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Concerning some Solutions of the Boundary Layer Equations in Hydrodynamics. By S. Goldstein, Ph.D., St John's College.

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I should like to make it clear at the outset that no complete solution of the boundary layer equations is given in this paper, a complete solution being understood to be one giving numerical values of the dependent variable (fluid velocity) over the whole range of the independent variables (for steady motion, the space coordinates) in question. I have attempted to exhibit the mathematical nature of certain solutions previously unknown, and have carried out part of the actual calculations—sufficient, in my opinion, to allow those who will, to judge the practicability of completing a solution for any definite example. (A summary is given at the end of the paper.)

1.

1.1. Consider a two-dimensional flow of an incompressible fluid of small viscosity (or, more generally, a flow at a high Reynolds number) along a straight or curved wall, or past an immersed cylindrical body. Let U be a representative velocity and d a representative length of the system considered. Let y_1 be distance measured normally from the boundary, x_1 distance along curves orthogonal to the normals (measured from the normal at the forward stagnation point for flow past a cylinder), u_1 and v_1 the components of fluid velocity in the directions of x_1 and y_1 increasing, p_1 the pressure, ρ the density, and ν the kinematic viscosity of the fluid. Further, let x be x_1/d, y be $R^{\frac{1}{2}}y_1/d$, u be u_1/U, v be $R^{\frac{1}{2}}v_1/U$, and p be $p_1/\rho U^2$, where R is Ud/ν. Then, on the assumption that a boundary layer exists in which the viscous terms in the equations of motion are of the same order of magnitude as the inertia terms, the approximate equations for determining a steady motion in the boundary layer are*

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \quad \text{.................}(1)$$

and

$$0 = -\frac{\partial p}{\partial y}, \quad \text{......................}(2)$$

* The equations were first given by Prandtl, *Verhandl. d. III intern. Math.-Kongresses*, Heidelberg, 1904; reprinted in *Vier Abhandlungen zur Hydrodynamik und Aerodynamik*, L. Prandtl and A. Betz, Göttingen, 1927. Prandtl's method is more fully given by Blasius, *Zeitschrift f. Math. u. Phys.* 56, 1 (1908). Concerning the derivation of the equations, see also v. Kármán, *Zeitschrift f. angewandte Math. und Mech.* (*Z.A.M.M.*) 1, 233 (1921); Polhausen, *Z.A.M.M.* 1, 252 (1921); Bairstow, *Journal Roy. Aeronautical Society*, 29, 3 (1925); and v. Mises, *Z.A.M.M.* 7, 425 (1927).

together with the equation of continuity,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \qquad (3)$$

For flow along a curved wall the error in each of the equations is of order $R^{-\frac{1}{2}}\sigma^{-1}$, where σ is the ratio of the radius of curvature of the boundary to the length d*; for flow along a straight wall the errors in (1) and (2) are of the order R^{-1}, and (3) is exact.

1.2. According to **1.1** (2) the pressure in the boundary layer is independent of y. Its value at any point is consequently the same as at the corresponding point outside the layer; and the pressure distribution immediately outside the layer is assumed to have been independently determined. The theoretical determination is a matter of difficulty†, since the flow outside the boundary layer is not everywhere irrotational‡. It is usual to find the pressure distribution experimentally, and then to use the experimental result so found to calculate the velocity distribution in the boundary layer, the frictional drag on the wall, and the point at which the forward flow leaves the wall. The results of the calculations are then compared with experiment.

1.3. The boundary conditions to be satisfied by u and v must next be considered. First, u and v must vanish at the wall, i.e. for $y=0$. In general, u is given for $x=0$. In addition, the fluid velocity must pass over smoothly into the velocity in the main stream. Thus u must become equal to V, and $\partial u/\partial y$ equal to 0, as we pass into the main stream. (Here V is the velocity immediately outside the boundary layer, divided by U.) These conditions cannot, in general, be satisfied for a finite value of y; we must take u to be asymptotically equal to V. Then if we define the limit of the boundary layer by requiring u to be equal to V to any prearranged degree of accuracy, this is attained for a finite value of y (though, of course, the value of y depends on the degree of accuracy required), and the solutions have the property that the difference of u from V is quite small for moderate values of y. Further,

* Terms of order $R^{-\frac{3}{2}}$ multiplied by the gradient of the curvature along the surface are also neglected. See Bairstow, *loc. cit.*

† For recent attempts, see Oseen's *Hydrodynamik* (Leipzig, 1927), including as appendix two lectures by Zeilon at the International Congress for Technical Mechanics, Zurich, 1926. Reference may also be made to Burgers, *Proc. Roy. Acad. Sci. Amsterdam*, 31, 433 (1928).

‡ In flow along a curved wall there is a region in which the pressure increases in the direction of motion; in this region the forward flow in the boundary layer is forced to leave the wall, and the fluid in it, having acquired vorticity, mixes in the main stream. See Prandtl, *loc. cit.* and *Journal Roy. Aeronautical Soc.* 31, 720 (1927). Also Blasius, *loc. cit.* Many popular and semi-popular expositions have been published, mainly in connection with the rotor ship (flow past rotating cylinders), and the effects of suction on the boundary layer. References are given in the *Vier Abhandlungen.*

equation **1.1** (1) shows that if, as y tends to infinity, u tends to a value independent of y, and $\partial u/\partial y$ and $\partial^2 u/\partial y^2$ tend to zero, then the limiting value of u is connected with the pressure by Bernoulli's equation. It is, then, sufficient to require that u should tend to a limit.

The velocity v is neglected in the boundary conditions at $x=0$ and $y=\infty$, and this introduces an error of order $R^{-\frac{1}{2}}$.

1.4. If the velocity and dimensions of the system are sufficiently large for any given fluid, then it has been experimentally established that at a sufficient distance downstream the flow in the boundary layer is turbulent. But the Reynolds numbers at which transitions to turbulent motion take place are large enough for the laminar motion to be calculated from the approximate equations*.

1.5. Complete mathematical solutions of the boundary layer equations for steady motion have been obtained for two problems only—for flow past an infinitely thin plate along the stream† and for converging flow between two non-parallel straight walls‡. For the flow past a cylindrical obstacle, the solution has been obtained by v. Kármán's approximate method§; in addition, the velocity has been expanded in a Taylor power series for x, the coefficients being functions of y to be determined from ordinary differential equations‖. An obvious, even if laborious, method of attempting to complete the solution would be to proceed step by step, using

* For experiments on flow along flat plates, containing measurements in the laminar and turbulent regions, and also in the region of transition, see Burgers and van der Hegge Zynen, *Mededeeling No. 5 uit het laboratorium voor aerodynamica en hydrodynamica der tecknische hoogeschool te Delft*; van der Hegge Zynen, *Mededeeling, No.* 6; Burgers, *Proceedings of the First International Congress for Applied Mechanics*, Delft (1924), p. 113; Hansen, *Z.A.M.M.* 8, 185 (1928). The flow past a circular cylinder in the critical Reynolds number interval has been examined by Fage, *Phil. Mag.* (7) 7, 253 (1929). The discovery that the flow in the boundary layer may become turbulent apparently dates back to Blasius, *Mitteilungen über Forschungsarbeiten herausg. vom Verein deutsch. Ing.* Heft 131, p. 1 (1913), and Prandtl, *Göttinger Nachrichten* (1914), p. 177.

† Blasius, *loc. cit.*

‡ Polhausen, *loc. cit.* The exact solution for converging or diverging flow between non-parallel straight walls was given by Hamel (*Jahresbericht der deutscher Math.-Vereinigung*, 25 (1916), p. 34) as a special case of flow in which the stream-lines are logarithmic spirals, which is the only possible form if they are to coincide with the stream-lines of a potential flow. See also Oseen, *Arkiv för Math.-Astron. och Fys.* 20, 1927, No. 14; Millikan, *Math. Ann.* 101 (1929), p. 446; and v. Kármán, *Vorträge aus dem Gebiete der Hydro- und Aerodynamik* (Innsbruck, 1922), p. 150.

§ Polhausen, *loc. cit.*

‖ Blasius, *loc. cit.*; Hiemenz, *Dinglers Polytechn. Journal*, Bd. 326 (1911). The latter used an experimentally determined pressure distribution. See Polhausen's remarks on Hiemenz's solution, and v. Mises's remarks on Polhausen's solution.

An approximate numerical solution has also been given by A. Thom, *Reports and Memoranda of the Aeronautical Research Committee*, No. 1176 (1928). The region in which the solufion holds appears to be almost the same as that for Hiemenz's solution. Measurements of the velocity distribution in the boundary layer at the surface of a circular cylinder are also given in the paper cited.

the Taylor series to give the solution up to a certain value of x, and then taking the value so calculated as a new initial value. This new initial value, u_0, will, in general, be given numerically; to continue the analysis we may approximate to u_0 by a polynomial, so that

$$u_0 = a_1 y + a_2 y^2 + \dots + a_n y^n. \quad \text{..............(1)}$$

In general, however, the solution with (1) as initial value is not free from singularities, and has not previously been found. If

$$-\frac{\partial p}{\partial x} = p_0 + p_1 x + p_2 x^2 + \dots, \quad \text{..............(2)}$$

the conditions for the absence of singularities in the solution are

$$2a_2 + p_0 = 0, \quad a_3 = 0, \quad 5!\,a_5 + 2a_1 p_1 = 0, \quad 6!\,a_6 = 2p_0 p_1, \dots \quad \text{......(3)}$$

Thus only a_1, a_4, a_7, and so on are at our disposal. We may then seek to determine these coefficients so that

$$u_0 = a_1 y - \tfrac{1}{2} p_0 y^2 + a_4 y^4 - \frac{2a_1 p_1}{5!} y^5 + \frac{2p_0 p_1}{6!} y^6 + a_7 y^7 + \dots \quad \text{......(4)}$$

as nearly as possible, and continue the analysis with (4) as initial value. There are several objections to this process. First, the determination of the coefficients in (4) is usually neither easy nor accurate. Second, if we determine n coefficients we have a polynomial of order $3n-2$. Third, with (4) as initial value, we find

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = a_1 + \frac{4!\,a_4}{a_1} x + \frac{7!\,a_1 a_7 + (4!\,a_4)^2}{4a_1^3} x^2 + \dots, \quad \text{...(5)}$$

and the presence of the factorials is unpleasant. It may well be, then, that the solution with (1) as initial value possesses advantages for practical calculation; in addition, it has theoretical mathematical interest. The singularities will invalidate the solution for practical purposes only in the immediate neighbourhood of $x = 0$.

Other problems of which mathematical solutions have not yet been given are that of finding the nature of the solution in the neighbourhood of the point at which the forward flow leaves the boundary, characterised by the condition that not only u, but also $\partial u/\partial y$, vanishes for $y = 0$; and also such a problem as that of the flow along an infinitely thin plate when the fluid flow has previously been disturbed. All these problems are included in that of finding the solution of the equations for a general initial value of u. Thus, if u_0 is the value of u for $x = 0$, and

$$u_0 = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \dots, \quad \text{..............(6)}$$

then for the first problem $a_0 = 0$ and $a_1 \neq 0$, for the second $a_0 = a_1 = 0$ and $a_2 \neq 0$, and for the third $a_0 \neq 0$. The solution for the first is given in § **2**, for the third in § **3**, and for the second in § **4**.

1.6. The boundary layer equations apply not only when the motion is being disturbed by the presence of a boundary, but also when a previously disturbed flow is recovering from the effect of the disturbance. Such an application is that of calculating the flow behind a flat plate along the stream. In this case the boundary conditions $u = 0$, $v = 0$ at $y = 0$ are replaced (because of symmetry) by $v = 0$, $\partial u/\partial y = 0$ at $y = 0$. The calculation is carried out in § **5**, and numerical values of the velocity are given. The extension to a general initial value of u and general pressure gradient is simple, but it has not been found possible to make allowance for a finite thickness of the plate.

2.

2.1. It is required to solve the equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + p_0 + p_1 x + p_2 x^2 + \ldots, \quad \ldots\ldots\ldots(1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

with the boundary conditions $u = v = 0$ at $y = 0$, u tends to a limit as y tends to infinity, and

$$u = u_0 = a_1 y + a_2 y^2 + a_3 y^3 + \ldots \qquad (a_1 \neq 0) \quad \ldots\ldots\ldots(3)$$

at $x = 0$.

2.2. A general transformation of the equations, of which a special case is required here, will be found useful later. Equation **2.1** (2) is integrated by taking

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and then we put

$$\xi = x^{\frac{1}{n}}, \quad \eta = yx^{-\frac{1}{n}}/n, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and

$$\psi = \xi^{n-1} f(\xi, \eta); \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

so that

$$\left.\begin{aligned} u &= \xi^{n-2} f_\eta / n \\ v &= -[(n-1)f + \xi f_\xi - \eta f_\eta]/n\xi \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots(4)$$

and f satisfies the equation

$$(n-2)f_\eta^2 + \xi f_\eta f_{\xi\eta} - (n-1) f f_{\eta\eta} - \xi f_\xi f_{\eta\eta} = f_{\eta\eta\eta} + n^3 \xi^{4-n} (p_0 + p_1 \xi^n + p_2 \xi^{2n} + \ldots). \quad \ldots\ldots(5)$$

2.3. To solve the equations of **2.1**, we take n equal to 3 and assume that

$$f = f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \dots \quad \dots\dots\dots\dots(1)$$

Then

$$u = \tfrac{1}{3}\xi(f_0' + \xi f_1' + \xi^2 f_2' + \dots), \quad \dots\dots\dots\dots(2)$$

and

$$\left.\begin{aligned} &f_0''' + 2f_0 f_0'' - f_0'^2 = 0 \\ &f_1''' + 2f_0 f_1'' - 3f_0' f_1' + 3f_0'' f_1 = -27p_0 \\ &f_2''' + 2f_0 f_2'' - 4f_0' f_2' + 4f_0'' f_2 = -3f_1 f_1'' + 2f_1'^2 \\ &f_3''' + 2f_0 f_3'' - 5f_0' f_3' + 5f_0'' f_3 = -4f_2 f_1'' + 5f_2' f_1' - 3f_2'' f_1 \\ &f_4''' + 2f_0 f_4'' - 6f_0' f_4' + 6f_0'' f_4 = -5f_3 f_1'' + 6f_3' f_1' - 3f_3'' f_1 \\ &\qquad\qquad\qquad\qquad + 3f_3'^2 - 4f_2'' f_2 - 27p_1 \end{aligned}\right\}, \quad \dots(3)$$

and so on, where dashes denote differentiations with respect to η.

The conditions $u = v = 0$ at $\eta = 0$ are equivalent to

$$f_r(0) = f_r'(0) = 0, \dots\dots\dots\dots(4)$$

while the condition that

$$u_0 = a_1(3\xi\eta) + a_2(3\xi\eta)^2 + \dots + a_r(3\xi\eta)^r + \dots \quad \dots\dots(5)$$

when $\eta \to \infty$ and $\xi \to 0$ requires that

$$\lim_{\eta \to \infty} f'_{r-1}(\eta)/\eta^r = \alpha_r, \quad \dots\dots\dots\dots(6)$$

where

$$\alpha_r = 3^{r+1} a_r. \quad \dots\dots\dots\dots(7)$$

The condition to be satisfied when y is infinite will be considered later (§ **2.4**).

The solution for f_0 can be written down at once. It is

$$f_0 = \tfrac{1}{2}\alpha_1 \eta^2. \quad \dots\dots\dots\dots(8)$$

The solution for f_1 is given in § **2.31**; in § **2.32** the complementary functions for the equation for f_r are considered; and the solution for f_2 is completed in § **2.33**.

2.31. If we put

$$z = \alpha_1^{\frac{1}{3}}\eta, \quad \dots\dots\dots\dots(1)$$

then

$$\frac{d^3 f_1}{dz^3} + z^2 \frac{d^2 f_1}{dz^2} - 3z\frac{df_1}{dz} + 3f_1 = -\pi_0/\alpha_1, \quad \dots\dots\dots(2)$$

where

$$\pi_0 = 27p_0. \quad \dots\dots\dots\dots(3)$$

The general solution having a double zero at the origin is

$$f_1 = \beta_1 g_1(z) - \pi_0 \alpha_1^{-1} z^3/3!, \quad \dots\dots\dots\dots(4)$$

where β_1 is an arbitrary constant, to be determined later from the condition at infinity, and

$$g_1(z) = \frac{z^2}{2!} + \frac{z^5}{5!} - \frac{2.4}{8!}z^8 + \frac{2.5.4.7}{11!}z^{11} - \frac{2.5.8.4.7.10}{14!}z^{14} + \dots \quad \dots\dots(5)$$

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$g_1(z)$ may be expressed in finite form in two ways. First,

$$g_1 = \tfrac{1}{2} z \int_0^z {}_1F_1(-\tfrac{1}{3};\, \tfrac{5}{3};\, -\tfrac{1}{3}z^3)\, dz, \quad \text{......(6)}$$

where

$${}_1F_1(a;\, b;\, x) = 1 + \frac{a}{1\,.\,b}\, x + \frac{a\,(a+1)}{2!\, b\,(b+1)}\, x^2 + \ldots, \quad \text{......(7)}$$

and has been studied by Barnes* and others. Another form is

$$g_1 = 3^{-\frac{4}{3}} \{\tfrac{1}{2}z^3 - 1\}\, \gamma(\tfrac{2}{3},\, \tfrac{1}{3}z^3) + \tfrac{1}{2}\,.\,3^{-\frac{2}{3}} z\gamma(\tfrac{1}{3},\, \tfrac{1}{3}z^3) + \tfrac{1}{6}z^2 \exp(-\tfrac{1}{3}z^3), \quad \text{......(8)}$$

where $\gamma(n, x)$ is the *incomplete gamma function*, namely,

$$\gamma(n, x) = \int_0^x t^{n-1} e^{-t}\, dt, \quad \text{......(9)}$$

so that

$$\Gamma(n) - \gamma(n, x) \sim e^{-x} x^{n-1} \left\{1 + \frac{n-1}{x} + \frac{(n-1)(n-2)}{x^2} + \ldots\right\}. \quad \text{......(10)}$$

The limit of $z^{-2} df_1/dz$ as z tends to infinity is now easily found to be

$$\tfrac{1}{2}\,.\,3^{-\frac{1}{3}}\,\Gamma(\tfrac{2}{3})\, \beta_1 - \tfrac{1}{2}\pi_0 \alpha_1^{-1}. \quad \text{......(11)}$$

Since from **2.3** (6) this must be equal to α_2/α_1, we must have

$$\begin{aligned}\beta_1 = 3^{\frac{1}{3}} \alpha_1^{-1} (\pi_0 + 2\alpha_2)/\Gamma(\tfrac{2}{3}) &= 3^{\frac{4}{3}} a_1^{-1} (p_0 + 2a_2)/\Gamma(\tfrac{2}{3}) \\ &= 3{\cdot}1954\, (p_0 + 2a_2)/a_1.\end{aligned} \quad \text{......(12)}$$

The asymptotic formula for f_1 is

$$f_1 \approx A_1\eta^3 + C_1\eta + D_1 + \beta_1 \alpha_1^{-\frac{7}{3}} \exp(-\tfrac{1}{3}\alpha_1\eta^3) \left\{\frac{1}{\eta^7} - \frac{16}{\alpha_1\eta^{10}} + \ldots\right\}, \quad \text{......(13)}$$

where

$$A_1 = 9a_2, \quad C_1 = \frac{3^{\frac{4}{3}}\,\Gamma(\tfrac{1}{3})}{2\Gamma(\tfrac{2}{3})} \frac{p_0 + 2a_2}{a_1^{\frac{2}{3}}} = 4{\cdot}2804\, \frac{p_0 + 2a_2}{a_1^{\frac{2}{3}}}, \quad D_1 = -\frac{p_0 + 2a_2}{a_1}. \quad \text{......(14)}$$

2.32. The equation for f_r has the form

$$\frac{d^3 f_r}{dz^3} + z^2 \frac{d^2 f_r}{dz^2} - (r+2)\, z \frac{df_r}{dz} + (r+2) f_r = G_r(z). \quad \text{...(1)}$$

* *Trans. Camb. Phil. Soc.* 20, 253–269 (1908).

Three independent complementary functions are z, g_r, and h_r, where

$$g_r = \frac{z^2}{2!} + \frac{r}{5!} z^5 + \frac{4r(r-3)}{8!} z^8 + \frac{4 \cdot 7r(r-3)(r-6)}{11!} z^{11} + \dots$$

$$= \tfrac{1}{2} z \int_0^z {}_1F_1 \left(-\frac{r}{3};\ \tfrac{5}{3};\ -\tfrac{1}{3} z^3 \right) dz, \quad \dots\dots(2)$$

and

$$h_r = 1 - \frac{r+2}{3!} z^3 - \frac{2(r+2)(r-1)}{6!} z^6 - \frac{2 \cdot 5(r+2)(r-1)(r-4)}{9!} z^9 - \dots$$

$$= 1 - z \int_0^z z^{-2} \left\{ {}_1F_1 \left(-\frac{r+2}{3};\ \tfrac{1}{3};\ -\tfrac{1}{3} z^3 \right) - 1 \right\} dz. \quad \dots\dots(3)$$

The asymptotic expansion of ${}_1F_1(a; b; -x)$ is*

$$\Gamma(b-a)\, {}_1F_1(a; b; -x) \sim \Gamma(b)\, x^{-a} \left\{ 1 + \frac{a(1-b+a)}{x} + \frac{a(a+1)(1-b+a)(2-b+a)}{2!\, x^2} + \dots \right\}. \quad \dots(4)$$

The asymptotic expansion of g_r is not immediately deducible, but may be found by the method used by Barnes. If we consider

$$\frac{1}{2\pi i} \int \frac{\Gamma(-s)\, \Gamma(s - \frac{1}{3} r)}{\Gamma(s + \frac{5}{3})} \frac{z^{3s+1}}{3^s (3s+1)} ds \quad \dots\dots(5)$$

taken round a contour consisting of a straight line from $-N - \frac{1}{2} - \infty i$ to $-N - \frac{1}{2} + \infty i$ and the part of a circle of infinite radius to the right of this line, we easily find that

$$g_r \sim \frac{z\, \Gamma(\frac{5}{3})}{2 \cdot 3^{\frac{1}{3} r}\, \Gamma\left(\frac{r+5}{3}\right)} \left\{ \frac{z^{r+1}}{r+1} + \frac{r(r+2)}{3} \frac{z^{r-2}}{r-2} + \frac{r(r-3)(r+2)(r-1)}{3 \cdot 6} \frac{z^{r-5}}{r-5} + \frac{r(r-3)(r-6)(r+2)(r-1)(r-4)}{3 \cdot 6 \cdot 9} \frac{z^{r-8}}{r-8} + \dots \right\}$$
$$+ \frac{3^{\frac{1}{3}}\, \Gamma(\frac{5}{3})\, \Gamma\left(-\frac{r+1}{3}\right)}{2 \Gamma(-\frac{1}{3} r)} z, \quad \dots(6)$$

when $r \neq 3n-1$ (n integral). When $r = 3n-1$, the term with

* Barnes, *loc. cit.*

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a zero denominator must be omitted, and the last term (the multiple of z) replaced by

$$\frac{(-)^n 3^{\frac{1}{3}} \Gamma(\frac{5}{3}) z}{2\Gamma(n+1)\Gamma(\frac{1}{3}-n)}[\log \tfrac{1}{3}z^3 + \psi(n+1) - \psi(\tfrac{4}{3}) - \psi(\tfrac{1}{3})]$$
$$= \frac{(-)^n 3^{\frac{1}{3}} \Gamma(\frac{5}{3}) z}{2\Gamma(n+1)\Gamma(\frac{1}{3}-n)}\left[3\log_e z + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \frac{1}{n} - 2 + 2\log_e 3 + \pi/\sqrt{3} + C\right], \dots(7)$$

where $\psi(x)$ is $\Gamma'(x)/\Gamma(x)$*, and C is Euler's constant, equal to 0·5772....

We can prove similarly that

$$h_r \sim \frac{-z\Gamma(\frac{1}{3})}{\Gamma\left(\frac{r}{3}+1\right)3^{\frac{r+2}{3}}}\left\{\frac{z^{r+1}}{r+1}+\dots\right\} + \frac{\Gamma(\frac{1}{3})\Gamma\left(-\frac{r+1}{3}\right)}{3^{\frac{1}{3}}\Gamma\left(-\frac{r+2}{3}\right)}z, \dots(8)$$

when $r \neq 3n-1$, and that, when $r = 3n-1$, the term with a zero denominator must be omitted, and the last term replaced by

$$\frac{(-)^n \Gamma(\frac{1}{3}) z}{3^{\frac{1}{3}}\Gamma(n+1)\Gamma(-\frac{1}{3}-n)}[\log \tfrac{1}{3}z^3 + \psi(n+1) - \psi(\tfrac{2}{3}) - \psi(\tfrac{1}{3})]$$
$$= \frac{(-)^n \Gamma(\frac{1}{3}) z}{3^{\frac{1}{3}}\Gamma(n+1)\Gamma(-\frac{1}{3}-n)}\left[3\log_e z + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \frac{1}{n} + 2\log_e 3 + C\right]. \dots(9)$$

The series in (8) is the same as in (6), and a certain linear combination of the solutions z, g_r, and h_r is asymptotically equal to zero. (This is obvious when $r \neq 3n-1$, and is easily proved when $r = 3n-1$.) This corresponds to the fact that the equation (1) (with the right-hand side omitted) has a normal solution of rank 3 at infinity, consisting of $\exp(-\frac{1}{3}z^3)$ multiplied by a descending series. The further discussion of the solutions is interesting but irrelevant.

The method of variation of parameters can now be used to show that the complete solution of (1) is

$$g_r \int^z {}_1F_1\left(\frac{r}{3}+1; \tfrac{1}{3}; \tfrac{1}{3}z^3\right) G_r dz + h_r \int^z \tfrac{1}{2}z^2 {}_1F_1\left(\frac{r+5}{3}; \tfrac{5}{3}; \tfrac{1}{3}z^3\right) G_r dz$$
$$+ z\int^z (g_r h_r' - g_r' h_r) G_r \exp(\tfrac{1}{3}z^3)\, dz, \dots(10)$$

but the application of this result is difficult.

* ψ is the recognised symbol for both the stream function in hydrodynamics and the logarithmic derivative of the gamma function. When it occurs in this paper with the second meaning, its numerical argument is always specified, so no confusion can be caused.

2.33. The equation for f_2 is **2.32** (1) with r equal to 2, and

$$G_2(z) = \beta_1{}^2\alpha_1{}^{-\frac{1}{3}}K_2(z) + \beta_1\pi_0\alpha_1{}^{-\frac{4}{3}}L_2(z), \quad \text{.........(1)}$$

where, with

$$\gamma_1 = \tfrac{1}{2}\,.\,3^{-\frac{4}{3}}\Gamma(\tfrac{2}{3}) = 0{\cdot}1565, \text{ and } \delta_1 = \tfrac{1}{2}\,.\,3^{-\frac{2}{3}}\Gamma(\tfrac{1}{3}) = 0{\cdot}6440, \quad \text{......(2)}$$

we have

$$K_2(z) = 2\left(\frac{dg_1}{dz}\right)^2 - 3g_1\frac{d^2g_1}{dz^2}$$

$$= \frac{z^2}{2} - \frac{13z^5}{2\,.\,3\,.\,4\,.\,5} + \frac{103z^8}{4\,.\,5\,.\,7\,.\,8\,.\,9} - \frac{3779z^{11}}{3\,.\,5\,.\,6\,.\,7\,.\,8\,.\,9\,.\,10\,.\,11} + \dots$$

$$\sim -6\gamma_1\delta_1z^2 + 36\gamma_1{}^2z + 2\delta_1{}^2 - \gamma_1\exp(-\tfrac{1}{3}z^3)\{3 + O(z^{-3})\}, \quad \text{......(3)}$$

and

$$L_2(z) = 3zg_1 - 2z^2\frac{dg_1}{dz} + \tfrac{1}{2}z^3\frac{d^2g_1}{dz^2}$$

$$= \frac{z^6}{2\,.\,4\,.\,5} - \frac{z^9}{6\,.\,7\,.\,8} + \frac{z^{12}}{4\,.\,9\,.\,10\,.\,11} - \frac{z^{15}}{3\,.\,9\,.\,12\,.\,13\,.\,14} + \dots$$

$$\sim \delta_1z^2 - 6\gamma_1z + \tfrac{1}{2}\exp(-\tfrac{1}{3}z^3)\{1 + O(z^{-3})\}. \quad \text{...........(4)}$$

The general solution having a double zero at the origin is

$$f_2 = \beta_2g_2 + \beta_1{}^2\alpha_1{}^{-\frac{1}{3}}k_2 + \beta_1\pi_0\alpha_1{}^{-\frac{4}{3}}l_2, \quad \text{...........(5)}$$

where β_2 is a constant to be determined later from the condition at infinity, g_2 is given by **2.32** (2) with r equal to 2,

$$k_2 = \frac{z^5}{5!} - \frac{17z^8}{8!} + \frac{888z^{11}}{11!} - \frac{92392z^{14}}{14!} + \dots, \quad \text{.........(6)}$$

and

$$l_2 = \frac{18z^9}{9!} - \frac{1800z^{12}}{12!} + \frac{279360z^{15}}{15!} - \frac{65197440z^{18}}{18!} + \dots. \quad \text{...(7)}$$

The complete solution is given by the right-hand side of (5) plus arbitrary multiples of z and h_2.

Now let

$$P(z) = z\log z - z + \frac{1}{3\,.\,6z^2} + \frac{4!}{3\,.\,6^2\,.\,9z^5} + \frac{7!}{3\,.\,6^2\,.\,9^2\,.\,12z^8} + \dots. \quad \text{......(8)}$$

Then the asymptotic expansion of g_2 is (see **2.32** (6) and (7))

$$g_2 \sim \frac{\Gamma(\frac{2}{3})}{3^{\frac{5}{3}}\Gamma(\frac{4}{3})}\left\{\tfrac{1}{4}z^4 + \frac{2z}{3}(1 + 2\log_e 3 + \pi/\sqrt{3} + C) + 2P\right\}. \quad \text{......(9)}$$

The formal asymptotic expansion satisfying the differential equation for f_2 and the condition **2.3** (6) is

$$f_2 \sim \alpha_3\alpha_1^{-\frac{4}{3}}\{\tfrac{1}{4}z^4 + 2P\} + \gamma_2' z$$
$$+ \beta_1^2\alpha_1^{-\frac{1}{3}}\{3\gamma_1\delta_1 z^2 + \tfrac{1}{2}\delta_1^2 - 12\gamma_1^2 P\}$$
$$+ \beta_1\pi_0\alpha_1^{-\frac{4}{3}}\{-\tfrac{1}{2}\delta_1 z^2 + 2\gamma_1 P\}. \quad \text{.........(10)}$$

The constant γ_2' is arbitrary. In addition there is a normal solution of rank 3, consisting of a factor $\exp(-\frac{1}{3}z^3)$ multiplying a descending series, which also contains an arbitrary constant. If we replace $\alpha_3\alpha_1^{-\frac{4}{3}}$ by a third arbitrary constant, we have the solution at infinity in its full generality. Comparing this with the complete solution obtained by adding arbitrary multiples of z and h_2 to (5), and remembering that h_2 is asymptotically equal to a multiple of g_2 plus a multiple of z, we deduce that there are constants S_2, T_2, γ_2 and δ_2, such that

$$k_2 - S_2 g_2 + \gamma_2 z \sim 3\gamma_1\delta_1 z^2 + \tfrac{1}{2}\delta_1^2 - 12\gamma_1^2 P, \quad \text{......(11)}$$

and

$$l_2 - T_2 g_2 + \delta_2 z \sim -\tfrac{1}{2}\delta_1 z^2 + 2\gamma_1 P. \quad \text{.........(12)}$$

The constants S_2 and T_2 were determined numerically from the formulae

$$S_2 \sim (k_2'' - 6\gamma_1\delta_1 + 12\gamma_1^2 P'')/g_2'', \quad \text{...........(13)}$$

and

$$T_2 \sim (l_2'' + \delta_1 - 2\gamma_1 P'')/g_2'', \quad \text{..............(14)}$$

where dashes denote differentiation with respect to z. The constants γ_2 and δ_2 were then found numerically from the formulae

$$\gamma_2 \sim 6\gamma_1\delta_1 z - 12\gamma_1^2 P' + S_2 g_2' - k_2', \quad \text{...........(15)}$$

and

$$\delta_2 \sim -\delta_1 z + 2\gamma_1 P' + T_2 g_2' - l_2'. \quad \text{...........(16)}$$

g_2' and g_2'' were calculated from the asymptotic expansion (9); k_2, k_2' and k_2'' were calculated from the series (6) for $z = 0{\cdot}6$, $0{\cdot}7$, $0{\cdot}8$, $0{\cdot}9$ and $1{\cdot}0$, and then computed for higher values of z by the numerical solution of the differential equation*. A similar procedure (except that the series was used up to the value $1{\cdot}3$ for z) was carried out for l_2. The limiting values of the right-hand sides of (13), (14), (15) and (16) were thus found, with the following results:

$$S_2 = 0{\cdot}044, \quad T_2 = 0{\cdot}203, \quad \gamma_2 = 0{\cdot}757, \quad \delta_2 = -0{\cdot}493. \quad \text{...(17)}$$

* The process adopted for the numerical solution of differential equations was that of Adams, described in Chap. XIV of Whittaker and Robinson's *Calculus of Observations*, and by Kriloff in the *Proceedings of the First International Congress for Applied Mechanics*, Delft (1924), p. 212. The method is so much less laborious than others in use (that of Runge and Kutta for example) that the numerical work in this paper was possible only because it was available.

The value of β_2 can now be found by comparing the asymptotic expansion of (5) with (10). The result is

$$\beta_2 = 3^{\frac{5}{3}}\Gamma(\tfrac{4}{3})\,\alpha_3\alpha_1^{-\frac{4}{3}}/\Gamma(\tfrac{2}{3}) - \beta_1^2\alpha_1^{-\frac{1}{3}}S_2 - \beta_1\pi_0\alpha_1^{-\frac{4}{3}}T_2$$
$$= 17{\cdot}804a_3a_1^{-\frac{4}{3}} - 0{\cdot}216(p_0+2a_2)^2a_1^{-\frac{7}{3}} - 0{\cdot}936p_0(p_0+2a_2)\,a_1^{-\frac{7}{3}}. \quad \text{......(18)}$$

The asymptotic expansion of f_2 is

$$f_2 \sim A_2\eta^4 + C_2\eta^2 + D_2'\eta\log\eta + D_2\eta + E_2 + \frac{G_2}{\eta^2} + \dots, \quad \text{...(19)}$$

where

$$\left.\begin{aligned}
A_2 &= 20\tfrac{1}{4}a_3,\\
C_2 &= \frac{3^{\frac{7}{3}}\Gamma(\frac{1}{3})}{2\Gamma(\frac{2}{3})}\,\frac{a_2(p_0+2a_2)}{a_1^{\frac{5}{3}}} = 12{\cdot}841\,\frac{a_2(p_0+2a_2)}{a_1^{\frac{5}{3}}},\\
D_2' &= 18a_3/a_1 - 6a_2(p_0+2a_2)/a_1^2,\\
D_2 &= a_1^{-1}\{6a_3 - 2a_2(p_0+2a_2)\}\log a_1\\
&\quad + 6a_3a_1^{-1}(4\log_e 3 - 2 + \pi/\sqrt{3} + C)\\
&\quad + a_1^{-2}a_2(p_0+2a_2)(6 - 4\log_e 3) - \gamma_2\beta_1^2\\
&\qquad - 3\delta_2\beta_1 a_1^{-1}p_0\\
&= a_1^{-1}\{6a_3 - 2a_2(p_0+2a_2)\}\log a_1 + 28{\cdot}712a_3a_1^{-1}\\
&\quad + 1{\cdot}606a_1^{-2}a_2(p_0+2a_2) - 7{\cdot}73a_1^{-2}(p_0+2a_2)^2\\
&\qquad + 4{\cdot}73a_1^{-2}p_0(p_0+2a_2),\\
E_2 &= \frac{3^{\frac{8}{3}}}{8}\left(\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}\right)^2\frac{(p_0+2a_2)^2}{a_1^{\frac{7}{3}}} = 1{\cdot}0179a_1^{-\frac{7}{3}}(p_0+2a_2).
\end{aligned}\right\} \quad \text{......(20)}$$

2.34. When η is small, the velocity may be calculated from **2.3** (2). In particular

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{1}{9}\left(\frac{d^2f_0}{d\eta^2} + \xi\frac{d^2f_1}{d\eta^2} + \xi^2\frac{d^2f_2}{d\eta^2} + \dots\right)$$
$$= a_1 + b_1\xi + c_1\xi^2 + d_1\xi^3 + \dots, \quad \text{...........(1)}$$

where

$$\left.\begin{aligned}
b_1 &= \frac{3^{\frac{2}{3}}(p_0+2a_2)}{\Gamma(\frac{2}{3})\,a_1^{\frac{1}{3}}} = 1{\cdot}536a_1^{-\frac{1}{3}}(p_0+2a_2),\\
c_1 &= \frac{3^{\frac{7}{3}}\Gamma(\frac{4}{3})\,a_3}{\Gamma(\frac{2}{3})\,a_1^{\frac{2}{3}}} - S_2\frac{3^{\frac{4}{3}}(p_0+2a_2)^2}{[\Gamma(\frac{2}{3})]^2a_1^{\frac{5}{3}}} - T_2\frac{3p_0(p_0+2a_2)}{\Gamma(\frac{2}{3})\,a_1^{\frac{5}{3}}}\\
&= 8{\cdot}560a_3a_1^{-\frac{2}{3}} - 0{\cdot}104a_1^{-\frac{5}{3}}(p_0+2a_2)^2\\
&\quad - 0{\cdot}450a_1^{-\frac{5}{3}}p_0(p_0+2a_2),\\
&\text{and}\\
d_1 &= 24a_4a_1^{-1} + \text{terms vanishing with } a_3 \text{ and } p_0+2a_2.
\end{aligned}\right\} \quad \text{......(2)}$$

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We are now in a position to consider the question discussed in § **1.5**—is the method a practicable one for step by step calculations? The answer must depend on the values of the coefficients a_1, a_2, a_3, etc. If the initial velocity u_0 can be represented by a cubic, or if the coefficients after a_3 are small, then it is likely that the terms already calculated will give a good approximation up to values of x sufficiently large for step by step calculations to be possible, though it would be safer to proceed one, or even two, stages further. If u_0 can be represented by a quartic, or if the coefficients after a_4 are small, it is imperative to proceed one stage further, and safer to proceed two or three stages further.

The solution for f_3 with general values of p_0 and of the coefficients a_0, a_1, a_2, etc., involves the numerical calculation of six particular integrals, and may be possible; but any such general calculation for f_4 will be difficult, and for f_5 appears to be out of the question without elaborate machinery or many co-workers. On the other hand, it is certainly not out of the question to calculate f_3, f_4 and f_5 when the coefficients have given numerical values, since each calculation requires the solution of only one equation.

On the whole, there seems a reasonable chance that step by step calculations by this method may be possible, even if laborious.

2.4. When η is large, f_r' increases rapidly with r, and formula **2.3** (2) is unusable. We must therefore use a different development for ψ. Now if we restrict ourselves to three terms, the equation **2.2** (3) for ψ, combined with **2.3** (1) and the asymptotic developments for f_1 and f_2, gives the formula

$$\begin{aligned}\psi = A_0\xi^2\eta^2 + \xi^3(A_1\eta^3 + C_1\eta + D_1)\\ + \xi^4(A_2\eta^4 + C_2\eta^2 + D_2'\eta\log\eta + D_2\eta + E_2 + G_2\eta^{-2} + \ldots),\end{aligned} \qquad \ldots\ldots(1)$$

where A_0 is $9a_1/2$, and the other coefficients are given by **2.31** (14) and **2.33** (20). If we assume that rearrangement is permissible, we can write this as

$$\begin{aligned}\psi = A_0(\tfrac{1}{3}y)^2 + A_1(\tfrac{1}{3}y)^3 + A_2(\tfrac{1}{3}y)^4 + \ldots\\ + \xi^2\{C_1(\tfrac{1}{3}y) + C_2(\tfrac{1}{3}y)^2 + \ldots\}\\ + \xi^3\{D_1 + (D_2 - D_2'\log 3)(\tfrac{1}{3}y) + D_2'y\log y + \ldots\}\\ - \xi^3\log\xi\,\{D_2'(\tfrac{1}{3}y) + \ldots\}\\ + \xi^4(E_2 + \ldots) + \ldots\end{aligned} \qquad \ldots\ldots(2)$$

We therefore assume, as a form valid for sufficiently large values of $yx^{-\frac{1}{3}}$, that

$$\psi = \psi_0(y) + \xi^2\frac{\psi_2(y)}{2!} + \xi^3\frac{\psi_3(y)}{3!} + \xi^3\log\xi\,\frac{\bar{\psi}_3(y)}{3!} + \xi^4\frac{\psi_4(y)}{4!} + \ldots, \qquad \ldots\ldots(3)$$

where $$\psi_0' = u_0 = a_1 y + a_2 y^2 + a_3 y^3 + \dots, \qquad \dots\dots\dots\dots(4)$$
and
$$\left.\begin{aligned} \psi_2 &= \tfrac{2}{3}C_1 y + \tfrac{2}{9}C_2 y^2 + \dots, \\ \psi_3 &= 6D_1 + 2(D_2 - D_2' \log 3)\, y + 2D_2' y \log y + \dots, \\ \bar{\psi}_3 &= -2D_2' y + \dots, \\ \psi_4 &= 24E_2 + \dots. \end{aligned}\right\} \quad \dots(5)$$
The substitution of (3) into **2.1** (1) gives
$$\left.\begin{aligned} &\psi_0'\psi_2' - \psi_0''\psi_2 = 0, \\ &\psi_0'\bar{\psi}_3' - \psi_0''\bar{\psi}_3 = 0, \\ &\psi_0'(\psi_3' + \tfrac{1}{3}\bar{\psi}_3') - \psi_0''(\psi_3 + \tfrac{1}{3}\bar{\psi}_3) = 6(\psi_0''' + p_0), \\ &\psi_0'\psi_4' - \psi_0''\psi_4 = 3(\psi_2\psi_2'' - \psi_2'^2), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \quad \dots(6)$$
where dashes denote differentiations with respect to y. Hence
$$\psi_2 = \text{constant}\ \psi_0' = \text{constant}\ (a_1 y + a_2 y^2 + \dots). \quad \dots\dots(7)$$
Comparison of the coefficient of y with that in the first equation of (5) gives
$$\psi_2 = \frac{2C_1}{3a_1}\psi_0', \dots\dots\dots\dots\dots\dots\dots\dots(8)$$
and then we verify that the coefficient of y^2 is $\frac{2}{9}C_2$.

Similarly $$\bar{\psi}_3 = -\frac{2D_2'}{a_1}\psi_0'. \dots\dots\dots\dots\dots\dots\dots(9)$$

Next $$\psi_3 + \tfrac{1}{3}\bar{\psi}_3 = 6\psi_0' \int^y \frac{\psi_0''' + p_0}{\psi_0'^2}\, dy, \qquad \dots\dots\dots\dots(10)$$
or, since $\bar{\psi}_3$ is a multiple of ψ_0',
$$\psi_3 = 6\psi_0' \int^y \frac{\psi_0''' + p_0}{\psi_0'^2}\, dy. \qquad \dots\dots\dots\dots\dots(11)$$
Now
$$\frac{\psi_0''' + p_0}{\psi_0'^2} = \frac{p_0 + 2a_2}{a_1^2 y^2}\left\{1 + \left(\frac{6a_3}{p_0 + 2a_2} - \frac{2a_2}{a_1}\right) y + O(y^2)\right\}. \qquad \dots\dots(12)$$
We therefore write
$$F(y) = \frac{\psi_0''' + p_0}{\psi_0'^2} - \frac{p_0 + 2a_2}{a_1^2 y^2} - \left\{\frac{6a_3}{a_1^2} - \frac{2a_2(p_0 + 2a_2)}{a_1^3}\right\}\frac{1}{y}, \qquad \dots\dots(13)$$
and
$$\psi_3 = 6\psi_0' \int_0^y F(y)\, dy - 6\psi_0' \left\{\frac{p_0 + 2a_2}{a_1^2 y} + \left(\frac{2a_2(p_0 + 2a_2)}{a_1^3} - \frac{6a_3}{a_1^2}\right)\log y\right\} + K\psi_0', \dots(14)$$

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where K is a constant to be determined. The expansion of (14) is

$$\psi_3 = -6\frac{p_0+2a_2}{a_1} + y\left\{Ka_1 - \frac{6a_2(p_0+2a_2)}{a_1^2}\right\}$$
$$+ y\log y\left\{\frac{36a_3}{a_1} - \frac{12a_2(p_0+2a_2)}{a_1^2}\right\} + O(y^2). \quad \ldots(15)$$

We verify that the constant term and the coefficient of $y\log y$ are $6D_1$ and $2D_2'$ respectively, and find that

$$Ka_1 = 2(D_2 - D_2'\log 3) - 6a_1^{-2}a_2(p_0+2a_2). \quad \ldots(16)$$

Finally, by using (8), we prove that

$$\psi_4 = \tfrac{4}{3}C_1^2 a_1^{-2}\psi_0'' + \text{constant}\,\psi_0', \quad \ldots\ldots\ldots\ldots(17)$$

and verify that the constant term is $24E_2$, but we cannot determine the constant from (5).

This solution does not satisfy the boundary conditions for $y=0$, and is valid only when $yx^{-\frac{1}{3}}$ is sufficiently large. On the other hand, it is simple to verify analytically that the boundary condition at infinity is satisfied as far as our series will take us, as indeed it must be from our method of calculation.

3.

3.1. It is required to solve the equations **2.1** (1) and (2), with the boundary conditions $u=v=0$ at $y=0$, u tends to a limit independent of y as y tends to infinity, and

$$u = u_0 = a_0 + a_1 y + a_2 y^2 + \ldots \qquad (a_0 \neq 0) \quad \ldots\ldots\ldots(1)$$

at $x=0$.

Take n equal to 2 in the equations of § **2.2**, and expand f in the form **2.3** (1). Then

$$u = \tfrac{1}{2}\{f_0'(\eta) + \xi f_1'(\eta) + \xi^2 f_2'(\eta) + \ldots\}, \quad \ldots\ldots\ldots(2)$$

where

$$\left.\begin{aligned} &f_0''' + f_0 f_0'' = 0,\\ &f_1''' + f_0 f_1'' - f_0' f_1' + 2f_0'' f_1 = 0,\\ &f_2''' + f_0 f_2'' - 2f_0' f_2' + 3f_0'' f_2 = f_1'^2 - 2f_1'' f_1 - 8p_0,\end{aligned}\right\} \ldots(3)$$

and so on.

The boundary conditions are

$$f_r(0) = f_r'(0) = 0, \text{ and } \lim_{\eta\to\infty} f_r'(\eta)/\eta^r = 2^{r+1}a_r. \quad \ldots\ldots(4)$$

3.11. The equation for f_0 has been studied by Blasius (*loc. cit.*) and Toepfer*, whose solution may be written

$$f_0 = \beta_0 \frac{\eta^2}{2!} - \beta_0^2 \frac{\eta^5}{5!} + 11\beta_0^3 \frac{\eta^8}{8!} - 375\beta_0^4 \frac{\eta^{11}}{11!} + \dots, \quad \dots(1)$$

and $$f_0 \sim A_0\eta + B_0 + \epsilon_0 \left\{\tfrac{1}{2}e^{-a_0\eta^2} - a_0^{\frac{1}{2}}\eta \int_{a_0^{\frac{1}{2}}\eta}^{\infty} e^{-u^2}\, du + \dots\right\}, \dots(2)$$

where $$\beta_0 = 1{\cdot}32824 a_0^{\frac{3}{2}}, \quad \dots\dots(3)$$

and $$A_0 = 2a_0, \quad B_0 = -1{\cdot}72075 a_0^{\frac{1}{2}}, \quad \epsilon_0 = 0{\cdot}923 a_0^{\frac{1}{2}}. \quad \dots\dots(4)$$

3.12. The solution for f_1 having a double zero at the origin is

$$f_1 = \beta_1 \left(\frac{\eta^2}{2!} - \frac{\beta_0\eta^5}{5!} + \dots\right), \quad \dots\dots(1)$$

and the solution satisfying the boundary condition at infinity is

$$f_1 \sim A_1\eta^2 + B_1\eta + C_1, \quad \dots\dots(2)$$

where $$A_1 = 2a_1, \quad B_1 = 2a_1 a_0^{-1} B_0 = -3{\cdot}4415 a_1 a_0^{-\frac{1}{2}}. \quad \dots\dots(3)$$

The constants β_1 and C_1 must be determined numerically (as in § **5**, for example).

The solutions for f_2, f_3 and so on are to be found in a similar way.

3.2. When η is large,

$$\begin{aligned}\psi = {} & A_0(\tfrac{1}{2}y) + A_1(\tfrac{1}{2}y)^2 + \dots \\ & + \xi\{B_0 + B_1(\tfrac{1}{2}y) + \dots\} \\ & + \xi^2\{C_1 + \dots\} + \dots . \quad \dots\dots(1)\end{aligned}$$

We therefore assume, as a form valid for large values of η, that

$$\psi = \psi_0 + \xi\psi_1 + \xi^2 \frac{\psi_2}{2!} + \dots \quad \dots\dots(2)$$

and find, by substituting in **2.1** (1), that

$$\left.\begin{aligned}&\psi_0'\psi_1' - \psi_0''\psi_1 = 0, \\ &\psi_0'\psi_2' - \psi_0''\psi_2 = 2(p_0 + \psi_0''') + \psi_1\psi_1'' - \psi_1'^2,\end{aligned}\right\} \quad \dots(3)$$

and so on.

Thus $$\psi_1 = \text{constant } \psi_0', \quad \dots\dots(4)$$

and comparison of the constant term with its value in (1) gives

$$\psi_1 = B_0\psi_0'/a_0.$$

We verify that the coefficient of y is $\frac{1}{2}B_1$.

* *Zeitschrift f. Math. u. Physik*, 60, 397–98 (1912).

The solution for ψ_2 is

$$\psi_2 = 2\psi_0' \int_0^y \frac{p_0 + \psi_0''}{\psi_0'^2}\, dy + B_0^2 a_0^{-2} \psi_0'' + K\psi_0', \quad \text{...(5)}$$

and comparison with (1) gives

$$Ka_0 + B_0^2 a_0^{-2} a_1 = 2C_1. \quad \text{..................(6)}$$

ψ_3, ψ_4 and so on can be found similarly when f_3, f_4 and so on have been found.

4.

4.1. It is required to solve **2.1** (1) and (2) with the same boundary conditions as before, except that*

$$u_0 = a_2 y^2 + a_3 y^3 + \dots \qquad (a_2 \neq 0). \quad \text{............(1)}$$

Take n equal to 4 in § **2.2**, and expand f as in **2.3** (1). Then

$$u = \tfrac{1}{4}\xi^2 \{f_0'(\eta) + \xi f_1'(\eta) + \xi^2 f_2'(\eta) + \dots\}, \quad \text{.........(2)}$$

and

$$\left.\begin{aligned} &f_0''' + 3f_0 f_0'' - 2f_0'^2 = \pi_0, \\ &f_1''' + 3f_0 f_1'' - 5f_0' f_1' + 4f_0'' f_1 = 0, \\ &f_2''' + 3f_0 f_2'' - 6f_0' f_1' + 5f_0'' f_2 = 3f_1'^2 - 4f_1 f_1'', \end{aligned}\right\} \quad \text{......(3)}$$

and so on, where $\pi_0 = -64p_0$. ....(4)

The boundary conditions are

$$f_r(0) = f_r'(0) = 0, \text{ and } \lim_{\eta \to \infty} f'_{r-2}(\eta)/\eta^r = 4^{r+1} a_r. \quad \text{...(5)}$$

4.11. The solution for f_0 with a double zero at the origin is

$$\begin{aligned} f_0 = {} & \frac{\beta_0 \eta^2}{2!} + \pi_0 \frac{\eta^3}{3!} + \beta_0^2 \frac{\eta^5}{5!} - 13\beta_0^3 \frac{\eta^8}{8!} - 18\pi_0 \beta_0^2 \frac{\eta^9}{9!} + 687\beta_0^4 \frac{\eta^{11}}{11!} \\ & + 3105\pi_0 \beta_0^3 \frac{\eta^{12}}{12!} + 3780\pi_0^2 \beta_0^2 \frac{\eta^{13}}{13!} - 77895\beta_0^5 \frac{\eta^{14}}{14!} \\ & - 752841\pi_0 \beta_0^4 \frac{\eta^{15}}{15!} + 2515725\pi_0^2 \beta_0^3 \frac{\eta^{16}}{16!} \\ & + (14648817\beta_0^6 - 2910600\pi_0^3 \beta_0^2) \frac{\eta^{17}}{17!} \\ & + 254083338\pi_0 \beta_0^6 \frac{\eta^{18}}{18!} + \dots, \end{aligned} \quad \text{............................(1)}$$

where β_0 is an arbitrary constant.

* A glance at equation **1.5** (5) shows that when, as here, a_1 is zero, the conditions **1.5** (3) for the absence of singularities are by no means sufficient. The conditions required are complicated. If we suppose u expanded in a power series in x, $u = u_0 + u_1 x + u_2 x^2 + \dots$, with u_1, u_2 as power series in y, $u_1 = b_1 y + b_2 y^2 + \dots$, $u_2 = c_1 y + c_2 y^2 + \dots$, we find that we must have

$$2a_2 + p_0 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = 0, \quad 6!\, a_6 = 2p_0 p_2, \quad a_7 = 0, \quad a_9 = 0,$$

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The solution satisfying the boundary condition at infinity is

$$f_0 \sim A_0\eta^3 + B_0\eta^2 + C_0\eta + D_0 + E_0\eta^{-1} + F_0\eta^{-2} + \dots, \quad \dots(2)$$

where

$$\left.\begin{array}{l} A_0 = 64a_2/3, \quad C_0 = B_0^2/3A_0, \quad D_0 = B_0^3/27A_0^2, \\ E_0 = (\pi_0 - 6A_0)/36A_0, \quad F_0 = -B_0E_0/3A_0, \dots, \end{array}\right\} \dots\dots(3)$$

and B_0 is arbitrary. β_0 and B_0 must again be determined numerically, but it is evident that the work will, in general, be much more difficult than in the other cases.

4.12. The problem* simplifies considerably if π_0 is zero; the numerical work is then similar to that carried out in § **5**. Hydrodynamically, this special case will not be interesting, since if the pressure gradient vanishes, there will be no return flow, and no separation of the forward flow from the wall.

4.2. If u_0 has a zero of order m at the origin, and p_0 is zero, we can find a solution by taking n equal to $m+2$ in § **2.2**. But if p_0 is not zero, the method will not give a solution when m is greater than 2. The problems with m greater than 2 do not appear to have any physical meaning.

5.

5.1. In this paragraph the laminar flow behind an infinitely thin plate of length l along the stream will be calculated numerically.

Let x_1 be distance from a plane perpendicular to the plate through its rear edge, y_1 distance from the plane of the plate, and u_1 and v_1 the components of velocity in the directions x_1 and y_1 increasing. Let R be $4Ul/\nu$, where U is the undisturbed velocity

and so on. Only $a_8, a_{12}, a_{16}, a_{20}, \dots$ are at our disposal. In addition $b_1, c_1, d_1, \dots$ are determined, not from the equations for $u_1, u_2, u_3, \dots$ respectively, but from the conditions for the absence of singularities in $u_2, u_3, u_4, \dots$ respectively. Further, there is an ambiguity of sign which can be decided only from physical considerations. A little more light, but not much, is shed on the matter by considering the equation in the form

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}\int^y \left[\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x}\right)\Big/ u^2\right] dy + \left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x}\right)\Big/ u.$$

* It is not suggested that the above solution of the problem is always valid. When f_0 has the asymptotic expansion given by **4.11** (2), in which A_0 must not vanish, then the equation for f_1 has three asymptotic solutions, giving f_1 of order η^4, of order η^2, and exponentially small respectively. The method used fails if the general solution for f_1 with a double zero at the origin does not involve the solution of order η^4 at infinity. This happens, for example, when $p_0 + 2a_2 = 0$ or $\frac{1}{2}\pi_0 = 64a_2$. The solution for f_0 is then $\pi_0\eta^3/3!$, and the general solution for f_1 with a double zero at the origin is any arbitrary multiple of η^2. This difficulty may occur, of course, in solving for any function f_r, and, though the matter may be tested by numerical computation in any given case for the first few of the f_r, a theoretical discussion of the general equation appears too difficult to be possible.

of the stream, x be $x_1/4l$, y be $R^{\frac{1}{2}}y_1/4l$, u be u_1/U and v be $R^{\frac{1}{2}}v_1/U$. The initial velocity distribution is given by*

$$u_0 = \tfrac{1}{2}d\zeta/dy, \quad \text{.........................(1)}$$

where

$$\frac{d^3\zeta}{dy^3} + \zeta\frac{d^2\zeta}{dy^2} = 0, \quad \text{.........................(2)}$$

ζ has a double zero at the origin and $d\zeta/dy$ tends to 2 as y tends to infinity.

u_0 may be expanded in a series, with the result

$$u_0 = a_1y + a_4y^4 + a_7y^7 + \ldots, \quad \text{.................(3)}$$

where

$$a_1 = \tfrac{1}{2}\alpha, \quad a_4 = -\tfrac{1}{2}\alpha^2/4!, \quad a_7 = 5{\cdot}5\alpha^3/7!, \quad a_{10} = -187{\cdot}5\alpha^4/10!, \ \ldots \quad \text{......(4)}$$

and

$$\alpha = 1{\cdot}32824. \quad \text{.........................(5)}$$

The flow outside the boundary layer is uniform, and the pressure gradient zero.

5.2. Take n equal to 3 in the equations of § **2.2**, and assume that

$$f = f_0(\eta) + \xi^3 f_3(\eta) + \xi^6 f_6(\eta) + \ldots, \quad \text{...........(1)}$$

so that

$$u = \tfrac{1}{3}\xi\{f_0'(\eta) + \xi^3 f_3'(\eta) + \xi^6 f_6'(\eta) + \ldots\}, \quad \text{.........(2)}$$

where

$$\left.\begin{aligned} &f_0''' + 2f_0f_0'' - f_0'^2 = 0,\\ &f_3''' + 2f_0f_3'' - 5f_0'f_3' + 5f_0''f_3 = 0,\\ &f_6''' + 2f_0f_6'' - 8f_0'f_6' + 8f_0''f_6 = 4f_3'^2 - 5f_3f_3'',\end{aligned}\right\} \quad \text{......(3)}$$

and so on.

The boundary conditions $v = 0$, $\partial u/\partial y = 0$ at $y = 0$ are equivalent to

$$f_r(0) = f_r''(0) = 0, \quad \text{.....................(4)}$$

while the condition $u = u_0$ at $x = 0$ requires that

$$\lim_{\eta\to\infty} f_{3r}'(\eta)/\eta^{3r+1} = \alpha_{3r+1}, \quad \text{.................(5)}$$

where

$$\alpha_{3r+1} = 3^{3r+2}a_{3r+1}. \quad \text{.....................(6)}$$

In particular,

$$f_0'/\eta \to \alpha_1 = 9\alpha/2 = 5{\cdot}97708, \quad f_3'/\eta^4 \to \alpha_4 = -3^5\alpha^2/2\,.\,4! = -8{\cdot}93136,$$

and

$$f_6'/\eta^7 \to \alpha_7 = 3^8\,.\,11\,.\,\alpha^3/2\,.\,7! = 16{\cdot}7777. \quad \text{.........(7)}$$

5.21. The required solution of the first equation of **5.2** (3) is

$$f_0 = \beta_0\eta + \beta_0^2\frac{\eta^3}{3!} - 2\beta_0^3\frac{\eta^5}{5!} + 10\beta_0^4\frac{\eta^7}{7!} - 56\beta_0^5\frac{\eta^9}{9!} + \ldots, \quad \text{......(1)}$$

* Blasius, *loc. cit.*

2-2

where β_0 is to be determined from **5.2** (7). Let $F_0(\eta)$ be the solution for which $F_0(0)$ and $F_0''(0)$ both vanish, and $F_0'(0)$ is 1. This solution is given by the right-hand side of (1) with β_0 equal to 1. Then

$$f_0(\eta) = \beta_0^{\frac{1}{3}} F_0(\beta_0^{\frac{1}{3}}\eta), \quad \text{......(2)}$$

so that

$$\lim_{\eta\to\infty} f_0'(\eta)/\eta = \beta_0^{\frac{2}{3}} \lim_{\eta\to\infty} F_0'(\eta)/\eta = \beta_0^{\frac{2}{3}} \lim_{\eta\to\infty} F_0''(\eta). \quad \text{...(3)}$$

Hence

$$\beta_0 = \{\alpha_1 / \lim_{\eta\to\infty} F_0''(\eta)\}^{\frac{3}{2}}. \quad \text{......(4)}$$

Asymptotically, the equation is satisfied by

$$\tfrac{1}{2}\gamma_0\eta'^2 + \epsilon_0 \exp(-\tfrac{1}{3}\gamma_0\eta'^3)\{(\gamma_0\eta'^2)^{-2} + \ldots\}, \quad \text{......(5)}$$

where

$$\eta' = \eta + \delta_0, \quad \text{......(6)}$$

and γ_0, δ_0 and ϵ_0 are arbitrary constants.

By numerical solution of the equation it was found that

$$F_0(\eta) \sim 0{\cdot}42356(\eta + 0{\cdot}65364)^2. \quad \text{......(7)}$$

Hence

$$\beta_0 = 3{\cdot}67869, \quad \text{......(8)}$$

and

$$f_0(\eta) \sim \tfrac{1}{2}\alpha_1\eta'^2, \quad \text{......(9)}$$

where η' is given by (6) and

$$\delta_0 = 0{\cdot}3408. \quad \text{......(10)}$$

The work was checked, and f_0, f_0' and f_0'' tabulated, by solving the differential equation numerically with the value of β_0 given by (8) as initial value for f_0'.

5.22. The required solution of the second equation of **5.2** (3) is $\beta_3 F_3(\eta)$, where

$$F_3(\eta) = \eta + 5\beta_0\frac{\eta^3}{3!} + 40\beta_0^3\frac{\eta^7}{7!} - 160\beta_0^4\frac{\eta^9}{9!} + \ldots, \quad \text{...(1)}$$

and β_3 is to be determined from **5.2** (7).

Asymptotically, the equation is satisfied by

$$\gamma_3\{\eta'^5 + 20\alpha_1^{-1}\eta'(\eta' - \delta_3)\}, \quad \text{......(2)}$$

where exponentially small terms have been neglected, γ_3 and δ_3 are arbitrary constants, and η' is given by **5.21** (6) with δ_0 equal to 0·3408 as in **5.21** (10).

By numerical solution of the differential equation it was found that

$$5F_3(\eta) \sim 2{\cdot}5219\{\eta'^5 + 20\alpha_1^{-1}\eta'(\eta' - 0{\cdot}3822)\}, \quad \text{......(3)}$$

so that $F_3'(\eta)/\eta^4$ tends to 2·5219. Hence

$$\beta_3 = -8{\cdot}93136/2{\cdot}5219 = -3{\cdot}5415, \quad \text{......(4)}$$

and

$$5f_3 \sim \alpha_4\{\eta'^5 + 20\alpha_1^{-1}\eta'(\eta' - \delta_3)\}, \quad \text{......(5)}$$

where

$$\delta_3 = 0{\cdot}3822. \quad \text{......(6)}$$

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The values of f_3, f_3' and f_3'' were tabulated by multiplying the values of F_3, F_3' and F_3'' by $-3{\cdot}5415$.

5.23. Let $\beta_3^2 G_6(\eta)$ be a particular integral of the third equation of **5.2** (3), and $F_6(\eta)$ the complementary function for which $F_6(0) = F_6''(0) = 0$ and $F_6'(0) = 1$.

Then

$$F_6(\eta) = \eta + 8\beta_0 \frac{\eta^3}{3!} + 24\beta_0^2 \frac{\eta^5}{5!} + 112\beta_0^3 \frac{\eta^7}{7!} + \ldots, \quad \ldots\ldots(1)$$

and we may take

$$G_6(\eta) = 4\frac{\eta^3}{3!} + 6\beta_0 \frac{\eta^5}{5!} + 132\beta_0^2 \frac{\eta^7}{7!} + \ldots \quad \ldots\ldots\ldots(2)$$

The required solution is then

$$f_6(\eta) = \beta_3^2 G_6(\eta) + \beta_6 F_6(\eta), \quad \ldots\ldots\ldots\ldots\ldots(3)$$

where β_6 is to be determined from **5.2** (7).

Asymptotically, the equation is satisfied by

$$\tfrac{1}{8}\gamma_6(\eta'^8 + 28\alpha_1^{-1}\eta'^5 + 280\alpha_1^{-2}\eta'^2) + \delta_6\eta'$$
$$+ 2\alpha_4^2\alpha_1^{-2}(\eta'^5 - 2\delta_3\eta'^4 + 2\alpha_1^{-1}\eta'^2 + 4\alpha_1^{-1}\delta_3^2), \ldots(4)$$

where exponentially small terms have been neglected, and γ_6 and δ_6 are arbitrary constants.

If this is to represent a solution satisfying **5.2** (7), then $\gamma_6 = \alpha_7$. Hence we have the following asymptotic equality for β_6:

$$\beta_6 \sim \{7\alpha_7(\eta'^6 + 10\alpha_1^{-1}\eta'^3 + 10\alpha_1^{-2}) + 8\alpha_4^2\alpha_1^{-2}(5\eta'^3 - 6\delta_3\eta'^2 + \alpha_1^{-1})$$
$$- \beta_3^2 G_6''(\eta)\}/F_6''(\eta). \quad \ldots\ldots(5)$$

$F_6(\eta)$ and $G_6(\eta)$, and the first two derivatives of each, were tabulated by numerical solution of the differential equation, and the limiting value of the right-hand side of (5) was thus found, with the result

$$\beta_6 = 8{\cdot}119. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

The value of δ_6 was then found by comparing the derivative of (4) with $\beta_3^2 G_6'(\eta) + \beta_6 F'(\eta)$ for sufficiently large values of η, with the result

$$\delta_6 = -8{\cdot}291. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

The asymptotic expansion of f_6 is given by (5) with this value of δ_6 and with γ_6 equal to α_7.

f_6' was tabulated from the equality $f_6' = \beta_3^2 G_6' + \beta_6 F_6'$.

5.24. For small values of η the ratio u may now be calculated from **5.2** (2). The first three coefficients have been tabulated; further coefficients were not calculated. It is impossible to form any theoretical estimate of the error introduced by stopping after three terms: the best we can do is to consider the magnitudes of the three terms separately. For this purpose the following table (Table I) is inserted. The values of u calculated from **5.2** (2) are shown in Table III, pp. 26, 27. In some cases, two values of u are recorded in this table. The upper one is then the value calculated from the first three terms of **5.2** (2); the lower one, in italics, is calculated by making an estimate of the probable effect of the terms after the third, combined with extrapolation and the need for a smooth passing over into the values of Table IV.

TABLE I. *Coefficients in the series* **5.2** (2) *for* u.

η	$\frac{1}{3}f_0'$	$-\frac{1}{3}f_3'$	$\frac{1}{3}f_6'$
0	1·2262	1·181	2·706
0·1	1·2486	1·289	3·192
0·2	1·3143	1·615	4·700
0·3	1·4186	2·160	7·394
0·4	1·5550	2·930	11·582
0·5	1·7157	3·942	17·782
0·6	1·8934	5·223	26·796
0·7	2·0816	6·820	39·807
0·8	2·2759	8·795	58·493
0·9	2·4731	11·225	85·148
1·0	2·6717	14·197	122·833
1·1	2·8707	17·807	175·566
1·2	3·0699	22·156	248·545
1·3	3·2691	27·354	348·433
1·4	3·4683	33·514	386·444

5.3. For large values of η,

$$\left.\begin{aligned} f_0(\eta) &\sim A_0\eta^2 + B_0\eta + C_0, \\ f_3(\eta) &\sim A_3\eta^5 + B_3\eta^4 + C_3\eta^3 + D_3\eta^2 + E_3\eta + F_3, \\ f_6(\eta) &\sim A_6\eta^8 + B_6\eta^7 + C_6\eta^6 + D_6\eta^5 + E_6\eta^4 + F_6\eta^3 + G_6\eta^2 + H_6\eta + I_6, \end{aligned}\right\} \quad \ldots\ldots(1)$$

where the A, B, C, etc. are known. Then, for large values of η,

$$\begin{aligned} \psi &= \xi^2(A_0\eta^2 + B_0\eta + C_0) + \xi^5(A_3\eta^5 + B_3\eta^4 + \ldots) + \xi^8(A_6\eta^8 + \ldots) + \ldots \\ &= A_0(\tfrac{1}{3}y)^2 + A_3(\tfrac{1}{3}y)^5 + \ldots + \xi\{B_0(\tfrac{1}{3}y) + B_3(\tfrac{1}{3}y)^4 + \ldots\} \\ &\quad + \xi^2\{C_0 + C_3(\tfrac{1}{3}y)^3 + \ldots\} + \ldots, \quad \ldots\ldots(2) \end{aligned}$$

if the legitimacy of rearrangement for large η be assumed. This leads to the assumption that for large values of η, ψ may be expanded in the form

$$\psi = \psi_0 + \xi\psi_1 + \xi^2\frac{\psi_2}{2!} + \xi^3\frac{\psi_3}{3!} + \xi^4\frac{\psi_4}{4!} + \ldots, \quad \ldots\ldots(3)$$

where ψ_0, ψ_1, etc. are functions of y satisfying the equations

$$\left.\begin{aligned}&\psi_0'\psi_1' - \psi_0''\psi_1 = 0,\\ &\psi_0'\psi_2' - \psi_0''\psi_2 = \psi_1\psi_1'' - \psi_1'^2,\\ &\psi_0'\psi_3' - \psi_0''\psi_3 = 6\psi_0''' + \psi_1\psi_2'' - 3\psi_1'\psi_2' + 2\psi_1''\psi_2,\end{aligned}\right\}\quad \ldots(4)$$

and so on, dashes denoting differentiations with respect to y. Now ψ_0' is equal to $\frac{1}{2}\zeta'$, where ζ satisfies **5.1** (2). By using this last equation the equations (4) may all be integrated in finite form. The constants of integration are found by expanding the integral in a series and comparing with (2). At each stage we can verify that the coefficients in the series not used to determine the constant of integration are the same as in (2). The derivatives of integrals so found are given below.

$$\left.\begin{aligned}
\psi_0' &= \tfrac{1}{2}\zeta',\\
\psi_1' &= \tfrac{1}{2}A\zeta'',\\
\frac{\psi_2'}{2!} &= \tfrac{1}{2}\frac{A^2}{2!}\zeta''',\\
\frac{\psi_3'}{3!} &= \tfrac{1}{2}\frac{A^3}{3!}\zeta^{(4)} - y\zeta'',\\
\frac{\psi_4'}{4!} &= \tfrac{1}{2}\frac{A^4}{4!}\zeta^{(5)} - Ay\zeta''' - B\zeta'',\\
\frac{\psi_5'}{5!} &= \tfrac{1}{2}\frac{A^5}{5!}\zeta^{(6)} - \frac{A^2}{2!}y\zeta^{(4)} - AB\zeta''',\\
\frac{\psi_6'}{6!} &= \tfrac{1}{2}\frac{A^6}{6!}\zeta^{(7)} - \frac{A^3}{3!}y\zeta^{(5)} - \frac{A^2B}{2!}\zeta^{(4)} + y^2\zeta''' + 3y\zeta'',\\
\frac{\psi_7'}{7!} &= \tfrac{1}{2}\frac{A^7}{7!}\zeta^{(8)} - \frac{A^4}{4!}y\zeta^{(6)} - \frac{A^3B}{3!}\zeta^{(5)} + Ay^2\zeta^{(4)}\\
&\qquad + (3A+2B)y\zeta''' + C\zeta'',\\
\frac{\psi_8'}{8!} &= \tfrac{1}{2}\frac{A^8}{8!}\zeta^{(9)} - \frac{A^5}{5!}y\zeta^{(7)} - \frac{A^4B}{4!}\zeta^{(6)} + \frac{A^2}{2!}y^2\zeta^{(5)}\\
&\qquad + \left(\frac{3A^2}{2!} + 2AB\right)y\zeta^{(4)} + (AC+B^2)\zeta'''.
\end{aligned}\right\}\quad \ldots\ldots(5)$$

A, B and C have been written for brevity instead of $3\delta_0$, $3\delta_0 - 1{\cdot}5\delta_3$, $9\delta_0 + \delta_6/3\alpha$ respectively, where δ_0 is given by **5.21** (10), δ_3 by **5.22** (6), δ_6 by **5.23** (7) and α by **5.1** (5).

ζ, ζ' and ζ'' were tabulated by numerical solution of the differential equation **5.1** (2) with α as initial value of ζ''; and ζ''' and higher derivatives were tabulated from the equation and

TABLE II. *Coefficients in the series* **5.3** (6) *for u.*

y	ψ_0'	ψ_1'	$-\psi_2'/2!$	$-\psi_3'/3!$	$\psi_4'/4!$	$\psi_5'/5!$	$\psi_6'/6!$	$\psi_7'/7!$	$\psi_8'/8!$
0	0	0·6790	0	0	−0·6367	0			
0·1	0·06641	0·6788	0·0023	0·1485	−0·6356	0·0139			
0·2	0·13276	0·6778	0·0092	0·2964	−0·6277	0·0553	+0·942	+1·402	−0·27
0·3	0·19893	0·6749	0·0206	0·4425	−0·6063	0·1231			
0·4	0·26470	0·6694	0·0363	0·5842	−0·5653	0·2147	+1·822	+1·045	−1·04
0·5	0·32978	0·6605	0·0559	0·7187	−0·4994	0·3253			
0·6	0·39378	0·6474	0·0787	0·8423	−0·4046	0·4470	+2·493	+0·155	−2·05
0·7	0·45625	0·6295	0·1039	0·9507	−0·2792	0·5692			
0·8	0·51675	0·6066	0·1303	1·0399	−0·1239	0·6797	+2·764	−1·286	−2·75
0·9	0·57475	0·5785	0·1566	1·1061	+0·0572	0·7648			
1·0	0·62976	0·5455	0·1812	1·1465	+0·2569	0·8112	+2·501	−2·966	−2·51
1·1	0·68130	0·5078	0·2028	1·1593	0·4656	0·8084			
1·2	0·72898	0·4664	0·2199	1·1442	0·6711	0·7504	+1·735	−4·288	−0·97
1·3	0·77245	0·4222	0·2314	1·1026	0·8610	0·6368			
1·4	0·81151	0·3763	0·2368	1·0375	1·0230	0·4740	+0·697	−4·661	+1·43
1·5	0·84605	0·3300	0·2356	0·9533	1·1473	0·2743			
1·6	0·87608	0·2845	0·2282	0·8553	1·2271	0·0545	−0·275	−3·870	+3·64
1·7	0·90175	0·2411	0·2153	0·7540	1·2592	−0·1663			
1·8	0·92332	0·2006	0·1978	0·6467	1·2449	−0·3693	−0·901	−2·239	+4·61
1·9	0·94112	0·1639	0·1773	0·5350	1·1894	−0·5393			
2·0	0·95552	0·1314	0·1548	0·4360	1·1002	−0·6651	−1·093	−0·438	+3·99
2·1	0·96696	0·1033	0·1319	0·3469	0·9871	−0·7420			
2·2	0·97589	0·0797	0·1097	0·2694	0·8601	−0·7704	−0·949	+0·921	+2·30
2·3	0·98271	0·0603	0·0890	0·2041	0·7285	−0·7581			
2·4	0·98781	0·0447	0·0705	0·1512	0·6009	−0·7076	−0·655	+1·568	+0·47
2·5	0·99155	0·0325	0·0546	0·1093	0·4825	−0·6355			
2·6	0·99426	0·0232	0·0413	0·0770	0·3771	−0·5489	−0·366	+1·588	−0·80
2·7	0·99616	0·0162	0·0305	0·0529	0·2878	−0·4582			
2·8	0·99749	0·0111	0·0220	0·0356	0·2141	−0·3706	−0·163	+1·256	−1·32
2·9	0·99838	0·0075	0·0156	0·0233	0·1557	−0·2909			
3·0	0·99898	0·0049	0·0108	0·0149	0·1108	−0·2221	−0·051	+0·838	−1·26
3·1	0·99936	0·0032	0·0073	0·0093	0·0767	−0·1648			

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equations obtained from it by repeated differentiation. Then ψ_0', ψ_1', etc. were tabulated. The results are given in the table above (Table II).

(ψ_9 could also have been determined from the solution for f_6, but the gain in accuracy in the calculation of the velocity would not have justified the labour.)

The ratio u was then computed from the formula

$$u = \psi_0' + \xi\psi_1' + \xi^2 \frac{\psi_2'}{2!} + \xi^3 \frac{\psi_3'}{3!} + \ldots \qquad \ldots\ldots\ldots(6)$$

The results are in Table IV, p. 28. ψ_6', ψ_7' and ψ_8' were not computed for odd values of $10y$. For these values of $10y$, u was obtained by interpolating either for the difference in the values obtained for two neighbouring values of ξ, or for the difference in the values obtained from six terms and from the full nine terms.

When two values of u for one pair of values of y and ξ occur in Table IV, the remarks on p. 22 concerning Table III apply.

When ξ is greater than or equal to 0·35 the last figure of u becomes uncertain. For these values of ξ entries in the table that would have been the same, to the accuracy obtainable, as the corresponding entries for ξ equal to 0·3, have been omitted.

The widening of the "boundary" layer at a sufficient distance behind the plate is so gradual that the process cannot, unfortunately, be followed by calculations of the accuracy obtained.

5.4. The only singularities occur at $x = 0$, and if the value of u for any other value of x be expanded in a series of ascending powers of y^2, there is no theoretical difficulty in continuing the solution by step-by-step calculations, or by successive approximation. But all sufficiently accurate methods are laborious, and I have been content to find a first approximation up to ξ equal to 0·7 by extrapolation—a process always dangerously liable to give inaccurate results.

The results are all collected in Fig. 1. The velocity along the axis is shown in Fig. 2.

My thanks are due to Prof. Prandtl, at whose suggestion the investigation was undertaken.

TABLE III. *Values of u from* **5.2** (2).

$\xi=0·05$		0·10		0·15		0·20		0·25	
$x_1/l=0·0005$		0·004		0·0135		0·032		0·0625	
y	u	y	u	y	u	y	u	y	u
0	0·061	0	0·123	0	0·183	0	0·243	0	0·302
0·015	0·062	0·03	0·125	0·045	0·187	0·06	0·248	0·075	0·307
0·03	0·066	0·06	0·131	0·09	0·196	0·12	0·260	0·15	0·323
0·045	0·071	0·09	0·142	0·135	0·212	0·18	0·280	0·225	0·347
0·06	0·078	0·12	0·155	0·18	0·232	0·24	0·306	0·30	0·378
0·075	0·086	0·15	0·171	0·225	0·255	0·30	0·337	0·375	0·415
0·09	0·095	0·18	0·189	0·27	0·281	0·36	0·371	0·45	0·455
0·105	0·104	0·21	0·207	0·315	0·309	0·42	0·406	0·525	0·496
0·12	0·114	0·24	0·227	0·36	0·337	0·48	0·442	0·60	0·538
0·135	0·124	0·27	0·246	0·405	0·365	0·54	0·478	0·675	0·580
0·15	0·133	0·30	0·266	0·45	0·394	0·60	0·513	0·75	0·620 } *0·618* }
0·165	0·143	0·33	0·285	0·495	0·422	0·66	0·548	0·825	0·659 } *0·658* }
0·18	0·153	0·36	0·305	0·54	0·450	0·72	0·582		
0·195	0·163	0·39	0·324	0·585	0·477	0·78	0·615		
0·21	0·173	0·42	0·344	0·63	0·504	0·84	0·646		

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TABLE III (*continued*).

$\xi=0{\cdot}30$		0·35		0·40		0·45		0·50		0·55	
$x_1/l=0{\cdot}108$		0·1715		0·256		0·3645		0·5		0·6655	
y	u	y	u	y	u	y	u	y	u	y	u
0	0·359	0	0·413	0	0·465 / *0·464*	0	0·513 / *0·511*	0	0·560 / *0·555*	0	0·608 / *0·594*
0·9	0·365	0·105	0·420	0·12	0·472 / *0·471*	0·135	0·521 / *0·518*	0·15	0·569 / *0·563*		
0·18	0·382	0·21	0·439	0·24	0·492 / *0·491*	0·27	0·543 / *0·530*	0·30	0·593 / *0·582*		
0·27	0·410	0·315	0·469 / *0·468*	0·36	0·524 / *0·521*	0·405	0·577 / *0·568*	0·45	0·632 / *0·607*		
0·36	0·445	0·42	0·508 / *0·507*	0·48	0·566 / *0·561*	0·54	0·623 / *0·604*				
0·45	0·487	0·525	0·553 / *0·551*	0·60	0·614 / *0·603*	0·675	0·677 / *0·650*				
0·54	0·532	0·63	0·602 / *0·598*	0·72	0·668 / *0·652*						
0·63	0·578	0·735	0·652 / *0·646*								
0·72	0·624	0·84	0·702 / *0·692*								
0·81	0·670 / *0·666*										

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TABLE IV. *Values of u from* **5.3** (6).

y	$\xi=0$	0·05	0·10	0·15	0·20	0·25	0·30	0·35	0·40	0·45
	$x_1/l=0$	0·0005	0·004	0·0135	0·032	0·0625	0·108	0·1715	0·256	0·3645
0	0									
0·1	0·066									
0·2	0·133	0·166								
0·3	0·199	0·233	0·266							
0·4	0·265	0·298	0·331	0·362						
0·5	0·330	0·363	0·395	0·425	0·453					
0·6	0·394	0·426	0·457	0·486	0·513					
0·7	0·456	0·487	0·517	0·544	0·570	0·593				
0·8	0·517	0·547	0·575	0·601	0·624	0·645	0·661 / *0·662*	0·674 / *0·675*	0·683 / *0·686*	0·690 / *0·696*
0·9	0·575	0·603	0·630	0·654	0·676	0·694	0·707	0·717	0·725	*0·732*
1·0	0·630	0·656	0·681	0·704	0·723	0·739	0·751	0·760	0·764	0·765 / *0·766*
1·1	0·681	0·706	0·729	0·749	0·766	0·780	0·790	0·797	0·799	0·799
1·2	0·729	0·752	0·772	0·791	0·806	0·817	0·826	0·830	0·831	
1·3	0·772	0·793	0·811	0·827	0·840	0·850	0·857	0·860	0·860	
1·4	0·812	0·830	0·846	0·860	0·871	0·879	0·884	0·886		
1·5	0·846	0·862	0·876	0·888	0·897	0·904	0·908	0·909		
1·6	0·876	0·890	0·902	0·911	0·919	0·925	0 928	0·928		
1·7	0·902	0·913	0·923	0·931	0·937	0·942	0·944			
1·8	0·923	0·933	0·941	0·947	0·952	0·955	0·957			
1·9	0·941	0·949	0·955	0·960	0·964	0·966	0·967			
2·0	0·956	0·962	0·967	0·971	0·974	0·975	0·976			
2·1	0·967	0·972	0·976	0·979	0·981	0·982				
2·2	0·976	0·980	0·983	0·985	0·986	0·987				
2·3	0·983	0·985	0·988	0·989	0·990	0·991				
2·4	0·988	0·990	0·991	0·993	0·993	0·994				
2·5	0·992	0·993	0·994	0·995	0·995	0·996				
2·6	0·994	0·995	0·996	0·997	0·997	0·997				
2·7	0·996	0·997	0·997	0·998	0·998	0·998				
2·8	0·997	0·998	0·998	0·999	0·999	0·999				
2·9	0·998	0·999	0·999	0·999	0·999	0·999				
3·0	0·999	0·999	0·999	0·999	1·000	1·000				
3·1	0·999	1·000	1·000	1·000						
3·2	1·000									

FIGS. 1 AND 2. THE VELOCITY DISTRIBUTION IN THE TWO-DIMENSIONAL LAMINAR FLOW OF INCOMPRESSIBLE VISCOUS FLUID AT HIGH REYNOLDS NUMBERS BEHIND A FLAT PLATE AT ZERO INCIDENCE IN A UNIFORM STREAM.

Curves are shown with u as abscissa and y as ordinate, drawn for various values of x_1/l. Here u is the ratio of the fluid velocity at any point to the undisturbed velocity, U, of the stream; l is the length of the plate; x_1 is distance from the plane perpendicular to the plate through its rear edge, and y is $\frac{1}{2}y_1(U/\nu l)^{\frac{1}{2}}$, where ν is the kinematic viscosity of the fluid, and y_1 is distance from the plane of the plate.

Those curves or parts of curves shown dotted were obtained by extrapolation.

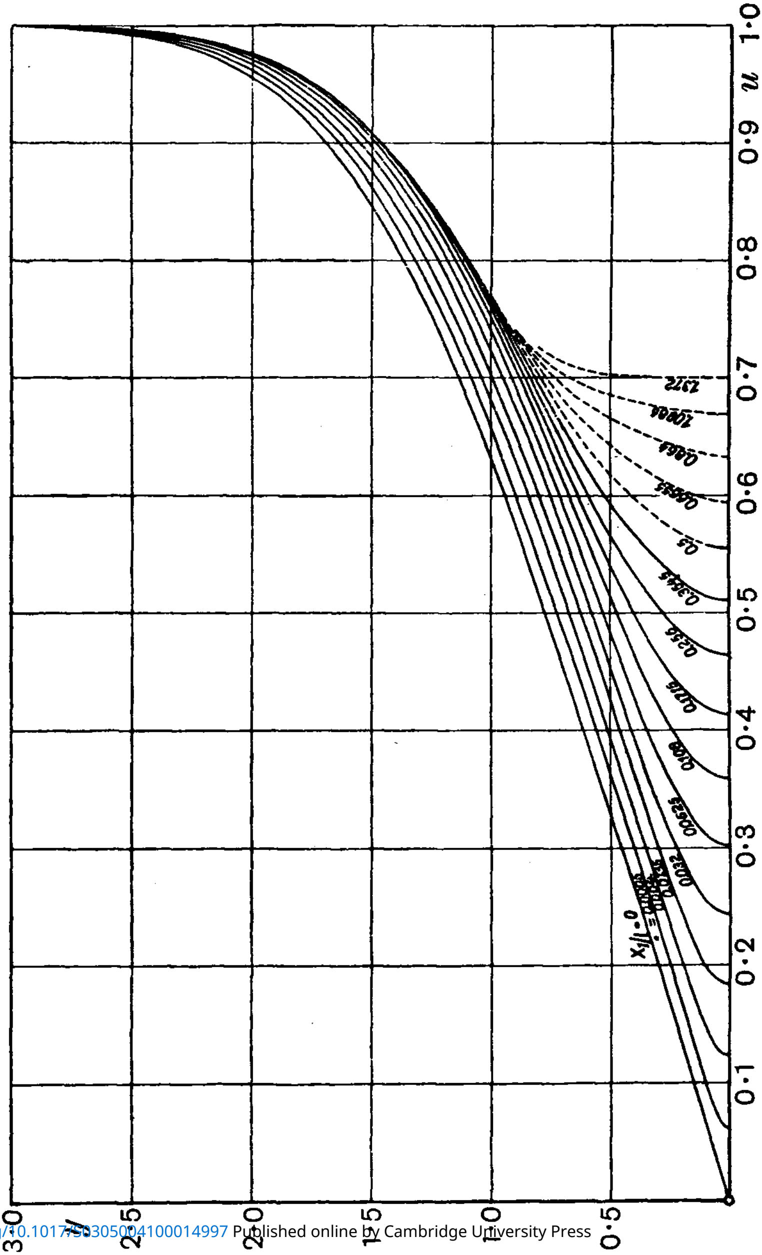

Fig. 1.

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Here the values of u along the axis, $y=0$, are shown plotted against x_1/l. The part dotted was obtained by extrapolation.

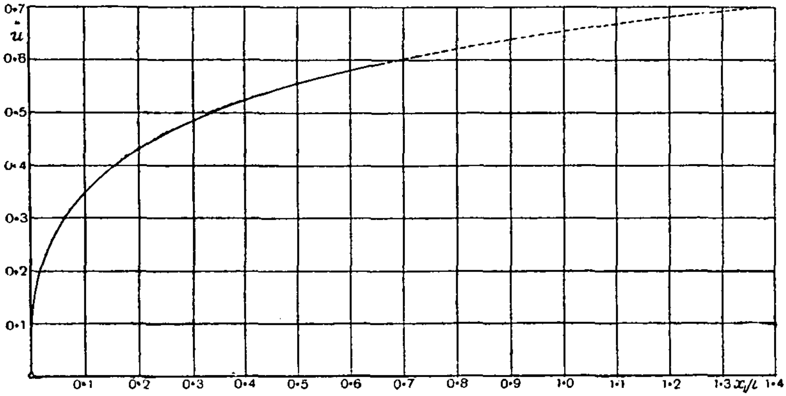

Fig. 2.

SUMMARY.

The boundary layer equations for a steady two-dimensional motion are solved for any given initial velocity distribution (distribution along a normal to the boundary wall, downstream of which the motion is to be calculated). This initial velocity distribution is assumed expressible as a polynomial in the distance from the wall. Three cases are considered: first, when in the initial distribution the velocity vanishes at the wall but its gradient along the normal does not; second, when the velocity in the initial distribution does not vanish at the wall; and third, when both the velocity and its normal gradient vanish at the wall (as at a point where the forward flow separates from the boundary). The solution is found as a power series in some fractional power of the distance along the wall, whose coefficients are functions of the distance from the wall to be found from ordinary differential equations. Some progress is made in the numerical calculation of these coefficients, especially in the first case. The main object was to find means for a step-by-step calculation of the velocity field in a boundary layer, and it is thought that such a procedure may possibly be successful even if laborious.

The same mathematical method is used to calculate the flow behind a flat plate along a stream. The results are shown in Figures 1 and 2, drawn from Tables III and IV.