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Abstract

We prove the test function conjecture of Kottwitz and the first named author for local models of Shimura varieties with parahoric level structure attached to Weil-restricted groups, as defined by B. Levin. Our result covers the (modified) local models attached to *all* connected reductive groups over p-adic local fields with $p \geqslant 5$. In addition, we give a self-contained study of relative affine Grassmannians and loop groups formed using general relative effective Cartier divisors in a relative curve over an arbitrary Noetherian affine scheme.

Contents

1. Introduction

Building upon the work of Pappas and Zhu [PZ13], Levin defines in [Lev16] candidates for parahoric local models of Shimura varieties for reductive groups of the form $\mathrm{Res}_{K/F}(G_0)$ where G_0 splits over a tamely ramified extension of K, and K/F is a finite (possibly wildly ramified) extension. The present paper is a follow-up of [HR18], in which we prove the test function conjecture for these local models. The method follows closely [HR18], and we only explain new arguments in detail, but repeat as much as necessary for the sake of readability. For a detailed introduction and further references we refer the reader to the introduction of [HR18].

Let us mention that the paper is supplemented in § 3 by a general study of relative affine Grassmannians and loop groups formed using a general Cartier divisor as in the work of Beilinson and Drinfeld [BD99]. This unifies the frameworks of [PZ13, Lev16] in mixed

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characteristic, of [Hei10, Zhu14, Zhu15, Ric16b] in equal characteristic, and of [FP15, Fed16] on the Grothendieck–Serre conjecture; cf. Examples 3.1 below. As an application, we identify the torus fixed points and their attractor and repeller loci in the sense of Drinfeld [Dri13] (cf. also [Hes81]) for these relative affine Grassmannians; cf. Theorem 3.17.

1.1 Formulation of the main result

Let p be a prime number. Let $F/\mathbb{Q}_p$ be a finite extension with residue field k_F of cardinality q. Let $\bar{F}/F$ be a separable closure, and denote by Γ_F the Galois group with inertia subgroup I_F and fixed geometric Frobenius lift $\Phi_F \in \Gamma_F$.

Let K/F be a finite extension, and let G_0 be a (connected) reductive K-group which splits over a tamely ramified extension. We are interested in the group of Weil restrictions $G = \mathrm{Res}_{K/F}(G_0)$ which is a reductive F-group but now possibly wildly ramified depending on K/F.

Let $\mathcal{G}$ be a parahoric $\mathcal{O}_F$-group scheme in the sense of Bruhat and Tits [BT84] with generic fiber G. Note that $\mathcal{G} = \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G}_0)$ for a unique parahoric $\mathcal{O}_K$-group scheme $\mathcal{G}_0$ with generic fiber G_0; cf. Corollary 4.8. We fix $\{\mu\}$ a (not necessarily minuscule) conjugacy class of geometric cocharacters in G defined over a finite (separable) extension E/F.

Attached to the triple $(G, \{\mu\}, \mathcal{G})$ is the (flat) *local model*

$$M_{\{\mu\}} = M_{(G,\{\mu\},\mathcal{G})},$$

which is a flat projective $\mathcal{O}_E$-scheme; cf. [PZ13] if $K = F$ and [Lev16] for general K/F (cf. also Definition 4.18). The generic fiber $M_{\{\mu\},E}$ is naturally the Schubert variety in the affine Grassmannian of G/E associated with the class $\{\mu\}$. The special fiber $M_{\{\mu\},k_E}$ is equidimensional, but neither irreducible nor a divisor with normal crossings in general.

Fix a prime number $\ell \neq p$, and fix $q^{-1/2} \in \bar{\mathbb{Q}}_\ell$ in order to define half Tate twists. Let d_μ be the dimension of the generic fiber $M_{\{\mu\},E}$, and denote the normalized intersection complex by

$$\mathrm{IC}_{\{\mu\}} \overset{\text{def}}{=} j_{!*}\bar{\mathbb{Q}}_\ell[d_\mu](d_\mu/2) \in D_c^b(M_{\{\mu\},E}, \bar{\mathbb{Q}}_\ell);$$

cf. § 5.2.1. Under the geometric Satake equivalence [Gin95, Lus83, BD99, MV07, Ric14, RZ15, Zhu17], the complex $\mathrm{IC}_{\{\mu\}}$ corresponds to the ${}^LG_E = G^\vee \rtimes \Gamma_E$-representation $V_{\{\mu\}}$ of highest weight $\{\mu\}$ defined in [Hai14, 6.1]; cf. [HR18, Corollary 3.12]. Note that we have $G^\vee = \mathrm{Ind}_{\Gamma_K}^{\Gamma_F}(G_0^\vee)$ as groups over $\bar{\mathbb{Q}}_\ell$ under which $V_{\{\mu\}} = \boxtimes_\psi V_{\mu_\psi}$; cf. Lemma 5.6.

Let E_0/F be the maximal unramified subextension of E/F, and let $\Phi_E = \Phi_{E_0} = \Phi_F^{[E_0:F]}$ and $q_E = q_{E_0} = q^{[E_0:F]}$. The semisimple trace of Frobenius function on the sheaf of nearby cycles

$$\tau_{\{\mu\}}^{\mathrm{ss}} \colon M_{\{\mu\}}(k_E) \to \bar{\mathbb{Q}}_\ell, \quad x \mapsto (-1)^{d_\mu}\, \mathrm{tr}^{\mathrm{ss}}(\Phi_E \mid \Psi_{M_{\{\mu\}}}(\mathrm{IC}_{\{\mu\}})_{\bar{x}})$$

is naturally a function in the center $\mathcal{Z}(G(E_0), \mathcal{G}(\mathcal{O}_{E_0}))$ of the parahoric Hecke algebra; cf. [PZ13, Theorem 10.14], [Lev16, Theorem 5.3.3] and § 6.3. We remark that $\tau_{\{\mu\}}^{\mathrm{ss}}$ lives in the center of the $\bar{\mathbb{Q}}_\ell$-valued Hecke algebra attached to *function field* analogues of $(G_{E_0}, \mathcal{G}_{\mathcal{O}_{E_0}}, E_0)$; we are implicitly identifying this with $\mathcal{Z}(G(E_0), \mathcal{G}(\mathcal{O}_{E_0}))$ via Lemma 4.12.

Our main result, *the test function conjecture for local models for Weil-restricted groups*, characterizes the function $\tau_{\{\mu\}}^{\mathrm{ss}}$, extending the main result of [HR18] to the Weil-restricted situation. It confirms that even for these local models, the local geometry of Shimura varieties at places of parahoric bad reduction can be related to automorphic-type data, as required by the Langlands–Kottwitz method.

Main Theorem. *Let $(G, \{\mu\}, \mathcal{G})$ be a triple as above. Let E/F be a finite separable extension over which $\{\mu\}$ is defined, and let E_0/F be the maximal unramified subextension. Then*

$$\tau^{\mathrm{ss}}_{\{\mu\}} = z^{\mathrm{ss}}_{\{\mu\}},$$

where $z^{\mathrm{ss}}_{\{\mu\}} = z^{\mathrm{ss}}_{\mathcal{G},\{\mu\}} \in \mathcal{Z}(G(E_0), \mathcal{G}(\mathcal{O}_{E_0}))$ is the unique function which acts on any $\mathcal{G}(\mathcal{O}_{E_0})$-spherical smooth irreducible $\bar{\mathbb{Q}}_\ell$-representation π by the scalar

$$\mathrm{tr}(s(\pi) \mid \mathrm{Ind}^{{}^L G_{E_0}}_{{}^L G_E}(V_{\{\mu\}})^{1 \rtimes I_{E_0}}),$$

where $s(\pi) \in [(G^\vee)^{I_{E_0}} \rtimes \Phi_{E_0}]_{\mathrm{ss}}/(G^\vee)^{I_{E_0}}$ is the Satake parameter for π [Hai15]. The function $q_{E_0}^{d_\mu/2}\tau^{\mathrm{ss}}_{\{\mu\}}$ takes values in $\mathbb{Z}$ and is independent of $\ell \neq p$ and $q^{1/2} \in \bar{\mathbb{Q}}_\ell$.

The construction of $s(\pi)$ is also reviewed in [HR18, § 7.2], and the values of $z^{\mathrm{ss}}_{\{\mu\}}$ are studied in [HR18, § 7.7]; cf. § 6.5. The definition of the local models $M_{\{\mu\}}$ depends on certain auxiliary choices (cf. Remark 4.19), but the function $\tau^{\mathrm{ss}}_{\{\mu\}}$ depends canonically only on the data $(G, \{\mu\}, \mathcal{G})$.

As an application of our main theorem we prove in § 7 the test function conjecture for (modified) local models attached to *all* groups and prime numbers $p \geqslant 5$. This relies on the fact that when $p \geqslant 5$ any adjoint reductive F-group is isomorphic to a product of Weil restrictions of scalars of tamely ramified groups; cf. (7.1) below. In Theorem 6.7 and § 6.3 we also show that the variant of the main theorem holds, where semisimple traces are replaced by traces with respect to any fixed lift Φ_E of geometric Frobenius.

1.2 Other results

Our methods can be used to obtain results on the fixed point (respectively, attractor and repeller) locus of $\mathbb{G}_m$-actions on fusion Grassmannians (cf. Theorem A below), and the special fiber of local models (cf. Theorem B below).

1.2.1 *Fusion Grassmannians.* Let F be any field, and let G be a reductive F-group. For each $n \geqslant 0$, there is the fusion Grassmannian $\mathrm{Gr}_{G,n} \to \mathbb{A}^n_F$ defined in [BD99] which parametrizes isomorphism classes of G-bundles on the affine line together with a trivialization away from n points. Given a cocharacter $\chi\colon \mathbb{G}_{m,F} \to G$, we obtain a fiberwise $\mathbb{G}_m$-action on the family $\mathrm{Gr}_{G,n} \to \mathbb{A}^n_F$, and we are interested in determining the diagram on the fixed point ind-scheme and attractor (respectively, repeller) ind-scheme

$$(\mathrm{Gr}_{G,n})^0 \leftarrow (\mathrm{Gr}_{G,n})^{\pm} \to \mathrm{Gr}_{G,n};$$

cf. (2.1). Let $M \subset G$ be the centralizer of χ, which is a Levi subgroup. The dynamic method promulgated in [CGP10] defines a pair of parabolic subgroups (P^+, P^-) in G such that $P^+ \cap P^- = M$; see the formulation of Theorem 3.17. The natural maps $M \leftarrow P^{\pm} \to G$ induce maps of fusion Grassmannians

$$\mathrm{Gr}_{M,n} \leftarrow \mathrm{Gr}_{P^{\pm},n} \to \mathrm{Gr}_{G,n}.$$

An extension of the method used in the proof of [HR18, Proposition 3.4] allows us to prove the following result.

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Theorem A. *For each $n \in \mathbb{Z}_{\geqslant 0}$, there is a commutative diagram of $\mathbb{A}^n_F$-ind-schemes*

$$\begin{array}{ccccc} \mathrm{Gr}_{M,n} & \longleftarrow & \mathrm{Gr}_{P^\pm,n} & \longrightarrow & \mathrm{Gr}_{G,n} \\ \simeq\downarrow & & \simeq\downarrow & & \downarrow\mathrm{id} \\ (\mathrm{Gr}_{G,n})^0 & \longleftarrow & (\mathrm{Gr}_{G,n})^\pm & \longrightarrow & \mathrm{Gr}_{G,n} \end{array}$$

where the vertical maps are isomorphisms.

Theorem A is a special case of Theorem 3.17 which applies to general reductive group schemes over $\mathbb{A}^n_F$ which are not necessarily defined over F. Let us point out that [HR18, Proposition 3.4] implies that Theorem A holds fiberwise. However, we do not know how to prove sufficiently good flatness properties of $\mathrm{Gr}_{G,n} \to \mathbb{A}^n_F$ in order to deduce the more general result from the fiberwise result.

The tensor structure on the constant term functors in geometric Langlands is constructed in [BD99, MV07]. In [Gai07, Rei12], it is explained how to use the nearby cycles to define the fusion structure used in the geometric Satake isomorphism. Theorem A together with [Ric19, Theorem 3.3] gives another way of constructing the tensor structure on the constant term functors, even without passing to the underlying reduced ind-schemes; cf. the proof of [HR18, Theorem 3.16].

1.2.2 *Special fibers of local models.* As in [HR18, § 6.3.1], we use the commutation of nearby cycles with constant terms to determine the irreducible components of the geometric special fiber $M_{\{\mu\},\bar{k}}$ of the local models. Recall that, by construction (cf. Definition 4.18), there is a closed embedding

$$M_{\{\mu\},\bar{k}} \hookrightarrow \mathcal{F}\ell_{\mathcal{G}^\flat,\bar{k}},$$

where $\mathcal{F}\ell_{\mathcal{G}^\flat}$ is the (partial) affine flag variety attached to the function field analogue $\mathcal{G}^\flat/k_F[\![u]\!]$ of $\mathcal{G}/\mathcal{O}_F$; cf. Theorem 4.13 and Proposition 4.15(ii). As envisioned by Kottwitz and Rapoport, the geometric special fiber $M_{\{\mu\},\bar{k}}$ should be the union of the Schubert varieties $\mathcal{F}l^{\leqslant w}_{\mathcal{G}^\flat,\bar{k}} \subset \mathcal{F}\ell_{\mathcal{G}^\flat,\bar{k}}$ where w ranges over the $\{\mu\}$-admissible set $\mathrm{Adm}^{\mathbf{f}}_{\{\mu\}} \subset W_{\mathbf{f}}\backslash W/W_{\mathbf{f}}$ where $\mathcal{G} = \mathcal{G}_{\mathbf{f}}$ and $W = W(G,F)$ denotes the Iwahori–Weyl group. Here we are identifying the Iwahori–Weyl groups attached to G/F and $G^\flat/k_F(\!(u)\!)$ by Lemma 4.11. The following result verifies their prediction; cf. Theorem 5.14.

Theorem B. *The smooth locus $(M_{\{\mu\}})^{\mathrm{sm}}$ is fiberwise dense in $M_{\{\mu\}}$, and on reduced subschemes a union of the Schubert varieties*

$$(M_{\{\mu\},\bar{k}})_{\mathrm{red}} = \bigcup_{w\in \mathrm{Adm}^{\mathbf{f}}_{\{\mu\}}} \mathcal{F}l^{\leqslant w}_{\mathcal{G}^\flat,\bar{k}}.$$

In particular, the geometric special fiber $M_{\{\mu\},\bar{k}}$ is generically reduced.

If $p \nmid |\pi_1(G_{\mathrm{der}})|$, then Theorem B is [PZ13, Theorem 9.3] for $K = F$, and [Lev16, Theorem 2.3.5] when $K \neq F$. We have removed this condition on p and thereby conclude that the Kottwitz–Rapoport strata in the special fiber are enumerated by the $\{\mu\}$-admissible set for all local models constructed in [PZ13, Lev16].

1.3 Overview

In § 2 we recall a few facts about $\mathbb{G}_m$-actions for convenience. The following § 3 studies relative affine Grassmannians formed using a general Cartier divisor. In § 4 we recall the definition of

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Weil-restricted local models and results from [Lev16] which are needed in the sequel. These results are applied in § 5 to study $\mathbb{G}_m$-actions on Beilinson–Drinfeld affine Grassmannians for Weil-restricted groups. In § 6 we formulate and prove the test function conjecture for Weil-restricted local models.

1.4 Conventions on ind-algebraic spaces

Let $\mathcal{O}$ be a ring, and denote by $\mathcal{O}$-Alg the category of $\mathcal{O}$-algebras equipped with the fpqc topology. An $\mathcal{O}$-space X is a sheaf on the site $\mathcal{O}$-Alg, and we denote the category of $\mathcal{O}$-spaces by $\mathrm{Sp}_{\mathcal{O}}$. As each object in the site $\mathcal{O}$-Alg is quasi-compact, the pretopology on $\mathcal{O}$-Alg is generated by finite covering families, and hence filtered colimits exist in $\mathrm{Sp}_{\mathcal{O}}$ and can be computed in the category of presheaves.

The category $\mathrm{Sp}_{\mathcal{O}}$ contains the category of $\mathcal{O}$-schemes $\mathrm{Sch}_{\mathcal{O}}$ as a full subcategory. An $\mathcal{O}$-algebraic space is a $\mathcal{O}$-space X such that $X \to X \times_{\mathcal{O}} X$ is relatively representable, and such that there exists an étale surjective map from a scheme $U \to X$. By a theorem of Gabber [StaPro, Tag 03W8] this agrees with the usual definition of algebraic spaces using étale or fppf sheaves.

The category of $\mathcal{O}$-algebraic spaces is denoted $\mathrm{AlgSp}_{\mathcal{O}}$. There are full embeddings $\mathrm{Sch}_{\mathcal{O}} \subset \mathrm{AlgSp}_{\mathcal{O}} \subset \mathrm{Sp}_{\mathcal{O}}$. A map of $\mathcal{O}$-spaces $X \to Y$ is called representable (respectively, schematic) if for every scheme $T \to Y$ the fiber product $X \times_Y T$ is representable by an algebraic space (respectively, scheme).

An $\mathcal{O}$-ind-algebraic space (respectively, $\mathcal{O}$-ind-scheme) is a contravariant functor

$$X\colon \mathcal{O}\text{-Alg} \to \text{Sets}$$

such that there exists a presentation as presheaves $X = \mathrm{colim}_i X_i$ where $\{X_i\}_{i\in I}$ is a filtered system of $\mathcal{O}$-algebraic spaces (respectively, $\mathcal{O}$-schemes) X_i with transition maps being (schematic) closed immersions. Since filtered colimits in $\mathrm{Sp}_{\mathcal{O}}$ can be computed in presheaves, every $\mathcal{O}$-ind-algebraic space (respectively, $\mathcal{O}$-ind-scheme) is an $\mathcal{O}$-space. The category of $\mathcal{O}$-ind-algebraic spaces (respectively, $\mathcal{O}$-ind-schemes) $\mathrm{IndAlgSp}_{\mathcal{O}}$ (respectively, $\mathrm{IndSch}_{\mathcal{O}}$) is the full subcategory of $\mathrm{Sp}_{\mathcal{O}}$ whose objects are $\mathcal{O}$-ind-algebraic spaces (respectively, $\mathcal{O}$-ind-schemes). If $X = \mathrm{colim}_i X_i$ and $Y = \mathrm{colim}_j Y_j$ are presentations of ind-algebraic spaces (respectively, ind-schemes), and if each X_i is quasi-compact, then as sets

$$\mathrm{Hom}_{\mathrm{Sp}_{\mathcal{O}}}(X, Y) = \lim_i \mathrm{colim}_j \mathrm{Hom}_{\mathrm{Sp}_{\mathcal{O}}}(X_i, Y_j),$$

because every map $X_i \to Y$ factors over some Y_j by quasi-compactness of X_i. The category $\mathrm{IndAlgSp}_{\mathcal{O}}$ (respectively, $\mathrm{IndSch}_{\mathcal{O}}$) is closed under fiber products, that is, $\mathrm{colim}_{(i,j)}(X_i \times_{\mathcal{O}} Y_j)$ is a presentation of $X \times_{\mathcal{O}} Y$. If $\mathcal{P}$ is a property of algebraic spaces (respectively, schemes), then an $\mathcal{O}$-ind-algebraic space (respectively, $\mathcal{O}$-ind-scheme) X is said to have ind-$\mathcal{P}$ if there exists a presentation $X = \mathrm{colim}_i X_i$ where each X_i has property $\mathcal{P}$. A map $f\colon X \to Y$ of $\mathcal{O}$-ind-algebraic spaces (respectively, $\mathcal{O}$-ind-schemes) is said to have property $\mathcal{P}$ if f is representable and, for all schemes $T \to Y$, the pullback $f \times_Y T$ has property $\mathcal{P}$. Note that every representable quasi-compact map of $\mathcal{O}$-ind-schemes is schematic.

2. Actions of $\mathbb{G}_m$ on ind-algebraic spaces

We recall the setup and some notation from [Dri13, Ric19]. Let $\mathcal{O}$ be a ring, and let X be an $\mathcal{O}$-algebraic space (or $\mathcal{O}$-ind-algebraic space) equipped with an action of $\mathbb{G}_m$ which is trivial on $\mathcal{O}$. There are the following three functors on the category of $\mathcal{O}$-algebras:

$$\begin{aligned} X^0 &\colon\ R \longmapsto \mathrm{Hom}_R^{\mathbb{G}_m}(R, X),\\ X^+ &\colon\ R \longmapsto \mathrm{Hom}_R^{\mathbb{G}_m}((\mathbb{A}^1_R)^+, X),\\ X^- &\colon\ R \longmapsto \mathrm{Hom}_R^{\mathbb{G}_m}((\mathbb{A}^1_R)^-, X), \end{aligned} \tag{2.1}$$

where $(\mathbb{A}^1_R)^+$ (respectively, $(\mathbb{A}^1_R)^-$) is $\mathbb{A}^1_R$ with the usual (respectively, opposite) $\mathbb{G}_m$-action. The functor X^0 is the functor of $\mathbb{G}_m$-fixed points in X, and X^+ (respectively, X^-) is called the attractor (respectively, repeller). Informally speaking, X^+ (respectively, X^-) is the space of points x such that the limit $\lim_{\lambda\to 0} \lambda \cdot x$ (respectively, $\lim_{\lambda\to\infty} \lambda \cdot x$) exists. The functors (2.1) come equipped with natural maps

$$X^0 \leftarrow X^{\pm} \to X, \tag{2.2}$$

where $X^{\pm} \to X^0$ (respectively, $X^{\pm} \to X$) is given by evaluating a morphism at the zero section (respectively, at the unit section). We say that the $\mathbb{G}_m$-action on an algebraic space X is étale (respectively, Zariski) locally linearizable if the $\mathbb{G}_m$-action lifts – necessarily uniquely – to an étale cover which is affine; cf. [Ric19, Definition 1.6]. We say that a $\mathbb{G}_m$-action on an S-ind-algebraic space X is étale (respectively, Zariski) locally linearizable if there is a $\mathbb{G}_m$-stable presentation with equivariant transition maps $X = \mathrm{colim}_i X_i$ where the $\mathbb{G}_m$-action on each X_i is étale (respectively, Zariski) locally linearizable. We use the following representability properties of the functors (2.1); cf. [HR18, Theorem 2.1].

Theorem 2.1. *Let $X = \mathrm{colim}_i X_i$ be an $\mathcal{O}$-ind-algebraic space equipped with an étale locally linearizable $\mathbb{G}_m$-action.*

(i) *The subfunctor $X^0 = \mathrm{colim}_i X_i^0$ is representable by a closed sub-ind-algebraic space of X.*

(ii) *The functor $X^{\pm} = \mathrm{colim}_i X_i^{\pm}$ is representable, and the map $X^{\pm} \to X$ is representable and quasi-compact. The map $X^{\pm} \to X^0$ is ind-affine with geometrically connected fibers and induces a bijection on connected components $\pi_0(X^{\pm}) \simeq \pi_0(X^0)$ of the underlying topological spaces.*

(iii) *If $X = \mathrm{colim}_i X_i$ is of ind-finite presentation (respectively, an ind-scheme; respectively, separated), so are X^0 and $X^{\pm}$.*

The proof is like that of [HR18, Theorem 2.1], using the representability results of [Ric19, Theorem 1.8]. We record the following lemma for later use.

Lemma 2.2. *For $n \in \mathbb{Z}_{>0}$, let $X_1, \dots, X_n$ be $\mathcal{O}$-algebraic spaces (or $\mathcal{O}$-ind-algebraic spaces) equipped with an étale locally linearizable $\mathbb{G}_m$-action. Then the diagonal $\mathbb{G}_m$-action on the product $\prod_{i=1}^n X_i$ is étale locally linearizable, and the canonical maps*

$$\Big(\prod_{i=1}^n X_i\Big)^0 \xrightarrow{\simeq} \prod_{i=1}^n X_i^0 \quad \text{and} \quad \Big(\prod_{i=1}^n X_i\Big)^{\pm} \xrightarrow{\simeq} \prod_{i=1}^n X_i^{\pm}$$

are isomorphisms.

Proof. If, for each i, the map $U_i \to X_i$ is an étale local linearization, then the product $\prod_{i=1}^n U_i \to \prod_{i=1}^n X_i$ is an étale local linearization. It is easy to check on the level of functors that the maps are isomorphisms. □

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3. Affine Grassmannians for Cartier divisors

In this section we give a self-contained treatment of affine Grassmannians for non-constant group schemes over relative curves which are formed using a formal neighborhood of a general Cartier divisor. This extends the work of Beilinson and Drinfeld [BD99], and is inspired by the work of Fedorov and Panin [FP15, Fed16] and Levin [Lev16].

3.1 Definitions and examples

Let $\mathcal{O}$ be a Noetherian ring. Let X be a smooth $\mathcal{O}$-curve, that is, the structure map $X \to \mathrm{Spec}(\mathcal{O})$ is of finite presentation and smooth of pure dimension 1. Let $D \subset X$ be a relative effective Cartier divisor which is finite and flat over $\mathcal{O}$. Let $\mathcal{G}$ be a smooth affine X-group scheme.

To the triple $(X, \mathcal{G}, D)$, we associate the functor $\mathrm{Gr}_{\mathcal{G}} = \mathrm{Gr}_{(X,\mathcal{G},D)}$ on the category of $\mathcal{O}$-algebras which assigns to every R the set of isomorphism classes of tuples $(\mathcal{F}, \alpha)$ with

$$\begin{cases} \mathcal{F} \text{ a } \mathcal{G}\text{-torsor on } X_R, \\ \alpha : \mathcal{F}|_{(X\backslash D)_R} \xrightarrow{\simeq} \mathcal{F}_0|_{(X\backslash D)_R} \text{ a trivialization,} \end{cases} \tag{3.1}$$

where $\mathcal{F}_0$ denotes the trivial $\mathcal{G}$-torsor. Fpqc descent for schemes affine over X_R implies that $\mathrm{Gr}_{\mathcal{G}}$ is an $\mathcal{O}$-space. As $\mathcal{G}$ is smooth affine and hence of finite presentation, the functor $\mathrm{Gr}_{\mathcal{G}}$ commutes with filtered colimits of $\mathcal{O}$-algebras. Further, if R is an $\mathcal{O}$-algebra, then as functors on R-Alg,

$$\mathrm{Gr}_{\mathcal{G}} \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(R) = \mathrm{Gr}_{\mathcal{G}}|_{R\text{-Alg}} = \mathrm{Gr}_{(X_R, \mathcal{G}_{X_R}, D_R)}. \tag{3.2}$$

If we replace D by a positive multiple nD for some $n \geqslant 1$, then $X\backslash D = X\backslash nD$, and hence, as $\mathcal{O}$-functors,

$$\mathrm{Gr}_{(X,\mathcal{G},D)} = \mathrm{Gr}_{(X,\mathcal{G},nD)}. \tag{3.3}$$

The following examples are special cases of the general setup.

Example 3.1. (i) *Affine Grassmannians/flag varieties.* Let $\mathcal{O} = F$ be a field, and let $D = \{x\}$ for some point $x \in X(F)$. Then on completed local rings $\mathcal{O}_x \simeq F[\![t_x]\!]$ where t_x denotes a local parameter at $x \in X$. If $\mathcal{G} = G \otimes_F X$ for some smooth affine F-group G, then $\mathrm{Gr}_G := \mathrm{Gr}_{\mathcal{G}}$ is (by the Beauville–Laszlo theorem [BL95]) the 'affine Grassmannian' formed using the local parameter t_x, that is, the ind-scheme given by the étale sheafification of the functor $R \mapsto G(R(\!(t_x)\!))/G(R[\![t_x]\!])$. In general, the functor $\mathrm{Gr}_{\mathcal{G}}$ is the 'twisted affine flag variety' for the group scheme $\mathcal{G} \otimes_X F[\![t_x]\!]$ in the sense of [PR08].

(ii) *Mixed characteristic.* Let $\mathcal{O} = \mathcal{O}_F$ be the valuation ring of a finite extension $F/\mathbb{Q}_p$. Let K/F be a finite totally ramified extension with uniformizer $\varpi \in K$. Let $X = \mathbb{A}^1_{\mathcal{O}_F}$ with global coordinate denoted by z, and let $D = \{Q = 0\}$, where $Q \in \mathcal{O}_F[z]$ is the minimal polynomial of ϖ over F (an Eisenstein polynomial). Let $\mathcal{G}$ be the X-group scheme constructed in [PZ13, Theorem 4.1] if $K = F$, and in [Lev16, Theorem 3.3.3] otherwise; here it is denoted by $\underline{\mathcal{G}}$ (see Theorem 4.13). Then $\mathrm{Gr}_{\mathcal{G}}$ is the $\mathcal{O}_F$-ind-scheme defined in [PZ13, Equation (6.11)] if $K = F$, and in [Lev16, Definition 4.1.1] otherwise; here we denote it by $\mathrm{Gr}_{\tilde{\mathcal{G}}}$ (see § 4.4.1).

(iii) *Equal characteristic.* Let F be a field, and let C be a smooth affine F-curve. Let $\mathcal{O} = \Gamma(C, \mathcal{O}_C)$ be the global sections, and let $X = C \times_F C = C_{\mathcal{O}}$. Let $\mathcal{G}_0$ be a smooth affine $\mathcal{O}$-group scheme, and let $\mathcal{G} = \mathcal{G}_0 \otimes_{\mathcal{O}} X$. Let $D := \Delta(C)$ be the diagonal divisor in X. If $C = \mathbb{A}^1_F$, then $\mathrm{Gr}_{\mathcal{G}}$ is the ind-scheme defined in [Zhu14, Equation (3.1.1)]. If $x \in C(F)$ is a point, and $\mathcal{O}_x \to \mathcal{O}$ denotes the completed local ring, then $\mathrm{Gr}_{\mathcal{G}} \otimes_{\mathcal{O}} \mathcal{O}_x$ is the ind-scheme defined in [Ric16b, Definition 2.3]. Let us remark that this is a special case of the general setup in [Hei10, § 2].

(iv) *Fusion Grassmannians.* Let F be a field, and let C be an affine curve over F. The dth symmetric product $C^{(d)}$ is by [SGA4, Example XVII, Proposition 6.3.9] the moduli space of degree-d effective Cartier divisors on C. Let $\mathrm{Spec}(\mathcal{O}) := C^{(d)}$, and let $D := C^{(d)}$ be the universal degree-d divisor on $X := C \times_F C^{(d)} = C_{\mathcal{O}}$. For a smooth affine F-group scheme G, let $\mathcal{G} = G \otimes_F X$. Then the ind-scheme $\mathrm{Gr}_{\mathcal{G}} \times_{\mathrm{Spec}(\mathcal{O})} C^d$ is the fusion Grassmannian defined in [BD99, 5.3.11].

(v) *Generically trivial bundles.* If $X = \mathbb{A}^1_{\mathcal{O}}$ and $\mathcal{G}$ is split reductive, then the functor $\mathrm{Gr}_{\mathcal{G}}$ in (3.1) is the moduli space of objects used in [Fed16, Theorem 2].

3.1.1 *Loop groups.* The functor $\mathrm{Gr}_{\mathcal{G}}$ is related to loop groups as follows. For an $\mathcal{O}$-algebra R, let $(X_R/D_R)^\wedge$ be the formal affine[1] scheme defined by D_R in X_R, and denote by $R[\![D]\!]$ its ring of regular functions. Explicitly, if $\mathcal{I}_R \subset \mathcal{O}_{X_R}$ is the ideal sheaf for D_R, then $(D_R, \mathcal{O}_{X_R}/\mathcal{I}_R^n)$ is an affine scheme $\mathrm{Spec}(A_n)$ for all $n \geqslant 1$, and $R[\![D]\!] \overset{\text{def}}{=} \varprojlim A_n = \varprojlim \Gamma(D_R, \mathcal{O}_{X_R}/\mathcal{I}_R^n)$. Let $\hat{D}_R = \mathrm{Spec}(R[\![D]\!])$ be the associated affine (true) scheme. The map $(X_R/D_R)^\wedge \to X_R$ uniquely extends to a map $p\colon \hat{D}_R \to X_R$ by [Bha16, Theorem 1.1],[2] and $p^{-1}(D_R) \simeq D_R$ defines a relative effective Cartier divisor on $\hat{D}_R$. Let $\hat{D}_R^o = \hat{D}_R \backslash D_R$. As D_R is a Cartier divisor in $\hat{D}_R$, it is locally principal, and hence the complement $\hat{D}_R^o := \mathrm{Spec}(R(\!(D)\!))$ is an affine scheme. The *(twisted) loop group* $L\mathcal{G} = L_D\mathcal{G}$ is the functor on the category of $\mathcal{O}$-algebras

$$L\mathcal{G}\colon R \mapsto \mathcal{G}(R(\!(D)\!)). \tag{3.4}$$

The *positive (twisted) loop group* $L^+\mathcal{G} = L^+_D\mathcal{G}$ is the functor on the category of $\mathcal{O}$-algebras

$$L^+\mathcal{G}\colon R \mapsto \mathcal{G}(R[\![D]\!]). \tag{3.5}$$

As every Cartier divisor is locally defined by a single non-zero divisor, we see that $L^+\mathcal{G} \subset L\mathcal{G}$ is a subgroup functor. Let us explain why these functors are representable in this generality.

Lemma 3.2. (i) *The functor $L^+\mathcal{G}$ (respectively, $L\mathcal{G}$) is representable by an affine scheme (respectively, ind-affine ind-scheme). In particular, $L^+\mathcal{G}$ and $L\mathcal{G}$ are $\mathcal{O}$-spaces.*

(ii) *The scheme $L^+\mathcal{G}$ is a faithfully flat affine $\mathcal{O}$-group scheme which is pro-smooth.*

Proof. Part (i) is true for every affine scheme $\mathcal{G}$ of finite presentation over $\mathcal{O}$: Let $\mathcal{G} = \mathbb{A}^1_{\mathcal{O}}$ first. Denote by I_D the invertible ideal defined by D in $\mathcal{O}[\![D]\!]$. By the preceding discussion, the ring $\mathcal{O}[\![D]\!]/I_D$ is isomorphic to the global sections of D and both are finite locally free $\mathcal{O}$-modules; cf. [StaPro, Tag 0B9C]. For any $a \in \mathbb{Z}$, we form I_D^a as an invertible $\mathcal{O}[\![D]\!]$-module. For $a \leqslant b$, denote by $E_{[a,b]}$ the $\mathcal{O}$-module I_D^a/I_D^b which is also finite locally free (hence reflexive) by an induction argument. As b varies, the set of $\mathcal{O}$-modules $\{E_{[a,b]}\}_{b \geqslant a}$ forms an inverse system, and $\mathcal{O}[\![D]\!] = \lim_{b \geqslant 0} E_{[0,b]}$ by definition. It follows that $I_D^a = \lim_{b \geqslant a} E_{[a,b]}$ for any $a \in \mathbb{Z}$. In particular, we get $\mathcal{O}(\!(D)\!) = \mathrm{colim}_a \lim_{b \geqslant a} E_{[a,b]}$. As $E_{[a,b]}$ is a reflexive $\mathcal{O}$-module, we get for every $\mathcal{O}$-algebra R that

$$E_{[a,b]} \otimes_{\mathcal{O}} R = \mathrm{Hom}_{\mathcal{O}\text{-Mod}}((E_{[a,b]})^*, R) = \mathrm{Hom}_{\mathcal{O}\text{-Sch}}(\mathrm{Spec}(R), \mathbb{V}_{[a,b]}), \tag{3.6}$$

[1] One can show that a formal completion $(X/X')^\wedge$ of a scheme X along an *affine* closed subscheme $X' \subset X$ is of the form $\mathrm{Spf}(A)$ for an admissible topological ring A.

[2] When X_R is quasi-projective, one can invoke the more elementary result of [BD99, 2.12.6].

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where $\mathbb{V}_{[a,b]} = \mathrm{Spec}(\mathrm{Sym}^{\otimes}(E_{[a,b]})^*)$ for every pair $b \geqslant a$. Taking limits shows that

$$\mathbb{A}^1_{\mathcal{O}}(R[\![D]\!]) = R[\![D]\!] = \lim_{b\geqslant 0}(E_{[0,b]} \otimes_{\mathcal{O}} R)$$

is identified with the R-points of the affine $\mathcal{O}$-scheme $\lim_{b\geqslant 0}\mathbb{V}_{[0,b]}$. The same argument shows that $R \mapsto \mathbb{A}^1_{\mathcal{O}}(R(\!(D)\!))$ is representable by the ind-affine ind-scheme $\mathrm{colim}_a \lim_{b\geqslant a}\mathbb{V}_{[a,b]}$. This gives part (i) in the case $\mathcal{G} = \mathbb{A}^1_{\mathcal{O}}$. For the general case, one verifies that the L^+-construction (respectively, L-construction) commutes with taking finite products and equalizers, and that finite products and equalizers are constructed termwise in the category of ind-schemes. Hence, the lemma follows for $L^+\mathbb{A}^n_{\mathcal{O}}$ (respectively, $L\mathbb{A}^n_{\mathcal{O}}$). A finite presentation $\mathcal{G} = \mathrm{Spec}(\mathcal{O}[t_1,\dots,t_n]/(f_1,\dots,f_m))$ realizes $\mathcal{G}$ as the equalizer of the two maps $\varphi, \psi\colon \mathbb{A}^n_{\mathcal{O}} \to \mathbb{A}^m_{\mathcal{O}}$ where φ is given by the functions $f_1,\dots,f_m$ and ψ is the composition of the structure map with the zero section. Hence, $L^+\mathcal{G}$ (respectively, $L\mathcal{G}$) is the equalizer of $L^+\varphi$ and $L^+\psi$ (respectively, $L\varphi$ and $L\psi$) in the category of schemes (respectively, ind-schemes). As equalizers define closed subschemes and $L^+\mathbb{A}^n_{\mathcal{O}}$ is affine (respectively, $L\mathbb{A}^n_{\mathcal{O}}$ ind-affine), (i) follows.

Part (ii) is true for every smooth affine $\mathcal{O}$-scheme $\mathcal{G}$, necessarily of finite presentation. For $n \geqslant 0$, let $D_n = \mathrm{Spec}(\mathcal{O}[\![D]\!]/I_D^{n+1})$ be the nth infinitesimal neighborhood of D in X. The Weil restriction of scalars $\mathcal{G}_n := \mathrm{Res}_{D_n/\mathcal{O}}(\mathcal{G} \times_X D_n)$ is a smooth affine $\mathcal{O}$-group scheme; cf. [BLR90, § 7.6, Theorem 4, Proposition 5]. For varying n, these groups fit into an inverse system $\mathcal{G}_m \to \mathcal{G}_n$ for $m \geqslant n$, and the natural map of functors

$$L^+\mathcal{G} \xrightarrow{\simeq} \lim_{n\geqslant 0}\mathcal{G}_n \tag{3.7}$$

is an isomorphism. This proves (ii), and the lemma follows. □

Remark 3.3. If nD is a positive multiple of D, then there is a canonical isomorphism $\mathcal{O}[\![D]\!] \xrightarrow{\simeq} \mathcal{O}[\![nD]\!]$ (respectively, $\mathcal{O}(\!(D)\!) \xrightarrow{\simeq} \mathcal{O}(\!(nD)\!)$). Indeed, as $I_{nD} = I_D^n \subset I_D$, the ring $\mathcal{O}[\![D]\!]$ is complete with respect to the I_{nD}-adic topology, and hence $\mathcal{O}[\![D]\!] \simeq \lim_{k\geqslant 0} R[\![D]\!]/I^k_{nD} = R[\![nD]\!]$.

Lemma 3.4. (i) *The loop group $L\mathcal{G}$ represents the functor on the category of $\mathcal{O}$-algebras which assigns to every R the set of isomorphism classes of triples $(\mathcal{F}, \alpha, \beta)$, where $\mathcal{F}$ is a $\mathcal{G}$-torsor on X_R, and $\alpha\colon \mathcal{F}|_{X_R\backslash D_R} \xrightarrow{\simeq} \mathcal{F}_0$ (respectively, $\beta\colon \mathcal{F}_0 \xrightarrow{\simeq} \mathcal{F}|_{\hat{D}_R}$) is a trivialization over $X_R\backslash D_R$ (respectively, $\hat{D}_R$).*

(ii) *The projection $L\mathcal{G} \to \mathrm{Gr}_{\mathcal{G}}$, $(\mathcal{F}, \alpha, \beta) \to (\mathcal{F}, \alpha)$ is a right $L^+\mathcal{G}$-torsor in the étale topology, and induces an isomorphism of sheaves $L\mathcal{G}/L^+\mathcal{G} \xrightarrow{\simeq} \mathrm{Gr}_{\mathcal{G}}$.*

Proof. Part (i) is deduced from the Beauville–Laszlo theorem [BL95]; cf. [BD99, § 2.12] for a further discussion (cf. also [PZ13, Lemma 6.1]). For (ii), it is enough to prove that the projection $L\mathcal{G} \to \mathrm{Gr}_{\mathcal{G}}$ admits sections étale locally.

Let R be an $\mathcal{O}$-algebra, and let $\mathcal{F} \to \hat{D}_R$ be a $\mathcal{G}$-torsor. We have to show that $\mathcal{F}$ is trivial étale locally on R, that is, admits a $\hat{D}_R$-section étale locally on R. By applying the lifting criterion for smoothness and an algebraization result for sections (algebraization is easy because $\mathcal{F}$ is affine), it is enough to show that the restriction $\mathcal{F}|_{D_R} \to D_R$ admits a section étale locally on R. Since the functor $\mathcal{F}|_{D_R}\colon R\text{-Alg} \to \mathrm{Sets}, B \mapsto \mathcal{F}(D_B)$ commutes with filtered colimits (because $\mathcal{F}$ is a scheme locally of finite presentation [StaPro, 01ZC]), we may assume without loss of generality that R is a strictly Henselian local $\mathcal{O}$-algebra. Now by assumption on D the R-algebra $R' := \Gamma(D_R, \mathcal{O}_{D_R})$ is finite, and hence a direct product of Henselian local rings $R' = R_1 \times \cdots \times R_n$; cf. [StaPro, 04GH]. As R is strictly Henselian, each R_i is strictly Henselian as well (because a finite extension of a separably closed field is separably closed). But each non-empty smooth scheme over a finite product of strictly Henselian local rings admits a section by Hensel's lemma. This finishes the proof. □

Lemma 3.4(ii) shows that there is a transitive action map

$$L\mathcal{G} \times_{\mathcal{O}} \mathrm{Gr}_{\mathcal{G}} \longrightarrow \mathrm{Gr}_{\mathcal{G}}. \tag{3.8}$$

Let us look at the fibers of (3.8) over $\mathcal{O}$.

Corollary 3.5. (i) *Let F be a field, and let $\mathcal{O} \to F$ be a ring morphism. The underlying reduced subscheme $D_{F,\mathrm{red}} \subset D_F$ is an effective Cartier divisor on X_F, and we write $D_{F,\mathrm{red}} = \sum_{i=1}^n D_i$ where D_i are distinct irreducible, that is, the D_i are closed points of X_F. There is a canonical isomorphism of F-spaces*

$$\mathrm{Gr}_{(X,\mathcal{G},D)} \otimes_{\mathcal{O}} F \xrightarrow{\simeq} \prod_{i=1}^n \mathrm{Gr}_{(X_F,\mathcal{G}_F,D_i)}$$

compatible with the action of $L\mathcal{G}_{(X,\mathcal{G},D)} \otimes_{\mathcal{O}} F \simeq \prod_{i=1}^n L\mathcal{G}_{(X_F,\mathcal{G}_F,D_i)}$.

(ii) *Let $\mathcal{O} = F$ be a field, and let $D = [x]$ be the divisor on X defined by a closed point $x \in X$. The residue field $K := \kappa(x)$ is a finite field extension, and we assume that K/F is separable. There is a canonical isomorphism of F-spaces*

$$\mathrm{Gr}_{(X,\mathcal{G},D)} \xrightarrow{\simeq} \mathrm{Res}_{K/F}(\mathrm{Gr}_{(X_K,\mathcal{G}_{X_K},D)})$$

compatible with the action of $L\mathcal{G}_{(X,\mathcal{G},D)} \simeq \mathrm{Res}_{K/F}(L\mathcal{G}_{(X_K,\mathcal{G}_{X_K},D)})$.

Proof. For (i), we may by (3.2) assume $\mathcal{O} = F$. It is immediate from Remark 3.3 that for any $\mathcal{O}$-algebra R, we have $R[\![D_{\mathrm{red}}]\!] \simeq R[\![D]\!]$ (respectively, $R(\!(D_{\mathrm{red}})\!) \simeq R(\!(D)\!)$). Further, there is a canonical isomorphism

$$R[\![D_{\mathrm{red}}]\!] \xrightarrow{\simeq} \prod_{i=1}^n R[\![D_i]\!] \quad \left(\text{respectively, } R(\!(D_{\mathrm{red}})\!) \xrightarrow{\simeq} \prod_{i=1}^n R(\!(D_i)\!)\right)$$

because X is of dimension 1, and hence $D_i \cap D_j = \emptyset$ for $i \neq j$. Part (i) follows from Lemma 3.4(ii).

For (ii), first note that if we consider D as the divisor on X_K defined by the K-point x, then $\mathrm{Gr}_{(X_K,\mathcal{G}_{X_K},D)}$ is the twisted affine Grassmannian over K; cf. Example 3.1(i). Let $\tilde{K}/F$ be the splitting field of K which is a finite Galois extension with Galois group $\tilde{\Gamma}$. There is a canonical isomorphism of $\tilde{K}$-algebras

$$K \otimes_F \tilde{K} \xrightarrow{\simeq} \prod_{\psi\colon K \hookrightarrow \tilde{K}} \tilde{K}, \quad a \otimes b \longmapsto (\psi(a) \cdot b)_\psi,$$

which is $\tilde{\Gamma}$-equivariant for the action $\gamma * (c_\psi)_\psi \mapsto (\gamma(c_\psi))_{\gamma\psi}$ on the target. Applying this isomorphism to $D \otimes_F \tilde{K}$, we obtain by (i) a $\tilde{\Gamma}$-equivariant isomorphism

$$\mathrm{Gr}_{(X,\mathcal{G},D)} \otimes_F \tilde{K} \xrightarrow{\simeq} \prod_{\psi} \mathrm{Gr}_{(X_K,\mathcal{G}_{X_K},D)} \otimes_{K,\psi} \tilde{K} \tag{3.9}$$

compatible with the actions of the loop groups. The canonical descent datum on the source in (3.9) induces a descent datum on the target of (3.9) which implies (ii). □

Let us point out some useful compatibility with Weil restriction of scalars.

COROLLARY 3.6. *Let $X' \to X$ be a finite flat surjective map of smooth quasi-projective $\mathcal{O}$-curves, and assume $\mathcal{G} = \mathrm{Res}_{X'/X}(\mathcal{G}')$ for a smooth affine X'-group scheme $\mathcal{G}'$. If $D' := D \times_X X'$, then the natural map is an isomorphism of $\mathcal{O}$-spaces*

$$\mathrm{Gr}_{(X',\mathcal{G}',D')} \xrightarrow{\simeq} \mathrm{Gr}_{(X,\mathcal{G},D)}, \quad (\mathcal{F}', \alpha') \mapsto (\mathrm{Res}_{X'/X}(\mathcal{F}'), \mathrm{Res}_{X'/X}(\alpha')). \tag{3.10}$$

Proof. Since $X' \to X$ is finite flat surjective, the closed subscheme $D' \subset X'$ is a relative effective Cartier divisor which is finite flat over $\mathcal{O}$. Hence, the functor $\mathrm{Gr}_{(X',\mathcal{G}',D')}$ is well defined. Using Lemma 3.4(ii), the map (3.10) is induced for any $\mathcal{O}$-algebra R by the canonical map of R-algebras

$$\mathrm{Spec}(R[\![D']\!]) \to \mathrm{Spec}(R[\![D]\!]) \times_X X' \quad (\text{respectively, } \mathrm{Spec}(R(\!(D')\!)) \to \mathrm{Spec}(R(\!(D)\!)) \times_X X').$$

If R is Noetherian, then the first map (hence the second map) is an isomorphism by [StaPro, 00MA] because $X' \to X$ is finite. In particular, (3.10) is an isomorphism for any Noetherian $\mathcal{O}$-algebra R. As both functors in (3.10) commute with filtered colimits of $\mathcal{O}$-algebras, the corollary follows. □

LEMMA 3.7. *Let $\mathcal{O}' \to \mathcal{O}$ be a finite étale map of Noetherian rings. Then the composition $X \to \mathrm{Spec}(\mathcal{O}) \to \mathrm{Spec}(\mathcal{O}')$ is a smooth curve as well, and there is a canonical isomorphism of functors*

$$\mathrm{Gr}_{(X/\mathcal{O}',\mathcal{G},D)} \simeq \mathrm{Res}_{\mathcal{O}/\mathcal{O}'}(\mathrm{Gr}_{(X/\mathcal{O},\mathcal{G},D)}).$$

Proof. If $T \to \mathrm{Spec}(\mathcal{O}')$ is a test scheme, then $X \times_{\mathrm{Spec}(\mathcal{O})} (\mathrm{Spec}(\mathcal{O}) \times_{\mathrm{Spec}(\mathcal{O}')} T) = X \times_{\mathrm{Spec}(\mathcal{O}')} T$. The lemma follows immediately from the definitions. □

3.1.2 *Basic representability properties.* The starting point is the following lemma, and we sketch its proof.

LEMMA 3.8. *If $\mathcal{G} = \mathrm{Gl}_{n,X}$, then the functor $\mathrm{Gr}_{\mathcal{G}}$ is representable by an ind-projective $\mathcal{O}$-ind-scheme.*

Proof. Let R be an $\mathcal{O}$-algebra. If $\mathcal{G} = \mathrm{Gl}_{n,X}$, then $\mathrm{Gr}_{\mathcal{G}}(R)$ classifies rank-n vector bundles $\mathcal{E}$ on X_R together with an isomorphism $\mathcal{E}|_{U_R} \simeq \mathcal{O}^n_{U_R}$ where $U_R := (X \backslash D)_R$. Let $\mathcal{I}_{D_R} \subset \mathcal{O}_{X_R}$ be the invertible ideal sheaf defined by $D_R \subset X_R$. For $N \geqslant 1$, let $\mathrm{Gr}_{\mathcal{G},N}$ be the $\mathcal{O}$-space whose R-valued points are rank-n vector bundles $\mathcal{E}$ on X_R such that, as $\mathcal{O}_{X_R}$-modules,

$$(\mathcal{I}^N_{D_R})^n \subset \mathcal{E} \subset (\mathcal{I}^{-N}_{D_R})^n.$$

Every vector bundle is locally free and by bounding the poles (respectively, zeros) of basis elements, one gets, as $\mathcal{O}$-spaces,

$$\mathrm{colim}_{N \geqslant 1} \mathrm{Gr}_{\mathcal{G},N} \xrightarrow{\simeq} \mathrm{Gr}_{\mathcal{G}}.$$

We claim that each $\mathrm{Gr}_{\mathcal{G},N}$ is representable by a Quot-scheme as follows. The $\mathcal{O}_{X_R}$-module $\mathcal{E}_{N,R} := (\mathcal{I}^{-N}_D / \mathcal{I}^N_D)^n \otimes_{\mathcal{O}} R$ is coherent and locally free over R. Let Quot_N be the $\mathcal{O}$-space whose R-points are coherent $\mathcal{O}_{X_R}$-module quotients $\mathcal{E}_{N,R} \twoheadrightarrow \mathcal{Q}$ which are locally free R-modules. The functor Quot_N is representable by a projective $\mathcal{O}$-scheme by the theory of Quot-schemes applied to the

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finite flat $\mathcal{O}$-scheme $2ND$, and the coherent $\mathcal{O}_{2ND} = \mathcal{O}_X/\mathcal{I}_D^{2N}$-module $\mathcal{E}_{N,\mathcal{O}}$. More precisely, in the notation of [Fan05, § 5.1.4], one has a finite disjoint union

$$\mathrm{Quot}_N = \coprod_{r\in\mathbb{Z}_{\geqslant 0}} \mathrm{Quot}^{r,\mathcal{O}_{2ND}}_{\mathcal{E}_{N,\mathcal{O}}/2ND/\mathrm{Spec}(\mathcal{O})},$$

and the representability result is then a theorem of Grothendieck [Fan05, § 5.5.2, Theorem 5.14]. Note that the structure sheaf $\mathcal{O}_{2ND}$ is relatively ample for $2ND \to \mathrm{Spec}(\mathcal{O})$ because the map is finite (cf. [StaPro, Tag 01VG, 28.35.6]). Concretely, Quot_N is the closed subscheme of the Grassmannian

$$\mathrm{Quot}_N \hookrightarrow \mathrm{Grass}(\mathcal{E}_{N,\mathcal{O}}),$$

which is cut out by the condition that the quotients are stable under the finitely many nilpotent operators $u_1, \ldots, u_n$ on $\mathcal{E}_{N,\mathcal{O}}$ induced by some presentation $2ND = \mathrm{Spec}(\mathcal{O}[u_1, \ldots, u_n]/J)$. Hence, to prove the lemma it is enough to show that, as functors,

$$\mathrm{Gr}_{\mathcal{G},N} \xrightarrow{\simeq} \mathrm{Quot}_N, \quad \mathcal{E} \longmapsto (\mathcal{I}_{D_R}^{-N})^n/\mathcal{E}. \tag{3.11}$$

We need to check that $\mathcal{Q} := (\mathcal{I}_{D_R}^{-N})^n/\mathcal{E}$ is a locally free R-module. This follows from the isomorphism as R-modules $\mathcal{O}_{U_R}^n/\mathcal{E} \simeq \bigoplus_{k\geqslant 0} \mathcal{I}_{D_R}^{-k-1}\mathcal{E}/\mathcal{I}_{D_R}^{-k}\mathcal{E}$, and the short exact sequence

$$0 \to (\mathcal{I}_{D_R}^{-N})^n/\mathcal{E} \to \mathcal{O}_{U_R}^n/\mathcal{E} \to \mathcal{O}_{U_R}^n/(\mathcal{I}_{D_R}^{-N})^n \to 0;$$

cf. also the argument in [Zhu17, Lemma 1.1.5]. Conversely, let $\mathcal{Q} \in \mathrm{Quot}_N(R)$, and define the coherent $\mathcal{O}_{X_R}$-module

$$\mathcal{E} \overset{\mathrm{def}}{=} \ker((\mathcal{I}_{D_R}^{-N})^n \to \mathcal{E}_{N,R} \to \mathcal{Q}).$$

We need to show that $\mathcal{E}$ is a rank-n vector bundle on X_R. Covering X_R with affine schemes, we may assume $X_R = \mathrm{Spec}(S)$ is affine. Let $\mathfrak{p} \subset S$ be a prime ideal lying over a prime ideal $\mathfrak{m} := \mathfrak{p} \cap R \subset R$. By [StaPro, Tag 00M] applied to the map of local rings $R_\mathfrak{m} \to S_\mathfrak{p}$ and the module $\mathcal{E}_\mathfrak{p}$ (note that $\mathcal{E}_\mathfrak{p}$ is still $R_\mathfrak{m}$-flat), to prove $\mathcal{E}_\mathfrak{p}$ is free over $S_\mathfrak{p}$ we are reduced to the case where R is a field. In the case where R is a field, $\mathcal{E} \subset (\mathcal{I}_{D_R}^{-N})^n$ is a torsion-free rank-n submodule, and since $X_R \to \mathrm{Spec}(R)$ is a smooth curve, $\mathcal{E}$ is a vector bundle. □

Remark 3.9. Using Lemma 3.4(ii), the set $\mathrm{Gr}_{\mathrm{Gl}_{n,X}}(\mathcal{O})$ can be identified with the set of $\mathcal{O}[\![D]\!]$-lattices in $\mathcal{O}(\!(D)\!)$, that is, in the notation of Lemma 3.2, the set of $\mathcal{O}[\![D]\!]$-submodules $M \subset \mathcal{O}(\!(D)\!)$ such that for some $N \gg 0$, $(I_D^N)^n \subset M \subset (I_D^{-N})^n$ and $(I_D^{-N})^n/M$ is a locally free $\mathcal{O}$-module.

Proposition 3.10. *If $\mathcal{G} \hookrightarrow G$ is a monomorphism of smooth X-affine X-group schemes such that the fppf quotient $G/\mathcal{G}$ is a X-quasi-affine scheme (respectively, X-affine scheme), then the map $\mathrm{Gr}_\mathcal{G} \to \mathrm{Gr}_G$ is representable by a quasi-compact immersion (respectively, closed immersion).*

Proof. Following the proof of [Zhu17, Proposition 1.2.6], it is enough to establish the analogue of [Zhu17, Lemma 1.2.7]. Let R an $\mathcal{O}$-algebra, and let $p\colon V \to \hat{D}_R$ be an affine scheme of finite presentation. Let s be a section of p over $\hat{D}_R^o$. We need to prove that the presheaf assigning, to any R-algebra R', the set of sections s' of p over $\hat{D}_{R'}$ such that $s'|_{\hat{D}_{R'}^o} = s|_{\hat{D}_{R'}^o}$, is representable by a closed subscheme of $\mathrm{Spec}(R)$. Indeed, choosing a closed embedding $V \subset \mathbb{A}^n_{\hat{D}_R}$ for some $n \gg 0$ and using that $R[\![D]\!] \subset R(\!(D)\!)$ is injective, we reduce to the case $V = \mathbb{A}^n_{\hat{D}_R}$. The presheaf in question is representable by the locus on $\mathrm{Spec}(R)$ where the class $\bar{s}$ of the section $s \in V(\hat{D}_R^o) = R(\!(D)\!)^n$

in $(R(\!(D)\!)/R[\![D]\!])^n$ vanishes. With the notation of Lemma 3.2, we have $\bar{s} \in E_{[-N,0]} \otimes_{\mathcal{O}} R$ for some $N \gg 0$. As $E_{[-N,0]}$ is a reflexive $\mathcal{O}$-module, we see that giving an element of $E_{[-N,0]} \otimes_{\mathcal{O}} R$ is equivalent to giving a map of R-schemes $\operatorname{Spec}(R) \to \mathbb{V}(E_{[-N,0]} \otimes_{\mathcal{O}} R)$. Then the presheaf in question is representable by the equalizer of the two maps corresponding to the elements $\bar{s}, 0 \in E_{[-N,0]} \otimes_{\mathcal{O}} R$ which is a closed subscheme of $\operatorname{Spec}(R)$. □

COROLLARY 3.11. (i) *If there exists a monomorphism $\mathcal{G} \hookrightarrow \mathrm{Gl}_{n,X}$ such that the fppf quotient is a X-quasi-affine scheme (respectively, an X-affine scheme), then $\mathrm{Gr}_{\mathcal{G}} = \operatorname{colim}_i \mathrm{Gr}_{\mathcal{G},i}$ is representable by a separated $\mathcal{O}$-ind-scheme of ind-finite type (respectively, separated ind-proper $\mathcal{O}$-ind-scheme). Each $\mathrm{Gr}_{\mathcal{G},i}$ can be chosen to be $L^+\mathcal{G}$-stable.*

(ii) *If in (i) the representation $\mathcal{G} \hookrightarrow \mathrm{Gl}_{n,X}$ exists étale locally on $\mathcal{O}$, then $\mathrm{Gr}_{\mathcal{G}} = \operatorname{colim}_i \mathrm{Gr}_{\mathcal{G},i}$ is a separated $\mathcal{O}$-ind-algebraic space of ind-finite presentation (respectively, separated ind-proper $\mathcal{O}$-ind-algebraic space). Each $\mathrm{Gr}_{\mathcal{G},i}$ can be chosen to be $L^+\mathcal{G}$-stable.*

(iii) *If $\mathcal{G} = G \otimes_{\mathcal{O}} X$ is constant and G is a reductive $\mathcal{O}$-group scheme, then $\mathrm{Gr}_{\mathcal{G}}$ is representable by an ind-proper $\mathcal{O}$-ind-algebraic space.*

Proof. Part (i) is immediate from Lemma 3.8 and Proposition 3.10.

For (ii), we use part (i) together with Lemma 3.12 below. Note that the diagonal of $\mathrm{Gr}_{\mathcal{G}}$ being representable by a closed immersion follows from the same property of $\mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}}}$ and the effectivity of descent for closed immersions. Further, if $\mathcal{O} \to \mathcal{O}'$ is étale, then the method of Lemma 3.12 shows that an $L^+\mathcal{G} \otimes_{\mathcal{O}} \mathcal{O}'$-stable presentation of $\mathrm{Gr}_{\mathcal{G}} \otimes_{\mathcal{O}} \mathcal{O}'$ induces an $L^+\mathcal{G}$-stable presentation of $\mathrm{Gr}_{\mathcal{G}}$ (because $L^+\mathcal{G}$ is affine and flat, and taking the scheme-theoretic image commutes with flat base change).

For (iii), note that after an étale cover $\mathcal{O} \to \mathcal{O}'$, the group scheme $G_{\mathcal{O}'} := G \otimes_{\mathcal{O}} \mathcal{O}'$ is split reductive and, in particular, linearly reductive. If we choose a closed immersion $G_{\mathcal{O}'} \hookrightarrow \mathrm{Gl}_{n,\mathcal{O}'}$, then the quotient $\mathrm{Gl}_{n,\mathcal{O}'}/G_{\mathcal{O}'}$ is representable by an affine scheme by [Alp14, Corollary 9.7.7], and (iii) follows from (ii). □

LEMMA 3.12. *Let X be an $\mathcal{O}$-space with schematic diagonal, and such that there exists a étale surjective (as sheaves) map of $\mathcal{O}$-spaces $U \to X$ with U an $\mathcal{O}$-ind-scheme. If either $U \to X$ is quasi-compact or U is quasi-separated, then X is an $\mathcal{O}$-ind-algebraic space.*

Proof. Given a presentation $U = \operatorname{colim}_{i \in I} U_i$ with U_i being schemes, we need to construct a presentation $X = \operatorname{colim}_{i \in I} X_i$ with X_i being algebraic spaces. For each i, consider $U_i \subset U \to X$. We define X_i' to be the scheme-theoretic image of the map

$$U_i \times_X U \subset U \times_X U \xrightarrow{p_2} U. \tag{3.12}$$

This is well defined for the following reason. Since $U_i \times_X U$ is a quasi-compact scheme, the map (3.12) factors through $U_j \subset U$ for some $j \gg i$. In either case, $U \to X$ quasi-compact or U quasi-separated, the map (3.12) is quasi-compact. By [StaPro, 01R8], the scheme-theoretic image behaves well for quasi-compact maps, and $X_i' \subset U_j$ is a quasi-compact closed subscheme. As scheme-theoretic images of quasi-compact maps commute with flat base change [StaPro, Tag 081I], the scheme X_i' is equipped with a descent datum relative to $U \to X$, and defines a closed $\mathcal{O}$-subspace $X_i \subset X$ together with an étale surjective map $X_i' \to X_i$. As $X_i \subset X$ is closed, the diagonal of X_i is schematic, and X_i is a quasi-compact algebraic space. By construction the X_i form a filtered direct system indexed by the poset I, and the canonical map $\operatorname{colim}_{i \in I} X_i \to X$ is an isomorphism (because $U \to X$ is a sheaf surjection, and $\operatorname{colim}_i X_i' = U$ by construction). □

Remark 3.13. It would be nice to give a proof of representability of $\mathrm{Gr}_{\mathcal{G}}$ which does not refer to the choice of an embedding $\mathcal{G} \hookrightarrow \mathrm{Gl}_{n,X}$.

3.2 The open cell

In this subsection and the next, we apply our methods to prove Theorem 3.17, a generalization of Theorem A from the introduction. The results are not used in the proof of our main theorem.

We specialize to the case where $X = \mathbb{A}^1_{\mathcal{O}}$, and where $\mathcal{G} = G \otimes_{\mathcal{O}} X$ is constant, that is, the base change of a smooth affine $\mathcal{O}$-group scheme G of finite presentation. In this case, we denote $L_D\mathcal{G}$ (respectively, $L_D^+\mathcal{G}$; respectively, $\mathrm{Gr}_{(X,\mathcal{G},D)}$) by $LG = L_DG$ (respectively, $L^+G = L_D^+G$; respectively, $\mathrm{Gr}_G = \mathrm{Gr}_{(X,G,D)}$).

Since $D \subset \mathbb{A}^1_{\mathcal{O}}$ is assumed to be finite over $\mathcal{O}$, the subscheme $D \subset \mathbb{P}^1_{\mathcal{O}}$ is closed and defines a relative effective Cartier divisor. In particular, Lemma 3.4(ii) (the Beauville–Laszlo lemma) implies that $\mathrm{Gr}_{(\mathbb{A}^1_{\mathcal{O}},G,D)} = \mathrm{Gr}_{(\mathbb{P}^1_{\mathcal{O}},G,D)}$ by extending torsors trivially to ∞.

The *negative loop group* is the functor on the category of $\mathcal{O}$-algebras

$$L^-G\colon R \mapsto G(\mathbb{P}^1_R \backslash D_R). \tag{3.13}$$

Then L^-G is an $\mathcal{O}$-space which is a subgroup functor $L^-G \subset LG$.

Lemma 3.14. *The functor L^-G is representable by an ind-affine ind-scheme locally of ind-finite presentation over $\mathcal{O}$.*

Proof. That the affine schemes are of finite presentation follows from the fact that L^-G commutes with filtered colimits (because G is of finite presentation). One verifies that L^- commutes with finite products and equalizers, and hence the proof of representability is reduced to the case $G = \mathbb{A}^1_{\mathcal{O}}$; cf. the proof of Lemma 3.2. We have to show that the functor on the category of $\mathcal{O}$-algebras R given by the global sections $R \mapsto \Gamma(\mathcal{O}_{\mathbb{P}^1_R\backslash D_R})$ is representable by an ind-affine ind-scheme. But, as R-modules, $\Gamma(\mathcal{O}_{\mathbb{P}^1_R\backslash D_R}) = \mathrm{colim}_n\,\Gamma(\mathcal{O}_{\mathbb{P}^1_R}(nD_R))$, and we claim that $\Gamma(\mathcal{O}_{\mathbb{P}^1_R}(nD_R))$ is finite locally free: indeed, this follows from the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1_R} \to \mathcal{O}_{\mathbb{P}^1_R}(nD_R) \to \mathcal{I}^{-1}_{nD_R}/\mathcal{O}_{\mathbb{P}^1_R} \to 0$$

and the vanishing of $H^1_{\mathrm{Zar}}(\mathbb{P}^1_R, \mathcal{O}_{\mathbb{P}^1_R})$. This proves the lemma. □

Now define $L^{--}G = \ker(L^-G \to G)$ for $g \mapsto g(\infty)$. Then the intersection $L^{--}G \cap L^+G$ is trivial inside LG, and we consider the orbit map

$$L^{--}G \longrightarrow \mathrm{Gr}_G, \quad g^- \longmapsto g^- \cdot e_0, \tag{3.14}$$

where $e_0 \in \mathrm{Gr}_G$ denotes the base point.

Lemma 3.15. *The map (3.14) is representable by an open immersion, and identifies $L^{--}G$ with those pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is the trivial torsor.*

Proof. The argument is the same as the deformation argument given in [HR18, Lemma 3.1], and we do not repeat it here. □

3.3 Geometry of $\mathbb{G}_m$-actions on Gr_G

We assume $X = \mathbb{A}^1_{\mathcal{O}}$, and $\mathcal{G} = G \otimes_{\mathcal{O}} X$ with G being a reductive $\mathcal{O}$-group scheme with connected (and hence geometrically connected) fibers. Let $\chi\colon \mathbb{G}_{m,\mathcal{O}} \to G$ be an $\mathcal{O}$-rational cocharacter. The cocharacter χ induces via the composition

$$\mathbb{G}_{m,\mathcal{O}} \subset L^+\mathbb{G}_{m,\mathcal{O}} \xrightarrow{L^+\chi} L^+G \subset LG \tag{3.15}$$

a left $\mathbb{G}_m$-action on the affine Grassmannian $\mathrm{Gr}_G \to \mathrm{Spec}(\mathcal{O})$. As in (2.2), we obtain maps of $\mathcal{O}$-spaces

$$(\mathrm{Gr}_G)^0 \leftarrow (\mathrm{Gr}_G)^{\pm} \to \mathrm{Gr}_G. \tag{3.16}$$

Let us mention the following lemma which implies the ind-representability of the spaces (3.16), in light of Theorem 2.1 and Corollary 3.11.

Lemma 3.16. *The $\mathbb{G}_m$-action on Gr_G is étale locally linearizable.*

Proof. After an étale cover $\mathcal{O} \to \mathcal{O}'$, there exists a closed immersion $\mathrm{Gr}_{G'_{\mathcal{O}}} \to \mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}'}}$ (cf. Proposition 3.11(iii)) which is $\mathbb{G}_m$-equivariant with respect to the action on $\mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}'}}$ given by the cocharacter $\mathbb{G}_{m,\mathcal{O}'} \overset{\chi}{\to} G_{\mathcal{O}'} \to \mathrm{Gl}_{n,\mathcal{O}'}$. The proof of Lemma 3.12 shows that an $L^+G_{\mathcal{O}'}$-stable presentation of $\mathrm{Gr}_{G_{\mathcal{O}'}}$ by quasi-compact schemes induces an L^+G-stable presentation of Gr_G by quasi-compact algebraic spaces. To prove the lemma it is enough to show that the $\mathbb{G}_m$-action on $\mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}'}}$ is Zariski locally linearizable, and we reduce to the case $\mathcal{O} = \mathcal{O}'$, $G = \mathrm{Gl}_{n,\mathcal{O}}$. By [Con14, Propositions 6.2.11 and 3.1.9], Zariski locally on $\mathcal{O}$ the cocharacter χ lies in a split maximal torus in $\mathrm{Gl}_{n,\mathcal{O}}$ which is $\mathcal{O}$-conjugate to the diagonal matrices in $\mathrm{Gl}_{n,\mathcal{O}}$, and hence is after conjugation with a permutation matrix dominant. In this way, we reduce to the case where χ is a standard dominant cocharacter given by $\lambda \mapsto \mathrm{diag}(\lambda^{a_1}, \dots, \lambda^{a_n})$ for some integers $a_1 \geqslant \cdots \geqslant a_n$. With the notation of Lemma 3.8, it is now immediate that the $\mathbb{G}_m$-action on $\mathrm{Quot}_N \subset \mathrm{Grass}(\mathcal{E}_{N,\mathcal{O}})$ is linear, and compatible with the transition maps for varying N. The lemma follows. □

Our aim is to express (3.16) in terms of group-theoretical data related to the cocharacter χ; cf. Theorem 3.17 below.

Let χ act on G via conjugation $(\lambda, g) \mapsto \chi(\lambda) \cdot g \cdot \chi(g)^{-1}$. The fixed points $M = G^0$ (respectively, the attractor $P^+ = G^+$; respectively, the repeller $P^- = G^-$) define a closed subgroup of G which is smooth of finite presentation over $\mathcal{O}$; cf. [Mar15]. The group M is the centralizer of χ, and is by the classical theory over a field a reductive $\mathcal{O}$-group scheme which is fiberwise connected (hence fiberwise geometrically connected). By (2.2) we have natural maps of $\mathcal{O}$-groups

$$M \leftarrow P^{\pm} \to G. \tag{3.17}$$

Theorem 3.17. *The maps (3.17) induce a commutative diagram of $\mathcal{O}$-ind-algebraic spaces*

$$\begin{array}{ccccc} \mathrm{Gr}_M & \longleftarrow & \mathrm{Gr}_{P^{\pm}} & \longrightarrow & \mathrm{Gr}_G \\ \downarrow{\scriptstyle \iota^0} & & \downarrow{\scriptstyle \iota^{\pm}} & & \downarrow{\scriptstyle \mathrm{id}} \\ (\mathrm{Gr}_G)^0 & \longleftarrow & (\mathrm{Gr}_G)^{\pm} & \longrightarrow & \mathrm{Gr}_G, \end{array} \tag{3.18}$$

where the vertical maps ι^0 and $\iota^{\pm}$ are isomorphisms.

Remark 3.18. (i) An interesting example to which Theorem 3.17 applies is the case of fusion Grassmannians $\mathrm{Gr}_G \to \mathbb{A}^n_F$; cf. Example 3.1(iv) with[3] $C = \mathbb{A}^1_F$. Hence, Theorem 3.17 implies Theorem A from the introduction. Note that the group G need not be defined over F, but can be a general reductive group scheme over the nth symmetric power $(\mathbb{A}^1_F)^{(n)}$. Changing the setup slightly, the group G could even be a general reductive group scheme over $\mathbb{A}^n_F$

[3] The case of general smooth F-curves C can be reduced to the special case of $\mathbb{A}^1_F$, but we do not need this in the present paper.

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(take $D = \mathrm{Spec}(\mathcal{O}) = \mathbb{A}^n_F$, $X = \mathbb{A}^n_F \times_F \mathbb{A}^1_F$ and consider the divisor $\mathbb{A}^n_F \to \mathbb{A}^n_F \times \mathbb{A}^1_F$, $(x_i)_i \mapsto ((x_i)_i, \sum_i x_i)$ for $i = 1, \dots, n$). (ii) Note that Theorem 3.17 also generalizes [HR18, Proposition 3.4] and justifies [HR18, sentence containing (3.33)].

3.3.1 *Construction of ι^0 and $\iota^\pm$.* The strategy of construction is the same as in [HR18], which we recall for the sake of readability.

As the $\mathbb{G}_m$-action on Gr_M is trivial, the natural map $\mathrm{Gr}_M \to \mathrm{Gr}_G$ factors as $\mathrm{Gr}_M \to (\mathrm{Gr}_G)^0 \to \mathrm{Gr}_G$, which defines ι^0. For the map $\iota^\pm$, we use the Rees construction explained in Heinloth [Hei18, 1.6.2]. The $\mathbb{G}_m$-action $P^\pm \times \mathbb{G}_{m,\mathcal{O}} \to P^\pm, (p,\lambda) \mapsto \chi(\lambda^\pm) \cdot p \cdot \chi(\lambda^\pm)^{-1}$ via conjugation extends via the monoid action of $\mathbb{A}^1$ on $(\mathbb{A}^1_{\mathcal{O}})^\pm$ in (2.1) to a monoid action

$$m_\chi \colon P^\pm \times \mathbb{A}^1_{\mathcal{O}} \longrightarrow P^\pm \tag{3.19}$$

such that $m_\chi(p, 0) \in M$. We let $\mathrm{gr}_\chi : P^\pm \times \mathbb{A}^1_{\mathcal{O}} \to P^\pm \times \mathbb{A}^1_{\mathcal{O}}$, $(p,\lambda) \mapsto (m_\chi(p,\lambda), \lambda)$, viewed as an $\mathbb{A}^1_{\mathcal{O}}$-group homomorphism. Then the restriction $\mathrm{gr}_\chi|_{\{1\}}$ is the identity, whereas $\mathrm{gr}_\chi|_{\{0\}}$ is the composition $P^\pm \to M \to P^\pm$. For a point $(\mathcal{F}^\pm, \alpha^\pm) \in \mathrm{Gr}_{P^\pm}(R)$, the Rees bundle is

$$\mathrm{Rees}_\chi(\mathcal{F}^\pm, \alpha^\pm) \overset{\mathrm{def}}{=} \mathrm{gr}_{\chi,*}(\mathcal{F}^\pm_{\mathbb{A}^1_R}, \alpha^\pm_{\mathbb{A}^1_R}) \in \mathrm{Gr}_{P^\pm}(\mathbb{A}^1_R), \tag{3.20}$$

where $\mathrm{gr}_{\chi,*}$ denotes the pushforward under the $\mathbb{A}^1$-group homomorphism. Then the restriction $\mathrm{Rees}_\chi(\mathcal{F}^\pm, \alpha^\pm)|_{\{1\}_R}$ is equal to $(\mathcal{F}^\pm, \alpha^\pm)$, whereas $\mathrm{Rees}_\chi(\mathcal{F}^\pm, \alpha^\pm)|_{\{0\}_R}$ is the image of $(\mathcal{F}^\pm, \alpha^\pm)$ under the composition $\mathrm{Gr}_{P^\pm} \to \mathrm{Gr}_M \to \mathrm{Gr}_{P^\pm}$. One checks that $\mathrm{Rees}_\chi(\mathcal{F}^\pm, \alpha^\pm)$ is $\mathbb{G}_m$-equivariant, and hence defines an R-point of $(\mathrm{Gr}_{P^\pm})^\pm$. As the Rees construction is functorial, we obtain a map of $\mathcal{O}$-spaces

$$\mathrm{Rees}_\chi : \mathrm{Gr}_{P^\pm} \to (\mathrm{Gr}_{P^\pm})^\pm, \tag{3.21}$$

which is inverse to the map $(\mathrm{Gr}_{P^\pm})^\pm \to \mathrm{Gr}_{P^\pm}$ given by evaluating at the unit section. We define the map $\mathrm{Gr}_{P^\pm} \to (\mathrm{Gr}_G)^\pm$ to be the composition $\mathrm{Gr}_{P^\pm} \simeq (\mathrm{Gr}_{P^\pm})^\pm \to (\mathrm{Gr}_G)^\pm$ where the latter map is deduced from the natural map $\mathrm{Gr}_{P^\pm} \to \mathrm{Gr}_G$. This constructs the commutative diagram (3.18).

We claim that the map ι^0 (respectively, $\iota^\pm$) is representable by a quasi-compact immersion. By [Con14, Theorem 2.4.1], the fppf quotient G/M is quasi-affine, and hence ι^0 is representable by a quasi-compact immersion by Proposition 3.10. Note that since M is reductive, the space Gr_M is ind-proper and hence ι^0 is even a closed immersion. For $\iota^\pm$, we use that quasi-compact immersions are of effective descent (cf. [StaPro, Tag 0247, 02JR]), and after passing to an étale ring extension of $\mathcal{O}$, we reduce to the case where G is linearly reductive. As in the proof of Corollary 3.11, we choose $G \hookrightarrow \mathrm{Gl}_{n,\mathcal{O}}$ such that $\mathrm{Gl}_{n,\mathcal{O}}/G$ is quasi-affine (or even affine). Let $Q^+ \subset \mathrm{Gl}_{n,\mathcal{O}}$ (respectively, $Q^- \subset \mathrm{Gl}_{n,\mathcal{O}}$) be the attractor (respectively, repeller) subgroup defined by the cocharacter $\mathbb{G}_{m,\mathcal{O}} \xrightarrow{\chi} G \to \mathrm{Gl}_{n,\mathcal{O}}$. Then we have $P^\pm = Q^\pm \times_{\mathrm{Gl}_{n,\mathcal{O}}} G$. The quotient $Q^\pm/P^\pm$ is an algebraic space of finite presentation over $\mathcal{O}$, and the map $i \colon Q^\pm/P^\pm \hookrightarrow \mathrm{Gl}_{n,\mathcal{O}}/G$ is a monomorphism of finite type (hence separated and quasi-finite, by [StaPro, Tag 0463, 59.27.10]). Thus, $Q^\pm/P^\pm$ is a scheme, and the map i is quasi-affine by Zariski's main theorem. In particular,

$Q^{\pm}/P^{\pm}$ is quasi-affine as well. Now there is a commutative diagram of $\mathcal{O}$-spaces

$$\begin{array}{ccccc} \mathrm{Gr}_{P^{\pm}} & \cdots\!\!\searrow & & \curvearrowright & \\ \downarrow & & (\mathrm{Gr}_G)^{\pm} & \longrightarrow & \mathrm{Gr}_G \\ & & \downarrow & & \downarrow \\ \mathrm{Gr}_{Q^{\pm}} & \xrightarrow{\simeq} & (\mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}}})^{\pm} & \longrightarrow & \mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}}}, \end{array} \tag{3.22}$$

constructed as follows. The map $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}}}$ is a quasi-compact immersion by Proposition 3.10, and as Gr_G is ind-proper, it is a closed immersion. Hence, the square is Cartesian by general properties of attractor (respectively, repeller) ind-schemes. This also constructs the dotted arrow in (3.22) which is the map $\iota^{\pm}$. Further, the map $\mathrm{Gr}_{Q^{\pm}} \to (\mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}}})^{\pm}$ is an isomorphism by Lemma 3.19 below. The map $\mathrm{Gr}_{P^{\pm}} \to \mathrm{Gr}_{Q^{\pm}}$ is a quasi-compact immersion because $Q^{\pm}/P^{\pm}$ is quasi-affine. Since $(\mathrm{Gr}_G)^{\pm} \to (\mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}}})^{\pm}$ is a closed immersion, the map $\iota^{\pm}$ is a quasi-compact immersion.

Lemma 3.19. *If $G = \mathrm{Gl}_{n,\mathcal{O}}$, then the maps ι^0 and $\iota^{\pm}$ are isomorphisms.*

Proof. As in the proof of Lemma 3.16, we reduce to the case where χ is a standard dominant cocharacter. Then χ corresponds to a $\mathbb{Z}$-grading on $V := \mathcal{O}^n$, say $V = \bigoplus_{i\in\mathbb{Z}} V_i$, compatible with the standard $\mathcal{O}$-basis of V. The group M (respectively, P^+/P^-) is a standard Levi (respectively, standard parabolic) of automorphisms of V preserving the grading (respectively, the ascending/descending filtration induced from the grading). In the description of Lemma 3.8, the subfunctor Gr_M (respectively, $\mathrm{Gr}_{P^{\pm}}$) consists of those vector bundles $\mathcal{E} \in \mathrm{Gr}_G(R)$ compatible with the grading (respectively, filtration induced by the grading) on $V \otimes_{\mathcal{O}} \mathcal{O}_{U_R}$. Likewise, the grading on V induces in the notation of Lemma 3.8 gradings on $\mathcal{E}_{N,\mathcal{O}} = V \otimes_{\mathcal{O}} (\mathcal{I}_D^{-N}/\mathcal{I}_D^{N})$ for each $N \geqslant 1$. As in Lemma 3.16, we have a closed $\mathbb{G}_m$-equivariant immersion, and hence the diagram of $\mathcal{O}$-schemes

$$\begin{array}{ccc} \mathrm{Quot}_N^0 & \longrightarrow & \mathrm{Grass}(\mathcal{E}_{N,\mathcal{O}})^0 \\ \downarrow & & \downarrow \\ \mathrm{Quot}_N & \longrightarrow & \mathrm{Grass}(\mathcal{E}_{N,\mathcal{O}}) \end{array}$$

is cartesian, and likewise on attractor (respectively, repeller) schemes. The equality $\mathrm{Grass}(\mathcal{E}_{N,\mathcal{O}})^0 = \prod_{i\in\mathbb{Z}} \mathrm{Grass}(V_i \otimes_{\mathcal{O}} (\mathcal{I}_D^{-N}/\mathcal{I}_D^{N}))$ is immediate, and one checks that $\mathrm{Grass}(V \otimes_{\mathcal{O}} (\mathcal{I}_D^{-N}/\mathcal{I}_D^{N}))^{\pm}$ is the subfunctor of those subspaces in $\mathcal{E}_{N,\mathcal{O}}$ compatible with the filtration. The lemma follows. □

3.3.2 *Proof of Theorem 3.17.* We need a lemma first. By functoriality of the loop group construction, the $\mathbb{G}_m$-action on G via χ-conjugation gives a $\mathbb{G}_m$-action on LG (respectively, L^+G; respectively, L^-G). There are natural monomorphisms on negative loop groups

$$L^-M \longrightarrow (L^-G)^0, \tag{3.23}$$

$$L^-P^{\pm} \longrightarrow (L^-G)^{\pm}. \tag{3.24}$$

Lemma 3.20. *The maps (3.23) and (3.24) are isomorphisms.*

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Proof. Replacing $\mathcal{O}$ by an étale cover, we may assume that there exists a closed embedding $G \hookrightarrow \mathrm{Gl}_{n,\mathcal{O}}$. By the proof of Lemma 3.14 (respectively, Lemma 3.2(i)), the induced map $L^-G \to L^- \mathrm{Gl}_{n,\mathcal{O}}$ is a closed immersion.

Let $\chi' \colon \mathbb{G}_{m,\mathcal{O}} \overset{\chi}{\to} G \to \mathrm{Gl}_{n,\mathcal{O}}$, and denote the fixed point group (respectively, attractor/repeller group) by L (respectively, $Q^{\pm}$). It is straightforward to check $L^-M = L^-G \cap L^-L$ (respectively, $L^-P^{\pm} = L^-G \cap L^-Q^{\pm}$) and $(L^-G)^0 = L^-G \cap (L^- \mathrm{Gl}_{n,\mathcal{O}})^0$ (respectively, $(L^-G)^{\pm} = L^-G \cap (L^- \mathrm{Gl}_{n,\mathcal{O}})^{\pm}$). Hence, we may assume $G = \mathrm{Gl}_{n,\mathcal{O}}$.

After passing to a Zariski cover of $\mathcal{O}$, we may assume that χ is a standard dominant cocharacter; cf. proof of Lemma 3.16. We have for every $\mathcal{O}$-algebra R,

$$(L^- \mathrm{Gl}_{n,\mathcal{O}})^0(R) = \{g \in G(\mathbb{P}^1_R \backslash D_R) \mid \forall S \in (\text{R-Alg}), \lambda \in \mathbb{G}_m(S)\colon\ \chi(\lambda) \cdot g \cdot \chi(\lambda)^{-1} = g\}.$$

Let $g \in (L^- \mathrm{Gl}_{n,\mathcal{O}})^0(R)$. To show $g \in (L^-M)(R)$, we can take $S = R[t, t^{-1}]$ to see that the desired entries in the matrix g vanish. The case of $(L^-G)^{\pm}$ is similar, and the lemma follows. □

First case. Let $\mathcal{O} = F$ be a field. By fpqc descent, we may assume that F is algebraically closed. Then $D_{\mathrm{red}} = \sum_{i=1}^d [x_i]$ for pairwise distinct points $x_i \in X(F)$. If $d = 1$, the maps ι^0 and $\iota^{\pm}$ are isomorphisms in light of Example 3.1(i) and [HR18, Proposition 3.4]. In general, by Corollary 3.5 each ind-scheme in (3.18) is a direct product of d copies (compatible with the maps) of classical affine Grassmannians formed using local parameters at x_i. The $\mathbb{G}_m$-action on the product via

$$\mathbb{G}_m \subset L^+_D\mathbb{G}_m \simeq L^+_{[x_1]}\mathbb{G}_m \times_F \cdots \times_F L^+_{[x_n]}\mathbb{G}_m$$

is the diagonal action, and we conclude using Lemma 2.2 and the case $d = 1$.

Second case. Let $\mathcal{O}$ be an Artinian local ring with maximal ideal $\mathfrak{m}$, and residue field F. Passing to the strict Henselization, we may assume that F is separably closed. The restriction of ι^0 (respectively, $\iota^{\pm}$) to the open cell $L^{--}M$ (respectively, $L^{--}P^{\pm}$) is an isomorphism by Lemma 3.20. By Lemma 3.15, there is the open subset

$$V_M \overset{\mathrm{def}}{=} \bigcup_m m \cdot L^{--}M \cdot e_0 \quad \left(\text{respectively, } V_{P^\pm} \overset{\mathrm{def}}{=} \bigcup_p p \cdot L^{--}P^{\pm} \cdot e_0\right),$$

of Gr_M (respectively, $\mathrm{Gr}_{P^\pm}$), where the union runs over all $m \in LM(\mathcal{O})$ (respectively, $p \in LP^{\pm}(\mathcal{O})$). The LM-equivariance (respectively, $LP^{\pm}$-equivariance) of ι^0 (respectively, $\iota^{\pm}$) implies that $\iota^0|_{V_M}$ (respectively, $\iota^{\pm}|_{V_{P^\pm}}$) is an isomorphism. As Gr_M (respectively, $\mathrm{Gr}_{P^\pm}$) is a nilpotent thickening of $\mathrm{Gr}_M \otimes_{\mathcal{O}} F$ (respectively, $\mathrm{Gr}_{P^\pm} \otimes_{\mathcal{O}} F$), it is enough to show that V_M (respectively, $V_{P^\pm}$) contains the special fiber. As G splits over F (because separably closed), the points $\mathrm{Gr}_M(F) \subset \mathrm{Gr}_M$ (respectively, $\mathrm{Gr}_{P^\pm}(F) \subset \mathrm{Gr}_{P^\pm}$) are dense, which follows from the density of $\mathbb{A}^n_F(F) \subset \mathbb{A}^n_F$ and the cellular structure of these spaces. Thus, it suffices to show that $\mathrm{Gr}_M(F) \subset V_M$ (respectively, $\mathrm{Gr}_{P^\pm}(F) \subset V_{P^\pm}$). In view of Lemma 3.4(ii), it suffices to show that the reduction map $LM(\mathcal{O}) \to LM(F)$ (respectively, $LP^{\pm}(\mathcal{O}) \to LP^{\pm}(F)$) is surjective. As $\mathcal{O}$ is Artinian, the ring $\mathcal{O}(\!(D)\!)$ is (semi-local) Artinian, and the reduction map $\mathcal{O}(\!(D)\!) \to F(\!(D)\!)$ is surjective with nilpotent kernel $\mathfrak{m}(\!(D)\!)$. Hence, the desired surjectivity follows from the formal lifting criterion using the smoothness of M (respectively, $P^{\pm}$). This handles the second case.

The general case. Passing to an étale extension of $\mathcal{O}$, we may assume that (3.18) is a diagram of ind-schemes; cf. Corollary 3.11. In view of (3.2), the closed immersion ι^0 (respectively, quasi-compact immersion $\iota^{\pm}$) is fiberwise bijective, and hence bijective. Now Theorem 3.17 follows from Lemma 3.21 below using the second case.

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Lemma 3.21. *Let $\mathcal{O}$ be a Noetherian ring, and let $\iota\colon Y \to Z$ be a quasi-compact immersion of finite type $\mathcal{O}$-schemes. If ι is set-theoretically bijective, and if, for every maximal ideal $\mathfrak{m} \subset \mathcal{O}$ and every $n \geqslant 1$, the reduction $\iota \otimes \mathcal{O}/\mathfrak{m}^n$ is an isomorphism, then ι is an isomorphism.*

Proof. By [StaPro, Tag 01QV], the map ι factors as an open immersion followed by a closed immersion: $Y \to \bar{Y} \to Z$. As ι is bijective, we have $Y = \bar{Y}$ and ι is a bijective closed immersion. Being an isomorphism is local on the target, and we may assume that $Z = \mathrm{Spec}(A)$ and hence $Y = \mathrm{Spec}(B)$ are affine. The map of $\mathcal{O}$-algebras $\iota^{\#}\colon A \to B$ is surjective (because closed immersion), and each element in $I := \ker(\iota^{\#})$ is nilpotent (because $\iota^{\#}$ is bijective on spectra). It is enough to show that for the localization $I_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \subset \mathcal{O}$. Without loss of generality, we may assume that $\mathcal{O}$ is local with maximal ideal $\mathfrak{m}$. If $\mathfrak{m}A = A$, that is, the fiber of Z over $\mathfrak{m}$ is empty, there is nothing to prove, and we may assume that $\mathfrak{m}A \subset A$ is a proper ideal. As $A/\mathfrak{m}^n A \to B/\mathfrak{m}^n B$ is an isomorphism for all $n \geqslant 1$, we have $I \subset \bigcap_{n\geqslant 1} \mathfrak{m}^n A$. But since $\mathfrak{m}^n A = (\mathfrak{m}A)^n$ and $\mathfrak{m}A \subset A$ is a proper ideal in a Noetherian ring, we have $\bigcap_{n\geqslant 1} \mathfrak{m}^n A = 0$ by Krull's intersection theorem. The lemma follows. □

4. Local models for Weil-restricted groups

In this section we collect a few properties of the Weil-restricted affine Grassmannians as constructed in [Lev16]. We provide proofs for several statements which appear to be well known but for which we could not find proofs in the literature.

4.1 Notation

Let $F/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_F$, and residue field k with q elements. Let K/F be a finite extension with ring of integers $\mathcal{O}_K$ and residue field k_0/k. Let K_0/F denote the maximal unramified subextension of K/F, with the same residue field k_0/k. Fix a uniformizer ϖ of K, and denote by $Q \in K_0[u]$ the minimal polynomial, that is, Q is the unique irreducible normalized polynomial with $Q(\varpi) = 0$. Note that $Q \in \mathcal{O}_{K_0}[u]$, and that $Q \equiv u^{[K:K_0]} \bmod \varpi$.

Let $\breve{F}$ (respectively, $\breve{K}$; respectively, $\breve{K}_0$) denote the completion of the maximal unramified extension of F (respectively, K; respectively, K_0) inside a fixed algebraic closure $\bar{F}$, and let $\sigma \in \mathrm{Aut}(\breve{F}/F)$ denote the Frobenius generator. We note that $\breve{F} = \breve{K}_0$.

In § 4.4 below, we specialize the general setup of § 3 to the case where $\mathcal{O} = \mathcal{O}_F$, $X = \mathbb{A}^1_{\mathcal{O}_{K_0}}$ is viewed as a smooth curve over $\mathcal{O}$, and D is defined by $\{Q = 0\}$. We first summarize some properties of parahoric groups for Weil-restricted groups (see § 4.2), and the group schemes $\underline{\mathcal{G}}_0$ over $X = \mathbb{A}^1_{\mathcal{O}_{K_0}}$ constructed in [PZ13, Lev16] (see § 4.3).

4.2 Parahoric group schemes for Weil-restricted groups

Let G_0 be a reductive K-group. Fix a maximal K-split torus A_0, and a maximal $\breve{K}$-split torus S_0 containing A_0 and defined over K. Let $M_0 = Z_{G_0}(A_0)$ denote the centralizer of A_0 which is a minimal K-Levi subgroup of G_0, and let $T_0 = Z_{G_0}(S_0)$ be the centralizer of S_0. Then T_0 is a maximal torus because $G_{0,\breve{K}}$ is quasi-split by Steinberg's theorem.

We are interested in parahoric subgroups of the Weil restriction of scalars $G := \mathrm{Res}_{K/F}(G_0)$. We will first need to classify the maximal F-split tori in G.

Lemma 4.1. *Suppose T_0 is any K-torus, so that $T = \mathrm{Res}_{K/F}(T_0)$ is an F-torus. Then there is a canonical isomorphism of groups*

$$X_*(T)_{\Gamma_F} = X_*(T_0)_{\Gamma_K}. \tag{4.1}$$

In particular, the F-split rank of T is the K-split rank of T_0.

Proof. Recall that T represents the functor on F-tori which sends the F-torus T' to

$$\mathrm{Hom}_{K\text{-tori}}(T' \otimes_F K, T_0) = \mathrm{Hom}_{\Gamma_K\text{-Mod}}(X_*(T'), X_*(T_0)) = \mathrm{Hom}_{\Gamma_F\text{-Mod}}(X_*(T'), \mathrm{Ind}_{\Gamma_K}^{\Gamma_F}(X_*(T_0))).$$

We deduce that $X_*(\mathrm{Res}_{K/F}(T_0)) = \mathrm{Ind}_{\Gamma_K}^{\Gamma_F}(X_*(T_0)) \cong X_*(T_0) \otimes_{\mathbb{Z}[\Gamma_K]} \mathbb{Z}[\Gamma_F]$ (since $[\Gamma_F : \Gamma_K] < \infty$). Then the H_0-version of Shapiro's lemma gives $(X_*(T_0) \otimes_{\mathbb{Z}[\Gamma_K]} \mathbb{Z}[\Gamma_F])_{\Gamma_F} = X_*(T_0)_{\Gamma_K}$, which implies the lemma. □

Under the bijection

$$\mathrm{Hom}_F(T', \mathrm{Res}_{K/F}(G_0)) = \mathrm{Hom}_K(T'_K, G_0), \tag{4.2}$$

$T' \to \mathrm{Res}_{K/F}(G_0)$ is injective if and only if the corresponding morphism $T'_K \to G_0$ is injective. Since any K-split torus is of the form T'_K for a unique F-split torus T', this shows that the rank of a maximal F-split torus in $\mathrm{Res}_{K/F}(G_0)$ is the same as the rank of a maximal K-split torus in G_0. For the maximal K-split torus $A_0 \subset G_0$, we write $A_0 = A_K$ for a unique F-split torus A. Using the canonical embedding $A \hookrightarrow \mathrm{Res}_{K/F}(A_K) = \mathrm{Res}_{K/F}(A_0)$, we see that A is the F-split component of $\mathrm{Res}_{K/F}(A_0)$ and also a maximal F-split torus in G.

From now on, we will abuse notation and denote by A the image of $A \hookrightarrow \mathrm{Res}_{K/F}(A_0) \hookrightarrow \mathrm{Res}_{K/F}(G_0) = G$ (even though A is not a Weil restriction of a torus). The discussion following (4.2) shows that $A_0 \mapsto A$ gives a bijection between maximal K-split tori in G_0 and maximal F-split tori in G.

Let us note that since S_0 is $\breve{K}$-split (and using $\breve{K}_0 \otimes_{K_0} K = \breve{K}$) there exists a unique subtorus $S'_0 \hookrightarrow \mathrm{Res}_{K/K_0}(S_0)$ which is a maximal $\breve{K}_0$-split torus in $\mathrm{Res}_{K/K_0}(G_0)$ defined over K_0 and of the same rank as S_0. We let S denote the image of $\mathrm{Res}_{K_0/F}(S'_0) \hookrightarrow \mathrm{Res}_{K_0/F}(\mathrm{Res}_{K/K_0}(S_0)) = \mathrm{Res}_{K/F} S_0$ which is a maximal $\breve{F}$-split torus in G defined over F.

LEMMA 4.2. *Letting $M = \mathrm{Res}_{K/F}(M_0)$ and $T = \mathrm{Res}_{K/F}(T_0)$, we have $M = Z_G(A)$ and $T = Z_G(S)$ as subgroups of $G = \mathrm{Res}_{K/F}(G_0)$.*

Proof. Both containments '$\subseteq$' are obvious. For '$\supseteq$' we note that the torus $A_{\bar{F}}$ is the diagonal torus inside $\prod_{K \hookrightarrow \bar{F}} A_0 \otimes_{K,\psi} \bar{F}$. Considering its centralizer inside $\prod_{K \hookrightarrow \bar{F}} G_0 \otimes_{K,\psi} \bar{F}$ proves $M = Z_G(A)$. Since $Z_G(S)$ is necessarily a maximal torus, the inclusion $T \subseteq Z_G(S)$ is also an equality. □

The correspondence $A \leftrightarrow A_0$ induces a correspondence between the apartments in the (extended) Bruhat–Tits buildings $\mathscr{B}(G, F)$ and $\mathscr{B}(G_0, K)$. We will show that there is a canonical isomorphism

$$\mathscr{B}(G, F) \simeq \mathscr{B}(G_0, K), \tag{4.3}$$

equivariant for the action of $G(F) = G_0(K)$, and compatible with an identification of apartments $\mathscr{A}(G, A, F) = \mathscr{A}(G_0, A_0, K)$.

The *Iwahori–Weyl group* $W = W(G, A, F)$ is the group

$$W \overset{\text{def}}{=} \mathrm{Norm}_G(A)(F)/M(F)_1, \tag{4.4}$$

where $M(F)_1$ is the unique parahoric subgroup of the minimal Levi M; cf. [HR08, Ric16a]. (By Lemma 4.2, M is a minimal F-Levi subgroup of G.) We define $\breve{W} = W(G, S, \breve{F})$ analogously. In the following we will use the identification $\breve{F} \otimes_F K = \prod_{[K_0:F]} \breve{K}$ which is σ-equivariant for the action $\sigma(a_j)_j = (\sigma a_{j-1})_j$ on the product.

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Lemma 4.3. *There is a canonical identification of Iwahori–Weyl groups*

$$W(G,A,F) = W(G_0,A_0,K) \quad \text{and} \quad W(G,S,\breve{F}) = W(G_0,S_0,\breve{F}\otimes_F K) = \prod_{[K_0:F]} W(G_0,S_0,\breve{K}).$$

Proof. As in Lemma 4.2, one shows $\mathrm{Norm}_G(A) = \mathrm{Res}_{K/F}(\mathrm{Norm}_{G_0}(A_0))$, and hence $\mathrm{Norm}_G(A)(F) = \mathrm{Norm}_{G_0}(A_0)(K)$. By Lemma 4.4 below, $M(F)_1 = M_0(K)_1$. The first equality follows and the second is similar. □

Lemma 4.4. *Let $G(F)_1 \subset G(F)$ denote the Kottwitz kernel, that is, $G(F)_1 = G(F) \cap G(\breve{F})_1$ with*

$$G(\breve{F})_1 = \ker(\kappa_G\colon G(\breve{F}) \to X^*(Z_{G^\vee}^{I_F})),$$

where κ_G is the Kottwitz homomorphism of [Kot97, § 7]. Then $G(\breve{F})_1 = \prod_{[K_0:F]} G_0(\breve{K})_1$ and $G(F)_1 = G_0(K)_1$.

Proof. The result is clear when G_0 is a torus: $G(\breve{F})_1$ and $\prod_{[K_0:F]} G_0(\breve{K})_1$ coincide with the unique maximal bounded subgroup of

$$G(\breve{F}) = G_0(\breve{F}\otimes_F K) = \prod_{[K_0:F]} G_0(\breve{K}).$$

Thus, by a variation of Lemma 4.1, $\kappa_G\colon G(\breve{F}) \to X_*(G)_{I_F}$ is the direct product over $[K_0 : K]$-many copies of $\kappa_{G_0}\colon G_0(\breve{K}) \to X_*(G_0)_{I_K}$. Clearly the result holds for $G_0 = G_{0,\mathrm{sc}}$ and hence for $G_{0,\mathrm{der}} = G_{0,\mathrm{sc}}$ by reduction to the torus case. Finally, the general case follows by the method of z-extensions as in the construction of κ_{G_0} [Kot97, § 7.4]. □

Lemma 4.5. *There is a canonical isomorphism of apartments $\mathscr{A}(G,S,\breve{F}) = \mathscr{A}(G_0,S_0,\breve{F}\otimes_F K)$ compatible with the action of the Iwahori–Weyl groups $W(G,S,\breve{F}) = W(G_0,S_0,\breve{F}\otimes_F K)$ and the action of the Frobenius σ.*

Proof. Let $\breve{\Sigma}_G$ (respectively, $\breve{\Sigma}_{G_0}$) denote the Bruhat–Tits échelonnage root system attached to (G,S) (respectively, (G_0,S_0)). Taking $T_0 = T_{0,\mathrm{sc}}$ in Lemma 4.1 and using [HR08, Lemma 15], we obtain an equality of coroot lattices

$$Q^\vee(\breve{\Sigma}_G) = X_*(T_{\mathrm{sc}})_{I_F} = \prod_{[K_0:F]} X_*(T_{0,\mathrm{sc}})_{I_K} = \prod_{[K_0:F]} Q^\vee(\breve{\Sigma}_{G_0}).$$

By considering minimal positive generators of these lattices, we deduce that $\breve{\Sigma}_G = \prod_{[K_0:F]} \breve{\Sigma}_{G_0}$. As all identifications are canonical this isomorphism is compatible with the action of σ on both sides. This gives the identification of affine root hyperplanes needed to prove the isomorphism of apartments

$$\mathscr{A}(G,S,\breve{F}) = \mathscr{A}(G_0,S_0,\breve{F}\otimes_F K).$$

The isomorphism is equivariant for $W(G,S,\breve{F}) = W(G_0,S_0,\breve{F}\otimes_F K)$ and σ. □

Proposition 4.6. *There is a canonical isomorphism $\mathscr{B}(G,F) \simeq \mathscr{B}(G_0,K)$, equivariant for the action of $G(F) = G_0(K)$, and compatible with an identification of apartments $\mathscr{A}(G,A,F) = \mathscr{A}(G_0,A_0,K)$.*

Proof. By construction $\mathscr{B}(G_0, \breve{K}) = (G_0(\breve{K}) \times \mathscr{A}(G_0, S_0, \breve{K}))/\sim$, where $(g,x) \sim (g',x')$ if there exists $n \in \mathrm{Norm}_{G_0}(S_0)(\breve{K})$ such that $n \cdot x = x'$ and $g^{-1}g'n \in U_x$. Here U_x is the subgroup of $G_0(\breve{K})$ generated by the affine root groups $U_{\alpha+r}$ associated to $\alpha + r$ with $\alpha(x) + r \geqslant 0$, for $(\alpha, r) \in \breve{\Sigma}_{G_0} \times \mathbb{Z}$. Because $\breve{\Sigma}_G = \prod_{[K_0:F]} \breve{\Sigma}_{G_0}$, the equivalence relation for G is the $[K_0 : F]$-fold product of the equivalence relation for G_0. Using Lemma 4.5, this proves

$$\mathscr{B}(G, \breve{F}) = \prod_{[K_0:F]} \mathscr{B}(G_0, \breve{K}) = \mathscr{B}(G, \breve{F} \otimes_F K),$$

equivariantly for σ, and the proposition follows by étale descent; cf. [BT84, § 5.1]. □

Let $\mathbf{f}$ be a facet of $\mathscr{A}(G, A, F)$, and denote by $\mathbf{f}_0$ the corresponding facet in $\mathscr{A}(G_0, A_0, K)$. Let $\mathcal{G}_{\mathbf{f}}$ (respectively, $\mathcal{G}_{\mathbf{f}_0}$) be the associated parahoric group scheme over $\mathcal{O}_F$ (respectively, $\mathcal{O}_K$).

Proposition 4.7. *There is a canonical isomorphism of $\mathcal{O}_F$-group schemes $\mathcal{G}_{\mathbf{f}} \simeq \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G}_{\mathbf{f}_0})$ inducing the identity on generic fibers.*

Proof. By the defining property of parahoric group schemes, it suffices to check that the group $\mathcal{H} := \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G}_{\mathbf{f}_0})$ is a smooth affine $\mathcal{O}_F$-group scheme of finite type with (geometrically) connected special fiber, with the property that $\mathcal{H}(\mathcal{O}_{\breve{F}})$ is the intersection of the Kottwitz kernel $G(\breve{F})_1$ with the pointwise fixer in $G(\breve{F})$ of $\mathbf{f}$ which we view as a subset of the building over $\breve{F}$; cf. [HR08]. The $\mathcal{O}_F$-group $\mathcal{H}$ is smooth affine and of finite type by general properties of Weil restriction of scalars; cf. [BLR90, § 7.6, Theorem 4, Proposition 5]. If $R = \mathcal{O}_K/\varpi^{[K:K_0]}$, then the special fiber is given by

$$\mathcal{H} \otimes_{\mathcal{O}_F} k = \mathrm{Res}_{R/k}(\mathcal{G}_{\mathbf{f}_0} \otimes_{\mathcal{O}_K} R),$$

which is a successive extension of smooth (geometrically) connected groups, and hence (geometrically) connected. As $\prod_{[K_0:F]} \breve{K} = K \otimes_F \breve{F}$ we have $\mathcal{H}(\mathcal{O}_{\breve{F}}) = \prod_{[K_0:F]} \mathcal{G}_{\mathbf{f}_0}(\mathcal{O}_{\breve{K}})$ which is the intersection of $\prod_{[K_0:F]} G_0(\breve{K})_1 = G(\breve{F})_1$ (Lemma 4.4) with the pointwise fixer in $\prod_{[K_0:F]} G_0(\breve{K})$ of $\mathbf{f}$, by Lemma 4.5. The proposition follows. □

Corollary 4.8. *Every parahoric $\mathcal{O}_F$-group scheme of G is of the form $\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G}_{\mathbf{f}_0})$ for a unique facet $\mathbf{f}_0 \subset \mathscr{B}(G, K)$.* □

The subgroup $W_{\mathbf{f}} = W_{\mathbf{f}}(G, A, F)$ of W associated with $\mathbf{f}$ is the group

$$W_{\mathbf{f}} \overset{\mathrm{def}}{=} (\mathrm{Norm}_G(A)(F) \cap \mathcal{G}_{\mathbf{f}}(\mathcal{O}_F))/M(F)_1.$$

The isomorphism $W(G, A, F) = W(G_0, A_0, K)$ induces $W_{\mathbf{f}}(G, A, F) = W_{\mathbf{f}_0}(G_0, A_0, K)$. Let us point out a consequence of Proposition 4.6 which is used later.

Corollary 4.9. *There is a canonical identification $\mathcal{Z}(G(F), \mathcal{G}_{\mathbf{f}}(\mathcal{O}_F)) = \mathcal{Z}(G_0(K), \mathcal{G}_{\mathbf{f}_0}(\mathcal{O}_K))$ of centers of parahoric Hecke algebras compatible with the Bernstein isomorphism of [Hai14, Theorem 11.10.1], where the Haar measures are normalized to give $\mathcal{G}_{\mathbf{f}}(\mathcal{O}_F) = \mathcal{G}_{\mathbf{f}_0}(\mathcal{O}_K)$ volume 1.*

Proof. In view of $\mathcal{G}_{\mathbf{f}}(\mathcal{O}_F) = \mathcal{G}_{\mathbf{f}_0}(\mathcal{O}_K)$, the equality of the centers is clear, and all that remains is to show the compatibility with the Bernstein isomorphism. This follows from the equality

$$\Lambda_M := M(F)/M(F)_1 = M_0(K)/M_0(K)_1 =: \Lambda_{M_0},$$

combined with the definition of Bernstein isomorphisms given by the integration formula (e.g. [Hai14, 11.11]) and the isomorphism of finite relative Weyl groups $W_0(G, A, F) = W_0(G_0, A_0, K)$ consistent with Lemma 4.3. □

4.3 Group schemes over $\mathbb{A}^1_{\mathcal{O}_{K_0}}$

Let G_0 be a reductive K-group which splits over a tamely ramified extension, and let $G := \mathrm{Res}_{K/F}(G_0)$. Fix a chain of subgroups $A_0 \subset S_0 \subset T_0 \subset M_0$ in G_0 as in § 4.2 with corresponding chain of subgroups $A \subset S \subset T \subset M$ in G. Further, fix a parabolic K-subgroup P_0 containing M_0 in G_0, and let $P := \mathrm{Res}_{K/F}(P_0)$ in G.

In [PZ13, § 3], a reductive $\mathcal{O}_K[u^{\pm}]$-group scheme $\underline{G}_0$ admitting a maximal torus, and with connected fibers, is constructed. As observed in [Lev16, § 3.1; Proposition 3.3], the group scheme $\underline{G}_0$ is defined over $\mathcal{O}_{K_0}[u^{\pm}]$ in the following sense.

Proposition 4.10. (i) *There exists a reductive $\mathcal{O}_{K_0}[u^{\pm}]$-group $\underline{G}_0$ together with a tuple of smooth closed $\mathcal{O}_{K_0}[u^{\pm}]$-subgroups $(\underline{A}_0, \underline{S}_0, \underline{T}_0, \underline{M}_0, \underline{P}_0)$ and an isomorphism of K-groups*

$$(\underline{G}_0, \underline{A}_0, \underline{S}_0, \underline{T}_0, \underline{M}_0, \underline{P}_0) \otimes_{\mathcal{O}_{K_0}[u^{\pm}], u\mapsto \varpi} K \;\simeq\; (G_0, A_0, S_0, T_0, M_0, P_0),$$

where $\underline{A}_0$ is a maximal $\mathcal{O}_{K_0}[u^{\pm}]$-split torus, $\underline{S}_0$ a maximal $\mathcal{O}_{\breve{K}_0}[u^{\pm}]$-split torus defined over $\mathcal{O}_{K_0}[u^{\pm}]$, $\underline{T}_0$ its centralizer, $\underline{M}_0$ the centralizer of $\underline{A}_0$ (a minimal Levi), and $\underline{P}_0$ a parabolic $\mathcal{O}_{K_0}[u^{\pm}]$-subgroup with Levi $\underline{M}_0$.

(ii) *The base change $\underline{G}_{\mathcal{O}_{\breve{K}_0}[u^{\pm}]}$ is quasi-split. In particular, $\underline{T}_0$ is a maximal torus.*

Proof. Let us recall some elements of the construction as needed later. Let $\tilde{K}/K$ be a tamely ramified Galois extension which splits G_0. After possibly enlarging $\tilde{K}$, we may assume:

(1) the group G_0 is quasi-split over the maximal unramified subextension $\tilde{K}_0$ of $\tilde{K}/F$;

(2) there exist a uniformizer $\tilde{\varpi} \in \tilde{K}$ and an integer $\tilde{e} \geqslant 1$ such that $\varpi = \tilde{\varpi}^{\tilde{e}}$, and therefore $\tilde{K} \xleftarrow{\sim} \tilde{K}_0[v]/Q(v^{\tilde{e}})$ via $\tilde{\varpi} \leftarrow\!\shortmid v$;

(3) $\tilde{K}_0$ contains a primitive $\tilde{e}$th root of unity (cf. [PZ13, § 3.1]).

There is a cocartesian diagram[4] of $\mathcal{O}_{K_0}$-algebras

$$\begin{array}{ccc} \mathcal{O}_{\tilde{K}_0}[v] & \xrightarrow{v\mapsto \tilde{\varpi}} & \tilde{K} \\ {\scriptstyle u\mapsto v^{\tilde{e}}}\uparrow & & \uparrow \\ \mathcal{O}_{K_0}[u] & \xrightarrow[u\mapsto \varpi]{} & K \end{array} \tag{4.5}$$

One can prove that $\mathcal{O}_{\tilde{K}_0}[v]/\mathcal{O}_{K_0}[u]$ is a ramified Galois cover with group isomorphic to $\tilde{\Gamma} := \mathrm{Gal}(\tilde{K}/K)$; for this we use that $\tilde{K}_0$ contains a primitive $\tilde{e}$th root of unity. As in [PZ13, § 3], the $\mathcal{O}_{K_0}[u^{\pm}]$-group scheme $\underline{G}_0$ is constructed in [Lev16, § 3.1] by descending a suitable choice of Chevalley model for $G_{0,\tilde{K}}$ along the étale ring extension $\mathcal{O}_{\tilde{K}_0}[v^{\pm}]/\mathcal{O}_{K_0}[u^{\pm}]$; see [PZ13, § 3] and [Lev16, § 3.1] for details. See also Example 4.14. □

Let us denote

$$(G_0^{\flat}, A_0^{\flat}, S_0^{\flat}, T_0^{\flat}, M_0^{\flat}, P_0^{\flat}) \overset{\text{def}}{=} (\underline{G}_0, \underline{A}_0, \underline{S}_0, \underline{T}_0, \underline{M}_0, \underline{P}_0) \otimes_{\mathcal{O}_{K_0}[u^{\pm}]} k_0(\!(u)\!). \tag{4.6}$$

Then $G_0^{\flat}$ is a reductive group over $K_0^{\flat} := k_0(\!(u)\!)$, and $(A_0^{\flat}, S_0^{\flat}, T_0^{\flat}, M_0^{\flat}, P_0^{\flat})$ are analogous to the corresponding groups above; cf. the discussion in [PZ13, 4.1.2–4.1.3] and [Lev16, 3.3]. Further, we obtain a canonical identification of the apartments

$$\mathscr{A}(G, A, F) = \mathscr{A}(G_0, A_0, K) = \mathscr{A}(\underline{G}_0, \underline{A}_0, \kappa(\!(u)\!)), \tag{4.7}$$

[4] This differs from [Lev16, § 3.1] in that Levin uses instead of $\tilde{K}_0$ the maximal unramified subextension of $\tilde{K}/K$; this seems to be a mistake (e.g. the diagram corresponding to (4.5) is not cocartesian).

for both $\kappa = K_0, k_0$; cf. [PZ13, 4.1.3] and [Lev16, Proposition 3.3.1 ff.]. In particular, we have $\mathscr{A}(G_0, A_0, K_0) = \mathscr{A}(G_0^\flat, A_0^\flat, K_0^\flat)$ for $\kappa = k_0$. Thus, we may think of $G_0^\flat$ as an equal characteristic analogue of G_0 of the same Dynkin type.

We now introduce the equal characteristic analogue $G^\flat$ of G by restriction of scalars along the unramified extension K_0/F: we define the sextuple of $k(\!(u)\!)$-groups

$$(G^\flat, A^\flat, S^\flat, T^\flat, M^\flat, P^\flat), \tag{4.8}$$

where $G^\flat = \mathrm{Res}_{K_0^\flat/F^\flat}(G_0^\flat)$ is a reductive group over $F^\flat := k(\!(u)\!)$, and likewise $(S^\flat, T^\flat, M^\flat, P^\flat)$ are obtained from $(S_0^\flat, T_0^\flat, M_0^\flat, P_0^\flat)$ by restriction of scalars along the unramified extension $K_0^\flat/F^\flat$. Here $A^\flat$ is the maximal $F^\flat$-split subtorus inside $\mathrm{Res}_{K_0^\flat/F^\flat}(A_0^\flat)$.

Combining (4.7) with Proposition 4.6, we obtain a canonical identification of the apartments

$$\mathscr{A}(G, A, F) = \mathscr{A}(G^\flat, A^\flat, F^\flat). \tag{4.9}$$

We shall use the following two results in § 6 below.

Lemma 4.11. *There is an identification of Iwahori–Weyl groups $W(G, A, F) = W(G^\flat, A^\flat, F^\flat)$ which is compatible with the action on the apartments under the identification (4.9).*

Proof. Over $\breve{F}$ we obtain a σ-equivariant isomorphism according to [PZ13, 4.1.2] and [Lev16, 3.3.0.1], compatible with the action on the apartments. The general case follows by taking σ-fixed points from [Ric16a, § 1.2] (cf. also [PZ13, 4.1.3] and [Lev16, Proposition 3.3.1 ii)]). □

Now let $\mathcal{G} = \mathcal{G}_{\mathbf{f}}$ be a parahoric $\mathcal{O}_F$-group scheme of G whose facet $\mathbf{f}$ is contained in $\mathscr{A}(G, A, F)$. Then under (4.9) we obtain a unique facet $\mathbf{f}^\flat \subset \mathscr{A}(G^\flat, A^\flat, F^\flat)$, and hence a parahoric $\mathcal{O}_{F^\flat}$-group scheme $\mathcal{G}^\flat := \mathcal{G}_{\mathbf{f}^\flat}$ of $G^\flat$.

Lemma 4.12. *There is a canonical identification $\mathcal{Z}(G(F), \mathcal{G}(\mathcal{O}_F)) = \mathcal{Z}(G^\flat(F^\flat), \mathcal{G}^\flat(\mathcal{O}_{F^\flat}))$ of centers of parahoric Hecke algebras, where the Haar measures are normalized to give $\mathcal{G}(\mathcal{O}_F)$ (respectively, $\mathcal{G}^\flat(\mathcal{O}_{F^\flat})$) volume 1.*

Proof. Applying Lemma 4.11 for M, we obtain an identification of abelian groups

$$\Lambda_M := M(F)/M(F)_1 = M^\flat(F^\flat)/M^\flat(F^\flat)_1 =: \Lambda_{M^\flat}, \tag{4.10}$$

where $M(F)_1$ (respectively, $M^\flat(F^\flat)_1$) is the unique parahoric group scheme of $M(F)$ (respectively, $M^\flat(F^\flat)$). The result follows via the Bernstein isomorphisms [Hai14, Theorem 11.10.1]

$$\mathcal{Z}(G^\flat(F^\flat), \mathcal{G}^\flat(\mathcal{O}_{F^\flat})) \simeq \bar{\mathbb{Q}}_\ell[\Lambda_{M^\flat}]^{W_0(G^\flat, A^\flat, F^\flat)} = \bar{\mathbb{Q}}_\ell[\Lambda_M]^{W_0(G,A,F)} \simeq \mathcal{Z}(G(F), \mathcal{G}(\mathcal{O}_F)),$$

noting that the finite relative Weyl groups of (G, A, F) and $(G^\flat, A^\flat, F^\flat)$ are isomorphic compatible with the action on $\Lambda_M = \Lambda_{M^\flat}$. □

Theorem 4.13. *Fix $(\underline{G}_0, \underline{A}_0, \underline{S}_0, \underline{T}_0)$ and $\mathcal{G}_{\mathbf{f}}$ with $\mathbf{f} \subset \mathscr{A}(G, A, F)$ as above. There exists a tuple of smooth affine $\mathcal{O}_{K_0}[u]$-group schemes $(\underline{\mathcal{G}}_0, \underline{\mathcal{A}}_0, \underline{\mathcal{S}}_0, \underline{\mathcal{T}}_0)$ with geometrically connected fibers satisfying the following properties.*

(i) *The restriction $(\underline{\mathcal{G}}_0, \underline{\mathcal{A}}_0, \underline{\mathcal{S}}_0, \underline{\mathcal{T}}_0)|_{\mathcal{O}_{K_0}[u^\pm]}$ is $(\underline{G}_0, \underline{A}_0, \underline{S}_0, \underline{T}_0)$ as $\mathcal{O}_{K_0}[u^\pm]$-groups.*

(ii) *The base change of $\underline{\mathcal{G}}_0$ under $\mathcal{O}_{K_0}[u] \to \mathcal{O}_K$, $u \mapsto \varpi$ is the parahoric group $\mathcal{G} = \mathcal{G}_{\mathbf{f}}$.*

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(iii) *The base change of* $\underline{\mathcal{G}}_0$ *under* $\mathcal{O}_{K_0}[u] \to \kappa[\![u]\!]$, $u \mapsto u$ *for both* $\kappa = K_0, k_0$ *is the parahoric group scheme for* $\underline{G}_{0,\kappa(\!(u)\!)}$ *attached to* $\mathbf{f}$ *under (4.7).*

(iv) *The group* $\underline{\mathcal{A}}_0$ *is a split* $\mathcal{O}_{K_0}[u]$*-torus,* $\underline{\mathcal{S}}$ *a* $\mathcal{O}_{K_0}[u]$*-torus which splits over* $\mathcal{O}_{\breve{F}}[u]$ *and* $\underline{\mathcal{T}}$ *is a smooth affine* $\mathcal{O}_{K_0}[u]$*-group scheme such that* $\underline{\mathcal{T}}_0 \otimes \kappa[\![u]\!]$ *is the neutral component of the lft (locally finite type) Néron model of* $\underline{T}_{0,\kappa(\!(u)\!)}$*, for* $\kappa = K_0, k_0$.

The group $\underline{\mathcal{G}}_0$ *is uniquely determined (up to unique isomorphism) by properties (i) and (iii) for* $\kappa = K_0$*, and so is the tuple* $(\underline{\mathcal{A}}_0, \underline{\mathcal{S}}_0, \underline{\mathcal{T}}_0)$ *using (iv).*

Proof. This is [Lev16, Theorem 3.3.3, Proposition 3.3.4]; cf. also [PZ13, Theorem 4.1, esp. 4.2.1] for the uniqueness assertion. □

Example 4.14. Suppose $G_0 = T_0$ is a tamely ramified torus over K. Let T_H be the split torus over $\mathcal{O}_{K_0}$ such that T_0 is given by a 1-cocycle

$$[\tau] \in H^1(\tilde{\Gamma}, \mathrm{Aut}(T_H \otimes_{\mathcal{O}_{K_0}} \tilde{K})).$$

Explicitly,

$$T_0 = (\mathrm{Res}_{\tilde{K}/K}(T_H \otimes_{\mathcal{O}_{K_0}} \tilde{K}))^{\tilde{\Gamma}}.$$

We let $T_H \otimes_{\mathcal{O}_{K_0}} \tilde{\mathcal{O}}_0[v]$ be the split torus over $\tilde{\mathcal{O}}_0[v] := \mathcal{O}_{\tilde{K}_0}[v]$ (cf. (4.5)), which is endowed with Galois actions $\tau(\gamma) \otimes \gamma$ for $\gamma \in \tilde{\Gamma}$ which we view as Galois descent data used to give a torus over $\mathcal{O}_{K_0}[u]$. Explicitly, we define $\underline{T}_0/\mathcal{O}_{K_0}[u^{\pm}]$ and $\underline{\mathcal{T}}_0/\mathcal{O}_{K_0}[u]$ by

$$\underline{T}_0 = (\mathrm{Res}_{\tilde{\mathcal{O}}_0[v^{\pm}]/\mathcal{O}_{K_0}[u^{\pm}]}(T_H \otimes_{\mathcal{O}_{K_0}} \tilde{\mathcal{O}}_0[v^{\pm}]))^{\tilde{\Gamma}},$$

and $\underline{\mathcal{T}}_0$ as the (fiberwise) neutral component of

$$(\mathrm{Res}_{\tilde{\mathcal{O}}_0[v]/\mathcal{O}_{K_0}[u]}(T_H \otimes_{\mathcal{O}_{K_0}} \tilde{\mathcal{O}}_0[v]))^{\tilde{\Gamma}}.$$

Write $\mathrm{Gal}(\tilde{K}/K) = \langle\gamma\rangle \rtimes \langle\sigma\rangle$ where γ generates the inertia subgroup and σ is a lift of a generator of $\mathrm{Gal}(\tilde{K}^{\mathrm{un}}/K)$ for $\tilde{K}^{\mathrm{un}}/K$ the maximal unramified subextension of $\tilde{K}/K$. Then $\underline{T}_0$ is realized as an $\mathrm{Aut}(\breve{K}_0/K_0)$-descent of

$$(\mathrm{Res}_{\mathcal{O}_{\breve{K}_0}[v^{\pm}]/\mathcal{O}_{\breve{K}_0}[u^{\pm}]}(T_H \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{\breve{K}_0}[v^{\pm}]))^{\gamma}.$$

This shows that the formation of $\underline{T}_0$ commutes with base change $\mathbb{A}^1_{E_0} \to \mathbb{A}^1_{K_0}$, where E_0/K_0 is any unramified extension. Similar remarks apply to $\underline{\mathcal{T}}_0$.

4.4 Affine Grassmannians and local models

We continue with the notation as in § 4.1. Recall that we fix a uniformizer $\varpi \in K$ with Eisenstein polynomial $Q \in \mathcal{O}_{K_0}[u]$. Let (G_0, A_0, S_0, T_0) be tamely ramified over K, and fix a spreading $(\underline{G}_0, \underline{A}_0, \underline{S}_0, \underline{T}_0)$ defined over $\mathcal{O}_{K_0}[u^{\pm}]$ as in Proposition 4.10. Let (G, A, S, T) be constructed from (G_0, A_0, S_0, T_0) by Weil restriction of scalars along K/F as in § 4.2. Choose a facet $\mathbf{f} \in \mathscr{A}(G, A, F)$, and let $\mathcal{G} := \mathcal{G}_{\mathbf{f}}$ be the corresponding parahoric $\mathcal{O}_F$-group scheme for G. Associated with these data, we have the tuple $(\underline{\mathcal{G}}_0, \underline{\mathcal{A}}_0, \underline{\mathcal{S}}_0, \underline{\mathcal{T}}_0)$ of smooth affine group schemes over $X := \mathrm{Spec}(\mathcal{O}_{K_0}[u])$ constructed in Theorem 4.13. Since $\mathcal{O}_{K_0}/\mathcal{O}_F$ is finite étale, we can view X as a smooth curve over $\mathcal{O}_F$. Let $D \subset X$ be the closed subscheme defined by $\{Q = 0\}$ viewed as a relative effective Cartier divisor over $\mathcal{O}_F$. We are interested in local models for the group $G = \mathrm{Res}_{K/F}(G_0)$ with level structure given by the parahoric $\mathcal{O}_F$-group $\mathcal{G} = \mathcal{G}_{\mathbf{f}} = \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G}_{\mathbf{f}_0})$; cf. Proposition 4.7.

4.4.1 *Affine Grassmannians for Weil-restricted groups.* The Beilinson–Drinfeld Grassmannian

$$\mathrm{Gr}_{\mathcal{G}} \overset{\mathrm{def}}{=} \mathrm{Gr}_{(X,\underline{\mathcal{G}}_0,D)} \tag{4.11}$$

from (3.1) specializes to [Lev16, Definition 4.1.1] for $K_0 = F$. By Lemma 3.7, we have

$$\mathrm{Gr}_{\mathcal{G}} = \mathrm{Gr}_{(X/\mathcal{O}_F,\underline{\mathcal{G}}_0,D)} = \mathrm{Res}_{\mathcal{O}_{K_0}/\mathcal{O}_F}(\mathrm{Gr}_{(X/\mathcal{O}_{K_0},\underline{\mathcal{G}}_0,D)}).$$

Hence, our definition of $\mathrm{Gr}_{\mathcal{G}}$ agrees with [Lev16, Proposition 4.1.8 ff.].

We think of (4.11) as being the Beilinson–Drinfeld Grassmannian associated with the parahoric $\mathcal{O}_F$-group scheme $\mathcal{G}$. Explicitly, $\mathrm{Gr}_{\mathcal{G}}$ is the functor on the category of $\mathcal{O}_F$-algebras R given by the isomorphism classes of tuples $(\mathcal{F}, \alpha)$ with

$$\begin{cases} \mathcal{F} \text{ a } \underline{\mathcal{G}}_0\text{-torsor on } \mathrm{Spec}((R \otimes_{\mathcal{O}_F} \mathcal{O}_{K_0})[u]), \\ \alpha\colon \mathcal{F}|_{\mathrm{Spec}((R\otimes_{\mathcal{O}_F}\mathcal{O}_{K_0})[u][1/Q])} \simeq \mathcal{F}^0|_{\mathrm{Spec}((R\otimes_{\mathcal{O}_F}\mathcal{O}_{K_0})[u][1/Q])} \text{ a trivialization,} \end{cases} \tag{4.12}$$

where $\mathcal{F}^0$ denotes the trivial torsor. If $Q = u - \varpi$ (i.e. $K = F$) then $\mathrm{Gr}_{\mathcal{G}}$ is the Beilinson–Drinfeld Grassmannian defined in [PZ13, 6.2.3; (6.11)].

For an $\mathcal{O}_F$-algebra R, we have the regular functions on the completion of X_R along D_R, namely the $\mathcal{O}_{K_0}[u]$-algebra $R[\![D]\!] = \lim_N (R \otimes_{\mathcal{O}_F} \mathcal{O}_{K_0})[u]/(Q^N)$, and likewise $R(\!(D)\!) = R[\![Q]\!][1/Q]$.With the notation of § 3.1.1, we have the loop group

$$L\mathcal{G}(R) \overset{\mathrm{def}}{=} L_D\underline{\mathcal{G}}_0(R) = \underline{\mathcal{G}}_0(R(\!(D)\!)),$$

and the positive loop group

$$L^+\mathcal{G}(R) \overset{\mathrm{def}}{=} L^+_D\underline{\mathcal{G}}_0(R) = \underline{\mathcal{G}}_0(R[\![D]\!]).$$

By Lemma 3.4, there is a natural isomorphism $L\mathcal{G}/L^+\mathcal{G} \simeq \mathrm{Gr}_{\mathcal{G}}$, and thus a transitive action morphism

$$L\mathcal{G} \times_{\mathcal{O}_F} \mathrm{Gr}_{\mathcal{G}} \longrightarrow \mathrm{Gr}_{\mathcal{G}}. \tag{4.13}$$

The following proposition is [Lev16, Propositions 4.1.6 and 4.1.8].

Proposition 4.15. (i) *The generic fiber of (4.13) is isomorphic to*

$$L_zG \times_F \mathrm{Gr}_G \longrightarrow \mathrm{Gr}_G, \tag{4.14}$$

where $L_zG(R) = G(R(\!(z)\!)) = G((K\otimes_F R)(\!(z)\!))$ is the loop group for $G = \mathrm{Res}_{K/F}(G)$ formed using the parameter $z := u - \varpi \in K[u]$, and Gr_G is, as in Example 3.1(i)*, the affine Grassmannian for the group $G \otimes_F F[\![z]\!]$, that is, the étale sheaf associated with the functor on F-algebras $R \mapsto G(R(\!(z)\!))/G(R[\![z]\!])$.*

(ii) *The special fiber of (4.13) is canonically isomorphic to*

$$L\mathcal{G}^\flat \times_{k_F} \mathcal{F}\ell_{\mathcal{G}^\flat} \longrightarrow \mathcal{F}\ell_{\mathcal{G}^\flat}, \tag{4.15}$$

where $L\mathcal{G}^\flat(R) = \mathcal{G}^\flat(R(\!(u)\!))$ is the twisted affine loop group for the parahoric $k_F[\![u]\!]$-group scheme $\mathcal{G}^\flat = \mathcal{G}_{\mathbf{f}^\flat}$ of $G^\flat$ as in (4.9), and $\mathcal{F}\ell_{\mathcal{G}^\flat}$ is the twisted affine flag variety for $\mathcal{G}^\flat/k_F[\![u]\!]$ defined in [PR08], that is, the étale-sheaf associated with the functor on k_F-algebras $R \mapsto \mathcal{G}^\flat(R(\!(u)\!))/\mathcal{G}^\flat(R[\![u]\!])$.

Proof. Let $L^+_{D/\mathcal{O}_{K_0}}\underline{\mathcal{G}}_0$ denote the positive loop group attached to D and the curve $\mathbb{A}^1_{\mathcal{O}_{K_0}}$, and let $L_{D/\mathcal{O}_{K_0}}\underline{\mathcal{G}}_0$ be the corresponding loop group. We then have

$$L\mathcal{G} = L_D\underline{\mathcal{G}}_0 = \mathrm{Res}_{\mathcal{O}_{K_0}/\mathcal{O}_F}(L_{D/\mathcal{O}_{K_0}}\underline{\mathcal{G}}_0),$$

and likewise for the positive loop group. We may use Theorem 4.13 to compute the generic and special fibers of the right-hand side. For example, if $\mathcal{G}_0^\flat := \underline{\mathcal{G}}_0 \otimes k_0[\![u]\!]$, then the special fiber of $L\mathcal{G}$ is

$$\mathrm{Res}_{k_0/k}(L\mathcal{G}_0^\flat) = L(\mathrm{Res}_{k_0[\![u]\!]/k[\![u]\!]}(\mathcal{G}_0^\flat)) = L\mathcal{G}^\flat, \tag{4.16}$$

and likewise for the positive loop group. This together with Lemma 3.7 reduces us to the case where $K_0 = F$. Then part (ii) is Corollary 3.5(i). For (i), note that the natural maps $\mathrm{Res}_{K/F}(L_zG_0) \to L_z\mathrm{Res}_{K/F}(G_0)$ and $\mathrm{Res}_{K/F}(\mathrm{Gr}_{G_0}) \to \mathrm{Gr}_{\mathrm{Res}_{K/F}(G_0)}$ are isomorphisms; cf. [PR08, (1.2)] and [Lev13, § 2.6]. Note that $Q(z+\varpi) \in zK[z]$. Hence, by induction on $n \geqslant 1$, the map $u \mapsto z+\varpi$ sets up an isomorphism $F[u]/(Q^n) \xrightarrow{\sim} K[z]/(z^n)$, and hence $F[\![u]\!] \xrightarrow{\sim} K[\![z]\!]$. Similarly, we remark that, for any F-algebra R, $u \mapsto z+\varpi$ gives an isomorphism $R[\![u]\!] \cong (R\otimes_F K)[\![z]\!]$. Let $\underline{\mathcal{G}}_{K[\![z]\!]} := \underline{\mathcal{G}}_0 \otimes_{\mathcal{O}_F[u]} K[\![z]\!]$, and denote by $\mathrm{Gr}_{\underline{\mathcal{G}}_{K[\![z]\!]}}$ the twisted affine Grassmannian for $\underline{\mathcal{G}}_{K[\![z]\!]}$; cf. Example 3.1(i). In view of Corollary 3.5, or the above remark, the generic fiber of (4.13) is canonically isomorphic to the action morphism

$$\mathrm{Res}_{K/F}(L\underline{\mathcal{G}}_{K[\![z]\!]}) \times_F \mathrm{Res}_{K/F}(\mathrm{Gr}_{\underline{\mathcal{G}}_{K[\![z]\!]}}) \longrightarrow \mathrm{Res}_{K/F}(\mathrm{Gr}_{\underline{\mathcal{G}}_{K[\![z]\!]}}).$$

Hence, as in [PZ13, § 6.2.6] and [Lev16, Proposition 4.1.6], it suffices to give an isomorphism of $K[\![z]\!]$-groups $\underline{\mathcal{G}}_{K[\![z]\!]} \simeq G_0 \otimes_K K[\![z]\!]$. But as u is invertible in $K[\![z]\!]$, we have $\underline{\mathcal{G}}_{K[\![z]\!]} = \underline{G}_0 \otimes_{\mathcal{O}_F[u^\pm]} K[\![z]\!]$. With the notation of (4.5), the group scheme $\underline{G}_0$ is constructed by descent from $\mathcal{O}_{\tilde{K}_0}[v^\pm]$ where it is a constant Chevalley group scheme. As in [PZ13, (6.9)], it is enough to give a commutative diagram of $\tilde{\Gamma}$-covers

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_{\tilde{K}_0}[v^\pm] \otimes_{\mathcal{O}_F[u^\pm]} K[\![z]\!]) & \xleftarrow{\ \simeq\ } & \mathrm{Spec}(\tilde{K}[\![z]\!]) \\ & \searrow^{\mathrm{pr}_2} \quad \swarrow^{\tau} & \\ & \mathrm{Spec}(K[\![z]\!]) & \end{array} \tag{4.17}$$

which matches the $\tilde{\Gamma}$-action on $\mathcal{O}_{\tilde{K}_0}[v^\pm]/\mathcal{O}_F[u^\pm]$ via (4.5) with the $\tilde{\Gamma}$-action on the coefficients in $\tilde{K}[\![z]\!]$ (see below for why this is enough). As in [PZ13, (6.9)], the isomorphism is given on rings by $v \mapsto \tilde{\varpi}\cdot(1+z)$ and $z \mapsto b\cdot z$ with

$$b := \frac{\varpi\cdot(1+z)^{\tilde{e}} - \varpi}{z} \in K[\![z]\!]^\times.$$

The map τ is the K-algebra morphism given by $z \mapsto b\cdot z$. (To see that the horizontal morphism is an isomorphism, observe that $K[\![z]\!] = K[\![bz]\!]$, and let $f(z) \in K[\![z]\!]$ be such that $f(bz) = (1+z)^{-1}$; then $v \otimes f(z) \mapsto \tilde{\varpi}$ and the morphism is surjective. One sees it is injective using an $\mathcal{O}_F[u]$-basis for $\mathcal{O}_{\tilde{K}_0}[v]$ of the form a_iv^j for $a_i \in \mathcal{O}_{\tilde{F}_0}$ to write any element in the source uniquely in the form $\sum_{i,j} a_iv^j \otimes f_{ij}$ for $f_{ij} \in K[\![z]\!]$. To see that diagram (4.17) suffices, note that the right oblique arrow is isomorphic via $\tilde{K}[\![z]\!] \xrightarrow{\sim} \tilde{K}[\![z]\!]$, $z \mapsto b\cdot z$ to the arrow $\mathrm{Spec}(\tilde{K}[\![z]\!]) \to \mathrm{Spec}(K[\![z]\!])$ induced by the inclusion $K[\![z]\!] \hookrightarrow \tilde{K}[\![z]\!]$.) □

Recall from [PZ13, Corollary 11.7] that there exists a closed immersion of X-groups $\underline{\mathcal{G}}_0 \hookrightarrow \mathrm{Gl}_{n,X}$ such that the quotient $\mathrm{Gl}_{n,X}/\underline{\mathcal{G}}_0$ is quasi-affine. Thus, the $\mathcal{O}_F$-space $\mathrm{Gr}_{\mathcal{G}} = \mathrm{Gr}_{(X,\underline{\mathcal{G}}_0,D)}$ is representable by a separated $\mathcal{O}_F$-ind-scheme of ind-finite type; cf. Corollary 3.11. We need the following stronger statement.

Theorem 4.16. *The Beilinson–Drinfeld Grassmannian* $\mathrm{Gr}_{\mathcal{G}} = \mathrm{colim}_i \mathrm{Gr}_{\mathcal{G},i}$ *is representable by an ind-projective* $\mathcal{O}_F$*-ind-scheme, and, for each* i*, the projective* $\mathcal{O}_F$*-scheme* $\mathrm{Gr}_{\mathcal{G},i}$ *can be chosen to be* $L^+\mathcal{G}$*-stable compatible with the transition maps.*

Proof. By Lemma 3.7 we reduce to the case where $K_0 = F$. Then the ind-projectivity is proven in [Lev16, Theorem 4.2.11, Proposition 5.1.5]. If G is unramified, the proof is considerably simpler; cf. [Lev16, Proposition 2.2.8]. The proof relies on the existence and properties of specialization morphisms

$$\mathrm{sp}\colon \mathrm{Gr}_{\mathcal{T}}(\bar{F}) \longrightarrow \mathrm{Gr}_{\mathcal{T}}(\bar{k}),$$

where $\mathrm{Gr}_{\mathcal{T}} \subset \mathrm{Gr}_{\mathcal{G}}$ is the part induced from the maximal torus; cf. Lemma 4.17 below. Levin constructs this map 'by hand' in [Lev16, Proposition 4.2.8]. We will follow a more conceptual approach which avoids constructing sp ahead of time and the calculations that entails. Our outline is as follows.

(a) Prove $\mathrm{Gr}_{\mathcal{T}} \to \mathrm{Spec}(\mathcal{O}_F)$ is ind-finite, using the method of [Ric16b, Lemma 2.20] (§ 4.4.2).
(b) Deduce existence of the specialization maps for $\mathcal{T}$ via the valuative criterion of properness, and prove the required compatibility with Kottwitz homomorphisms (§ 4.4.3).
(c) Use (b) to show that each local model has non-empty special fiber and deduce by [Ric16b, Lemma 2.22] that each local model is proper (§ 4.4.5).
(d) Conclude that $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spec}(\mathcal{O}_F)$ is ind-proper (§ 4.4.6).

In view of Lemma 3.8 and Corollary 3.11, the ind-properness of $\mathrm{Gr}_{\mathcal{G}}$ implies the theorem. Steps (a)–(d) are explicated in the next several subsections, and with them the proof is concluded. □

4.4.2 $\mathrm{Gr}_{\mathcal{T}}$ *is ind-finite.* Recall that we have already reduced to the case $K_0 = F$ so that K/F is totally ramified. Without loss of generality, we further assume that $F = \breve{F}$, $\mathcal{O}_F = \mathcal{O}_{\breve{F}}$. Here we use that the formation of the affine Grassmannian (4.11) and the group scheme $\underline{\mathcal{T}}_0$ from Theorem 4.13 is compatible with unramified base change; cf. also Example 4.14. We show that $\mathrm{Gr}_{\mathcal{T}} := \mathrm{Gr}_{(X,\underline{\mathcal{T}}_0,D)}$ is ind-proper over $\mathcal{O}_F$, where $X = \mathbb{A}^1_{\mathcal{O}_F}$ and $D = \{Q = 0\}$. It is then ind-finite, since this holds fiberwise by Proposition 4.15. We proceed in two steps as follows.

Step 1. First assume that $T = \mathrm{Res}_{K/F}(T_0)$, where T_0 is an *induced* K-torus which splits over a tamely ramified extension. Then T_0 is isomorphic to a finite product of K-tori of the form $T_1 := \mathrm{Res}_{K_1/K}(\mathbb{G}_m)$, where K_1/K is a tamely ramified finite field extension. Note that K_1/K is totally ramified by our assumption $F = \breve{F}$. Accordingly, the $\mathbb{A}^1_{\mathcal{O}_F}$-group scheme $\underline{\mathcal{T}}_0$ is isomorphic to a finite product of $\mathbb{A}^1_{\mathcal{O}_F}$-group schemes of the form

$$\underline{\mathcal{T}}_1 := \mathrm{Res}_{\mathcal{O}_F[v]/\mathcal{O}_F[u]}(\mathbb{G}_m),$$

where $v^{[K_1:K]} = u$. After fixing a uniformizer $\varpi_1 \in K_1$ with $(\varpi_1)^{[K_1:K]} = \varpi$ (possible because $F = \breve{F}$), this can be verified using Example 4.14 (use that, in this case, $T_H \otimes \tilde{\mathcal{O}}_0[v] \cong (\mathbb{G}_{m,\tilde{\mathcal{O}}_0[v]})^{[K_1:K]}$ with $\mathrm{Gal}(K_1/K)$ acting via the permutation of the factors). Likewise, the affine Grassmannian $\mathrm{Gr}_{\mathcal{T}}$ is a finite $\mathcal{O}_F$-product of the affine Grassmannians $\mathrm{Gr}_{(X,\underline{\mathcal{T}}_1,D)}$, where $X = \mathbb{A}^1_{\mathcal{O}_F}$ and

$D = \{Q(u) = 0\}$. Hence, we reduce to the case where $\underline{\mathcal{T}}_0 = \underline{\mathcal{T}}_1$, that is, $T = \mathrm{Res}_{K_1/K}(\mathbb{G}_m)$. By Corollary 3.6, there is an equality of ind-schemes

$$\mathrm{Gr}_{(X,\underline{\mathcal{T}}_0,D)} = \mathrm{Gr}_{(X',\mathbb{G}_m,D')},$$

where $X' = \mathbb{A}^1_{\mathcal{O}_F} = \mathrm{Spec}(\mathcal{O}_F[v])$ and $D' = \{Q(v^{[K_1:K]}) = 0\}$. We reduce to the case $X = X'$, $\underline{\mathcal{T}}_0 = \mathbb{G}_m$ and $D = D'$. Then $\mathrm{Gr}_{(X,\mathbb{G}_m,D)}$ is ind-projective (hence ind-proper) by Lemma 3.8.

Step 2. Now let $T = \mathrm{Res}_{K/F}(T_0)$, where T_0 is a K-torus which splits over a tamely ramified extension. As in [Kot97, § 7], we choose a surjection of K-tori $T_1 \to T_0$ where T_1 is induced, and where the kernel $T_2 := \ker(T_1 \to T)$ is a K-torus. Note that T_1 can be chosen to split over a tamely ramified extension (and so does T_2 as well). The proof of [KP18, Proposition 2.2.2] adapts to our setup, and the map $T_1 \to T_0$ extends to a map of X-groups $\underline{\mathcal{T}}_1 \to \underline{\mathcal{T}}_0$ with kernel $\underline{\mathcal{T}}_2$ an X-group scheme extending T_2. (Instead of using [KP18], one can also deduce this by making use of the prescription given in Example 4.14.) We claim that the resulting map of $\mathcal{O}_F$-ind-schemes

$$\mathrm{Gr}_{\mathcal{T}_1} = \mathrm{Gr}_{(X,\underline{\mathcal{T}}_1,D)} \longrightarrow \mathrm{Gr}_{(X,\underline{\mathcal{T}}_0,D)} = \mathrm{Gr}_{\mathcal{T}} \tag{4.18}$$

is surjective on the underlying topological spaces. Clearly, this can be tested on the fibers of (4.18) over $\mathcal{O}_F$ which are determined by Proposition 4.15. The geometric generic fiber of (4.18) is isomorphic (on the underlying topological spaces) to the map of discrete groups $X_*(\mathrm{Res}_{K/F}(T_1)) \to X_*(\mathrm{Res}_{K/F}(T_0))$ which is surjective because $T_1 \to T_0$ is surjective and its kernel T_2 is a torus (that is, connected). The geometric special fiber of (4.18) is under the Kottwitz map isomorphic to $X_*(T_1^\flat)_{I_{k(\!(u)\!)}} \to X_*(T_0^\flat)_{I_{k(\!(u)\!)}}$ which is induced by $T_1^\flat := \underline{\mathcal{T}}_1 \otimes k(\!(u)\!) \to \underline{\mathcal{T}}_0 \otimes k(\!(u)\!) =: T_0^\flat$. This map is isomorphic to $X_*(T_1)_{I_K} \to X_*(T_0)_{I_K}$ which follows by applying the Kottwitz map to the identification (4.10). As in [Kot97, § 7 (7.2.5)], the desired surjectivity now follows from T_2 being a K-torus. By Step 1, the $\mathcal{O}_F$-scheme $\mathrm{Gr}_{\mathcal{T}_1}$ is ind-proper and maps surjectively onto the separated ind-scheme $\mathrm{Gr}_{\mathcal{T}}$ which is therefore ind-proper as well. This concludes § 4.4.2.

4.4.3 *The specialization map.* Once $\mathrm{Gr}_{\mathcal{G}}$ is known to be ind-proper, by the valuative criterion for properness there exists a specialization map

$$\mathrm{sp} : \mathrm{Gr}_G(\bar{F}) = \mathrm{Gr}_{\mathcal{G}}(\bar{F}) \longrightarrow \mathrm{Gr}_{\mathcal{G}}(\bar{k}) = \mathcal{F}\ell_{\mathcal{G}^\flat}(\bar{k}). \tag{4.19}$$

In the case where $G_0 = T_0$ is a maximal torus, and hence $\underline{\mathcal{G}}_0 = \underline{\mathcal{T}}_0$ is as in Theorem 4.13(iv), we therefore know the existence of the specialization map. It is made explicit in [PZ13, Lemma 9.8] and [Lev16, Proposition 4.2.8]. Recall the following result for later use (which compared to [Lev16] is proved in a more conceptual way here).

Lemma 4.17. *Let Γ_F denote the Galois group of F, and likewise Γ_k and Γ_{k_0}. There is a commutative diagram of abelian groups*

$$\begin{array}{ccc} \mathrm{Gr}_{\mathrm{Res}_{K/F}(T_0)}(\bar{F}) & \xrightarrow{\simeq} & X_*(\mathrm{Res}_{K/F}(T_0)) \\ \downarrow{\scriptstyle \mathrm{sp}} & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathcal{F}\ell_{\mathcal{T}^\flat}(\bar{k}) & \xrightarrow{\simeq} & X_*(\mathrm{Res}_{K/F}(T_0))_{I_F}, \end{array} \tag{4.20}$$

which is Galois equivariant for the Γ_F-action on the top covering the Γ_k-action on the bottom.

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Proof. The top arrow is the natural isomorphism, and the map pr is the canonical projection to the coinvariants. Let us construct the bottom arrow. Note that $X_*(\mathrm{Res}_{K/F}(T_0)) = \mathrm{Ind}_{\Gamma_K}^{\Gamma_F}(X_*(T_0))$ is an induced Galois module by the proof of Lemma 4.1. Shapiro's lemma induces a Γ_k-equivariant isomorphism

$$\mathrm{Ind}_{\Gamma_{k_0}}^{\Gamma_k}(X_*(T_0)_{I_K}) \xrightarrow{\simeq} X_*(\mathrm{Res}_{K/F}(T_0))_{I_F}. \tag{4.21}$$

(For any $\mathbb{Z}[\Gamma_K]$-module M, we have $(M \otimes_{\mathbb{Z}[\Gamma_K]} \mathbb{Z}[\Gamma_F])_{I_F} = M_{I_K} \otimes_{\mathbb{Z}[\Gamma_{k_0}]} \mathbb{Z}[\Gamma_k]$ canonically.) Further, the Kottwitz map (cf. [Kot97, § 7]) applied to (4.10) in the case of $T_0(\breve{K})$ (respectively, $T_0^\flat(\bar{k}(\!(u)\!))$) induces a Γ_{k_0}-equivariant isomorphism $X_*(T_0^\flat)_{I_{K_0^\flat}} \simeq X_*(T_0)_{I_K}$. Applying the induction functor, we deduce a Γ_k-equivariant isomorphism

$$\mathrm{Ind}_{\Gamma_{k_0}}^{\Gamma_k}(X_*(T_0^\flat)_{I_{K_0^\flat}}) \xrightarrow{\simeq} \mathrm{Ind}_{\Gamma_{k_0}}^{\Gamma_k}(X_*(T_0)_{I_K}). \tag{4.22}$$

Finally, we use $\mathcal{F}\ell_{\mathcal{T}^\flat} = \mathrm{Res}_{k_0/k}(\mathcal{F}\ell_{\mathcal{T}_0^\flat})$ (cf. (4.16)) together with the isomorphism induced by the Kottwitz map

$$\mathcal{F}\ell_{\mathcal{T}_0^\flat}(\bar{k}) = T_0^\flat(\bar{k}(\!(u)\!))/\mathcal{T}_0^\flat(\bar{k}[\![u]\!]) \xrightarrow{\simeq} X_*(T_0^\flat)_{I_{K_0^\flat}},$$

which is Γ_{k_0}-equivariant as well. This induces the Γ_k-equivariant isomorphism

$$\mathcal{F}\ell_{\mathcal{T}^\flat}(\bar{k}) \xrightarrow{\simeq} \mathrm{Ind}_{\Gamma_{k_0}}^{\Gamma_k}(X_*(T_0^\flat)_{I_{K_0^\flat}}). \tag{4.23}$$

The bottom arrow in (4.20) is defined to be the composition of (4.21), (4.22) and (4.23).

All that remains is to prove the commutativity, which is a reformulation of [Lev16, Proposition 4.2.8]: the composition pr with the inverse of (4.21) is the map given by $\mu' \mapsto \bar{\lambda}_{\mu'}$ in the notation of [Lev16]. We show the commutativity as follows. The diagram is compatible with unramified extensions, and we reduce to the case $K_0 = F$. Changing notation, we may now assume that $F = \breve{F}$, $k = \bar{k}$. The diagram (4.20) is functorial in the tamely ramified K-torus T_0. Arguing as in § 4.4.2 Step 2, we choose an induced tamely ramified K-torus $T_1 \twoheadrightarrow T_0$ with kernel being a torus. Each item in the diagram for T_1 maps surjectively onto each item in the diagram for T_0, and we reduce to the case where $T_0 = T_1$ is an induced tamely ramified K-torus. Arguing as in § 4.4.2 Step 1, the torus T_0 is a product of K-tori of the form $\mathrm{Res}_{K_1/K}(\mathbb{G}_m)$ with K_1/K being totally (tamely) ramified. Accordingly, each item in the diagram (4.20) splits as a product compatible with the maps, and we reduce to the case where $T_0 = \mathrm{Res}_{K_1/K}(\mathbb{G}_m)$. Replacing the pair (X, D) with the pair (X', D') as in § 4.4.2 Step 1, we reduce further to the case where $T_0 = \mathbb{G}_m$. In this case, we have for the (global) loop group

$$L\mathbb{G}_m(\mathcal{O}_{\bar{F}}) = (L\mathbb{G}_m)_{(X,\mathbb{G}_m,D)}(\mathcal{O}_{\bar{F}}) = \mathcal{O}_{\bar{F}}(\!(Q)\!)^\times,$$

where $Q \in \mathcal{O}_F[u]$ is the minimal polynomial of $\varpi \in K$ over F. Writing $Q = (u - a_1) \cdot \ldots \cdot (u - a_d)$ for $d = [K : F]$ and pairwise distinct elements $a_1, \ldots, a_d \in \mathcal{O}_{\bar{F}}$, we compute for the generic fiber

$$(L\mathbb{G}_m)_{(X,\mathbb{G}_m,D)}(\bar{F}) = \prod_{i=1,\ldots,n} \bar{F}(\!(u - a_i)\!)^\times.$$

For $i = 1, \ldots, d$, let v_i be the $(u - a_i)$-adic valuation of $\bar{F}(\!(u - a_i)\!)$. The specialization map (4.19) is explicitly given by the map

$$\prod_{i=1,\ldots,d} \bar{F}(\!(u - a_i)\!)^\times / \bar{F}[\![u - a_i]\!]^\times \to k(\!(u)\!)^\times / k[\![u]\!]^\times, \quad (x_1, \ldots, x_d) \mapsto u^{\sum_{i=1}^d v_i(x_i)},$$

where we use that $Q \equiv u^{[K:F]} \bmod \varpi$. One checks that (4.20) commutes for $T_0 = \mathbb{G}_m$. which finishes the proof of the lemma. □

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4.4.4 *Local models for Weil-restricted groups.* We now recall the definition of local models for the pair $(G, \mathcal{G}) = (\mathrm{Res}_{K/F}(G_0), \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G}_{\mathbf{f}_0}))$. Let $\{\mu\}$ be a $G(\bar{F})$-conjugacy class of geometric cocharacters with reflex field E/F. For a representative $\mu \in \{\mu\}$, the associated *Schubert variety* is the reduced $L_z^+ G_{\bar{F}}$-orbit closure

$$\mathrm{Gr}_G^{\leqslant\{\mu\}} \overset{\mathrm{def}}{=} \overline{L_z^+ G_{\bar{F}} \cdot z^\mu \cdot e_0} \subset \mathrm{Gr}_{G,\bar{F}}. \tag{4.24}$$

The $\bar{F}$-scheme $\mathrm{Gr}_G^{\leqslant\{\mu\}}$ is defined over the reflex field $E = E(\{\mu\})$, that is, the field of definition of $\{\mu\}$ which is a finite extension of F, and is a (geometrically irreducible) projective E-variety.

The following definition is [PZ13, Definition 7.1] if K/F is tamely ramified, and [Lev16, Definition 4.2.1] in general; cf. [Lev16, Proposition 4.2.4].

Definition 4.18. The local model $M_{\{\mu\}} = M(\underline{G}_0, \mathcal{G}_{\mathbf{f}}, \{\mu\}, \varpi)$ is the scheme-theoretic closure of the locally closed subscheme

$$\mathrm{Gr}_G^{\leqslant\{\mu\}} \hookrightarrow \mathrm{Gr}_G \otimes_F E \hookrightarrow \mathrm{Gr}_{\mathcal{G}} \otimes_{\mathcal{O}_F} \mathcal{O}_E,$$

where $\mathrm{Gr}_G^{\leqslant\{\mu\}}$ is as in (4.24).

By definition, the local model $M_{\{\mu\}}$ is a closed flat $L^+\mathcal{G}_{\mathcal{O}_E}$-invariant subscheme of $(\mathrm{Gr}_{\mathcal{G}} \otimes_{\mathcal{O}_F} \mathcal{O}_E)_{\mathrm{red}}$ which is uniquely determined up to unique isomorphism by the data $(\underline{G}_0, \mathcal{G}_{\mathbf{f}}, \{\mu\}, \varpi)$. Its generic fiber $M_{\{\mu\}} \otimes E = \mathrm{Gr}_{G,E}^{\leqslant\{\mu\}}$ is a (geometrically irreducible) variety, and the special fiber $M_{\{\mu\}} \otimes k_E$ is equidimensional; cf. [GW10, Theorem 14.114]. By Proposition 4.15, the map $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spec}(\mathcal{O}_F)$ is fiberwise ind-proper, and hence the map $M_{\{\mu\}} \to \mathrm{Spec}(\mathcal{O}_E)$ is fiberwise proper. Note that there is a closed embedding into the flag variety

$$M_{\{\mu\}} \otimes k_E \hookrightarrow \mathrm{Gr}_{\mathcal{G}} \otimes_{\mathcal{O}_F} k_E = \mathcal{F}\ell_{\mathcal{G}^\flat, k_E}, \tag{4.25}$$

which identifies the reduced locus $(M_{\{\mu\}} \otimes k_E)_{\mathrm{red}}$ with a union of Schubert varieties in $\mathcal{F}\ell_{\mathcal{G}^\flat, k_E}$.

Remark 4.19. The local model $M_{\{\mu\}}$ should up to unique isomorphism only depend on the data $(G, \mathcal{G}, \{\mu\})$. The uniqueness of $M_{\{\mu\}}$ is a separate question, and not of importance for the present paper. We refer the reader to [PZ13, Remark 3.2] for remarks on the uniqueness of $\underline{G}_0$, and to [Lev16, Remark 4.2.5] for remarks on the independence of $M_{\{\mu\}}$ on the choice of the uniformizer $\varpi \in K$. In the recent preprint [HPR18, Theorem 2.7], it is shown the ind-scheme $\mathrm{Gr}_{\mathcal{G}}$ for $K = F$ depends up to equivariant isomorphism only on the data $(G, \mathcal{G})$. So $M_{\{\mu\}}$ for $K = F$ depends up to equivariant isomorphism only on the data $(G, \mathcal{G}, \{\mu\})$. Note that [HPR18, Conjecture 2.12] uniquely characterizes $M_{\{\mu\}}$ for $K = F$ in the case where $\{\mu\}$ is minuscule.

4.4.5 *Each local model is proper.* For every conjugacy class $\{\mu\}$, we need to show that the local model $M_{\{\mu\}}$ is proper over $\mathcal{O}_E$, where $E = E(\{\mu\})$ is the reflex field. In view of [Ric16b, Lemma 2.20] and the discussion after Definition 4.18, all that remains is to show that the special fiber of $M_{\{\mu\}}$ is non-empty. The inclusion $\underline{\mathcal{T}}_0 \subset \underline{\mathcal{G}}_0$ induces a map of $\mathcal{O}_F$-ind-schemes

$$\mathrm{Gr}_{\mathcal{T}} = \mathrm{Gr}_{(X, \underline{\mathcal{T}}_0, D)} \to \mathrm{Gr}_{(X, \underline{\mathcal{G}}_0, D)} = \mathrm{Gr}_{\mathcal{G}}. \tag{4.26}$$

In the notation of Proposition 4.15, the geometric generic fiber $M_{\{\mu\}}(\bar{F})$ contains the element

$$\mu \in \mathrm{Gr}_T(\bar{F}) = \mathrm{Gr}_{\mathcal{T}}(\bar{F}),$$

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for any representative $\mu \in X_*(\tilde{T})$ of $\{\mu\}$. As $\mathrm{Gr}_{\mathcal{T}}$ is ind-finite (hence ind-proper) by § 4.4.2, the element $\mu \in \mathrm{Gr}_{\mathcal{T}}(\bar{F})$ uniquely extends to a point $\tilde{\mu} \in \mathrm{Gr}_{\mathcal{T}}(\mathcal{O}_{\bar{F}})$ by the valuative criterion for properness. Composed with (4.26), this defines a point (still denoted) $\tilde{\mu} \in \mathrm{Gr}_{\mathcal{G}}(\mathcal{O}_{\bar{F}})$. Since $M_{\{\mu\}} \subset \mathrm{Gr}_{\mathcal{G},\mathcal{O}_E}$ is a closed subscheme, we have

$$\tilde{\mu} \in M_{\{\mu\}}(\bar{F}) \cap \mathrm{Gr}_{\mathcal{G}}(\mathcal{O}_{\bar{F}}) = M_{\{\mu\}}(\mathcal{O}_{\bar{F}}), \tag{4.27}$$

and its special fiber $\bar{\mu} := \tilde{\mu}_{\bar{k}} \in M_{\{\mu\}}(\bar{k})$ is non-empty. This concludes § 4.4.5.

4.4.6 *Conclusion of proof of Theorem 4.16.* We need to show that $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spec}(\mathcal{O}_F)$ is ind-proper. It suffices to prove that the map $(\mathrm{Gr}_{\mathcal{G}} \otimes \mathcal{O}_{\bar{F}})_{\mathrm{red}} \to \mathrm{Spec}(\mathcal{O}_{\bar{F}})$ is ind-proper. In view of § 4.4.5, we have to show that the closed immersion

$$\bigcup_{\{\mu\}} (M_{\{\mu\},\mathcal{O}_{\bar{F}}})_{\mathrm{red}} \subset (\mathrm{Gr}_{\mathcal{G}} \otimes \mathcal{O}_{\bar{F}})_{\mathrm{red}} \tag{4.28}$$

is an equality. Here $\{\mu\}$ ranges over all $G(\bar{F})$-conjugacy classes of geometric cocharacters. As both ind-schemes in (4.28) are reduced, one can check the equality on the underlying topological spaces. As in [Ric16b, § 2.5] (respectively, [Lev16, Theorem 4.2.11]), this follows from Lemma 4.17 combined with (4.25) and (4.27). This concludes § 4.4.6, and hence the proof of Theorem 4.16.

5. Actions of $\mathbb{G}_m$ on Weil-restricted affine Grassmannians

5.1 Geometry of $\mathbb{G}_m$-actions on affine Grassmannians

Fix the data and notation as in § 4.4. In particular, we denote the group schemes over $X = \mathbb{A}^1_{\mathcal{O}_{K_0}}$ by $(\underline{\mathcal{G}}_0, \underline{\mathcal{A}}_0, \underline{\mathcal{S}}_0, \underline{\mathcal{T}}_0)$.

5.1.1 *Main geometric result.* Let $\chi\colon \mathbb{G}_{m,K} \to A_0 \subset G_0$ be a cocharacter which acts on G_0 by conjugation. As in (3.17), the centralizer is a Levi subgroup $M_0 \subset G_0$, and the attractor (respectively, repeller) subgroup P_0^+ (respectively, P_0^-) is a parabolic subgroup with $P_0^+ \cap P_0^- = M_0$. Further, we have semidirect product decompositions $P_0^{\pm} = M_0 \rtimes N_0^{\pm}$ defined over K.

Via the fixed isomorphism $\underline{\mathcal{G}}_{0,K} \simeq G$ compatible with $\underline{\mathcal{A}}_{0,K} \simeq A_0$, we may view χ as a cocharacter of $\underline{\mathcal{A}}_{0,K}$. As X is connected and $\underline{\mathcal{A}}$ a split torus, χ extends uniquely to a cocharacter also denoted by

$$\chi\colon \mathbb{G}_{m,X} \longrightarrow \underline{\mathcal{A}}_0 \subset \underline{\mathcal{G}}_0. \tag{5.1}$$

Hence, the cocharacter χ acts by conjugation on $\underline{\mathcal{G}}_0$ via the rule $\mathbb{G}_{m,X} \times_X \underline{\mathcal{G}}_0 \to \underline{\mathcal{G}}_0$, $(\lambda, g) \mapsto \chi(\lambda)\cdot g \cdot \chi(\lambda)^{-1}$. Using the dynamic method promulgated in [CGP10], the functors (2.1) define X-subgroup schemes of $\underline{\mathcal{G}}_0$ given by the fixed points $\underline{\mathcal{M}}_0 = \underline{\mathcal{G}}_0^{0,\chi}$, and the attractor $\underline{\mathcal{P}}_0^+ = \underline{\mathcal{G}}_0^{+,\chi}$ (respectively, the repeller $\underline{\mathcal{P}}_0^- = \underline{\mathcal{G}}_0^{-,\chi}$). Note that $\underline{\mathcal{M}}_0$ is by definition the schematic centralizer of χ in $\underline{\mathcal{G}}_0$.

Lemma 5.1. (i) *The X-group schemes $\underline{\mathcal{M}}_0$ and $\underline{\mathcal{P}}_0^{\pm}$ are smooth closed subgroup schemes of $\underline{\mathcal{G}}_0$ with geometrically connected fibers.*

(ii) *The centralizer $\underline{\mathcal{M}}_0$ is a parahoric X-group scheme for M_0 in the sense of Theorem 4.13.*

(iii) *There is a semidirect product decomposition as X-group schemes $\underline{\mathcal{P}}_0^{\pm} = \underline{\mathcal{M}}_0 \ltimes \underline{\mathcal{N}}_0^{\pm}$ where $\underline{\mathcal{N}}_0^{\pm}$ is a smooth affine group scheme with geometrically connected fibers.*

(iv) *The fixed isomorphism $\underline{G}_{0,K} \simeq G_0$ induces isomorphisms of $K[\![z]\!]$-groups $\underline{\mathcal{M}}_{0,K[\![z]\!]} \simeq M_0 \otimes_K K[\![z]\!]$, and $\underline{\mathcal{P}}^{\pm}_{0,K[\![z]\!]} \simeq P_0^{\pm} \otimes_K K[\![z]\!]$ compatible with the semidirect product decomposition in (iii).*

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Proof. The method of [HR18, Lemma 5.16] extends to give (i), (ii) and (iii) of the lemma. Part (iv) is immediate from the construction of χ and the proof of Proposition 4.15(i). $\square$

By (2.2), there are natural maps of X-group schemes

$$\underline{\mathcal{M}}_0 \leftarrow \underline{\mathcal{P}}_0^{\pm} \rightarrow \underline{\mathcal{G}}_0. \tag{5.2}$$

The maps (5.2) induce, by functoriality of Beilinson–Drinfeld Grassmannians, maps of $\mathcal{O}_F$-spaces

$$\mathrm{Gr}_{\mathcal{M}} \leftarrow \mathrm{Gr}_{\mathcal{P}^{\pm}} \rightarrow \mathrm{Gr}_{\mathcal{G}}, \tag{5.3}$$

where $\mathrm{Gr}_{\mathcal{G}} := \mathrm{Gr}_{(X,\underline{\mathcal{G}}_0,D)}$ (respectively, $\mathrm{Gr}_{\mathcal{M}} := \mathrm{Gr}_{(X,\underline{\mathcal{M}}_0,D)}$; respectively, $\mathrm{Gr}_{\mathcal{P}^{\pm}} := \mathrm{Gr}_{(X,\underline{\mathcal{P}}_0^{\pm},D)}$) by notational convention. In light of [PZ13, Corollary 11.7] and Corollary 3.11(i), the functors in (5.3) are representable by separated $\mathcal{O}_F$-ind-schemes of ind-finite type. Note that, by Theorem 4.16(i) and Lemma 5.1(ii), the $\mathcal{O}_F$-ind-schemes $\mathrm{Gr}_{\mathcal{G}}$ and $\mathrm{Gr}_{\mathcal{M}}$ are even ind-projective. The $\mathcal{O}_F$-ind-scheme $\mathrm{Gr}_{\mathcal{P}}$ is never ind-projective besides the trivial cases.

By functoriality of the loop group, we obtain via the composition

$$\mathbb{G}_{m,\mathcal{O}_F} \subset L_D^+\mathbb{G}_{m,X} \xrightarrow{L_D^+\chi} L_D^+\underline{\mathcal{A}}_0 \subset L_D^+\underline{\mathcal{G}}_0 \tag{5.4}$$

a $\mathbb{G}_{m,\mathcal{O}_F}$-action on $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spec}(\mathcal{O}_F)$.

Lemma 5.2. *The $\mathbb{G}_m$-action on $\mathrm{Gr}_{\mathcal{G}}$ is Zariski locally linearizable.*

Proof. By [PZ13, Corollary 11.7] there exists an monomorphism of X-groups $\underline{\mathcal{G}}_0 \hookrightarrow \mathrm{Gl}_{n,X}$ such that the fppf quotient $\mathrm{Gl}_{n,X}/\underline{\mathcal{G}}_0$ is quasi-affine. Hence, the induced monomorphism $\iota\colon \mathrm{Gr}_{\mathcal{G}} \hookrightarrow \mathrm{Gr}_{\mathrm{Gl}_{n,X}}$ is representable by a quasi-compact immersion (cf. Proposition 3.10) which is even a closed immersion because $\mathrm{Gr}_{\mathcal{G}}$ is ind-proper; cf. Theorem 4.16. The map ι is $\mathbb{G}_m$-equivariant for the cocharacter $\mathbb{G}_{m,X} \xrightarrow{\chi} \underline{\mathcal{G}}_0 \to \mathrm{Gl}_{n,X}$, and we reduce to the case $\underline{\mathcal{G}}_0 = \mathrm{Gl}_{n,X}$. By [Con14, Proposition 6.2.11] (use $\mathrm{Pic}(X) = 0$), the cocharacter $\chi\colon \mathbb{G}_{m,X} \to \mathrm{Gl}_{n,X}$ is conjugate to a cocharacter with values in the standard diagonal torus, and hence defined over $\mathcal{O}_F$. The lemma follows from the proof of Lemma 3.16. $\square$

In light of Theorems 4.16 and 2.1, we obtain maps of separated $\mathcal{O}_F$-ind-schemes

$$(\mathrm{Gr}_{\mathcal{G}})^0 \leftarrow (\mathrm{Gr}_{\mathcal{G}})^{\pm} \rightarrow \mathrm{Gr}_{\mathcal{G}}. \tag{5.5}$$

The following theorem compares (5.3) with (5.5).

Theorem 5.3. *The maps induce a commutative diagram of $\mathcal{O}_F$-ind-schemes*

$$\begin{array}{ccccc} \mathrm{Gr}_{\mathcal{M}} & \longleftarrow & \mathrm{Gr}_{\mathcal{P}^{\pm}} & \longrightarrow & \mathrm{Gr}_{\mathcal{G}} \\ \downarrow{\scriptstyle \iota^0} & & \downarrow{\scriptstyle \iota^{\pm}} & & \downarrow{\scriptstyle \mathrm{id}} \\ (\mathrm{Gr}_{\mathcal{G}})^0 & \longleftarrow & (\mathrm{Gr}_{\mathcal{G}})^{\pm} & \longrightarrow & \mathrm{Gr}_{\mathcal{G}}, \end{array} \tag{5.6}$$

where the maps ι^0 and $\iota^{\pm}$ satisfy the following properties.

(i) *In the generic fiber, the diagram is isomorphic to (5.7) below, and the maps ι_F^0 and $\iota_F^{\pm}$ are isomorphisms.*

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(ii) *In the special fiber, the diagram is isomorphic to (5.8) below, and the maps ι_k^0 and $\iota_k^\pm$ are closed immersions which are open immersions on the underlying reduced loci.*

(iii) *The maps ι^0 and $\iota^\pm$ are closed immersions which are open immersions on the underlying reduced loci.*

The diagram is constructed as follows. The fppf quotient $\underline{\mathcal{G}}_0/\underline{\mathcal{M}}_0$ is quasi-affine by [Con14, Theorem 2.4.1], which implies that the map $\mathrm{Gr}_{\mathcal{M}} \to \mathrm{Gr}_{\mathcal{G}}$ as in the proof of Lemma 5.2 is representable by a closed immersion. Since the $\mathbb{G}_m$-action on $\mathrm{Gr}_{\mathcal{M}}$ is trivial, the map factors as $\mathrm{Gr}_{\mathcal{M}} \to (\mathrm{Gr}_{\mathcal{G}})^0 \to \mathrm{Gr}_{\mathcal{G}}$, and we obtain the closed immersion ι^0.

The map $\iota^\pm$ is given via a Rees construction in terms of the moduli description (4.12); cf. § 3.3.1. Alternatively, if we choose a monomorphism of X-groups $\underline{\mathcal{G}}_0 \hookrightarrow \mathrm{Gl}_{n,X}$ such that $\mathrm{Gl}_{n,X}/\underline{\mathcal{G}}_0$ is quasi-affine (cf. [PZ13, Corollary 11.7]), then the same argument as in (3.22) applies, and we conclude that $\iota^\pm$ is representable by a quasi-compact immersion. We do not repeat the argument here, but instead refer the reader to § 3.3.1 for details. This constructs the commutative diagram (5.6).

Proof of Theorem 5.3. (i) In the generic fiber, by (4.14) and Lemma 5.1(iv), (5.6) is the commutative diagram of F-ind-schemes

$$\begin{array}{ccccc} \mathrm{Gr}_M & \longleftarrow & \mathrm{Gr}_{P^\pm} & \longrightarrow & \mathrm{Gr}_G \\ \downarrow{\scriptstyle \iota_F^0} & & \downarrow{\scriptstyle \iota_F^\pm} & & \downarrow{\scriptstyle \mathrm{id}} \\ (\mathrm{Gr}_G)^0 & \longleftarrow & (\mathrm{Gr}_G)^\pm & \longrightarrow & \mathrm{Gr}_G \end{array} \tag{5.7}$$

where $G = \mathrm{Res}_{K/F}(G_0)$ (respectively, $M = \mathrm{Res}_{K/F}(M_0)$; respectively, $P^\pm = \mathrm{Res}_{K/F}(P_0^\pm)$). The $\mathbb{G}_m$-action on the diagram is induced by the L_z^+-construction applied to the cocharacter

$$\tilde{\chi}\colon \mathbb{G}_{m,F} \subset \mathrm{Res}_{K/F}(\mathbb{G}_{m,K}) \xrightarrow{\mathrm{Res}_{K/F}(\chi)} \mathrm{Res}_{K/F}(A_0) \subset \mathrm{Res}_{K/F}(G_0) = G,$$

combined with the inclusion $\mathbb{G}_{m,F} \subset L_z^+\mathbb{G}_{m,F}$. We claim that the conjugation action of $\tilde{\chi}$ on G gives the group of fixed points $M = G^{0,\tilde{\chi}}$ and the attractor (respectively, repeller) group $P^+ = G^{+,\tilde{\chi}}$ (respectively, $P^- = G^{-,\tilde{\chi}}$). Indeed, the canonical maps of F-subgroups of G,

$$\begin{aligned} \mathrm{Res}_{K/F}(M_0) &\hookrightarrow G^{0,\tilde{\chi}}, \\ \mathrm{Res}_{K/F}(P_0^\pm) &\hookrightarrow G^{\pm,\tilde{\chi}} \end{aligned}$$

are isomorphisms. By descent, it is enough to prove this after passing to $\bar{F}$. But $G \otimes_F \bar{F} \simeq \prod_{K \hookrightarrow \bar{F}} G_0 \otimes_{K,\psi} \bar{F}$, where the $\mathbb{G}_m$-action induced by $\tilde{\chi}$ is the diagonal action on the product. Lemma 2.2 implies the claim. Part (i) follows from [HR18, Proposition 3.4] applied to the pair $(G, \tilde{\chi})$.

(ii) In the special fiber, (5.6) is the commutative diagram of k-ind-schemes

$$\begin{array}{ccccc} \mathcal{F}\ell_{\mathcal{M}^\flat} & \longleftarrow & \mathcal{F}\ell_{\mathcal{P}^{\flat\pm}} & \longrightarrow & \mathcal{F}\ell_{\mathcal{G}^\flat} \\ \downarrow{\scriptstyle \iota_k^0} & & \downarrow{\scriptstyle \iota_k^\pm} & & \downarrow{\scriptstyle \mathrm{id}} \\ (\mathcal{F}\ell_{\mathcal{G}^\flat})^0 & \longleftarrow & (\mathcal{F}\ell_{\mathcal{G}^\flat})^\pm & \longrightarrow & \mathcal{F}\ell_{\mathcal{G}^\flat} \end{array} \tag{5.8}$$

The $\mathbb{G}_m$-action on the diagram is given as follows. Base-changing (5.1) along $\mathcal{O}_{K_0}[u] \to k_0[\![u]\!]$ and taking restriction of scalars along $k_0[\![u]\!]/k[\![u]\!]$, we obtain the cocharacter

$$\chi^\flat\colon \mathbb{G}_{m,k[\![u]\!]} \subset \mathrm{Res}_{k_0[\![u]\!]/k[\![u]\!]}(\mathbb{G}_{m,k_0[\![u]\!]}) \subset \mathcal{G}^\flat,$$

which factors through $\mathcal{A}^\flat \subset \mathcal{G}^\flat$, the natural $k[\![u]\!]$-extension of the maximal $k(\!(u)\!)$-split torus $A^\flat \subset G^\flat$ in (4.8). Then $\mathbb{G}_{m,k}$ acts on the diagram via $\chi^\flat$ after applying the L^+-construction combined with the inclusion $\mathbb{G}_{m,k} \subset L^+\mathbb{G}_{m,k[\![u]\!]}$. Since taking fixed points (respectively, attractors; respectively, repellers) commutes with base change [Ric19, (1.3)] and is compatible with restriction of scalars along étale extensions, we have $\mathcal{M}^\flat = (\mathcal{G}^\flat)^{0,\chi^\flat}$ and $\mathcal{P}^{\flat,\pm} = (\mathcal{G}^\flat)^{\pm,\chi^\flat}$. Part (ii) follows from [HR18, Proposition 4.7] applied to the pair $(\mathcal{G}^\flat, \chi^\flat)$.

(iii) This follows as in [HR18, Theorem 5.5, 5.17] using Proposition 5.5 below, and we sketch the argument for convenience. With the notation of Proposition 5.5, the map ι^0 (respectively, $\iota^\pm$) factors as a set-theoretically bijective quasi-compact immersion

$$\iota^{0,c}\colon \mathrm{Gr}_{\mathcal{M}} \to (\mathrm{Gr}_{\mathcal{G}})^{0,c} \quad (\text{respectively, } \iota^{\pm,c}\colon \mathrm{Gr}_{\mathcal{P}^\pm} \to (\mathrm{Gr}_{\mathcal{G}})^{\pm,c}),$$

where $(\mathrm{Gr}_{\mathcal{G}})^{0,c}$ (respectively, $(\mathrm{Gr}_{\mathcal{G}})^{\pm,c}$) is an open and closed $\mathcal{O}_F$-sub-ind-scheme of $(\mathrm{Gr}_{\mathcal{G}})^0$ (respectively, $(\mathrm{Gr}_{\mathcal{G}})^\pm$). But any such map $\iota^{0,c}$ (respectively, $\iota^{\pm,c}$) is a closed immersion which is an isomorphism on the underlying reduced loci; cf. [HR18, Lemma 5.8]. □

We record the following properties.

Lemma 5.4. (i) *The map $(\mathrm{Gr}_{\mathcal{G}})^\pm \to \mathrm{Gr}_{\mathcal{G}}$ is schematic.*

(ii) *The map $(\mathrm{Gr}_{\mathcal{G}})^\pm \to (\mathrm{Gr}_{\mathcal{G}})^0$ is ind-affine with geometrically connected fibers, and induces an isomorphism on the group of connected components $\pi_0((\mathrm{Gr}_{\mathcal{G}})^\pm) \simeq \pi_0((\mathrm{Gr}_{\mathcal{G}})^0)$.*

Proof. These are general properties of attractors in ind-schemes endowed with étale locally linearizable $\mathbb{G}_m$-actions; cf. Lemma 5.2, and Theorem 2.1(ii) or [HR18, Theorem 2.1(ii)]. □

The following proposition decomposes the image of the maps ι^0 and $\iota^\pm$ into connected components, and will be important in what follows.

Proposition 5.5. *Let either $N = N^+$ or $N = N^-$. There exists an open and closed $\mathcal{O}_F$-ind-subscheme $(\mathrm{Gr}_{\mathcal{G}})^{0,c}$ (respectively, $(\mathrm{Gr}_{\mathcal{G}})^{\pm,c}$) of $(\mathrm{Gr}_{\mathcal{G}})^0$ (respectively, $(\mathrm{Gr}_{\mathcal{G}})^\pm$) together with a disjoint decomposition, depending up to sign on the choice of N, as $\mathcal{O}_F$-ind-schemes*

$$(\mathrm{Gr}_{\mathcal{G}})^{0,c} = \coprod_{m\in\mathbb{Z}} (\mathrm{Gr}_{\mathcal{G}})^0_m \quad \left(\text{respectively, } (\mathrm{Gr}_{\mathcal{G}})^{\pm,c} = \coprod_{m\in\mathbb{Z}} (\mathrm{Gr}_{\mathcal{G}})^\pm_m\right),$$

which has the following properties.

(i) *The map $\iota^0\colon \mathrm{Gr}_{\mathcal{M}} \to (\mathrm{Gr}_{\mathcal{G}})^0$ (respectively, $\iota^\pm\colon \mathrm{Gr}_{\mathcal{P}^\pm} \to (\mathrm{Gr}_{\mathcal{G}})^\pm$) factors through $(\mathrm{Gr}_{\mathcal{G}})^{0,c}$ (respectively, $(\mathrm{Gr}_{\mathcal{G}})^{\pm,c}$), inducing a closed immersion $\iota^{0,c}\colon \mathrm{Gr}_{\mathcal{M}} \to (\mathrm{Gr}_{\mathcal{G}})^{0,c}$ (respectively, $\iota^{\pm,c}\colon \mathrm{Gr}_{\mathcal{P}^\pm} \to (\mathrm{Gr}_{\mathcal{G}})^{\pm,c}$) which is an isomorphism on reduced loci.*

(ii) *The complement $(\mathrm{Gr}_{\mathcal{G}})^0 \backslash (\mathrm{Gr}_{\mathcal{G}})^{0,c}$ (respectively, $(\mathrm{Gr}_{\mathcal{G}})^\pm \backslash (\mathrm{Gr}_{\mathcal{G}})^{\pm,c}$) has empty generic fiber, that is, is concentrated in the special fiber.*

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Proof. The proof closely follows [HR18, Propositions 5.7 and 5.21]. We recall some steps of the construction. Let us denote $\mathcal{O} := \mathcal{O}_F$ and $\breve{\mathcal{O}} := \mathcal{O}_{\breve{F}}$. Let $\pi_1(M) = X_*(T)/X_*(T_{M_{\mathrm{sc}}})$ be the algebraic fundamental group of M in the sense of [Bor98], and denote by $\pi_1(M)_{I_F}$ the coinvariants. By [PR08, Theorem 5.1], the group of connected components is given by

$$\pi_0(\mathcal{F}\ell_{\mathcal{M}^\flat,\bar{k}}) = \pi_1(M^\flat)_{I_{k(u)}} = \pi_1(M)_{I_F},$$

where the last equality follows from the proof of Lemma 4.11. Note that $\pi_1(M)_{I_F} = \mathrm{Ind}_{\Gamma_{k_0}}^{\Gamma_k} \pi_1(M_0)_{I_K}$; cf. (4.21). Since $\mathrm{Gr}_{\mathcal{M},\breve{\mathcal{O}}} \to \mathrm{Spec}(\breve{\mathcal{O}})$ is ind-proper and $\breve{\mathcal{O}}$ is Henselian, the natural map

$$\pi_0(\mathrm{Gr}_{\mathcal{M},\breve{\mathcal{O}}}) \xrightarrow{\simeq} \pi_0(\mathcal{F}\ell_{\mathcal{M}^\flat,\bar{k}})$$

is an isomorphism by [SGA4½, Arcata; IV-2; Proposition 2.1]. This shows that there is a decomposition into connected components

$$\mathrm{Gr}_{\mathcal{M},\breve{\mathcal{O}}} = \coprod_{\bar{\nu}\in\pi_1(M)_{I_F}} (\mathrm{Gr}_{\mathcal{M},\breve{\mathcal{O}}})_{\bar{\nu}} \tag{5.9}$$

such that $(\mathrm{Gr}_{\mathcal{M},\breve{\mathcal{O}}})_{\bar{\nu}} \otimes \bar{k} \simeq (\mathcal{F}\ell_{\mathcal{M}^\flat,\bar{k}})_{\bar{\nu}}$. By Lemma 4.17, the generic fiber decomposes as $(\mathrm{Gr}_{\mathcal{M},\breve{\mathcal{O}}})_{\bar{\nu}} \otimes \bar{F} \simeq \coprod_{\nu\mapsto\bar{\nu}} (\mathrm{Gr}_{M,\bar{F}})_\nu$, where $\nu \in \pi_1(M)$ runs over the elements which map to $\bar{\nu}$ under the reduction map $\pi_1(M) \to \pi_1(M)_{I_F}$.

By Theorem 5.3(i) and (ii), it is easy to see that the closed immersion $\iota^0 \colon \mathrm{Gr}_{\mathcal{M},\breve{\mathcal{O}}} \to (\mathrm{Gr}_{\mathcal{G},\breve{\mathcal{O}}})^0$ is open on the underlying topological spaces (e.g. its image is closed under generization), that is, the image identifies each connected component of $\mathrm{Gr}_{\mathcal{M},\breve{\mathcal{O}}}$ with a connected component of $(\mathrm{Gr}_{\mathcal{G},\breve{\mathcal{O}}})^0$. Using Lemma 5.4(ii), we get an inclusion

$$\pi_1(M)_{I_F} = \pi_0(\mathrm{Gr}_{\mathcal{M},\breve{\mathcal{O}}}) \subset \pi_0((\mathrm{Gr}_{\mathcal{G},\breve{\mathcal{O}}})^0) = \pi_0((\mathrm{Gr}_{\mathcal{G},\breve{\mathcal{O}}})^\pm).$$

For $\bar{\nu} \in \pi_1(M)_{I_F}$, we denote the corresponding connected component of $(\mathrm{Gr}_{\mathcal{G},\breve{\mathcal{O}}})^0$ (respectively, $(\mathrm{Gr}_{\mathcal{G},\breve{\mathcal{O}}})^\pm$) by $(\mathrm{Gr}_{\mathcal{G}})^0_{\bar{\nu}}$ (respectively, $(\mathrm{Gr}_{\mathcal{G}})^\pm_{\bar{\nu}}$).

Let ρ denote the half-sum of the roots in $\mathrm{Res}_{K/F}(N)_{\bar{F}}$ with respect to $\mathrm{Res}_{K/F}(T)_{\bar{F}}$. For $\pi_1(M) \ni \nu \mapsto \bar{\nu} \in \pi_1(M)_{I_F}$, and $\dot{\nu} \in X_*(\mathrm{Res}_{K/F}(T))$ a lift of ν, we define the integer $n_\nu := \langle 2\rho, \dot{\nu}\rangle$ (respectively, $n_{\bar{\nu}} := \langle 2\rho, \dot{\nu}\rangle$) which is well defined independent of the choice of $\dot{\nu}$; cf. [HR18, (3.19)]. Note that we have $n_\nu = n_{\bar{\nu}}$ for all $\nu \mapsto \bar{\nu}$ by definition. For fixed $m \in \mathbb{Z}$, we consider the disjoint union

$$(\mathrm{Gr}_{\mathcal{G}})^0_m \overset{\mathrm{def}}{=} \coprod_{\bar{\nu}} (\mathrm{Gr}_{\mathcal{G}})^0_{\bar{\nu}} \quad \left(\text{respectively, } (\mathrm{Gr}_{\mathcal{G}})^\pm_m \overset{\mathrm{def}}{=} \coprod_{\bar{\nu}} (\mathrm{Gr}_{\mathcal{G}})^\pm_{\bar{\nu}}\right),$$

where the disjoint sum is indexed by all $\bar{\nu} \in \pi_1(M)_{I_F}$ such that $n_{\bar{\nu}} = m$. The Galois action preserves the integers $n_{\bar{\nu}}$, and hence the ind-scheme $(\mathrm{Gr}_{\mathcal{G}})^0_m$ (respectively, $(\mathrm{Gr}_{\mathcal{G}})^\pm_m$) is defined over $\mathcal{O}$. Note that $(\mathrm{Gr}_{\mathcal{G}})^\pm_m$ is the preimage of $(\mathrm{Gr}_{\mathcal{G}})^0_m$ along $(\mathrm{Gr}_{\mathcal{G}})^\pm \to (\mathrm{Gr}_{\mathcal{G}})^0$. We obtain a decomposition as $\mathcal{O}$-ind-schemes

$$(\mathrm{Gr}_{\mathcal{G}})^{0,c} \overset{\mathrm{def}}{=} \coprod_{m\in\mathbb{Z}} (\mathrm{Gr}_{\mathcal{G}})^0_m \quad \left(\text{respectively, } (\mathrm{Gr}_{\mathcal{G}})^{\pm,c} \overset{\mathrm{def}}{=} \coprod_{m\in\mathbb{Z}} (\mathrm{Gr}_{\mathcal{G}})^\pm_m\right).$$

Properties (i) and (ii) are immediate from the construction. □

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5.2 Cohomology of $\mathbb{G}_m$-actions on affine Grassmannians

The conventions are the same as in [HR18, § 3.4]. We fix a prime $\ell \neq p$ and an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. We fix once and for all $q^{1/2} \in \bar{\mathbb{Q}}_\ell$ and the square root of the cyclotomic character $\mathrm{cycl}\colon \Gamma_F \to \bar{\mathbb{Z}}_\ell^\times$ which maps any lift of the geometric Frobenius Φ_F to $q^{-1/2}$. The Tate twists are normalized such that the geometric Frobenius Φ_F acts on $\bar{\mathbb{Q}}_\ell(-1/2)$ by $q^{1/2}$.

For a separated ind-scheme $X = \mathrm{colim}_i X_i$ of finite type over a field (e.g. F) or a discrete valuation ring (e.g. $\mathcal{O}_F$), we denote the bounded derived category $D_c^b(X) = D_c^b(X, \bar{\mathbb{Q}}_\ell)$ of $\bar{\mathbb{Q}}_\ell$-complexes with constructible cohomologies by

$$D_c^b(X) \overset{\text{def}}{=} \mathrm{colim}_i\, D_c^b(X_i, \bar{\mathbb{Q}}_\ell).$$

There is the full abelian subcategory $\mathrm{Perv}(X) \subset D_c^b(X)$ of perverse sheaves; cf. [Zhu17, A.1] in the setting of ind-schemes, for example. For a complex $\mathcal{A} \in D_c^b(X)$, we denote for any $n \in \mathbb{Z}$ the shifted and twisted complex by

$$\mathcal{A}\langle n\rangle \overset{\text{def}}{=} \mathcal{A}[n](n/2).$$

Let us briefly recall the nearby cycles functor. Let $S = \mathrm{Spec}(\mathcal{O}_F)$ with open (respectively, closed) point $\eta = \mathrm{Spec}(F)$ (respectively, $s = \mathrm{Spec}(k)$). Let $\bar{\eta} := \mathrm{Spec}(\bar{F}) \to \eta$ (respectively, $\bar{s} := \mathrm{Spec}(\bar{k}) \to s$) denote the geometric point with Galois group $\Gamma = \mathrm{Gal}(\bar{\eta}/\eta)$. Let $\bar{S}$ denote the integral closure of S in $\bar{\eta}$. This gives rise to the 7-tuple $(S, \eta, s, \bar{S}, \bar{\eta}, \bar{s}, \Gamma)$. Now if X is an $\mathcal{O}_F$-ind-scheme of ind-finite type, there is by [SGA7, Example XIII] (cf. also [Ill94, App]) the functor of nearby cycles

$$\Psi_X\colon D_c^b(X_\eta) \longrightarrow D_c^b(X_s \times_S \eta), \tag{5.10}$$

where $D_c^b(X_s \times_S \eta)$ denotes the bounded derived category of $\bar{\mathbb{Q}}_\ell$-sheaves on $X_{\bar{s}}$ with constructible cohomologies, and with a continuous action of Γ compatible with the action on $X_{\bar{s}}$. The nearby cycles preserve perversity and restrict to a functor $\Psi_X\colon \mathrm{Perv}(X_\eta) \to \mathrm{Perv}(X_s \times_S \eta)$. We refer the reader to [PZ13, § 10] for the extension to ind-schemes.

For a map of $\mathcal{O}_F$-ind-schemes $f\colon X \to Y$, the nearby cycles are functorial in the obvious way; cf. [SGA7, Example XIII, 1.2.7–1.2.9]. Further, if f is a nilpotent thickening, that is, a closed immersion defined by an nilpotent ideal sheaf, then f induces $\Psi_X \simeq \Psi_Y$.

5.2.1 *Geometric Satake for Weil restrictions.* Recall the geometric Satake equivalence from [Gin95, Lus83, BD99, MV07, Ric14, RZ15, Zhu17]. We work under the same conventions as in [HR18, § 3.4], and we refer the reader to that work for more details.

Let G_0 be a reductive group over K. We are interested in the geometric Satake isomorphism for the group $G = \mathrm{Res}_{K/F}(G_0)$. For a conjugacy class $\{\mu\}$ of geometric cocharacters in G, denote the inclusion of the open $L_z^+ G_{\bar{F}}$-orbit by

$$j\colon \mathrm{Gr}_G^{\{\mu\}} \hookrightarrow \mathrm{Gr}_G^{\leqslant\{\mu\}};$$

cf. (4.24). The map j is defined over the reflex field $E = E(\{\mu\})$. We define the normalized intersection complex by

$$\mathrm{IC}_{\{\mu\}} \overset{\text{def}}{=} j_{!*}\bar{\mathbb{Q}}_\ell\langle d_\mu\rangle \in P(\mathrm{Gr}_{G,E}), \tag{5.11}$$

where d_μ denotes the dimension of $\mathrm{Gr}_G^{\leqslant\{\mu\}}$. The category $P_{L_z^+G}(\mathrm{Gr}_G)$ of L_z^+G-equivariant perverse sheaves (cf. [Zhu17, A.1] for equivariant perverse sheaves on ind-schemes) is generated by the

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intersection complexes (5.11) and local systems concentrated on the base point $e_0 \in \mathrm{Gr}_G(F)$. More precisely, every indecomposable object in $P_{L_z^+G}(\mathrm{Gr}_G)$ is of the form

$$\Big(\bigoplus_{\gamma\in\Gamma_F/\Gamma_E} \mathrm{IC}_{\gamma\cdot\{\mu\}}\Big)\otimes\mathcal{L}, \tag{5.12}$$

where $\mathcal{L}$ is a $\bar{\mathbb{Q}}_\ell$-local system on $e_0 = \mathrm{Spec}(F)$. The *Satake category* Sat_G is the full subcategory of $P_{L_z^+G}(\mathrm{Gr}_G)$ generated by objects (5.12) where the local system $\mathcal{L}$ is trivial over a finite field extension $\tilde{F}/F$.

We view Γ_F as a pro-algebraic group, and we let $\mathrm{Rep}^{\mathrm{alg}}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F)$ be the category of algebraic $\bar{\mathbb{Q}}_\ell$-representations of Γ_F, that is, finite-dimensional representations which factor through a finite quotient of Γ_F. There is the Tate twisted global cohomology functor

$$\begin{aligned}\omega\colon \mathrm{Sat}_G &\longrightarrow \mathrm{Rep}^{\mathrm{alg}}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F)\\ \mathcal{A} &\longmapsto \bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i(\mathrm{Gr}_{G,\bar F},\mathcal{A}_{\bar F})(i/2).\end{aligned} \tag{5.13}$$

By the geometric Satake equivalence, the functor ω can be upgraded to an equivalence of abelian tensor categories

$$\omega\colon \mathrm{Sat}_G \xrightarrow{\simeq} \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^LG), \tag{5.14}$$

where ${}^LG = G^\vee \rtimes \Gamma_F$ denotes the L-group viewed as a pro-algebraic group over $\bar{\mathbb{Q}}_\ell$. The tensor structure on Sat_G is given by the convolution of perverse sheaves; cf. § 5.5 below. The normalized intersection complex $\mathrm{IC}_{\{\mu\}}$ is an object in the category Sat_{G_E}, and its cohomology $\omega(\mathrm{IC}_{\{\mu\}})$ is under the geometric Satake equivalence (5.14) the ${}^LG_E := G^\vee\rtimes\Gamma_E$-representation $V_{\{\mu\}}$ of highest weight $\{\mu\}$ defined in [Hai14, § 6.1]; cf. [HR18, Corollary 3.12].

Let us describe the dual group $G^\vee = \mathrm{Res}_{K/F}(G_0)^\vee$ and the representation $V_{\{\mu\}}$ explicitly in terms of $G_0^\vee$. Of course, $G^\vee$ is canonically isomorphic to the product $\prod_{K\hookrightarrow\bar F} G_0^\vee$, but the Galois action does not respect the factors in general.

Let $\underline{\mathrm{Hom}}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F, G_0^\vee)$ be the sheaf of $\bar{\mathbb{Q}}_\ell$-scheme morphisms where again Γ_F is viewed as a pro-algebraic group. Then $\underline{\mathrm{Hom}}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F, G_0^\vee)$ is a group functor, and the pro-algebraic group Γ_K acts on $\underline{\mathrm{Hom}}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F, G_0^\vee)$ via $\bar{\mathbb{Q}}_\ell$-group automorphisms by the rule $(\gamma * f)(g) = \gamma(f(\gamma^{-1}g))$. Following [Bor79, I.5], we define the induced group as the Γ_K-fixed point sheaf

$$I^{\Gamma_F}_{\Gamma_K}(G_0^\vee) \overset{\mathrm{def}}{=} \underline{\mathrm{Hom}}^{\Gamma_K}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F, G_0^\vee), \tag{5.15}$$

which is a group functor. Note that choosing any finite extension $\tilde K/K$ which is Galois over F and splits G_0, we get an isomorphism of $\bar{\mathbb{Q}}_\ell$-groups

$$\underline{\mathrm{Hom}}^{\Gamma_{\tilde K/K}}_{\bar{\mathbb{Q}}_\ell}(\Gamma_{\tilde K/F}, G_0^\vee) \xrightarrow{\simeq} I^{\Gamma_F}_{\Gamma_K}(G_0^\vee), \tag{5.16}$$

where $\Gamma_{\tilde K/K} = \mathrm{Gal}(\tilde K/K)$ (respectively, $\Gamma_{\tilde K/F} = \mathrm{Gal}(\tilde K/F)$). In particular, $I^{\Gamma_F}_{\Gamma_K}(G_0^\vee)$ is an algebraic group, and is the colimit indexed by the filtered direct system (5.16) indexed by the splitting fields $\tilde K$. In this way, we get, as in [Bor79, I.5], an Γ_F-equivariant isomorphism of algebraic $\bar{\mathbb{Q}}_\ell$-groups

$$G^\vee \simeq I^{\Gamma_F}_{\Gamma_K}(G_0^\vee). \tag{5.17}$$

Let us turn to the representation $V_{\{\mu\}}$. We write the conjugacy class as $\{\mu\} = (\{\mu_\psi\})_\psi$ according to $G_{\bar F} \simeq \prod_{\psi\colon K\hookrightarrow\bar F} G_0\otimes_{\psi,K}\bar F$. The reflex field E of $\{\mu\}$ is the intersection (inside $\bar F$)

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of the reflex fields E_ψ of $\{\mu_\psi\}$. For each ψ, let $V_{\{\mu_\psi\}}$ denote the representation of $G_0^\vee$ of highest weight $\{\mu_\psi\}$, where we view $\{\mu_\psi\}$ as a Weyl orbit in the dual torus $X^*(T^\vee)$. The following lemma is immediate from the construction, and left to the reader.

Lemma 5.6. *The* $\prod_\psi G_0^\vee$*-representation* $\boxtimes_\psi V_{\{\mu_\psi\}}$ *uniquely extends to the* ${}^LG_E = G^\vee \rtimes \Gamma_E$ *representation* $V_{\{\mu\}}$ *defined above.* □

5.2.2 *Constant terms commute with nearby cycles.* We proceed with the notation as in § 5.1, and view the cocharacter χ as in (5.1). Combining Theorem 5.3 and Proposition 5.5 from the previous section, we have constructed a commutative diagram of $\mathcal{O}_F$-ind-schemes

$$\begin{array}{ccccc} \mathrm{Gr}_{\mathcal{M}} & \xleftarrow{q^\pm} & \mathrm{Gr}_{\mathcal{P}^\pm} & \xrightarrow{p^\pm} & \mathrm{Gr}_{\mathcal{G}} \\ \downarrow{\scriptstyle \iota^{0,c}} & & \downarrow{\scriptstyle \iota^{\pm,c}} & & \downarrow{\scriptstyle \mathrm{id}} \\ (\mathrm{Gr}_{\mathcal{G}})^{0,c} & \longleftarrow & (\mathrm{Gr}_{\mathcal{G}})^{\pm,c} & \longrightarrow & \mathrm{Gr}_{\mathcal{G}} \end{array} \tag{5.18}$$

The generic fiber of (5.18) is (5.7), and the special fiber of (5.18) is (5.8). The maps $\iota^{0,c}\colon \mathrm{Gr}_{\mathcal{M}} \hookrightarrow (\mathrm{Gr}_{\mathcal{G}})^{0,c}$ and $\iota^{\pm,c}\colon \mathrm{Gr}_{\mathcal{P}^\pm} \hookrightarrow (\mathrm{Gr}_{\mathcal{G}})^{\pm,c}$ are nilpotent thickenings by Proposition 5.5, and we may and do identify their derived categories of ℓ-adic complexes. Then there is a natural isomorphism of functors $D_c^b(\mathrm{Gr}_M) \to D_c^b(\mathcal{F}\ell_{\mathcal{M}^\flat} \times_S \eta)$,

$$\Psi_{\mathrm{Gr}_{\mathcal{M}}} \simeq \Psi_{(\mathrm{Gr}_{\mathcal{G}})^{0,c}}. \tag{5.19}$$

We write $\Psi_{\mathcal{G}} = \Psi_{\mathrm{Gr}_{\mathcal{G}}}$ (respectively, $\Psi_{\mathcal{M}} = \Psi_{\mathrm{Gr}_{\mathcal{M}}}$) in what follows. Since $\iota^{0,c}$ and $\iota^{\pm,c}$ are nilpotent thickenings, Proposition 5.5 gives us a decomposition

$$q^\pm = \coprod_{m\in\mathbb{Z}} q_m^\pm\colon\ \mathrm{Gr}_{\mathcal{P}^\pm} = \coprod_{m\in\mathbb{Z}} \mathrm{Gr}_{\mathcal{P}^\pm,m} \longrightarrow \coprod_{m\in\mathbb{Z}} \mathrm{Gr}_{\mathcal{M},m} = \mathrm{Gr}_{\mathcal{M}},$$

according to the choice of the parabolic $P^\pm$. We use the generic and the special fiber of diagram (5.18) to define normalized geometric constant term functors as follows.

Definition 5.7. We define the functor $\mathrm{CT}_M : D_c^b(\mathrm{Gr}_G) \to D_c^b(\mathrm{Gr}_M)$ (respectively, $\mathrm{CT}_{\mathcal{M}^\flat} : D_c^b(\mathcal{F}\ell_{\mathcal{G}^\flat} \times_S \eta) \to D_c^b(\mathcal{F}\ell_{\mathcal{M}^\flat} \times_S \eta)$) as the shifted pull-push functor

$$\mathrm{CT}_M \overset{\mathrm{def}}{=} \bigoplus_{m\in\mathbb{Z}} (q_{m,\eta}^+)_!(p_\eta^+)^*\langle m\rangle \quad \Big(\text{respectively, } \mathrm{CT}_{\mathcal{M}^\flat} \overset{\mathrm{def}}{=} \bigoplus_{m\in\mathbb{Z}} (q_{m,s}^+)_!(p_s^+)^*\langle m\rangle\Big).$$

As in [HR18, Theorem 6.1, (6.11)], the functorialities of nearby cycles give a transformation of functors $D_c^b(\mathrm{Gr}_G) \to D_c^b(\mathcal{F}\ell_{\mathcal{M}^\flat} \times_S \eta)$ as

$$\mathrm{CT}_{\mathcal{M}^\flat} \circ \Psi_{\mathcal{G}} \longrightarrow \Psi_{\mathcal{M}} \circ \mathrm{CT}_M. \tag{5.20}$$

Theorem 5.8. *The transformation (5.20) is an isomorphism of functors* $\mathrm{Sat}_G \to D_c^b(\mathcal{F}\ell_{\mathcal{M}^\flat} \times_S \eta)$*. In particular, for every* $\mathcal{A} \in \mathrm{Sat}_G$*, the complex* $\mathrm{CT}_{\mathcal{M}^\flat} \circ \Psi_{\mathcal{G}}(\mathcal{A})$ *is naturally an object in the category* $\mathrm{Perv}_{L^+\mathcal{M}^\flat}(\mathcal{F}\ell_{\mathcal{M}^\flat} \times_S \eta)$*.*

Proof. Every object in Sat_G is $\mathbb{G}_m$-equivariant. In view of Theorem 5.3 and (5.18), the extension of the method used in [HR18, Theorem 6.5] to this more general situation is obvious. We do not repeat the arguments. □

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5.3 Constant terms for tori

We aim to make Theorem 5.8 more explicit in the special case where $M = T$ is a torus; cf. Theorem 5.12 and Corollary 5.13. We retain the notation of § 5.2.

Let $\mathrm{Sat}_{\mathcal{T}^\flat} \subset \mathrm{Perv}_{L^+\mathcal{T}^\flat}(\mathcal{F}\ell_{\mathcal{T}^\flat} \times_S \eta)$ denote the semisimple full subcategory defined as in [Ric16a, Definition 5.10]. By [Ric16a, Theorem 5.11], the category $\mathrm{Sat}_{\mathcal{T}^\flat}$ has a Tannakian structure with tensor structure given by the convolution product, and with fiber functor given by the global sections functor $\omega_{\mathcal{T}^\flat}\colon \mathrm{Sat}_{\mathcal{T}^\flat} \to \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F)$, $\mathcal{A} \mapsto \mathrm{H}^0(\mathcal{F}\ell_{\mathcal{T}^\flat,\bar{k}}, \mathcal{A}_{\bar{k}})$. Note that $\mathcal{F}\ell_{\mathcal{T}^\flat}$ is ind-finite, and hence there is no higher cohomology and the convolution product is given by the usual tensor product. Further, for every $\mathcal{A} \in \mathrm{Sat}_{\mathcal{T}^\flat}$ the Γ_F-action on $\omega_{\mathcal{T}^\flat}(\mathcal{A})$ factors by definition through a finite quotient.

Lemma 5.9. *The functor $\omega_{\mathcal{T}^\flat}$ can be upgraded to an equivalence of Tannakian categories*

$$\mathrm{Sat}_{\mathcal{T}^\flat} \xrightarrow{\simeq} \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^LT_r),$$

where ${}^LT_r = (T^\vee)^{I_F} \rtimes \Gamma_F$ viewed as a pro-algebraic subgroup of ${}^LT$. Here the subscript $(\cdot)_r$ stands for 'ramified'.

Proof. By Lemma 4.17, there are $\mathrm{Gal}(\bar{k}/k)$-isomorphisms of abelian groups

$$\mathcal{F}\ell_{\mathcal{T}^\flat}(\bar{k}) \simeq X_*(T^\flat)_{I_{k(u)}} \simeq X_*(T)_{I_F} \simeq X^*((T^\vee)^{I_F}),$$

where the equivariance of the last isomorphism holds by construction of the dual torus. This induces a $\mathrm{Gal}(\bar{k}/k)$-equivariant isomorphism of $\bar{k}$-schemes

$$(\mathcal{F}\ell_{\mathcal{T}^\flat,\bar{k}})_{\mathrm{red}} \simeq \underline{X^*((T^\vee)^{I_F})}. \tag{5.21}$$

By definition, the objects in $\mathrm{Sat}_{\mathcal{T}^\flat}$ are finite-dimensional $\bar{\mathbb{Q}}_\ell$-vector spaces on (5.21) (viewed as complexes concentrated in cohomological degree 0) together with an action of Γ_F which is equivariant over the base, and which factors through a finite quotient. The lemma follows from this description. □

The following proposition is the analogue of [PZ13, Theorems 10.18, 10.23] in the special case of a torus.

Proposition 5.10. *There is a commutative diagram of Tannakian categories*

$$\begin{array}{ccc} \mathrm{Sat}_T & \xrightarrow{\Psi_{\mathcal{T}}} & \mathrm{Sat}_{\mathcal{T}^\flat} \\ {\scriptstyle\omega_T}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\omega_{\mathcal{T}^\flat}}\downarrow{\scriptstyle\simeq} \\ \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^LT) & \xrightarrow{\mathrm{res}} & \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^LT_r), \end{array}$$

where res *denotes the restriction of representations along the inclusion ${}^LT_r \subset {}^LT$.*

Proof. This is a reformulation of Lemma 4.17 as follows. In view of (5.14) and Lemma 5.9 the diagram is well defined, and it suffices to prove the commutativity. Let $f\colon \mathrm{Gr}_T \to \mathrm{Spec}(\mathcal{O}_F)$ denote the structure map. Since f is ind-proper, there is a Γ_F-equivariant isomorphism

$$\Psi_{\mathcal{O}_F} \circ f_{\bar{\eta},*} \xrightarrow{\simeq} f_{\bar{s},*} \circ \Psi_{\mathcal{T}},$$

and passing to the zeroth cohomology defines a Γ_F-equivariant isomorphism $\alpha\colon \mathrm{res}\circ\omega_T \simeq \omega_{\mathcal{T}^\flat}\circ\Psi_{\mathcal{T}}$. We have to show that α is a map of ${}^LT_r$-representations. As we already know the Γ_F-equivariance, it is enough to check that α is a map of $(T^\vee)^{I_F}$-representations, that is, respects the grading by $X^*((T^\vee)^{I_F}) = X_*(T)_{I_F}$ on the underlying $\bar{\mathbb{Q}}_\ell$-vector spaces. By (5.9), we have a decomposition into connected components

$$\mathrm{Gr}_{\mathcal{T}}\otimes\mathcal{O}_{\breve F} = \coprod_{\bar\nu\in X_*(T)_{I_F}} (\mathrm{Gr}_{\mathcal{T},\mathcal{O}_{\breve F}})_{\bar\nu},$$

where $(\mathrm{Gr}_{\mathcal{T},\mathcal{O}_{\breve F}})_{\bar\nu}\otimes\bar k = \{\bar\nu\}$ and $(\mathrm{Gr}_{\mathcal{T},\mathcal{O}_{\breve F}})_{\bar\nu}\otimes\bar F = \coprod_{\nu\mapsto\bar\nu}\{\nu\}$ on the underlying reduced subschemes; cf. also Lemma 4.17. The proposition follows from the fact that nearby cycles of a disjoint sum are computed as the sum of the single components. □

Remark 5.11. It would be interesting to see whether the analogue of Proposition 5.10 for more general *very* special parahoric group schemes as in [PZ13, Theorems 10.18, 10.23] holds true.

Combining Proposition 5.10 with Theorem 5.8, we arrive as in [HR18, § 6.2] at the following theorem which is the analogue of [AB09, Theorem 4] in our situation.

THEOREM 5.12. (i) *For every $\mathcal{A}\in\mathrm{Sat}_G$, one has $\mathrm{CT}_{\mathcal{T}^\flat}\circ\Psi_{\mathcal{G}}(\mathcal{A})\in\mathrm{Sat}_{\mathcal{T}^\flat}$.*

(ii) *The functor $\mathrm{CT}_{\mathcal{T}^\flat}\circ\Psi_{\mathcal{G}}\colon\mathrm{Sat}_G\to\mathrm{Sat}_{\mathcal{T}^\flat}$ admits a unique structure of a tensor functor together with an isomorphism $\omega_{\mathcal{T}^\flat}\circ\mathrm{CT}_{\mathcal{T}^\flat}\circ\Psi_{\mathcal{G}}\simeq\omega_G$. Under the geometric Satake equivalence, it corresponds to the restriction of representations $\mathrm{res}\colon\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^LG)\to\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^LT_r)$ along the inclusion ${}^LT_r\subset{}^LG$.* □

We now apply Theorem 5.12 in a special case. For more details, we refer to [HR18, § 6.2.1] which is analogous. Assume $F=\breve F$, and hence that K/F is totally ramified. Let $\chi\colon\mathbb{G}_{m,K}\to A_0\subset G_0$ be a regular cocharacter (i.e. its centralizer $M_0=T_0$ is a maximal torus), and let the parahoric $\mathcal{O}_K$-group scheme $\mathcal{G}_0$ be an Iwahori. Hence, $\mathcal{G}=\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G}_0)$ is an Iwahori $\mathcal{O}_F$-group scheme as well; cf. Proposition 4.7. There is a decomposition into connected components

$$(\mathcal{F}\ell_{\mathcal{G}^\flat})^+ = \coprod_{w\in W}(\mathcal{F}\ell_{\mathcal{G}^\flat})^+_w,$$

where $W=W(G,A,F)=W(G_0,A_0,K)$ is the Iwahori–Weyl group; cf. Lemma 4.3. Let $\Lambda_T = T(F)/\mathcal{T}(\mathcal{O}_F)\subset W$ be the subset of 'translation' elements. Let $X_*(T)_{I_F}\simeq\Lambda_T, \bar\lambda\mapsto t^{\bar\lambda}$ be the isomorphism given by the Kottwitz map. Let $2\rho\in X^*(T)$ be the sum of the positive roots contained in the positive Borel B^+ of $G_{\bar F}$ determined by χ. Then the integer $\langle 2\rho,\bar\lambda\rangle := \langle 2\rho,\lambda\rangle$ is well defined independent of the choice of $\lambda\in X_*(T)$ with $\lambda\mapsto\bar\lambda$.

COROLLARY 5.13. *Let $V\in\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(G^\vee)$, and denote by $\mathcal{A}_V\in\mathrm{Sat}_{G,\bar F}$ the object with $\omega_{G,\bar F}(\mathcal{A}_V) = V$. As $\bar{\mathbb{Q}}_\ell$-vector spaces, the compactly supported cohomology groups are given by the equality*

$$\mathbb{H}^i_c((\mathcal{F}\ell_{\mathcal{G}^\flat})^+_w,\Psi_{\mathcal{G}}(\mathcal{A}_V)) = \begin{cases} V(\bar\lambda) & \text{if } w=t^{\bar\lambda} \text{ and } i=\langle 2\rho,\bar\lambda\rangle,\\ 0 & \text{otherwise,}\end{cases}$$

where $V(\bar\lambda)$ is the $\bar\lambda$-weight space in $V|_{(T^\vee)^{I_F}}$. □

5.4 Special fibers of local models

Levin proved in [Lev16, Theorem 2.3.5] the analogue of the following theorem in the special case where $p \nmid |\pi_1(G_{\mathrm{der}})|$. As in [HR18, §§ 6.2 and 6.3], Corollary 5.13 can be used to obtain this result on the special fibers of local models, with no hypothesis on p. We do not need this result for the proof of our main theorem, but include it for completeness: together with the corresponding result in [HR18], we can conclude that the admissible sets $\mathrm{Adm}^{\mathbf{f}}_{\{\mu\}}$ parametrize the strata in the special fiber of $M_{\{\mu\}}$ *for all known local models* $M_{\{\mu\}}$.

The following theorem is precisely the analogue of [HR18, Theorem 6.12] in the current Weil restriction setting. We will assume for simplicity here that $K_0 = F$; a similar result holds without this assumption. Since $\breve{K} = K\breve{F}$, we may work over $F = \breve{F}$, so that $K = \breve{K}$ and $k = \bar{k}$. The special fiber $M_{\{\mu\},k}$ and the relevant Schubert varieties live in the affine flag variety attached to *equal characteristic* analogues $G^\flat = G^\flat_{k(\!(u)\!)}$, $A^\flat = S^\flat$ defined in (4.8), and by Lemmas 4.3 and 4.11 there is an identification of Iwahori–Weyl groups

$$W = W(G, A, F) = W(G_0, A_0, K) = W(G^\flat, A^\flat, k(\!(u)\!)).$$

For $w \in W$, we define the Schubert variety $\mathcal{F}l^{\leqslant w}_{\mathcal{G}^\flat}$ exactly as in [HR18, § 3.2].

Theorem 5.14. *The smooth locus* $(M_{\{\mu\}})^{\mathrm{sm}}$ *is fiberwise dense in* $M_{\{\mu\}}$*, and on reduced subschemes*

$$(M_{\{\mu\},k})_{\mathrm{red}} = \bigcup_{w \in \mathrm{Adm}^{\mathbf{f}}_{\{\mu\}}} \mathcal{F}l^{\leqslant w}_{\mathcal{G}^\flat}.$$

In particular, the special fiber $M_{\{\mu\},k}$ *is generically reduced.*

Proof. We may imitate the proof of [HR18, Theorem 6.12]. First we follow the method of [Ric16b, Lemma 3.12] to prove $\mathrm{Adm}^{\mathbf{f}}_{\{\mu\}} \subseteq \mathrm{Supp}^{\mathbf{f}}_{\{\mu\}} := \mathrm{Supp}\,\Psi_{\mathcal{G}_{\mathbf{f}}}(\mathrm{IC}_{\{\mu\}})$, using our Lemma 4.17 in place of [Ric16b, Lemma 2.21].

Also as in [HR18, Theorem 6.12], we reduce to the case where $\mathbf{f} = \mathbf{a}$. Then is it enough to show that if $w \in \mathrm{Supp}^{\mathbf{a}}_{\{\mu\}}$ is maximal, then $w \in \mathrm{Adm}^{\mathbf{a}}_{\{\mu\}}$. Now we choose a regular cocharacter $\chi\colon \mathbb{G}_{m,K} \to A_0 \subset G_0$, and use Corollary 5.13 as follows. As $\bar{\mathbb{Q}}_\ell$-vector spaces, we have

$$\mathbb{H}^*_c((\mathcal{F}\ell_{\mathcal{G}^\flat})^+_w, \Psi_{\mathcal{G}}(\mathrm{IC}_{\{\mu\}})) \neq 0,$$

because $\mathcal{F}\ell^{\leqslant w}_{\mathcal{G}^\flat} \cap (\mathcal{F}\ell_{\mathcal{G}^\flat})^+_w \subset \mathcal{F}\ell^w_{\mathcal{G}^\flat}$ is non-empty by [HR18, Lemma 6.10], and because up to shift and twist $\Psi_{\mathcal{G}}(\mathrm{IC}_{\{\mu\}})|_{\mathcal{F}\ell^w_{\mathcal{G}^\flat}} = \bar{\mathbb{Q}}^d_\ell$ for some $d > 0$ since $w \in \mathrm{Supp}^{\mathbf{a}}_{\{\mu\}}$ is maximal. Thus, Corollary 5.13 applies to show $w = t^{\bar{\lambda}}$ for some $\bar{\lambda} \in X_*(T)_{I_F}$ which is a weight in $V_{\{\mu\}}|_{(G^\vee)^{I_F}}$. As in [HR18, Theorem 6.12], we can conclude that $w = t^{\bar{\lambda}} \in \mathrm{Adm}^{\mathbf{a}}_{\{\mu\}}$ by citing [Hai18, Theorem 4.2 and (7.11-12)]. □

5.5 Central sheaves

We recall some facts on central sheaves which will be used in what follows. We proceed with the notation as in § 4.4. Let $\mathrm{Perv}_{L^+\mathcal{G}^\flat}(\mathcal{F}\ell_{\mathcal{G}^\flat} \times_S \eta)$ be the category of $L^+\mathcal{G}^\flat$-equivariant perverse sheaves compatible with a continuous Galois action; cf. [PZ13, Definition 10.3].

Recall that for objects in $\mathrm{Perv}_{L^+\mathcal{G}^\flat}(\mathcal{F}\ell_{\mathcal{G}^\flat} \times_S \eta)$ there is the convolution product defined by Lusztig [Lus83]. Consider the convolution diagram

$$\mathcal{F}\ell_{\mathcal{G}^\flat} \times \mathcal{F}\ell_{\mathcal{G}^\flat} \xleftarrow{q} L\mathcal{G}^\flat \times \mathcal{F}\ell_{\mathcal{G}^\flat} \xrightarrow{p} L\mathcal{G}^\flat \times^{L^+\mathcal{G}^\flat} \mathcal{F}\ell_{\mathcal{G}^\flat} =: \mathcal{F}\ell_{\mathcal{G}^\flat} \tilde{\times} \mathcal{F}\ell_{\mathcal{G}^\flat} \xrightarrow{m} \mathcal{F}\ell_{\mathcal{G}^\flat}.$$

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For $\mathcal{A},\mathcal{B}\in \mathrm{Perv}_{L^+\mathcal{G}^\flat}(\mathcal{F}\ell_{\mathcal{G}^\flat}\times_S\eta)$, let $\mathcal{A}\,\tilde{\times}\,\mathcal{B}$ be the (unique up to canonical isomorphism) complex on $\mathcal{F}\ell_{\mathcal{G}^\flat}\,\tilde{\times}\,\mathcal{F}\ell_{\mathcal{G}^\flat}$ such that $q^*(\mathcal{A}\boxtimes\mathcal{B})\simeq p^*(\mathcal{A}\,\tilde{\times}\,\mathcal{B})$. By definition

$$\mathcal{A}\star\mathcal{B}\overset{\mathrm{def}}{=} m_*(\mathcal{A}\,\tilde{\times}\,\mathcal{B})\in D^b_c(\mathcal{F}\ell_{\mathcal{G}^\flat}\times_S\eta,\bar{\mathbb{Q}}_\ell). \tag{5.22}$$

In the following, we consider $P_{L^+\mathcal{G}^\flat}(\mathcal{F}\ell_{\mathcal{G}^\flat})$ as a full subcategory of $P_{L^+\mathcal{G}^\flat}(\mathcal{F}\ell_{\mathcal{G}^\flat}\times_S\eta)$.

Let $W=W(G,A,K)=W(G^\flat,A^\flat,F^\flat)$ be the associated Iwahori–Weyl group; cf. Lemma 4.11. For each $w\in W$, the associated Schubert variety $\mathcal{F}\ell^{\leqslant w}_{\mathcal{G}^\flat}\subset\mathcal{F}\ell_{\mathcal{G}^\flat}$ is defined over k_F. Let $j\colon \mathcal{F}\ell^{w}_{\mathcal{G}^\flat}\hookrightarrow\mathcal{F}\ell^{\leqslant w}_{\mathcal{G}^\flat}$, and denote by $\mathrm{IC}_w=j_{!*}(\bar{\mathbb{Q}}_\ell[\dim(\mathcal{F}\ell^{w}_{\mathcal{G}^\flat})])$ the intersection complex. We have the functor of nearby cycles

$$\Psi_{\mathcal{G}}\colon\ \mathrm{Perv}_{L^+_zG}(\mathrm{Gr}_G)\longrightarrow \mathrm{Perv}_{L^+\mathcal{G}^\flat}(\mathcal{F}\ell_{\mathcal{G}^\flat}\times_S\eta).$$

The next theorem follows from [PZ13, Theorem 10.5] if K/F is tamely ramified, and from [Lev16, Theorem 5.2.10] in general.

Theorem 5.15 (Gaitsgory, Zhu, Pappas and Zhu, Levin). *For each $\mathcal{A}\in\mathrm{Perv}_{L^+_zG}(\mathrm{Gr}_G)$, and $w\in W$, both convolutions $\Psi_{\mathcal{G}}(\mathcal{A})\star\mathrm{IC}_w$, $\mathrm{IC}_w\star\Psi_{\mathcal{G}}(\mathcal{A})$ are objects in $P_{L^+\mathcal{G}^\flat}(\mathcal{F}\ell_{\mathcal{G}^\flat}\times_S\eta)$, and as such there is a canonical isomorphism*

$$\Psi_{\mathcal{G}}(\mathcal{A})\star\mathrm{IC}_w\simeq\mathrm{IC}_w\star\Psi_{\mathcal{G}}(\mathcal{A}).$$

Proof. If $\mathcal{A}=\mathrm{IC}_{\{\mu\}}$ where $\{\mu\}$ is a class which is defined over F, then the theorem is a special case of [Lev16, Theorem 5.2.10] which follows the method of [PZ13, Theorem 10.5]. However, the proof given there works for general objects $\mathcal{A}\in P_{L^+_zG}(\mathrm{Gr}_G)$, and only uses that the support $\mathrm{Supp}(\mathcal{A})$ is finite-dimensional and defined over F. We do not repeat the arguments here. □

6. Test functions for Weil-restricted local models

6.1 Preliminaries

Recall that we let $G=\mathrm{Res}_{K/F}(G_0)$ and ${}^LG=G^\vee\rtimes\Gamma_F$. Recall that $\{\mu\}$ is defined over a field E, a separable field extension of F which is a possibly non-trivial extension of the reflex field, and that E_0/F is the maximal unramified subextension of E/F. We have $V_{\{\mu\}}\in\mathrm{Rep}({}^LG_E)$ and $I(V_{\{\mu\}})\in\mathrm{Rep}({}^LG_{E_0})$, where $I(V):=\mathrm{Ind}^{{}^LG_{E_0}}_{{}^LG_E}(V)$ for $V\in\mathrm{Rep}({}^LG_E)$. Writing $\mathcal{G}:=\mathcal{G}_{\mathbf{f}}$ and $\mathcal{G}_0:=\mathcal{G}_{\mathbf{f_0}}$, the parahoric group scheme of $G=\mathrm{Res}_{K/F}(G_0)$ is given by $\mathcal{G}=\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G}_0)$ by Corollary 4.8.

Because the representation $I(V_{\{\mu\}})$ is 'defined over E_0' (not F), it is convenient to reformulate the test function conjecture after base-changing all geometric objects from $\mathcal{O}_F$ to $\mathcal{O}_{E_0}$. This ultimately allows us to reduce to the case where $E_0=F$ (see the end of this subsection, and § 6.3 below). The next few lemmas are ingredients toward this reduction.

Lemma 6.1. *The following statements hold.*

(i) *We may write $K_0\otimes_F E=\prod_j E_j$ and $K_0\otimes_F E_0=\prod_j E_{j,0}$, where E_j/K_0 is a finite extension of fields with maximal unramified subextension $E_{j,0}/K_0$, and where j ranges over the finite index set of Γ_{K_0}-orbits of F-embeddings $E_0\to\bar{K}_0$, that is, over the set $\Gamma_{E_0}\backslash\Gamma_F/\Gamma_{K_0}$. Similarly, for rings of integers we have $\mathcal{O}_{K_0}\otimes_{\mathcal{O}_F}\mathcal{O}_{E_0}=\prod_j\mathcal{O}_{E_{j,0}}$. Furthermore, the inertia groups satisfy $I_E=I_{E_j}\subset I_{E_0}=I_{E_{j,0}}$.*

(ii) *$K\otimes_F E_0=\prod_j KE_{j,0}$.*

(iii) *The canonical map $\Gamma_{E_0}\backslash\Gamma_F/\Gamma_K\to\Gamma_{E_0}\backslash\Gamma_F/\Gamma_{K_0}$ is a bijection.*

Proof. Write $K_0 \otimes_F E_0 = \prod_j E_{j,0}$ and $E = E_0[X]/(Q)$, where Q is an Eisenstein polynomial over $\mathcal{O}_{E_0}$. Each extension $E_{j,0}/F$ is unramified, and so Q remains an Eisenstein polynomial in the overfield $E_{j,0}$ of E_0. As $K_0 \otimes_F E = \prod_j E_{j,0}[X]/(Q) =: \prod_j E_j$, it follows that $E_j/E_{j,0}$ is totally ramified and that $E_j = EE_{j,0}$, from which it follows that E_j/E is unramified and hence $I_E = I_{E_j}$.

Since K/K_0 is totally ramified, $K \otimes_{K_0} E_{j,0} = KE_{j,0}$, which implies (ii).

Abstractly $K_0 \otimes_F E$ is a product of fields indexed by the set $\Gamma_E \backslash \Gamma_F / \Gamma_{K_0}$, and this set coincides with $\Gamma_{E_0} \backslash \Gamma_F / \Gamma_{K_0}$ by the above argument. Interchanging the roles of E and K, we also get the bijection in (iii). □

Lemma 6.2. *We have* $G_{E_0} = \prod_j \mathrm{Res}_{KE_{j,0}/E_0} G_{0,KE_{j,0}}$ *and* $\mathcal{G}_{\mathcal{O}_{E_0}} = \prod_j \mathrm{Res}_{\mathcal{O}_{KE_{j,0}}/\mathcal{O}_{E_0}} \mathcal{G}_{0,\mathcal{O}_{KE_{j,0}}}$.

Proof. This is a consequence of the compatibility of Weil restriction of scalars with base change along the ring extension $F \to E_0$ (respectively, $\mathcal{O}_F \to \mathcal{O}_{E_0}$) and Lemma 6.1(i) and (ii). □

By [Bor79, I.5] (cf. (5.17)) there are natural identifications

$$G^\vee = I_K^F(G_0^\vee),$$
$${}^L G = I_K^F(G_0^\vee) \rtimes \Gamma_F,$$

where we abbreviate $I_K^F := I_{\Gamma_K}^{\Gamma_F}$ for the induction functor. Using Lemma 6.2, we obtain the following lemma.

Lemma 6.3. *We have an identification*

$${}^L G_{E_0} = \left(\prod_j I_{E_{j,0}}^{E_0} I_{KE_{j,0}}^{E_{j,0}} (G_{0,KE_{j,0}}^\vee) \right) \rtimes \Gamma_{E_0}.$$

Let $X = \mathbb{A}^1_{\mathcal{O}_{K_0}} = \mathrm{Spec}(\mathcal{O}_{K_0}[u])$ and $D = \{Q = 0\}$, viewed as a relative effective Cartier divisor on X which is finite and flat over $\mathrm{Spec}(\mathcal{O}_F)$. The following lemma helps us to determine $\mathrm{Gr}_{(X/\mathcal{O}_F, \underline{\mathcal{G}}_0, D)} \otimes_{\mathcal{O}_F} \mathcal{O}_{E_0}$; it handles the special case where K/F is totally ramified.

Lemma 6.4. *Assume $K_0 = F$, let $K' = E_0 K$, which is the maximal unramified subextension of KE/K, and let $\mathcal{O}_{K'} = \mathcal{O}_K \otimes_{\mathcal{O}_F} \mathcal{O}_{E_0}$ be its ring of integers. Since $\mathcal{O}_{K_0} = \mathcal{O}_F$, note $\underline{\mathcal{G}}_0$ is defined over $\mathcal{O}_F[u]$. Then we have identifications:*

(i) $\underline{\mathcal{G}}_0 \otimes_{\mathcal{O}_F[u]} \mathcal{O}_{E_0}[u] = \underline{\mathcal{G}_{\mathcal{O}_{K'}}}_0 =: \underline{\mathcal{G}}_{0,\mathcal{O}_{E_0}}$;

(ii) $(L_D \underline{\mathcal{G}}_0) \otimes_{\mathcal{O}_F} \mathcal{O}_{E_0} = L_{D_{\mathcal{O}_{E_0}}} \underline{\mathcal{G}}_{0,\mathcal{O}_{E_0}}$ *(and similarly for L_D^+);*

(iii) $\mathrm{Gr}_{(X, \underline{\mathcal{G}}_0, D)} \otimes_{\mathcal{O}_F} \mathcal{O}_{E_0} = \mathrm{Gr}_{(X_{\mathcal{O}_{E_0}}, \underline{\mathcal{G}}_{0,\mathcal{O}_{E_0}}, D_{\mathcal{O}_{E_0}})}$.

Proof. Part (i) follows because the formation of $\underline{G}_0$ and $\underline{\mathcal{G}}_0$ as in [Lev13, Proposition 3.1.2 and Theorem 3.3.3] is compatible with change of base $\mathcal{O}_F[u^\pm] \to \mathcal{O}_{E_0}[u^\pm]$ (respectively, $\mathcal{O}_F[u] \to \mathcal{O}_{E_0}[u]$); see also Example 4.14. Part (ii) follows formally from part (i) and the identities $R[\![D_{\mathcal{O}_{E_0}}]\!] = \varprojlim_n R[u]/Q^n = R[\![D]\!]$ (respectively, $R(\!(D_{\mathcal{O}_{E_0}})\!) = (\varprojlim_n R[u]/Q^n)[1/Q] = R(\!(D)\!)$) for $\mathcal{O}_{E_0}$-algebras R. Part (iii) follows from part (ii) and Lemma 3.4(ii). □

Proposition 6.5. *In the notation above, there are canonical isomorphisms*

$$\begin{aligned} \mathrm{Gr}_{(X/\mathcal{O}_F, \underline{\mathcal{G}}_0, D)} \otimes_{\mathcal{O}_F} \mathcal{O}_{E_0} &= \prod_j \mathrm{Res}_{\mathcal{O}_{E_{j,0}}/\mathcal{O}_{E_0}} (\mathrm{Gr}_{(X/\mathcal{O}_{K_0}, \underline{\mathcal{G}}_0, D)} \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{E_{j,0}}) \\ &= \prod_j \mathrm{Res}_{\mathcal{O}_{E_{j,0}}/\mathcal{O}_{E_0}} (\mathrm{Gr}_{(X_{\mathcal{O}_{E_{j,0}}}/\mathcal{O}_{E_{j,0}}, \underline{\mathcal{G}}_{0,\mathcal{O}_{E_{j,0}}}, D \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{E_{j,0}})}). \end{aligned}$$

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Proof. The first equality is proved using Lemma 3.7. The second equality follows by applying Lemma 6.4(iii) to each factor indexed by j, replacing the data (F, K, E_0) with $(K_0, K, E_{j,0})$. □

Recall that $\underline{\mathcal{G}}_0$ is defined using the following data: the totally ramified extension K/K_0, the K_0-group G_0, the facet $\mathbf{f}_0$, and the choice of spreading $\underline{G}_0/\mathcal{O}_{K_0}[u^{\pm}]$; and the generic fiber of $\mathrm{Gr}_{(X/\mathcal{O}_F, \underline{\mathcal{G}}, D)}$ is the affine Grassmannian for $G = \mathrm{Res}_{K/F} G_0$, by Proposition 4.15. By contrast $\underline{\mathcal{G}}_{0,\mathcal{O}_{E_{j,0}}}$ is defined from the data: the totally ramified extension $KE_{j,0}/E_{j,0}$, the $E_{j,0}$-group $G_{0,E_{j,0}}$, the facet $\mathbf{f}_0$, and the spreading $\underline{G}_{0,\mathcal{O}_{E_{j,0}}[u^{\pm}]}$; and the generic fiber of $\mathrm{Gr}_{(X_{\mathcal{O}_{E_{j,0}}}/\mathcal{O}_{E_{j,0}},\, \underline{\mathcal{G}}_{0,\mathcal{O}_{E_{j,0}}},\, D\otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{E_{j,0}})}$ is the affine Grassmannian for $\mathrm{Res}_{KE_{j,0}/E_{j,0}} G_{0,KE_{j,0}}$. So when restricting attention to the part inside the Weil restriction in the jth factor, we are in the situation '$F = E_0$' and '$K_0 = F$'. Our next goal is to show how we may effectively reduce our problem to the study of each $\mathrm{Gr}_{(X_{\mathcal{O}_{E_{j,0}}}/\mathcal{O}_{E_{j,0}},\, \underline{\mathcal{G}}_{0,\mathcal{O}_{E_{j,0}}},\, D\otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{E_{j,0}})}$ separately.

6.2 Statement of theorem

Given an irreducible algebraic representation V of ${}^L G$, we define the parity $d_V \in \mathbb{Z}/2\mathbb{Z}$ as in [HR18, (7.11)]. Then we define the function $\tau^{\mathrm{ss}}_{\mathcal{G},V}$ on $\mathrm{Gr}_{\mathcal{G}}(k_F)$ by the identity

$$\tau^{\mathrm{ss}}_{\mathcal{G},V} = (-1)^{d_V}\, \mathrm{tr}^{\mathrm{ss}}(\Phi \mid \Psi_{\mathrm{Gr}_{\mathcal{G}}}(\mathrm{Sat}(V))). \tag{6.1}$$

We extend this definition to general representations V of ${}^L G$ (not necessarily irreducible) by linearity. By Theorem 5.15, Lemma 4.12, and Corollary 4.9, we may view $\tau^{\mathrm{ss}}_{\mathcal{G},V}$ as an element in the Hecke algebra $\mathcal{Z}(G(F), \mathcal{G}(\mathcal{O}_F))$. Given any algebraic representation V, we also define $z^{\mathrm{ss}}_{\mathcal{G},V} \in \mathcal{Z}(G(F), \mathcal{G}(\mathcal{O}_F))$ to be the unique function such that, if π is an irreducible smooth representation of $G(F)$ on a $\bar{\mathbb{Q}}_\ell$-vector space such that $\pi^{\mathcal{G}(\mathcal{O}_F)} \neq 0$, then $z^{\mathrm{ss}}_{\mathcal{G},V}$ acts on $\pi^{\mathcal{G}(\mathcal{O}_F)}$ by the scalar $\mathrm{tr}(s(\pi) \mid V^{1\rtimes I_F})$, where $s(\pi)$ is the Satake parameter of π as defined in [Hai15].

Theorem 6.6. *For $(G, \mathcal{G}, V)$ as above, we have the equality $\tau^{\mathrm{ss}}_{\mathcal{G},V} = z^{\mathrm{ss}}_{\mathcal{G},V}$.*

Recall that taking inertia invariants does not commute in general with forming tensor products of representations. Because of the products and unramified Weil restrictions appearing in Lemma 6.3 and Proposition 6.5, it is problematic to reduce our main theorem to Theorem 6.6. Instead we need a variant of Theorem 6.6 *without semisimplifying the trace, for a fixed lift Φ of geometric Frobenius*. This is formulated as follows. For each fixed lift Φ, we define a function $z^{\Phi}_{\mathcal{G},V} \in \mathcal{Z}(G(F), \mathcal{G}(\mathcal{O}_F))$; similarly, we define a function $\tau^{\Phi}_{\mathcal{G},V}$ on $\mathrm{Gr}_{\mathcal{G}}(k_F)$ (see Appendix A). By the same arguments due to Gaitsgory, Pappas and Zhu, and Levin cited above, this function can be viewed as an element of $\mathcal{Z}(G(F), \mathcal{G}(\mathcal{O}_F))$.

Theorem 6.7. *For $(G, \mathcal{G}, V)$ as above and for every fixed choice of lift Φ of geometric Frobenius, we have the equality $\tau^{\Phi}_{\mathcal{G},V} = z^{\Phi}_{\mathcal{G},V}$.*

In fact we will prove Theorem 6.7, and we deduce Theorem 6.6 immediately by Lemma A.1. However, we will not require Theorem 6.6, but only Theorem 6.7, to prove our main theorem.

6.3 Reducing the main theorem to Theorem 6.7

Following the method of [HR18, 7.3], we show that the main theorem is a consequence of Theorem 6.7 as follows. Recall that $V_{\{\mu\}}$ is a representation of ${}^L G_E = G^\vee \rtimes \Gamma_E$, the L-group of $\mathrm{Res}_{K/F}(G_0) \otimes_F E$, and that $I(V_{\{\mu\}})$ is a representation of ${}^L G_{E_0} = G^\vee \rtimes \Gamma_{E_0}$, the L-group of $\mathrm{Res}_{K/F}(G_0) \otimes_F E_0$. Arguing as in [HR18, § 7.3], up to the sign $(-1)^{d_\mu}$ the function $\tau^{\mathrm{ss}}_{\{\mu\}}$ is identified with the function in $\mathcal{Z}(G(E_0), \mathcal{G}(\mathcal{O}_{E_0}))$

$$\mathrm{tr}^{\mathrm{ss}}(\Phi_E \mid \Psi_{\mathrm{Gr}_{\mathcal{G},\mathcal{O}_E}}(\mathrm{IC}_{\{\mu\}})) = \mathrm{tr}^{\mathrm{ss}}(\Phi_{E_0} \mid \Psi_{\mathrm{Gr}_{\mathcal{G},\mathcal{O}_{E_0}}}(\mathrm{Sat}(I(V_{\{\mu\}})))). \tag{6.2}$$

Also, $z^{\mathrm{ss}}_{\{\mu\}}$ acts on $\pi^{\mathcal{G}(\mathcal{O}_{E_0})} \neq 0$ by

$$\mathrm{tr}(s(\pi) \mid V_{\{\mu\}}^{I_E}) = \mathrm{tr}(s(\pi) \mid I(V_{\{\mu\}})^{I_{E_0}}). \tag{6.3}$$

Therefore, to prove the main theorem, it suffices to prove $\tau^{\mathrm{ss}}_{\mathcal{G}_{\mathcal{O}_{E_0}}, I(V_{\{\mu\}})} = z^{\mathrm{ss}}_{\mathcal{G}_{\mathcal{O}_{E_0}}, I(V_{\{\mu\}})}$. All irreducible constituents of $I(V_{\{\mu\}})$ have the same parity, so we may replace $I(V_{\{\mu\}})$ with an arbitrary irreducible representation V of ${}^L G_{E_0}$. By Lemma A.1, it suffices to prove

$$\tau^{\Phi_{E_0}}_{\mathcal{G}_{\mathcal{O}_{E_0}}, V} = z^{\Phi_{E_0}}_{\mathcal{G}_{\mathcal{O}_{E_0}}, V} \tag{6.4}$$

for every fixed lift Φ_{E_0} of geometric Frobenius.

Now ${}^L G_{E_0} = (\prod_j I^{E_0}_{E_{j,0}} (\mathrm{Res}_{KE_{j,0}/E_{j,0}} G_{0,KE_{j,0}})^\vee) \rtimes \Gamma_{E_0}$ by Lemma 6.3. Because $E_{j,0}/E_0$ is unramified (cf. Lemma 6.1(i)), any irreducible representation V is built up, as explained in the Appendix A, from irreducible representations of ${}^L(\mathrm{Res}_{KE_{j,0}/E_{j,0}} G_{0,KE_{j,0}})$. There is a parallel description of the corresponding perverse sheaves on the generic fiber of $\mathrm{Gr}_{\mathcal{G},\mathcal{O}_{E_0}}$, thanks to Proposition 6.5. Using Lemmas A.2 and A.3, we easily see that (6.4) will be proved if we can prove Theorem 6.7 for any irreducible representation of ${}^L(\mathrm{Res}_{KE_{j,0}/E_{j,0}} G_{0,KE_{j,0}})$ and corresponding nearby cycles along $\mathrm{Gr}_{(X_{\mathcal{O}_{E_{j,0}}}/\mathcal{O}_{E_{j,0}}, \underline{\mathcal{G}}_{0,\mathcal{O}_{E_{j,0}}}, D \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_{E_{j,0}})}$.

Therefore, we may assume $F = E_0$ henceforth, and we have seen that in order to prove the main theorem it is enough to prove Theorem 6.7, and in fact we may even assume $K_0 = F$ (i.e. that K/F is totally ramified), and that V is an irreducible representation of ${}^L G$. By the argument of [HR18, Lemma 7.7], we may also assume that $V|_{G^\vee \rtimes I_F}$ is irreducible, whenever convenient.

6.4 Proof of Theorem 6.7

As above, we will assume $V|_{G^\vee \rtimes I_F}$ is irreducible. Following the proof of [HR18, Theorem 7.6], there are three main steps:

(1) reduction to minimal F-Levi subgroups of G;

(2) reduction from anisotropic mod center groups to quasi-split groups;

(3) proof for quasi-split groups.

The proofs work exactly the same way as in [HR18], with only a few additional remarks. For Step 1, we use Lemma 4.2 to ensure that a minimal F-Levi subgroup of G is of the form $M = \mathrm{Res}_{K/F}(M_0)$, for M_0 a minimal K-Levi subgroup of G; in light of Theorem 5.15 and Theorem 5.8, the argument of [HR18] goes through to reduce us to proving Theorem 6.7 for M, that is, for $\mathrm{Gr}_{(X, \underline{\mathcal{M}}_0, D)}$.

For Step 2, we assume G is F-anisotropic mod center and we observe that if G_0^* is a K-quasi-split inner form of G_0, then $G^* = \mathrm{Res}_{K/F}(G_0^*)$ is an F-quasi-split inner form of G. More to the point, $\mathrm{Gr}_{(X, \underline{\mathcal{G}}_0, D)}$ and $\mathrm{Gr}_{(X, \underline{\mathcal{G}}_0^*, D)}$ become isomorphic over $\breve{\mathcal{O}}_F$ and hence we may think of them as the same geometrically, with differing Galois actions Φ and Φ^* of the geometric Frobenius element; applying the argument of [HR18], we reduce to proving Theorem 6.7 for G^*, that is, for $\mathrm{Gr}_{(X, \underline{\mathcal{G}}_0^*, D)}$.

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For Step 3, we apply Step 1 to G^*, and we are reduced to proving the theorem for a torus of the form $T^* = \mathrm{Res}_{K/F}(T_0^*)$, that is, for $\mathrm{Gr}_{(X,\mathcal{T}_0^*,D)}$. The theorem for any torus $T = \mathrm{Res}_{K/F}(T_0)$ is easy. Let us explain, following the method of [HR18, § 7.6]. Let V be a representation of $T^\vee \rtimes \Gamma_F$ such that $V|_{T^\vee \rtimes I_F}$ is irreducible. As in [HR18, Definition 7.11], let $\omega_V \in \pi_1(T)_{I_F}^{\Phi}$ be the common image of the $T^\vee$-weights in $V|_{T^\vee}$. Then ω_V can be viewed as the unique k-rational point in the support of $\Psi(\mathrm{Sat}(V|_{T^\vee \rtimes I_F}))$, and also as the element indexing the unique coset in the support of $z^{\Phi}_{\mathcal{T},V}$. Further, by Proposition 5.10, we have an identification of ${}^L T_r = (T^\vee)^{I_F} \rtimes \Gamma_F$-modules

$$\mathbb{H}^*(\Psi_{\mathrm{Gr}_{\mathcal{T}}}(\mathrm{Sat}(V))) = \mathbb{H}^*(\mathrm{Gr}_{T,\bar{F}}, \mathrm{Sat}(V))|_{{}^L T_r}.$$

By the Grothendieck–Lefschetz fixed point theorem, it suffices to prove

$$z^{\Phi}_{\mathcal{T},V}(\omega_V) = \mathrm{tr}(\Phi \mid V) = \mathrm{tr}(\Phi \mid \mathbb{H}^*(\mathrm{Gr}_{T,\bar{F}}, \mathrm{Sat}(V))).$$

The second equality comes from the identifications $\mathbb{H}^*(\mathrm{Gr}_{T,\bar{F}}, \mathrm{Sat}(V)) = \mathbb{H}^0(\mathrm{Gr}_{T,\bar{F}}, \mathrm{Sat}(V)) = V$ as ${}^L T$-representations under the Satake correspondence. Therefore we need to prove the first equality. Note that all the weights in V are I_F-conjugate, and $z^{\Phi}_{\mathcal{T},V}$ acts on a weakly unramified character $\chi : T(F)/T(F)_1 \to \bar{\mathbb{Q}}_\ell^\times$ by the scalar

$$\mathrm{tr}(s^{\Phi}(\chi) \mid V) = \mathrm{tr}(\chi \rtimes \Phi \mid V) = \chi(\omega_V)\, \mathrm{tr}(\Phi \mid V),$$

the second equality holding since $s^{\Phi}(\chi) \in (T^\vee)^{I_F} \rtimes \Phi$. Thus $z^{\Phi}_{\mathcal{T},V} = \mathrm{tr}(\Phi \mid V)\, \mathbb{1}_{\omega_V}$, as desired. This completes the proof of Step 3 and therefore of Theorem 6.7. □

6.5 Values of test functions

As in the main theorem of [HR18], the function $q_{E_0}^{d_\mu/2} z^{\mathrm{ss}}_{\mathcal{G},\{\mu\}}$ takes values in $\mathbb{Z}$ and is independent of $\ell \neq p$. The proof given in [HR18, § 7.7] uses only general facts about the Bernstein functions and related combinatorics, and applies equally well to all groups, including those which are Weil-restricted groups such as G.

7. Test functions for modified local models

The aim of this final section is to formulate and prove the test function conjecture for all reductive groups and all primes $p \geqslant 5$ using the modified local models introduced in [HPR18, § 2.6]. This is a consequence of our main theorem and some geometric results in [HR19]; cf. Corollary 7.4 below.

7.1 Modified local models

We denote by G a reductive group over a non-archimedean local field F of mixed characteristic $(0, p)$. We fix an isomorphism

$$G_{\mathrm{ad}} \;\simeq\; \prod_{j \in J} \mathrm{Res}_{K_j/F}(G_j), \tag{7.1}$$

where each K_j/F is a finite field extension, and each G_j is an absolutely simple, reductive K_j-group. We assume that each G_j is tamely ramified. This is only a restriction for $p = 2, 3$: whenever $p \geqslant 5$ this assumption is automatically satisfied by the classification (cf. [Tit77, § 1.12; § 4]; see also [PR08, § 7.a]).

We further fix a facet $\mathbf{f} \subset \mathscr{B}(G, F)$ which corresponds to facets $\mathbf{f}_j \subset \mathscr{B}(G_j, K_j)$, $j \in J$, under the identifications

$$\mathscr{B}(G, F) = \mathscr{B}(G_{\mathrm{ad}}, F) = \prod_{j \in J} \mathscr{B}(G_j, K_j)$$

deduced from Proposition 4.6 applied to each pair $(G_j, K_j/F)$. We denote by $\mathcal{G} = \mathcal{G}_{\mathbf{f}}$ over $\mathcal{O}_F$ and by $\mathcal{G}_j = \mathcal{G}_{\mathbf{f}_j}$ over $\mathcal{O}_{K_j}$ the associated parahoric group schemes.

We fix a uniformizer $\varpi_j \in K_j$ and an $\mathcal{O}_{K_{j,0}}[u^{\pm}]$-extension $\underline{G}_{j,0}$ of G_j where $K_{j,0}/F$ denotes the maximal unramified subextension in K_j/F. Each geometric conjugacy class of cocharacters $\{\mu\}$ in G induces a geometric conjugacy class $\{\mu_{\mathrm{ad}}\}$ in G_{ad} and hence for each $j \in J$ a geometric conjugacy class $\{\mu_j\}$ in $\mathrm{Res}_{K_j/F}(G_j)$. We note that the reflex field E of $\{\mu\}$ is naturally an overfield of the reflex field E_j of $\{\mu_j\}$.

DEFINITION 7.1. The modified local model $\mathbb{M}_{\mathcal{G}}(G, \{\mu\}) = \mathbb{M}(K_j/F, \underline{G}_{j,0}, \mathbf{f}_j, \{\mu_j\}, \varpi_j\,;\, j \in J)$ is the $\mathcal{O}_E$-product of the $\mathcal{O}_E$-schemes

$$\prod_{j \in J} M^{\mathrm{norm}}_{\{\mu_j\}} \otimes_{\mathcal{O}_{E_j}} \mathcal{O}_E,$$

where $M_{\{\mu_j\}} = M(K_j/F, \underline{G}_{j,0}, \mathbf{f}_j, \{\mu_j\}, \varpi_j)$ is the local model over $\mathcal{O}_{E_j}$ as in § 4.4.4, and $M^{\mathrm{norm}}_{\{\mu_j\}} \to M_{\{\mu_j\}}$ denotes its normalization.

For convenience we summarize the results on the modified local models obtained in [PZ13, Lev16, HPR18, HR19].

THEOREM 7.2. (i) *If G splits over a tamely ramified extension of F, then $\mathbb{M}_{\mathcal{G}}(G, \{\mu\})$ is isomorphic to the modified local model defined in [HPR18, § 2.6].*

(ii) *A morphism of local model triples $(G', \{\mu'\}, \mathcal{G}') \to (G, \{\mu\}, \mathcal{G})$ with $G'_{\mathrm{ad}} \simeq G_{\mathrm{ad}}$ satisfying the tameness assumption in (7.1) induces an isomorphism of $\mathcal{O}_{E'}$-schemes*

$$\mathbb{M}_{\mathcal{G}'}(G', \{\mu'\}) \xrightarrow{\simeq} \mathbb{M}_{\mathcal{G}}(G, \{\mu\}) \otimes_{\mathcal{O}_E} \mathcal{O}_{E'}.$$

(iii) *The modified local model $\mathbb{M}_{\mathcal{G}}(G, \{\mu\})$ is normal with geometrically reduced special fiber, and if $p > 2$ it is Cohen–Macaulay as well.*

Proof. As in [HPR18, § 2.6] we choose for each $j \in J$ a suitable z-extension $\tilde{G}_j \to G_j$ whose derived group $\tilde{G}_{j,\mathrm{der}}$ is simply connected. Then the geometric conjugacy class $\{\mu_j\}$ in $\mathrm{Res}_{K_j/F}(G_j)$ can be lifted to $\{\tilde{\mu}_j\}$ in $\mathrm{Res}_{K_j/F}(\tilde{G}_j)$ with the same reflex field $\tilde{E}_j = E_j$; cf. [HPR18, § 2.6]. Denote by $\tilde{\mathbf{f}}_j$ the facet of $\tilde{G}_j$ corresponding to $\mathbf{f}_j$. This induces a morphism of $\mathcal{O}_{E_j}$-schemes on Weil-restricted local models

$$M_{\{\tilde{\mu}_j\}} := M(K_j/F, \underline{\tilde{G}}_{j,0}, \tilde{\mathbf{f}}_j, \{\tilde{\mu}_j\}, \varpi_j) \to M(K_j/F, \underline{G}_{j,0}, \mathbf{f}_j, \{\mu_j\}, \varpi_j) =: M_{\{\mu_j\}}, \tag{7.2}$$

where $\underline{\tilde{G}}_{j,0} \to \underline{G}_{j,0}$ is an $\mathcal{O}_{K_{j,0}}[u^{\pm}]$-extension of $\tilde{G}_j \to G_j$. Now for tamely ramified extensions K_j/F the Weil-restricted local models agree by [Lev16, Proposition 4.2.4] with the local models of [PZ13]. Further, the morphism (7.2) is a finite, birational, universal homeomorphism by [HR19, Corollary 2.3]. Since $M_{\{\tilde{\mu}_j\}}$ is normal by [PZ13, Theorem 9.1] (see also [Lev16, Theorem 4.2.7]), the map induces an isomorphism $M_{\{\tilde{\mu}_j\}} \simeq M^{\mathrm{norm}}_{\{\tilde{\mu}_j\}}$ on normalizations, and the former are the modified local models of [HPR18, § 2.6]. Extending scalars to $\mathcal{O}_E$ and taking the product over

$j \in J$ implies part (i). Part (iii) follows from [HR19, Corollary 2.5]; see also the references cited there. For part (ii) we remark that the morphism is a finite, birational, universal homeomorphism by the same reasoning as in (i), and that the target is normal: its generic fiber is normal by definition and its special fiber is reduced by (iii). As the local model is flat and of finite type, the target is normal by Serre's criterion; cf. [PZ13, Proposition 9.2]. □

Remark 7.3. As in Remark 4.19 one can show that $\mathbb{M}_{\mathcal{G}}(G, \{\mu\})$ depends up to equivariant isomorphism only on the data $(G, \{\mu\}, \mathcal{G})$, which justifies the notation.

We also record the following property which is important for the proof of the test function conjecture.

Corollary 7.4. *The product of the normalization morphisms*

$$\mathbb{M}_{\mathcal{G}}(G, \{\mu\}) = \prod_{j \in J} M^{\mathrm{norm}}_{\{\mu_j\}} \otimes_{\mathcal{O}_{E_j}} \mathcal{O}_E \to \prod_{j \in J} M_{\{\mu_j\}} \otimes_{\mathcal{O}_{E_j}} \mathcal{O}_E$$

is finite, birational and a universal homeomorphism. In particular, this morphism induces an equivalence on the associated étale topoi of source and target [StaPro, 03SI].

Proof. This is immediate from the isomorphism $M_{\{\tilde{\mu}_j\}} \simeq M^{\mathrm{norm}}_{\{\tilde{\mu}_j\}}$, $j \in J$, in the proof of Theorem 7.2(i), together with [HR19, Corollary 2.3]. □

7.2 Test functions

Let $(G, \{\mu\}, \mathcal{G})$ be a triple as above, where G/F satisfies the tameness assumption in (7.1). Denote by $\mathbb{M}_{\{\mu\}} = \mathbb{M}_{\mathcal{G}}(G, \{\mu\})$ the modified local model as in Definition 7.1. For a finite extension E/F over which $\{\mu\}$ is defined, we consider the associated semisimple trace of Frobenius function on the sheaf of nearby cycles

$$\tau^{\mathrm{ss}}_{\{\mu\}} \colon \mathbb{M}_{\{\mu\}}(k_E) \to \bar{\mathbb{Q}}_\ell, \quad x \mapsto (-1)^{d_\mu} \operatorname{tr}^{\mathrm{ss}}(\Phi_E \mid \Psi_{\mathbb{M}_{\{\mu\}}}(\mathrm{IC}_{\{\mu\}})_{\bar{x}}),$$

where $\mathrm{IC}_{\{\mu\}}$ denotes the normalized intersection complex of the generic fiber of $\mathbb{M}_{\{\mu\}}$. As an application of our main theorem we deduce the test function conjecture for modified local models.

Theorem 7.5. *Let $(G, \{\mu\}, \mathcal{G})$ be as above, and denote by E/F an extension which contains the reflex field of $\{\mu\}$. Let E_0/F be the maximal unramified subextension. Then $\tau^{\mathrm{ss}}_{\{\mu\}}$ naturally defines an element in $\mathcal{Z}(G(E_0), \mathcal{G}(\mathcal{O}_{E_0}))$, and one has*

$$\tau^{\mathrm{ss}}_{\{\mu\}} = z^{\mathrm{ss}}_{\{\mu\}},$$

where $z^{\mathrm{ss}}_{\{\mu\}} \in \mathcal{Z}(G(E_0), \mathcal{G}(\mathcal{O}_{E_0}))$ is the unique function which acts on any $\mathcal{G}(\mathcal{O}_{E_0})$-spherical smooth irreducible $\bar{\mathbb{Q}}_\ell$-representation π by the scalar

$$\operatorname{tr}(s(\pi) \mid \operatorname{Ind}^{{}^LG_{E_0}}_{{}^LG_E}(V_{\{\mu\}})^{1 \rtimes I_{E_0}}),$$

where $s(\pi) \in [(G^\vee)^{I_{E_0}} \rtimes \Phi_{E_0}]_{\mathrm{ss}}/(G^\vee)^{I_{E_0}}$ is the Satake parameter for π [Hai15]. The function $q_{E_0}^{d_\mu/2} \tau^{\mathrm{ss}}_{\{\mu\}}$ takes values in $\mathbb{Z}$ and is independent of $\ell \neq p$ and $q^{1/2} \in \bar{\mathbb{Q}}_\ell$.

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Proof. First, we will show that $\tau^{\mathrm{ss}}_{\{\mu\}}$ naturally defines an element of $\mathcal{Z}(G(E_0), \mathcal{G}(\mathcal{O}_{E_0}))$ for which we need to prove the analogue of (6.2). As in (7.1) we denote by $(H_j, \{\mu_j\}, \mathcal{H}_j)$, $j \in J$, the local model triple where $H_j := \mathrm{Res}_{K_j/F}(G_j)$ and $\mathcal{H}_j := \mathrm{Res}_{\mathcal{O}_{K_j}/\mathcal{O}_F}(\mathcal{G}_j)$. Note that $G_{\mathrm{ad}} = \prod_j H_j$ and $\mathcal{G}_{\mathrm{ad}} = \prod_j \mathcal{H}_j$. For the next statement, observe that $\mathrm{IC}_{\{\mu\}} = \mathrm{Sat}(V_{\{\mu_{\mathrm{ad}}\}})$ under the equivalence of étale topoi of Corollary 7.4 for the generic fibers $\mathbb{M}_{\{\mu\},E}$ and $\prod_j M_{\{\mu_j\}} \otimes_{\mathcal{O}_{E_j}} E \subset \mathrm{Gr}_{\mathcal{G}_{\mathrm{ad}},\mathcal{O}_{E_0}} \otimes_{\mathcal{O}_{E_0}} E = \mathrm{Gr}_{G_{\mathrm{ad}},E}$.

LEMMA 7.6. *We have*

$$\mathrm{tr}^{\mathrm{ss}}(\Phi_E \mid \Psi_{\mathbb{M}_{\{\mu\}}}(\mathrm{IC}_{\{\mu\}})) = \mathrm{tr}^{\mathrm{ss}}(\Phi_{E_0} \mid \Psi_{\mathrm{ad}}(\mathrm{Sat}(I(V_{\{\mu_{\mathrm{ad}}\}})))),$$

where Ψ_{ad} *denotes the nearby cycles functor for* $\mathrm{Gr}_{\mathcal{G}_{\mathrm{ad}},\mathcal{O}_{E_0}} = \prod_j \mathrm{Gr}_{\mathcal{H}_j,\mathcal{O}_{E_0}}$.

Proof. The argument follows the proof of (6.2) as in [HR18, § 7.3]. Consider the finite morphism

$$f\colon \mathrm{Gr}_{\mathcal{G}_{\mathrm{ad}},\mathcal{O}_{E_0}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E \longrightarrow \mathrm{Gr}_{\mathcal{G}_{\mathrm{ad}},\mathcal{O}_{E_0}}.$$

Abbreviate the nearby cycles for $\mathrm{Gr}_{\mathcal{G}_{\mathrm{ad}},\mathcal{O}_{E_0}} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E$ by $\Psi_{\mathrm{ad},E}$. We have

$$\begin{aligned}\mathrm{tr}^{\mathrm{ss}}(\Phi_E \mid \Psi_{\mathrm{ad},E}(\mathrm{Sat}(V_{\{\mu_{\mathrm{ad}}\}}))) &= \mathrm{tr}^{\mathrm{ss}}(\Phi_{E_0} \mid f_{\bar{s},*}\Psi_{\mathrm{ad},E}(\mathrm{Sat}(V_{\{\mu_{\mathrm{ad}}\}}))) \\ &= \mathrm{tr}^{\mathrm{ss}}(\Phi_{E_0} \mid \Psi_{\mathrm{ad}}(f_{\eta,*}\mathrm{Sat}(V_{\{\mu_{\mathrm{ad}}\}}))) \\ &= \mathrm{tr}^{\mathrm{ss}}(\Phi_{E_0} \mid \Psi_{\mathrm{ad}}(\mathrm{Sat}(I(V_{\{\mu_{\mathrm{ad}}\}})))).\end{aligned}$$

We used the analogue of [Hai18, Lemma 8.1] for the first equality, proper base change for the second equality and [HR18, Proposition 3.14] for the final equality. By the topological invariance of the étale site in Corollary 7.4 we have

$$\mathrm{tr}^{\mathrm{ss}}(\Phi_E \mid \Psi_{\mathbb{M}_{\{\mu\}}}(\mathrm{IC}_{\{\mu\}})) = \mathrm{tr}^{\mathrm{ss}}(\Phi_E \mid \Psi_{\mathrm{ad},E}(\mathrm{Sat}(V_{\{\mu_{\mathrm{ad}}\}}))),$$

which proves the lemma. □

By Lemma 7.6 (combined with Theorem 5.15), the test function

$$\tau^{\mathrm{ss}}_{\{\mu\}} = (-1)^{d_\mu}\mathrm{tr}^{\mathrm{ss}}(\Phi_{E_0} \mid \Psi_{\mathrm{ad}}(\mathrm{Sat}(I(V_{\{\mu_{\mathrm{ad}}\}})))) \tag{7.3}$$

naturally defines an element of $\mathcal{Z}(G_{\mathrm{ad}}(E_0), \mathcal{G}_{\mathrm{ad}}(\mathcal{O}_{E_0}))$. By § 7.3 below, the natural projection $p\colon G \to G_{\mathrm{ad}}$ induces a canonical morphism of algebras

$$p_*\colon \mathcal{Z}(G(E_0), \mathcal{G}(\mathcal{O}_{E_0})) \to \mathcal{Z}(G_{\mathrm{ad}}(E_0), \mathcal{G}_{\mathrm{ad}}(\mathcal{O}_{E_0})).$$

There is a disjoint union

$$G(E_0) = \coprod_{\omega \in \pi_1(G)_I} G(E_0)_\omega, \tag{7.4}$$

where $G(E_0)_\omega = \kappa^{-1}_{G,E_0}(\omega)$ is the fiber of the Kottwitz morphism $\kappa_{G,E_0}\colon G(E_0) \to \pi_1(G)_I$ with $I = I_{E_0} = I_F$ (use E_0/F unramified), and likewise for G replaced by G_{ad}. The key observation is proved in Lemma 7.7 below: if $\omega \mapsto \omega_{\mathrm{ad}}$ under the map $\pi_1(G)_I \to \pi_1(G_{\mathrm{ad}})_I$, then p_* restricts to an isomorphism

$$\mathcal{Z}(G(E_0), \mathcal{G}(\mathcal{O}_{E_0}))_\omega \xrightarrow{\simeq} \mathcal{Z}(G_{\mathrm{ad}}(E_0), \mathcal{G}_{\mathrm{ad}}(\mathcal{O}_{E_0}))_{\omega_{\mathrm{ad}}} \tag{7.5}$$

on the functions whose support is contained in $G(E_0)_\omega$ (respectively, in $G_{\mathrm{ad}}(E_0)_{\omega_{\mathrm{ad}}}$). We apply this to $\omega = \omega_{\{\mu\}}$, the image of $\{\mu\}$ inside $\pi_1(G)_I$. To explain, note that the representation

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$I(V_{\{\mu\}})$ of ${}^L G_{E_0}$ need not be irreducible, but all its $B^\vee$-highest $T^\vee$ are I_{E_0}-conjugate to μ. Similar remarks apply to the restriction $I(V_{\{\mu_{\rm ad}\}})$ of $I(V_{\{\mu\}})$ to ${}^L(G_{\rm ad})_{E_0}$; note as well that $\omega_{\{\mu\}} \mapsto \omega_{\{\mu_{\rm ad}\}}$. Geometrically, this means that the support of (7.3) is contained in the connected component of $\mathrm{Gr}_{\mathcal{G}_{\rm ad}} \otimes_{\mathcal{O}_F} k_{E_0}$ indexed by $\omega_{\{\mu_{\rm ad}\}}$, and hence it belongs to $\mathcal{Z}(G_{\rm ad}(E_0), \mathcal{G}_{\rm ad}(\mathcal{O}_{E_0}))_{\omega_{\{\mu_{\rm ad}\}}}$. Finally, via (7.5) we see that $\tau^{\rm ss}_{\{\mu\}}$ identifies with an element of $\mathcal{Z}(G(E_0), \mathcal{G}(\mathcal{O}_{E_0}))_{\omega_{\{\mu\}}} \subset \mathcal{Z}(G(E_0), \mathcal{G}(\mathcal{O}_{E_0}))$.

Next we prove the equality of $\tau^{\rm ss}_{\{\mu\}} = z^{\rm ss}_{\{\mu\}}$ as elements of $\mathcal{Z}(G(E_0), \mathcal{G}(\mathcal{O}_{E_0}))$. Their values then satisfy the required integrality properties independently of the choice of $\ell \neq p$ and $q^{1/2} \in \bar{\mathbb{Q}}_\ell$ by § 6.5. It is enough to show the equality $\tau^{\rm ss}_{\{\mu_{\rm ad}\}} = z^{\rm ss}_{\{\mu_{\rm ad}\}}$, that is, the equality of the two functions when they are viewed as elements of $\mathcal{Z}(G_{\rm ad}(E_0), \mathcal{G}_{\rm ad}(\mathcal{O}_{E_0}))$ via (7.5). Here $\tau^{\rm ss}_{\{\mu_{\rm ad}\}}$ is just a relabeling of (7.3) and we are using Lemma 7.7 below with $V = I(V_{\{\mu\}})$ and $V_{\rm ad} = I(V_{\{\mu_{\rm ad}\}})$ to justify that $z^{\rm ss}_{\{\mu\}} \mapsto z^{\rm ss}_{\{\mu_{\rm ad}\}}$ under (7.5).

Fix a lift of geometric Frobenius $\Phi = \Phi_{E_0}$. We will show that $\tau^\Phi_{\{\mu_{\rm ad}\}} = z^\Phi_{\{\mu_{\rm ad}\}}$; cf. Appendix A. The result will then follow from Lemma A.1 by averaging over the different lifts Φ.

For each $j \in J$ as in (7.1), we denote by $\tau^\Phi_{\{\mu_j\}}$ (respectively, $z^\Phi_{\{\mu_j\}}$) the central functions associated with the local model triple $(H_j, \{\mu_j\}, \mathcal{H}_j)$ over the extension E/F. Parallel to (7.3) we have

$$\tau^\Phi_{\{\mu_j\}} = (-1)^{d_{\mu_j}} \mathrm{tr}(\Phi \mid \Psi_j(\mathrm{Sat}(I(V_{\{\mu_j\}})))),$$

where Ψ_j stands for the nearby cycles functor for $\mathrm{Gr}_{\mathcal{H}_j} \otimes_{\mathcal{O}_F} \mathcal{O}_E$. The tameness assumption in (7.1) guarantees that Theorem 6.7, and in particular (6.4), is applicable, and we deduce the equality

$$\tau^\Phi_{\{\mu_j\}} = z^\Phi_{\{\mu_j\}}$$

as functions in $\mathcal{Z}(H_j(E_0), \mathcal{H}_j(\mathcal{O}_{E_0}))$ for all $j \in J$. Recall that $\mathrm{IC}_{\{\mu\}} = \mathrm{Sat}(V_{\{\mu_{\rm ad}\}})$ can be expressed as the external tensor product $\boxtimes_j \mathrm{Sat}(V_{\{\mu_j\}}) = \boxtimes_j \mathrm{IC}_{\{\mu_j\}}$ on the generic fiber $\mathrm{Gr}_{\mathcal{G}_{\rm ad}, \mathcal{O}_{E_0}} \otimes_{\mathcal{O}_{E_0}} E = \prod_j \mathrm{Gr}_{H_j, E}$. Since the formation of the non-semisimplified functions commutes with direct products of groups by Lemma A.2, we get the equality

$$\tau^\Phi_{\{\mu_{\rm ad}\}} = \prod_j \tau^\Phi_{\{\mu_j\}} = \prod_{j \in J} z^\Phi_{\{\mu_j\}} = z^\Phi_{\{\mu_{\rm ad}\}}$$

inside $\mathcal{Z}(G_{\rm ad}(E_0), \mathcal{G}_{\rm ad}(\mathcal{O}_{E_0})) = \prod_j \mathcal{Z}(H_j(E_0), \mathcal{H}_j(\mathcal{O}_{E_0}))$. We have also used the equality $\sum_j d_{\mu_j} \equiv d_\mu \bmod 2$. This completes the proof of the theorem. □

7.3 Passing to adjoint groups

In this section we do not need any tameness assumptions – the arguments hold for general groups. We work over the unramified extension E_0/F. Let Z be the center of G. Let A, S, T, M be as usual for the E_0-group G, and denote by $A_{\rm ad}, S_{\rm ad}, T_{\rm ad}, M_{\rm ad}$ their images under the canonical map $p: G \to G_{\rm ad} = G/Z$. Recall that $M_{\rm ad}$ is not the adjoint group of M, but is a minimal E_0-Levi subgroup of $G_{\rm ad}$. Let $\mathcal{G}$ (respectively, $\mathcal{G}_{\mathbf{a}}$) be the parahoric group $\mathcal{O}_{E_0}$-scheme with generic fiber G_{E_0} attached to a facet $\mathbf{f}$ (respectively, an alcove $\mathbf{a}$ with $\mathbf{f} \subset \bar{\mathbf{a}}$). Let $\mathcal{G}_{\rm ad}$ (respectively, $\mathcal{G}_{\rm ad, \mathbf{a}}$) be their analogues for $G_{\rm ad}$. The morphism $p\colon G(E_0) \to G_{\rm ad}(E_0)$ sends $\mathcal{G}(\mathcal{O}_{E_0})$ into $\mathcal{G}_{\rm ad}(\mathcal{O}_{E_0})$, and so by the étoffé property of $\mathcal{G}$, p extends to an $\mathcal{O}_{E_0}$-morphism $p: \mathcal{G} \to \mathcal{G}_{\rm ad}$; cf. [BT84, 1.7]. Similarly, we have $p: \mathcal{G}_{\mathbf{a}} \to \mathcal{G}_{\rm ad, \mathbf{a}}$.

Let $\mathcal{O}$ stand for either $\mathcal{O}_{E_0}$ or $\mathcal{O}_{\breve{E}_0}$, with K its fraction field. In general, the natural maps $p\colon \mathcal{G}(\mathcal{O}) \to \mathcal{G}_{\rm ad}(\mathcal{O})$ and $G(K)/\mathcal{G}(\mathcal{O}) \to G_{\rm ad}(K)/\mathcal{G}_{\rm ad}(\mathcal{O})$ are not surjective. Nevertheless, using [BT84, 5.2.4], if $\mathcal{M}_{\rm ad}$ (respectively, $\mathcal{T}_{\rm ad}$) denotes the unique parahoric group scheme for $M_{\rm ad}$ (respectively, $T_{\rm ad}$), one can check that $p(\mathcal{G}(\mathcal{O}_{E_0})) \cdot \mathcal{M}_{\rm ad}(\mathcal{O}_{E_0}) = \mathcal{G}_{\rm ad}(\mathcal{O}_{E_0})$ (respectively,

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$p(\mathcal{G}(\mathcal{O}_{\breve{E}_0})) \cdot \mathcal{T}_{\rm ad}(\mathcal{O}_{\breve{E}_0}) = \mathcal{G}_{\rm ad}(\mathcal{O}_{\breve{E}_0}))$. This implies that p takes any $\mathcal{G}(\mathcal{O})$-orbit in $G(K)/\mathcal{G}(\mathcal{O})$ onto a $\mathcal{G}_{\rm ad}(\mathcal{O})$-orbit in $G_{\rm ad}(K)/\mathcal{G}_{\rm ad}(\mathcal{O})$, since $\mathcal{M}_{\rm ad}(\mathcal{O}_{E_0})$ (respectively, $\mathcal{T}_{\rm ad}(\mathcal{O}_{\breve{E}_0})$) is normalized by the group $N_{G_{\rm ad}}(A_{\rm ad})(E_0)$ (respectively, $N_{G_{\rm ad}}(S_{\rm ad})(\breve{E}_0)$) giving rise to representatives of those $\mathcal{G}_{\rm ad}(\mathcal{O})$-orbits.

Write $\mathcal{O} = \mathcal{O}_{E_0}$ and $\breve{\mathcal{O}} = \mathcal{O}_{\breve{E}_0}$. We make some abbreviations: Iwahori subgroups $\breve{I} = \mathcal{G}_{\mathbf{a}}(\breve{\mathcal{O}})$ and $I = \mathcal{G}_{\mathbf{a}}(\mathcal{O})$; parahoric subgroups $\breve{J} = \mathcal{G}(\breve{\mathcal{O}})$ and $J = \mathcal{G}(\mathcal{O})$. Similarly define their analogues for $G_{\rm ad}$: $\breve{I}_{\rm ad}$, $I_{\rm ad}$, $\breve{J}_{\rm ad}$, and $J_{\rm ad}$. Let W_0 denote the finite relative Weyl group for G/E_0. There are decompositions of the Iwahori–Weyl groups over E_0:

$$\begin{aligned} W(G) = W &\cong X^*(Z(M^\vee)^{I_{E_0}})^\Phi \rtimes W_0 \cong W_{\rm sc} \rtimes \Omega_{\mathbf{a}}, \\ W(G_{\rm ad}) = W_{\rm ad} &\cong X^*(Z((M_{\rm ad})^\vee)^{I_{E_0}})^\Phi \rtimes W_0 \cong W_{\rm sc} \rtimes \Omega_{\mathbf{a}_{\rm ad}}, \end{aligned} \tag{7.6}$$

and $p\colon W \to W_{\rm ad}$ is compatible with these decompositions. In general, p maps $\Omega_{\mathbf{a}}$ to $\Omega_{\mathbf{a}_{\rm ad}}$, but neither surjectively nor injectively. However, for each fixed $\omega \in \Omega_{\mathbf{a}}$ with image $\omega_{\rm ad} \in \Omega_{\mathbf{a}_{\rm ad}}$, we obtain an isomorphism

$$p : W_{\rm sc} \rtimes \omega \xrightarrow{\sim} W_{\rm sc} \rtimes \omega_{\rm ad}. \tag{7.7}$$

Recall that $\mathcal{H}(G(E_0), I)$ is generated by $T_w^G = 1_{I\dot{w}I}$, where $\dot{w} \in N_G(T)(E_0)$ is a lift of $w \in W(G)$ along the Kottwitz homomorphism κ_{G,E_0}. The algebra $\mathcal{H}(G(E_0), I)$ has an Iwahori–Matsumoto presentation, that is, it is isomorphic to an affine Hecke algebra over $\mathbb{Z}[v, v^{-1}]$ with possibly unequal parameters, after a specialization $v \mapsto \sqrt{q_{E_0}} \in \bar{\mathbb{Q}}_\ell$; cf. [Hai14, 11.3.2] or [Ros15] for a proof in this generality. The map $T_w^G \mapsto T_{p(w)}^{G_{\rm ad}}$ respects the braid and quadratic relations, hence gives an algebra homomorphism

$$\mathcal{H}(G(E_0), I) \to \mathcal{H}(G_{\rm ad}(E_0), I_{\rm ad}).$$

Let e_J and $e_{J_{\rm ad}}$ be the idempotents corresponding to J and $J_{\rm ad}$; these both correspond to the same set of reflections in $W_{\rm sc}$ (those which fix $\mathbf{f}$). Therefore, using the usual relations in the Iwahori–Hecke algebra to understand the products of such idempotents by the standard generators T_w^G, we see that the map

$$e_J T_w^G e_J \mapsto e_{J_{\rm ad}} T_{w_{\rm ad}}^{G_{\rm ad}} e_{J_{\rm ad}},$$

where $w_{\rm ad} = p(w)$, determines a homomorphism of algebras

$$p_*\colon \mathcal{H}(G(E_0), J) \to \mathcal{H}(G_{\rm ad}(E_0), J_{\rm ad}).$$

The homomorphism above preserves centers; to see this one uses the Bernstein presentation of $\mathcal{H}(G(E_0), I)$ and $\mathcal{H}(G_{\rm ad}(E_0), I_{\rm ad})$ (we refer to [Ros15] for a proof of the Bernstein presentation in this most general setting). Put $\Lambda_M = X^*(Z(M^\vee)^{I_{E_0}})^\Phi$. The Bernstein presentations reflect the decomposition $W = \Lambda_M \rtimes W_0$ of (7.6), and, for each W_0-orbit $\bar{\lambda} \subset \Lambda_M$, there is a basis element $z_{\bar{\lambda}} \in \mathcal{Z}(G(E_0), I)$; cf. [Hai14, 11.10.2]. The map sends $z_{\bar{\lambda}}$ to $z_{\bar{\lambda}_{\rm ad}}$. To see this, we write out each of $z_{\bar{\lambda}}$ and $z_{\bar{\lambda}_{\rm ad}}$ in terms of the standard basis elements T_w^G and $T_{p(w)}^{G_{\rm ad}}$. (Use that the alcove walk description of Bernstein functions depends only on the combinatorics of $W_{\rm sc}$; cf. [HR12, § 14.2].) Hence the diagram

$$\begin{array}{ccc} \mathcal{Z}(G(E_0), J) & \longrightarrow & \mathcal{Z}(G_{\rm ad}(E_0), J_{\rm ad}) \\ \wr\downarrow & & \wr\downarrow \\ \bar{\mathbb{Q}}_\ell[X^*(Z(M^\vee)^{I_{E_0}})^\Phi]^{W_0} & \longrightarrow & \bar{\mathbb{Q}}_\ell[X^*(Z((M_{\rm ad})^\vee)^{I_{E_0}})^\Phi]^{W_0} \end{array} \tag{7.8}$$

commutes, where the vertical arrows are the Bernstein isomorphisms of [Hai14, 11.10.1].

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Now, the top arrow is neither injective nor surjective in general. To remedy this, we establish the analogue of (7.7) on the level of centers. Assume $V \in \mathrm{Rep}({}^L G_{E_0})$ has the following property:

$$\text{all the } B^\vee\text{-highest } T^\vee\text{-weights in } V|_{G^\vee} \text{ are } I_{E_0}\text{-conjugate} \qquad (*)$$

(e.g. $V|_{G^\vee \rtimes I_{E_0}}$ is irreducible). Then the restriction $V_{\mathrm{ad}} := V|_{{}^L(G_{\mathrm{ad}})_{E_0}}$ has the same property. Let $\omega_V \in \pi_1(G)_{I_{E_0}}^{\Phi_E}$ be the common image of the $\lambda_i \in X_*(T)$ appearing as $B^\vee$-highest $T^\vee$-weights in $V|_{G^\vee}$, and define $\omega_{V_{\mathrm{ad}}}$ similarly. Note that $\omega_V \mapsto \omega_{V_{\mathrm{ad}}}$ under $\pi_1(G)_{I_{E_0}} \to \pi_1(G_{\mathrm{ad}})_{I_{E_0}}$, that is, $(\omega_V)_{\mathrm{ad}} = \omega_{V_{\mathrm{ad}}}$.

Let $\mathcal{H}(G(E_0), J)_{\omega_V}$ be the subspace generated by the elements $e_J T_w^G e_J$, $w \in W_{\mathrm{sc}} \rtimes \omega_V$. Define $\mathcal{Z}(G(E_0), J)_{\omega_V} = \mathcal{H}(G(E_0), J)_{\omega_V} \cap \mathcal{Z}(G(E_0), J)$. As in (7.4) these are the functions supported on $G(E_0)_{\omega_V} \subset G(E_0)$.

Lemma 7.7. *Assume V satisfies property $(*)$; for example, $V = I(V_{\{\mu\}})$. Then the map $p_* \colon \mathcal{Z}(G(E_0), J) \to \mathcal{Z}(G_{\mathrm{ad}}(E_0), J_{\mathrm{ad}})$ induces a vector space isomorphism*

$$\mathcal{Z}(G(E_0), J)_{\omega_V} \xrightarrow{\sim} \mathcal{Z}(G_{\mathrm{ad}}(E_0), J_{\mathrm{ad}})_{\omega_{V_{\mathrm{ad}}}} \tag{7.9}$$

taking $z^\Phi_{\mathcal{G},V}$ to $z^\Phi_{\mathcal{G}_{\mathrm{ad}},V_{\mathrm{ad}}}$.

Proof. Since $p(z_{\bar\lambda}) = z_{\bar\lambda_{\mathrm{ad}}}$ for $\bar\lambda$ (respectively, $\bar\lambda_{\mathrm{ad}}$) ranging over W_0-orbits in $\Lambda_M \cap (W_{\mathrm{sc}} \rtimes \omega_V)$ (respectively, in $\Lambda_{M_{\mathrm{ad}}} \cap (W_{\mathrm{sc}} \rtimes \omega_{V_{\mathrm{ad}}})$), the first statement is clear. All that remains is to show that $p_*(z^\Phi_{\mathcal{G},V}) = z^\Phi_{\mathcal{G}_{\mathrm{ad}},V_{\mathrm{ad}}}$.

Using the construction of Satake parameters ([Hai15, § 9], [Hai17]), the map $G \to G_{\mathrm{ad}}$ induces a commutative diagram (here for notational convenience we write $\widehat{G}$ in place of $G^\vee$):

$$\begin{array}{ccccccc}
(Z(\widehat{M})^{I_{E_0}})_\Phi/W_0 & \longrightarrow & (\widehat{T^*}^{I_{E_0}})_{\Phi^*}/W_0^* & \longrightarrow & (\widehat{T^*}^{I_E})_{\Phi^*}/W_{0,E}^* & = & [\widehat{G^*}^{I_E} \rtimes \Phi^*]_{\mathrm{ss}}/\widehat{G^*}^{I_E} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
(Z(\widehat{M_{\mathrm{ad}}})^{I_{E_0}})_\Phi/W_0 & \longrightarrow & (\widehat{T^*_{\mathrm{ad}}}^{I_{E_0}})_{\Phi^*}/W_0^* & \longrightarrow & (\widehat{T^*_{\mathrm{ad}}}^{I_E})_{\Phi^*}/W_{0,E}^* & = & [\widehat{G^*_{\mathrm{ad}}}^{I_E} \rtimes \Phi^*]_{\mathrm{ss}}/\widehat{G^*_{\mathrm{ad}}}^{I_E}
\end{array}$$

Fix $V \in \mathrm{Rep}({}^L G_E)$. Starting with the regular function $g^* \rtimes \Phi^* \mapsto \mathrm{tr}(g^* \rtimes \Phi^* \mid V)$ on the variety in the upper right corner, we pull back along the diagram to get a regular function on the lower left-hand corner. Pulling back in one way yields $z^\Phi_{\mathcal{G}_{\mathrm{ad}},V_{\mathrm{ad}}}$, and pulling back in the other way, by (7.8), yields $p_*(z^\Phi_{\mathcal{G},V})$. □

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Appendix A. Non-semisimplified trace

A.1 Basic definitions

Let $V \in \mathrm{Rep}({}^L G)$, and let $\Phi \in \mathrm{Gal}(\bar F/F)$ denote a fixed lift of geometric Frobenius. Let $\mathcal{G}/\mathcal{O}_F$ denote a parahoric group scheme. Define $z^\Phi_{\mathcal{G},V} \in \mathcal{Z}(G(F), \mathcal{G}(\mathcal{O}_F))$ to be the unique function such that, if π is an irreducible smooth representation of $G(F)$ on a $\bar{\mathbb{Q}}_\ell$-vector space such

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that $\pi^{\mathcal{G}(\mathcal{O}_F)} \neq 0$, then $z^{\Phi}_{\mathcal{G},V}$ acts on $\pi^{\mathcal{G}(\mathcal{O}_F)}$ by the scalar

$$\mathrm{tr}(s^{\Phi}(\pi) \mid V),$$

where $s^{\Phi}(\pi) \in [(G^{\vee})^{I_F} \rtimes \Phi]_{\mathrm{ss}}/(G^{\vee})^{I_F}$ is the Satake parameter of π as defined in [Hai15], relative to the fixed choice of Φ.

Similarly, if V is irreducible with parity $d_V \in \mathbb{Z}/2\mathbb{Z}$ as in [HR18, (7.11)], we define the function $\tau^{\Phi}_{\mathcal{G},V}$ on $\mathrm{Gr}_{\mathcal{G}}(k_F)$ by the identity

$$\tau^{\Phi}_{\mathcal{G},V} = (-1)^{d_V}\, \mathrm{tr}(\Phi \mid \Psi_{\mathrm{Gr}_{\mathcal{G}}}(\mathrm{Sat}(V))).$$

We extend by linearity to define $\tau^{\Phi}_{\mathcal{G},V}$ for all V. By the same arguments which proved Theorem 5.15, Lemma 4.12 and Corollary 4.9, we may view $\tau^{\Phi}_{\mathcal{G},V}$ as an element in the Hecke algebra $\mathcal{Z}(G(F), \mathcal{G}(\mathcal{O}_F))$.

A.2 Averaging over inertia

Choose any normal finite-index subgroup $I_1 \subseteq I_F$ having the property that $1 \rtimes I_1$ acts trivially on V and I_1 acts purely unipotently on all cohomology stalks of $\Psi_{\mathrm{Gr}_{\mathcal{G}}}(\mathrm{Sat}(V))$.

Lemma A.1. *Let $\dot{\gamma} \in I_F$ range over a set of lifts of the elements $\gamma \in I_F/I_1$. Then*

$$\tau^{\mathrm{ss}}_{\mathcal{G},V} = \frac{1}{|I_F/I_1|} \sum_{\dot{\gamma}} \tau^{\Phi\dot{\gamma}}_{\mathcal{G},V}, \tag{A.1}$$

$$z^{\mathrm{ss}}_{\mathcal{G},V} = \frac{1}{|I_F/I_1|} \sum_{\dot{\gamma}} z^{\Phi\dot{\gamma}}_{\mathcal{G},V}. \tag{A.2}$$

Consequently, $\tau^{\mathrm{ss}}_{\mathcal{G},V} = z^{\mathrm{ss}}_{\mathcal{G},V}$, if $\tau^{\Phi}_{\mathcal{G},V} = z^{\Phi}_{\mathcal{G},V}$ for all lifts Φ of Frobenius.

Proof. Let H be a finite group acting on a finite-dimensional $\bar{\mathbb{Q}}_\ell$-vector space $\mathcal{V}$. Let Φ denote an arbitrary linear operator on $\mathcal{V}$. Then we have the identity

$$\mathrm{tr}(\Phi \mid \mathcal{V}^H) = \frac{1}{|H|} \sum_{h \in H} \mathrm{tr}(\Phi \circ h \mid \mathcal{V}). \tag{A.3}$$

Now in a cohomology stalk of $\Psi_{\mathrm{Gr}_{\mathcal{G}}}(\mathrm{Sat}(V))$, choose a $\mathrm{Gal}(\bar{F}/F)$-stable filtration on which I_F acts through a finite quotient on the associated graded module, denoted $\mathcal{V}$. Then (A.3) yields (A.1). Similarly, using that $1 \rtimes I_F$ already acts through a finite quotient on V, we obtain (A.2). □

A.3 Products and unramified Weil restrictions

A.3.1 *Products.* Let G_j, $j = 1, \ldots, n$, be connected reductive groups with corresponding parahoric groups $\mathcal{G}_j$. Write $G = \prod_j G_j$ and $\mathcal{G} = \prod_j \mathcal{G}_j$.

Suppose V_j are representations of ${}^L G_j$. We form the dual group ${}^L(\prod_j G_j) = (\prod_j G^{\vee}) \rtimes \Gamma$, with $\Gamma = \mathrm{Gal}(\bar{F}/F)$ acting diagonally on the factors.

Lemma A.2. *Let $V = \boxtimes_j V_j$, the representation of ${}^L(\prod_j G_j)$ with Γ acting diagonally in the obvious manner. Then we have equalities of functions in $\mathcal{Z}(G(F), \mathcal{G}(\mathcal{O}_F)) = \prod_j \mathcal{Z}(G_j(F), \mathcal{G}(\mathcal{O}_F))$:*

$$\tau^{\Phi}_{\mathcal{G},V} = \prod_j \tau^{\Phi}_{\mathcal{G}_j,V_j}, \quad z^{\Phi}_{\mathcal{G},V} = \prod_j z^{\Phi}_{\mathcal{G}_j,V_j}.$$

A.3.2 *Unramified Weil restrictions.* Let F_n/F be a finite unramified extension of degree n, and let G_0 be a connected reductive F_n-group; let $G = \mathrm{Res}_{F_n/F} G_0$. Then ${}^L G$ identifies with the induced group ${}^L G = (I^F_{F_n} G_0^{\vee}) \rtimes \Gamma_F$, where Γ_F acts on the $\bar{\mathbb{Q}}_\ell$-group $I^F_{F_n} G_0^{\vee} \cong \prod_{j=0}^{n-1} G_0^{\vee}$ in

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the obvious way. Explicitly, as a Γ_n-group $I_{F_n}^F G_0^\vee = \prod_{j=0}^{n-1} {}^{\Phi^{-j}}G_0^\vee$, where ${}^{\Phi^0}G_0^\vee$ is $G_0^\vee$ endowed with the given action of Γ_n, and ${}^{\Phi^{-j}}G_0^\vee$ is the same group but with Γ_n acting through the given action precomposed with the automorphism $\gamma_n \mapsto \Phi^j\gamma_n\Phi^{-j}$. The action of Φ on $I_{F_n}^F G_0^\vee$ is given by $(g_0, g_1, \dots, g_{n-1}) \mapsto (g_1, g_2, \dots, g_{n-1}, \Phi^n(g_0))$.

An irreducible algebraic representation of ${}^LG$ is of the form $V = \boxtimes_{j=0}^{n-1} {}^{\Phi^{-j}}V_0$, where V_0 is an irreducible representation of ${}^LG_0 = G_0^\vee \rtimes \Gamma_n$, and Γ_n acts on ${}^{\Phi^{-j}}V_0$ by precomposing the given action on V_0 with the automorphism $\gamma_n \mapsto \Phi^j\gamma_n\Phi^{-j}$. Then, as before, $\Gamma_F = \langle\Phi\rangle\Gamma_{F_n}$ operates as follows: Γ_{F_n} acts 'diagonally' on vectors of the form $v_0 \boxtimes v_1 \boxtimes \cdots \boxtimes v_{n-1}$, and Φ sends such a vector to the vector $v_1 \boxtimes v_2 \boxtimes \cdots \boxtimes v_{n-1} \boxtimes \Phi^n(v_0)$.

LEMMA A.3. *We have the identity*

$$\mathrm{tr}(\Phi \mid V) = \mathrm{tr}(\Phi^n \mid V_0),$$

and this implies the identities

$$\tau^{\Phi}_{\mathcal{G},V} = \tau^{\Phi^n}_{\mathcal{G}_0,V_0}, \quad z^{\Phi}_{\mathcal{G},V} = z^{\Phi^n}_{\mathcal{G}_0,V_0}$$

in $\mathcal{Z}(G(F), \mathcal{G}(\mathcal{O}_F)) = \mathcal{Z}(G_0(F_n), \mathcal{G}_0(\mathcal{O}_{F_n}))$.

Proof. The first identity is a special case of a result of Saito and Shintani; cf. [Fen20, Lemma 6.12]. The other assertions follow from this one. □

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Thomas J. Haines tjh@math.umd.edu
Department of Mathematics, University of Maryland,
College Park, MD 20742-4015, USA

Timo Richarz richarz@mathematik.tu-darmstadt.de
Fachbereich Mathematik, TU Darmstadt,
Schlossgartenstrasse 7, 64289 Darmstadt, Germany

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