Ergod. Th. & Dynam. Sys. (2009), **29**, 1693–1703
doi:10.1017/S0143385708000928

The Morse–Bott inequalities via a dynamical systems approach

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(*Received* 1 *September* 2007 *and accepted in revised form* 8 *August* 2008)

Abstract. Let $f : M \to \mathbb{R}$ be a Morse–Bott function on a compact smooth finite-dimensional manifold M. The polynomial Morse inequalities and an explicit perturbation of f defined using Morse functions f_j on the critical submanifolds C_j of f show immediately that $MB_t(f) = P_t(M) + (1+t)R(t)$, where $MB_t(f)$ is the Morse–Bott polynomial of f and $P_t(M)$ is the Poincaré polynomial of M. We prove that $R(t)$ is a polynomial with non-negative integer coefficients by showing that the number of gradient flow lines of the perturbation of f between two critical points $p, q \in C_j$ of relative index one coincides with the number of gradient flow lines between p and q of the Morse function f_j. This leads to a relationship between the kernels of the Morse–Smale–Witten boundary operators associated to the Morse functions f_j and the perturbation of f. This method works when M and all the critical submanifolds are oriented or when $\mathbb{Z}_2$ coefficients are used.

1. *Introduction*

Let $h : M \to \mathbb{R}$ be a Morse function on a compact smooth manifold of dimension m. The Morse inequalities say that

$$\nu_n - \nu_{n-1} + \cdots + (-1)^n \nu_0 \geq b_n - b_{n-1} + \cdots + (-1)^n b_0$$

for all $n \in \{0, \ldots, m\}$ (with equality holding when $n = m$) where, for each k, ν_k is the number of critical points of h of index k and b_k is the kth Betti number of M. These inequalities follow from the fact that the Morse function h determines a CW-complex X whose cellular homology is isomorphic to the singular homology of M (the k-cells of X are in bijective correspondence with the critical points of index k).

In [**14**], Witten introduced the idea that the Morse inequalities can be studied by deforming the de Rham differential on differential forms by the differential of the Morse function. This led him to consider a chain complex whose chains are generated by the critical points of the Morse function and whose differential is defined by counting the gradient flow lines between critical points of relative index one (the so-called Morse–Smale–Witten chain complex); see also [**9**]. The homology of this complex is called the 'Morse homology', and the Morse homology theorem asserts that the Morse homology is isomorphic to the singular homology (see [**3**] and [**13**]). For an excellent exposition of Witten's ideas, see [**11**].

Bismut [**6**] and Helffer and Sjöstrand [**10**] have given rigorous mathematical derivations of Witten's analytical ideas. In this way, they were able to prove the Morse inequalities and the Morse–Bott inequalities, which relate the Betti numbers of M and the Betti numbers of the critical submanifolds of a Morse–Bott function on M, without using the Morse homology theorem.

In this paper, we present a proof of the Morse–Bott inequalities that uses ideas from dynamical systems. The proof makes use of the Morse inequalities, the Morse homology theorem, and an explicit perturbation technique that produces a Morse–Smale function h which is arbitrarily close to a given Morse–Bott function f [**1, 2**]. In outline, the perturbation h of f is constructed as follows. We first fix a Riemannian metric on M and apply the Kupka–Smale theorem to choose Morse–Smale functions f_j on the critical submanifolds C_j for $j = 1, \ldots, l$. Next, we extend the Morse–Smale functions f_j to tubular neighborhoods T_j of the critical submanifolds and define

$$h = f + \varepsilon\bigg(\sum_{j=1}^{l} \rho_j f_j\bigg),$$

where ρ_j is a bump function on T_j and $\varepsilon > 0$. We then apply a well-known folk theorem (whose proof can be found in [**1**, §2.12]) which says that we can perturb the Riemannian metric on M outside of the union of the tubular neighborhoods T_j, $j = 1, \ldots, l$, so that h satisfies the Morse–Smale transversality condition with respect to the perturbed metric.

Once we know that h and f_j ($j = 1, \ldots, l$) satisfy the Morse–Smale transversality condition, we can compare the Morse–Smale–Witten chain complex of h to the Morse–Smale–Witten chain complexes of f_j, for $j = 1, \ldots, l$. This allows us to show that the coefficients of the polynomial $R(t)$ in Theorem 8 are non-negative. The proof works when the manifold M and all the critical submanifolds are oriented or when $\mathbb{Z}_2$ coefficients are used.

2. *The Morse–Smale–Witten chain complex*

In this section we briefly recall the construction of the Morse–Smale–Witten chain complex and the Morse homology theorem. For more details, see [**3**].

Let $\mathrm{Cr}(f) = \{p \in M \mid df_p = 0\}$ denote the set of critical points of a smooth function $f : M \to \mathbb{R}$ on a smooth m-dimensional manifold M. A critical point $p \in \mathrm{Cr}(f)$ is said to be *non-degenerate* if and only if the Hessian $H_p(f)$ is non-degenerate. The *index* λ_p of a non-degenerate critical point p is the dimension of the subspace of T_pM where $H_p(f)$

is negative definite. If all the critical points of f are non-degenerate, then f is called a *Morse function*.

If $f : M \to \mathbb{R}$ is a Morse function on a finite-dimensional compact smooth Riemannian manifold (M, g), then the *stable manifold* $W^s(p)$ and the *unstable manifold* $W^u(p)$ of a critical point $p \in \mathrm{Cr}(f)$ are defined by

$$W^s(p) = \left\{ x \in M \;\middle|\; \lim_{t\to\infty} \varphi_t(x) = p \right\},$$
$$W^u(p) = \left\{ x \in M \;\middle|\; \lim_{t\to-\infty} \varphi_t(x) = p \right\},$$

where φ_t is the one-parameter group of diffeomorphisms generated by minus the gradient vector field, i.e. $-\nabla f$. The index of p coincides with the dimension of $W^u(p)$. The stable/unstable manifold theorem for a Morse function says that the tangent space at p splits as

$$T_p M = T_p^s M \oplus T_p^u M,$$

where the Hessian is positive definite on $T_p^s M \stackrel{\text{def}}{=} T_p W^s(p)$ and negative definite on $T_p^u M \stackrel{\text{def}}{=} T_p W^u(p)$; moreover, the stable and unstable manifolds of p are surjective images of smooth embeddings

$$E^s : T_p^s M \to W^s(p) \subseteq M,$$
$$E^u : T_p^u M \to W^u(p) \subseteq M.$$

Hence, $W^s(p)$ is a smoothly embedded open disk of dimension $m - \lambda_p$, and $W^u(p)$ is a smoothly embedded open disk of dimension λ_p.

If the stable and unstable manifolds of a Morse function $f : M \to \mathbb{R}$ all intersect transversally, then the function f is said to be *Morse–Smale*. Note that for a given Morse function $f : M \to \mathbb{R}$, one can choose a Riemannian metric on M so that f is Morse–Smale with respect to the chosen metric (see [**1**, Theorem 2.20]). Moreover, if f is Morse–Smale, then $W(q, p) = W^u(q) \cap W^s(p)$ is an embedded submanifold of M of dimension $\lambda_q - \lambda_p$, and when $\lambda_q = \lambda_p + 1$, one can use Palis's λ-lemma to prove that the number of gradient flow lines from q to p is finite.

If we assume that M is oriented and choose an orientation for each of the unstable manifolds of f, then there is an induced orientation on the stable manifolds. Thus, we can define an integer $n(q, p)$ associated to any two critical points p and q of relative index one by counting the number of gradient flow lines from q to p with signs determined by the orientations. If M is not orientable, then we can still define $n(q, p) \in \mathbb{Z}_2$ by counting the number of gradient flow lines from q to p mod 2.

The *Morse–Smale–Witten chain complex* is defined to be the chain complex $(C_*(f), \partial_*)$, where $C_k(f)$ is the free abelian group generated by the critical points q of index k (tensored with $\mathbb{Z}_2$ when $n(q, p)$ is defined as an element of $\mathbb{Z}_2$) and the boundary operator $\partial_k : C_k(f) \to C_{k-1}(f)$ is given by

$$\partial_k(q) = \sum_{p\in \mathrm{Cr}_{k-1}(f)} n(q, p)p.$$

THEOREM 1. (Morse homology theorem) *The pair* $(C_*(f), \partial_*)$ *is a chain complex. If* M *is orientable and the boundary operator is defined with* $n(q, p) \in \mathbb{Z}$*, then the homology of* $(C_*(f), \partial_*)$ *is isomorphic to the singular homology* $H_*(M; \mathbb{Z})$*. If the boundary operator is defined with* $n(q, p) \in \mathbb{Z}_2$*, then the homology of* $(C_*(f), \partial_*)$ *is isomorphic to the singular homology* $H_*(M; \mathbb{Z}_2)$*.*

Note that the Morse homology theorem implies that the homology of $(C_*(f), \partial_*)$ is independent of the Morse–Smale function $f : M \to \mathbb{R}$, the Riemannian metric, and the orientations.

3. *The Morse inequalities*

Let M be a compact smooth manifold of dimension m. When M is orientable, we define the kth *Betti number* of M, denoted by b_k, to be the rank of the kth homology group $H_k(M; \mathbb{Z})$ modulo its torsion subgroup. When M is not orientable, we define b_k to be $\dim H_k(M; \mathbb{Z}_2)$. Let $f : M \to \mathbb{R}$ be a Morse function on M and, for each $k = 0, \ldots, m$, let ν_k be the number of critical points of f of index k. As a consequence of the Morse homology theorem, we have

$$\nu_k \geq b_k$$

for $k = 0, \ldots, m$, since $\nu_k = \operatorname{rank} C_k(f)$ and $H_k(M; \mathbb{Z})$ (or $H_k(M; \mathbb{Z}_2)$) is a quotient of $C_k(f)$. These inequalities are known as the *weak Morse inequalities*.

Definition 2. The *Poincaré polynomial* of M is defined to be

$$P_t(M) = \sum_{k=0}^{m} b_k t^k,$$

and the *Morse polynomial* of f is defined to be

$$M_t(f) = \sum_{k=0}^{m} \nu_k t^k.$$

The Morse inequalities stated in the introduction are known as the *strong Morse inequalities*. The strong Morse inequalities are equivalent to the following *polynomial Morse inequalities* (for a proof, see [**3**, Lemma 3.43]). It is this version of the Morse inequalities that we shall generalize to Morse–Bott functions.

THEOREM 3. (Polynomial Morse inequalities) *For any Morse function* $f : M \to \mathbb{R}$ *on a smooth manifold* M*, we have*

$$M_t(f) = P_t(M) + (1 + t)R(t),$$

where $R(t)$ *is a polynomial with non-negative integer coefficients, that is,* $R(t) = \sum_{k=0}^{m-1} r_k t^k$ *with* $r_k \in \mathbb{Z}$ *satisfying* $r_k \geq 0$ *for all* $k \in \{0, \ldots, m-1\}$.

Although Theorem 3 is a standard fact, here we give a detailed proof using the Morse–Smale–Witten chain complex, because this proof provides an explicit formula $r_k = \nu_{k+1} - z_{k+1}$ for the coefficients of the polynomial $R(t)$ which will be useful in proving Theorem 8.

Proof of Theorem 3. Let $f : M \to \mathbb{R}$ be a Morse function on a finite-dimensional compact smooth manifold M, and choose a Riemannian metric on M for which f is a Morse–Smale function. Let $C_k(f)$ denote the kth chain group in the Morse–Smale–Witten chain complex of f (with coefficients in either $\mathbb{Z}$ or $\mathbb{Z}_2$), and let $\partial_k : C_k(f) \to C_{k-1}(f)$ denote the kth Morse–Smale–Witten boundary operator. The rank of $C_k(f)$ is equal to the number ν_k of critical points of f of index k and, by the Morse homology theorem (Theorem 1), the rank of $H_k(C_*(f), \partial_*)$ is equal to b_k, the kth Betti number of M, for all $k \in \{0, \ldots, m\}$.

Let $z_k = \text{rank ker } \partial_k$ for $k = 0, \ldots, m$. The exact sequence

$$0 \to \ker \partial_k \to C_k(f) \xrightarrow{\partial_k} \text{im } \partial_k \to 0$$

implies that $\nu_k = z_k + \text{rank im } \partial_k$ for all $k \in \{0, \ldots, m\}$, and

$$0 \to \text{im } \partial_{k+1} \to \ker \partial_k \to H_k(C_*(f), \partial_*) \to 0$$

implies that $b_k = z_k - \text{rank im } \partial_{k+1}$ for all $k \in \{0, \ldots, m\}$. Hence,

$$\begin{aligned}
M_t(f) - P_t(M) &= \sum_{k=0}^{m} \nu_k t^k - \sum_{k=0}^{m} b_k t^k \\
&= \sum_{k=0}^{m} (z_k + \text{rank im } \partial_k) t^k - \sum_{k=0}^{m} (z_k - \text{rank im } \partial_{k+1}) t^k \\
&= \sum_{k=0}^{m} (\text{rank im } \partial_k + \text{rank im } \partial_{k+1}) t^k \\
&= \sum_{k=0}^{m} (\nu_k - z_k + \nu_{k+1} - z_{k+1}) t^k \\
&= \sum_{k=0}^{m} (\nu_k - z_k) t^k + \sum_{k=0}^{m-1} (\nu_{k+1} - z_{k+1}) t^k \\
&= t \sum_{k=1}^{m} (\nu_k - z_k) t^{k-1} + \sum_{k=1}^{m} (\nu_k - z_k) t^{k-1} \quad (\text{since } \nu_0 = z_0) \\
&= (t+1) \sum_{k=1}^{m} (\nu_k - z_k) t^{k-1}.
\end{aligned}$$

Therefore, $M_t(f) = P_t(M) + (1+t)R(t)$, where $R(t) = \sum_{k=0}^{m-1} (\nu_{k+1} - z_{k+1}) t^k$. Note that $\nu_{k+1} - z_{k+1} \geq 0$ for $k = 0, \ldots, m-1$ because z_{k+1} is the rank of a subgroup of $C_{k+1}(f)$ and $\nu_{k+1} = \text{rank } C_{k+1}(f)$. □

4. *The Morse–Bott inequalities*

Let $f : M \to \mathbb{R}$ be a smooth function whose critical set $\text{Cr}(f)$ contains a submanifold C of positive dimension. Pick a Riemannian metric on M and use it to split $T_*M|_C$ as

$$T_*M|_C = T_*C \oplus \nu_*C,$$

where T_*C is the tangent space of C and ν_*C is the normal bundle of C. Let $p \in C$, $V \in T_pC$ and $W \in T_pM$, and let $H_p(f)$ denote the Hessian of f at p. We have

$$H_p(f)(V, W) = V_p \cdot (\tilde{W} \cdot f) = 0,$$

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since $V_p \in T_pC$ and any extension of W to a vector field $\tilde{W}$ satisfies $df(\tilde{W})|_C = 0$. Therefore, the Hessian $H_p(f)$ induces a symmetric bilinear form $H_p^\nu(f)$ on ν_pC.

Definition 4. A smooth function $f : M \to \mathbb{R}$ on a smooth manifold M is called a *Morse–Bott function* if and only if the set of critical points $\mathrm{Cr}(f)$ is a disjoint union of connected submanifolds and, for each connected submanifold $C \subseteq \mathrm{Cr}(f)$, the bilinear form $H_p^\nu(f)$ is non-degenerate for all $p \in C$.

Often, one says that the Hessian of a Morse–Bott function f is non-degenerate in the direction normal to the critical submanifolds.

For a proof of the following lemma, see [**3**, §3.5] or [**4**].

LEMMA 5. (Morse–Bott Lemma) *Let $f : M \to \mathbb{R}$ be a Morse–Bott function and let $C \subseteq \mathrm{Cr}(f)$ be a connected component. For any $p \in C$, there exist a local chart of M around p and a local splitting $\nu_*C = \nu_*^-C \oplus \nu_*^+C$ which identifies a point $x \in M$ in its domain with (u, v, w), where $u \in C$, $v \in \nu_*^-C$ and $w \in \nu_*^+C$, such that within this chart f assumes the form*

$$f(x) = f(u, v, w) = f(C) - |v|^2 + |w|^2.$$

Definition 6. Let $f : M \to \mathbb{R}$ be a Morse–Bott function on a finite-dimensional smooth manifold M, and let C be a critical submanifold of f. For any $p \in C$, let λ_p denote the index of $H_p^\nu(f)$. This integer is the dimension of ν_p^-C and is locally constant by the preceding lemma. If C is connected, then λ_p is constant throughout C and we call $\lambda_p = \lambda_C$ the *Morse–Bott index* of the connected critical submanifold C.

Let $f : M \to \mathbb{R}$ be a Morse–Bott function on a finite-dimensional compact smooth manifold, and assume that

$$\mathrm{Cr}(f) = \coprod_{j=1}^{l} C_j,$$

where $C_1, \ldots, C_l$ are disjoint connected critical submanifolds.

Definition 7. The Morse–Bott polynomial of f is defined to be

$$MB_t(f) = \sum_{j=1}^{l} P_t(C_j)t^{\lambda_j},$$

where λ_j is the Morse–Bott index of the critical submanifold C_j and $P_t(C_j)$ is the Poincaré polynomial of C_j.

Note. Clearly, $MB_t(f)$ reduces to $M_t(f)$ when f is a Morse function. Also, if M or any of the critical submanifolds are not orientable, then $\mathbb{Z}_2$ coefficients are used to compute all the Betti numbers.

The following result is due to Bott. In fact, Bott proved a more general version of this result (without the assumption that the critical submanifolds are orientable) by using homology with local coefficients in an orientation bundle [**7, 8**].

THEOREM 8. (Morse–Bott inequalities) *Let $f : M \to \mathbb{R}$ be a Morse–Bott function on a finite-dimensional oriented compact smooth manifold, and assume that all the critical* *submanifolds of f are orientable. Then there exists a polynomial $R(t)$ with non-negative integer coefficients such that*

$$MB_t(f) = P_t(M) + (1+t)R(t).$$

If either M or some of the critical submanifolds are not orientable, then this equation holds when $\mathbb{Z}_2$ coefficients are used to define the Betti numbers.

5. *A perturbation technique and the proof of the main theorem*

To prove Theorem 8, we will use the following perturbation technique based on [**2**], the Morse–Bott lemma and a folk theorem proved in [**1**]. This construction produces an explicit Morse–Smale function $h : M \to \mathbb{R}$ which is arbitrarily close to a given Morse–Bott function $f : M \to \mathbb{R}$ such that $h = f$ outside of a neighborhood of the critical set $\mathrm{Cr}(f)$.

For $j = 1, \ldots, l$, let T_j be a small tubular neighborhood around each connected component $C_j \subseteq \mathrm{Cr}(f)$, having local coordinates (u, v, w) consistent with those from the Morse–Bott lemma (Lemma 5). By 'small' we mean that each T_j is contained in the union of the domains of the charts from the Morse–Bott lemma; for $i \neq j$, $T_i \cap T_j = \emptyset$ and f decreases by at least three times $\max\{\mathrm{var}(f, T_j) \mid j = 1, \ldots, l\}$ along any gradient flow line from T_i to T_j, where $\mathrm{var}(f, T_j) = \sup\{f(x) \mid x \in T_j\} - \inf\{f(x) \mid x \in T_j\}$; and, if $f(C_i) \neq f(C_j)$, then

$$\mathrm{var}(f, T_i) + \mathrm{var}(f, T_j) < \tfrac{1}{3}|f(C_i) - f(C_j)|.$$

Pick a Riemannian metric on M such that the charts from the Morse–Bott lemma are isometries with respect to the standard Euclidean metric on $\mathbb{R}^m$, and then pick positive Morse functions $f_j : C_j \to \mathbb{R}$ that are Morse–Smale with respect to the restriction of the Riemannian metric to C_j, for $j = 1, \ldots, l$. The Morse–Smale functions $f_j : C_j \to \mathbb{R}$ exist by the Kupka–Smale theorem (see, for instance, [**3**, Theorem 6.6.]).

For every $j = 1, \ldots, l$, extend f_j to a function on T_j by making f_j constant in the direction normal to C_j, i.e. by letting f_j be constant in the v and w coordinates coming from the Morse–Bott lemma. Let $\tilde{T}_j \subset T_j$ be a smaller tubular neighborhood of C_j with the same coordinates as T_j, and let ρ_j be a smooth non-increasing bump function which is constant in the u coordinates, equal to 1 on $\tilde{T}_j$, and equal to 0 outside of T_j. Choose $\varepsilon > 0$ small enough so that

$$\sup_{T_j - \tilde{T}_j} \varepsilon \|\nabla \rho_j f_j\| < \inf_{T_j - \tilde{T}_j} \|\nabla f\|$$

for all $j \in \{1, \ldots, l\}$, and define

$$h = f + \varepsilon\left(\sum_{j=1}^{l} \rho_j f_j\right).$$

The function $h : M \to \mathbb{R}$ is a Morse function close to f, and the critical points of h are exactly the critical points of the f_j, for $j = 1, \ldots, l$. Moreover, if $p \in C_j$ is a critical point of $f_j : C_j \to \mathbb{R}$ of index λ_p^j, then p is a critical point of h of index $\lambda_p^h = \lambda_j + \lambda_p^j$.

We now apply a well-known folk theorem (whose proof can be found in [**1**, §2.12]) which says that we can perturb the Riemannian metric on M outside of the union of the tubular neighborhoods T_j, $j = 1, \ldots, l$, so that h satisfies the Morse–Smale transversality condition with respect to the perturbed metric. In summary, we have achieved the following conditions.

(1) The gradient ∇f is equal to ∇h outside of the union of the tubular neighborhoods T_j, $j = 1, \ldots, l$. Moreover, the tubular neighborhoods T_j are chosen small enough so that if $f(C_i) \leq f(C_j)$ for some $i \neq j$, then there are no gradient flow lines of h from C_i to C_j.

(2) The charts from the Morse–Bott lemma are isometries with respect to the metric on M and the standard Euclidean metric on $\mathbb{R}^m$.

(3) In the local coordinates (u, v, w) of a tubular neighborhood T_j, $f = f(C) - |v|^2 + |w|^2$, ρ_j depends only on the v and w coordinates, and f_j depends only on the u coordinates. In particular, $\nabla f \perp \nabla f_j$ on T_j by the previous condition.

(4) The function h satisfies $h = f + \varepsilon f_j$ on the tubular neighborhood $\tilde{T}_j$.

(5) The gradient ∇f dominates $\varepsilon \nabla \rho_j f_j$ on $T_j - \tilde{T}_j$.

(6) The functions $h : M \to \mathbb{R}$ and $f_j : C_j \to \mathbb{R}$ satisfy the Morse–Smale transversality condition for all $j \in \{1, \ldots, l\}$.

(7) For every $n = 0, \ldots, m$, we have the following description of the nth Morse–Smale–Witten chain group of h in terms of the Morse–Smale–Witten chain groups of the f_j, $j = 1, \ldots, l$:

$$C_n(h) = \bigoplus_{\lambda_j + k = n} C_k(f_j).$$

Now, let ∂_*^h denote the Morse–Smale–Witten boundary operator of h, and let $\partial_*^{f_j}$ denote the Morse–Smale–Witten boundary operator of f_j, for $j = 1, \ldots, l$.

LEMMA 9. *If $p, q \in C_j$ are critical points of $f_j : C_j \to \mathbb{R}$ of relative index one, then the coefficients $n(q, p)$ used to define the Morse–Smale–Witten boundary operator are the same for ∂_*^h and $\partial_*^{f_j}$ (assuming, in the case where C_j is orientable, that the appropriate orientation of C_j has been chosen).*

Proof. We have $h = f + \varepsilon f_j$ on $\tilde{T}_j$ by condition (4) above. This implies that $\nabla h = \nabla f + \varepsilon \nabla f_j$ on $\tilde{T}_j$, and that $\nabla h = \varepsilon \nabla f_j$ on the critical submanifold C_j. Moreover, by conditions (3) and (5), a gradient flow line of h cannot begin and end in C_j unless the entire flow line is contained in C_j. Thus, if p and q are both in C_j, the flows connecting them along the gradient flow lines of h and of f_j are the same, and the numbers $n(q, p)$ in the complex of h and of f_j are the same as long as the orientations are chosen appropriately. □

We now order the connected critical submanifolds $C_1, \ldots, C_l$ by height, i.e. in such a way that $f(C_i) \leq f(C_j)$ whenever $i \leq j$. For $\alpha \in C_n(h)$ and C_j a connected critical submanifold, we denote by α_j the chain obtained from α by retaining only those critical points belonging to C_j. By condition (7), any $\alpha \neq 0$ can be written uniquely as

$$\alpha = \alpha_{j_1} + \cdots + \alpha_{j_r},$$

where $j_1 < \cdots < j_r$ and $\alpha_{j_i} \in C_{k_i}(f_{j_i}) - \{0\}$ with $\lambda_{j_i} + k_i = n$. We will refer to α_{j_r} as the 'top part' of the chain α.

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COROLLARY 10. *If α is a non-zero element in* $\ker \partial_*^h$, *then the top part of α is a non-zero element in the kernel of $\partial_*^{f_j}$ for some j; that is, using the above notation for $\alpha \in \ker \partial_n^h$, we have $\alpha_{j_r} \in \ker \partial_{k_r}^{f_{j_r}}$ where $\lambda_{j_r} + k_r = n$.*

Proof. Let $\alpha = \sum_i n_i q_i \in \ker \partial_n^h$, that is,

$$0 = \partial_n^h(\alpha) = \sum_i n_i \partial_n^h(q_i) = \sum_i n_i \sum_{p \in \mathrm{Cr}_{n-1}(h)} n(q_i, p)p = \sum_{p \in \mathrm{Cr}_{n-1}(h)} \left(\sum_i n_i n(q_i, p)\right) p.$$

Then $\sum_i n_i n(q_i, p) = 0$ for all $p \in \mathrm{Cr}_{n-1}(h)$ and, in particular, for $p \in \mathrm{Cr}_{n-1}(h) \cap C_{j_r}$. For any $p \in \mathrm{Cr}_{n-1}(h) \cap C_{j_r}$, we have $f(q_i) \leq f(p)$ for all q_i in the sum of α such that $q_i \notin C_{j_r}$; thus there are no gradient flow lines of h from q_i to p, by condition (1). Therefore, $n(q_i, p) = 0$ if $p \in \mathrm{Cr}_{n-1}(h) \cap C_{j_r}$ and $q_i \notin C_{j_r}$, i.e.

$$0 = \sum_i n_i n(q_i, p) = \sum_{q_i \in C_{j_r}} n_i n(q_i, p)$$

when $p \in \mathrm{Cr}_{n-1}(h) \cap C_{j_r}$. Lemma 9 then implies that

$$\begin{aligned}\partial_{k_r}^{f_{j_r}}(\alpha_{j_r}) = \sum_{q_i \in C_{j_r}} n_i \partial_{k_r}^{f_{j_r}}(q_i) &= \sum_{q_i \in C_{j_r}} n_i \sum_{p \in \mathrm{Cr}_{k_r-1}(f_{j_r})} n(q_i, p)p \\ &= \sum_{p \in \mathrm{Cr}_{k_r-1}(f_{j_r})} \left(\sum_{q_i \in C_{j_r}} n_i n(q_i, p)\right) p = 0,\end{aligned}$$

where $\lambda_{j_r} + k_r = n$ and $\mathrm{Cr}_{k_r-1}(f_{j_r}) = \mathrm{Cr}_{n-1}(h) \cap C_{j_r}$. □

Proof of Theorem 8. Let f, h, f_j $(j = 1, \ldots, l)$ and the Riemannian metric on M be as above. Let $M_t(f_j)$ denote the Morse polynomial of $f_j : C_j \to \mathbb{R}$, and write $c_j = \dim C_j$ for $j = 1, \ldots, l$. Note that the relation $\lambda_p^h = \lambda_j + \lambda_p^j$ implies that

$$M_t(h) = \sum_{j=1}^{l} M_t(f_j) t^{\lambda_j}.$$

The polynomial Morse inequalities (Theorem 3) say that

$$M_t(h) = P_t(M) + (1+t)R_h(t)$$

and

$$M_t(f_j) = P_t(C_j) + (1+t)R_j(t),$$

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where $R_h(t)$ and $R_j(t)$, $j = 1, \ldots, l$, are polynomials with non-negative integer coefficients. Now,

$$\begin{aligned} MB_t(f) &= \sum_{j=1}^{l} P_t(C_j)t^{\lambda_j} \\ &= \sum_{j=1}^{l}\big(M_t(f_j) - (1+t)R_j(t)\big)t^{\lambda_j} \\ &= \sum_{j=1}^{l} M_t(f_j)t^{\lambda_j} - (1+t)\sum_{j=1}^{l} R_j(t)t^{\lambda_j} \\ &= M_t(h) - (1+t)\sum_{j=1}^{l} R_j(t)t^{\lambda_j} \\ &= P_t(M) + (1+t)R_h(t) - (1+t)\sum_{j=1}^{l} R_j(t)t^{\lambda_j}. \end{aligned}$$

In the proof of the polynomial Morse inequalities we saw that

$$R_j(t) = \sum_{k=1}^{c_j}(\nu_k^j - z_k^j)t^{k-1},$$

where ν_k^j is the rank of the group $C_k(f_j)$ and z_k^j is the rank of the kernel of the boundary operator $\partial_k^{f_j} : C_k(f_j) \to C_{k-1}(f_j)$ in the Morse–Smale–Witten chain complex of $f_j : C_j \to \mathbb{R}$. Hence,

$$MB_t(f) = P_t(M) + (1+t)\sum_{n=1}^{m}(\nu_n^h - z_n^h)t^{n-1} - (1+t)\sum_{j=1}^{l}\sum_{k=1}^{c_j}(\nu_k^j - z_k^j)t^{\lambda_j+k-1},$$

where ν_n^h denotes the rank of the chain group $C_n(h)$ and z_n^h denotes the rank of the kernel of the Morse–Smale–Witten boundary operator $\partial_n^h : C_n(h) \to C_{n-1}(h)$. Since the critical points of h coincide with the critical points of the functions f_j $(j = 1, \ldots, l)$ and a critical point $p \in C_j$ of f_j of index λ_p^j is a critical point of h of index $\lambda_j + \lambda_p^j$, we have

$$\sum_{n=1}^{m}(\nu_n^h - z_n^h)t^{n-1} - \sum_{j=1}^{l}\sum_{k=1}^{c_j}(\nu_k^j - z_k^j)t^{\lambda_j+k-1} = \sum_{j=1}^{l}\sum_{k=1}^{c_j} z_k^j t^{\lambda_j+k-1} - \sum_{n=1}^{m} z_n^h t^{n-1}.$$

Therefore

$$MB_f(t) = P_t(M) + (1+t)R(t),$$

where

$$R(t) = \sum_{j=1}^{l}\sum_{k=1}^{c_j} z_k^j t^{\lambda_j+k-1} - \sum_{n=1}^{m} z_n^h t^{n-1} = \sum_{n=1}^{m}\bigg(\sum_{\lambda_j+k=n} z_k^j - z_n^h\bigg)t^{n-1}.$$

To see that $\sum_{\lambda_j+k=n} z_k^j \geq z_n^h$ for all $n = 1, \ldots, m$, we apply Corollary 10 as follows. If $z_n^h > 0$, then we can choose a non-zero element $\beta_1 \in \ker \partial_n^h$. By Corollary 10, the 'top

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part' of β_1 is a non-zero element in ker $\partial_k^{f_j}$ for some k and j such that $\lambda_j + k = n$. If $z_n^h = 1$, then we are done. If $z_n^h > 1$, then we can find an element $\beta_2 \in \ker \partial_n^h$ that is not in the group generated by β_1 and, by adding a multiple of β_1 to β_2 if necessary, we can choose β_2 such that the top part of β_2 is a non-zero element that is not in the subgroup generated by the top part of β_1. Continuing in this fashion, we can find generators for ker ∂_n^h whose top parts are independent elements in

$$\bigoplus_{\lambda_j + k = n} \ker \partial_k^{f_j},$$

that is,

$$\sum_{\lambda_j + k = n} (\text{rank } \ker \partial_k^{f_j}) \geq \text{rank } \ker \partial_n^h. \qquad \square$$

Acknowledgement. We would like to thank William Minicozzi for several insightful conversations concerning Riemannian metrics and gradient flows.

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