 & & 0 \end{bmatrix} = t^{-1} w t.$$

Up to L-conjugacy, the G-conjugates of y lying in $\mathfrak{l}$ are y and $-y = w \cdot y$, and so $L\backslash G(y, \mathfrak{l})/G(y) = \{1, w\}$.

2B. $\nu = -\mu$. Let $z = \mathrm{diag}\,(-\mu, -\mu, \mu, \mu)$, so that $y = w \cdot z$. Then $\overline{G}(y) = \mathrm{Ad}_w\,(\overline{G}(z))$, and $\overline{G}(z) = \overline{L}$ as in Case 1A. Since $\mathrm{Ad}_w^{-1}\,(\overline{U}) \cap \overline{L} = \overline{V'}$, and $\mathrm{Ad}_w^{-1}\,(\overline{V}) \cap \overline{L} = \overline{U'}$, we have $\overline{U}(y) = \mathrm{Ad}_w\,(\overline{V'})$ and $\overline{V}(y) = \mathrm{Ad}_w\,(\overline{U'})$. The argument of Case 1A gives $L\backslash G(y, \mathfrak{l})/G(y) = \{1, w, wt\}$.

Case 3. $x = \left[\begin{smallmatrix} \alpha & \beta \\ \mu\beta & \alpha \end{smallmatrix}\right]$, $\mu \in \mathbb{k}$ non-square, $\alpha \in \mathbb{k}$, $\beta \in \mathbb{k}^\times$.

In this case $M_2(\mathbb{k})(x) = \left\{\left[\begin{smallmatrix} \alpha_1 & \beta_1 \\ \mu\beta_1 & \alpha_1 \end{smallmatrix}\right] \mid \alpha_1, \beta_1 \in \mathbb{k}\right\}$ is a quadratic field extension of $\mathbb{k}$, which we shall denote by $\mathbb{k}_2$. There are two subcases to consider.

3A. $\alpha \neq 0$. Similar arguments to those of case 1A (considering the eigenvalues in $\mathbb{k}_2$) show that $\overline{G}(y) = \overline{L}(y)$, while $L\backslash G(y, \mathfrak{l})/G(y) = \{1, s\}$.

3B. $\alpha = 0$. Since $\mathrm{tr}\,(x) = 0$, Lemma A.1 implies that $\mathrm{Ad}_{\Sigma^{-1}} : \mathrm{GL}_4\,(\mathbb{k})(y) \to \mathrm{GL}_2\,(\mathbb{k}_2)$ is an isomorphism. Observing that $\mathrm{Ad}_{\Sigma^{-1}}\,(j) = \left[\begin{smallmatrix} & \sigma^{-1} \\ \sigma^{-1} & \end{smallmatrix}\right]$, and that $\Sigma^t = \Sigma^{-1}$, we find that the isomorphism $\mathrm{Ad}_{\Sigma^{-1}}$ sends $\overline{G}(y)$ to

$$\mathrm{Ad}_{\Sigma^{-1}}\,(\overline{G}(y)) = \{g \in \mathrm{GL}_2\,(\mathbb{k}_2) \mid g^* g = 1\},$$

where

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} \sigma d^t \sigma^{-1} & \sigma b^t \sigma^{-1} \\ \sigma c^t \sigma^{-1} & \sigma a^t \sigma^{-1} \end{bmatrix}.$$

The map $a \mapsto \sigma a^t \sigma^{-1}$ is a nontrivial $\Bbbk$-algebra automorphism of $\Bbbk_2$, and so is equal to the nontrivial element $a \mapsto a^{|\Bbbk|}$ in $\mathrm{Gal}\,(\Bbbk_2/\Bbbk)$.

Let $\overline{\Bbbk}$ denote an algebraic closure of $\Bbbk_2$. The above computations show that $\mathrm{Ad}_{\Sigma^{-1}}$ restricts to an isomorphism from $\overline{G}(y)$ to the (unitary) group $\mathrm{GU}_2\,(\Bbbk)$ of fixed points of the automorphism

$$\mathrm{GL}_2\,(\overline{\Bbbk}) \to \mathrm{GL}_2\,(\overline{\Bbbk}), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d^{|\Bbbk|} & b^{|\Bbbk|} \\ c^{|\Bbbk|} & a^{|\Bbbk|} \end{bmatrix}^{-1}.$$

The subgroup $\overline{L}(y)$ corresponds under this isomorphism to the non-split rational maximal torus of diagonal matrices $\{\mathrm{diag}\,(a, a^{-|\Bbbk|}) \mid a \in \Bbbk_2^\times\}$ in $\mathrm{GU}_2\,(\Bbbk)$, while $\overline{U}(y)$ and $\overline{V}(y)$ correspond to the unipotent radicals of rational Borel subgroups of upper- / lower-triangular matrices. The Weyl group of $\mathrm{GU}_2\,(\Bbbk)$ with respect to its diagonal torus is generated by the matrix $[\,_{-1}\ {}^{-1}\,] = \mathrm{Ad}_{\Sigma^{-1}}\,(s)$.

The argument of case 1A shows that all of the G-conjugates of y lying in L are already L-conjugate, and so we have $L\backslash G(y, \mathfrak{l})/G(y) = \{1\}$.

Case 4. $x = [\,^{\mu}\ {}^{1}_{\mu}\,]$.

We have $M_2(\Bbbk)(x) = \{[\,^{\alpha}\ {}^{\beta}_{\alpha}\,] \mid \alpha, \beta \in \Bbbk\} \cong \Bbbk[\varepsilon]/(\varepsilon^2)$. There are two subcases to consider.

4A. $\mu \neq 0$. Arguing as in case 1A once again, we find that $\overline{G}(y) = \overline{L}(y)$, while $L\backslash G(y, \mathfrak{l})/G(y) = \{1, s\}$.

4B. $\mu = 0$. Arguing as in case 3B, we find that the isomorphism

$$\mathrm{Ad}_{\Sigma^{-1}} : \mathrm{GL}_4\,(\Bbbk)(y) \to \mathrm{GL}_2\,(\Bbbk[x])$$

of Lemma A.1 restricts to an isomorphism between $\overline{G}(y)$ and the group $Q \subset \mathrm{GL}_2\,(\Bbbk[x])$ of fixed points of the involution

$$\mathrm{GL}_2\,(\Bbbk[x]) \to \mathrm{GL}_2\,(\Bbbk[x]), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d^\# & b^\# \\ c^\# & a^\# \end{bmatrix}^{-1}$$

where $\#$ denotes the $\Bbbk$-automorphism $x \mapsto -x$ of $\Bbbk[x]$. We have furthermore

$$\mathrm{Ad}_{\Sigma^{-1}}\,(\overline{L}(y)) = \left\{ \begin{bmatrix} a & \\ & a^{-\#} \end{bmatrix} \in \mathrm{GL}_2\,(\Bbbk[x]) \;\middle|\; a \in \Bbbk[x]^\times \right\} =: H,$$

$$\mathrm{Ad}_{\Sigma^{-1}}\,(\overline{U}(y)) = \left\{ \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} \in \mathrm{GL}_2\,(\Bbbk[x]) \;\middle|\; b \in x\Bbbk[x] \right\} =: X, \quad \text{and}$$

$$\mathrm{Ad}_{\Sigma^{-1}}\,(\overline{V}(y)) = \left\{ \begin{bmatrix} 1 & \\ c & 1 \end{bmatrix} \in \mathrm{GL}_2\,(\Bbbk[x]) \;\middle|\; c \in x\Bbbk[x] \right\} =: Y.$$

Let S denote the two-element subgroup of $\overline{G}(y)$ generated by s, and let R denote the subgroup $\mathrm{Ad}_{\Sigma^{-1}}\,(S)$ of Q; thus R is the two-element group generated by $r = \mathrm{Ad}_{\Sigma^{-1}}\,(s) = [\,_{-1}\ {}^{-1}\,]$.

The subgroups X and Y commute in Q, because $x^2 = 0$. Since H normalises X and Y, this implies that the product XHY is a subgroup of Q, equal to $(X \times Y) \rtimes H$. Explicitly,

$$XHY = \left\{ \begin{bmatrix} a & b \\ c & a^{-\#} \end{bmatrix} \;\middle|\; a \in \Bbbk[x]^{\times},\ b, c \in x\Bbbk[x] \right\},$$

i.e. the group of $q \in Q$ such that q is diagonal modulo x.

Now, for each $q \in Q$, the reduction of q modulo x is a fixed point of the involution

$$\mathrm{GL}_2(\Bbbk) \to \mathrm{GL}_2(\Bbbk), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d & b \\ c & a \end{bmatrix}^{-1}$$

and so q modulo x is either of the form $[\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}]$ or $[\begin{smallmatrix} & b \\ b^{-1} & \end{smallmatrix}]$. Thus the homomorphism

$$Q \to \{\pm 1\}, \quad q \mapsto \det(q \text{ modulo } x)$$

has kernel XHY, and is split by the homomorphism

$$\{\pm 1\} \to Q, \quad -1 \mapsto r.$$

This gives a decomposition $Q = ((X \times Y) \rtimes H) \rtimes R$. Since conjugation by r preserves H and permutes X and Y, we may rewrite this decomposition as $Q = (X \times Y) \rtimes (H \rtimes R)$. Applying Ad_{Σ} gives

$$\overline{G}(y) = (\overline{U}(y) \times \overline{V}(y)) \rtimes (\overline{L}(y) \rtimes S).$$

As in Case 3B we have $G \cdot y \cap \mathfrak{l} = L \cdot y$, and so $L \backslash G(y, \mathfrak{l})/G(y) = \{1\}$.

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