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Non-smooth saddle-node bifurcations II: Dimensions of strange attractors

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Abstract. We study the geometric and topological properties of strange non-chaotic attractors created in non-smooth saddle-node bifurcations of quasiperiodically forced interval maps. By interpreting the attractors as limit objects of the iterates of a continuous curve and controlling the geometry of the latter, we determine their Hausdorff and box-counting dimension and show that these take distinct values. Moreover, the same approach allows us to describe the topological structure of the attractors and to prove their minimality.

1. *Introduction*

One of the most intriguing phenomena in dynamical systems is the existence of strange attractors and the fact that these intricate structures already occur for relatively simple deterministic systems given by low-dimensional maps and flows. The discovery of paradigm examples like the Hénon or the Lorenz attractor has given great impetus to the field. Usually, strange attractors are associated with chaotic dynamics. However, this is not always the case, and in a seminal paper [**15**] Grebogi, Ott, Pelikan and Yorke demonstrated that such objects may also occur in systems which do not allow for chaotic motion—in the sense of positive topological entropy—for structural reasons. Their heuristic and numerical arguments were later confirmed in a rigorous analysis by Keller [**23**]. The class of systems considered in [**15**, **23**] were *quasiperiodically forced (qpf) monotone interval maps*. These are skew product transformations of the form

$$f: \mathbb{T}^d \times X \to \mathbb{T}^d \times X, \quad (\theta, x) \mapsto (\theta + \omega, \tilde{f}(\theta, x)), \tag{1.1}$$

where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $X \subseteq \mathbb{R}$ is an interval (possibly non-compact). Here, $\mathbb{T}^d$ is referred to as the *base*. We assume that the *rotation vector* ω is totally irrational and that for each $\theta \in \mathbb{T}^d$ the *fibre map* $\tilde{f}(\theta, \cdot)$ is monotonously increasing†.

The specific examples in [**15**, **23**] belong to the class of so-called *pinched skew products*, which are characterized by the fact that for some $\theta \in \mathbb{T}^d$ the fibre map $\tilde{f}(\theta, \cdot)$ is constant and consequently the whole fibre $\{\theta\} \times X$ is mapped to a single point [**13**]. This greatly simplifies their analysis, but at the same time it gives them a certain toy model character. In particular, pinched skew products are not invertible and can therefore not be the time-one maps of flows, which are of main interest from the applied point of view. Nevertheless, it was later confirmed both numerically (e.g. [**10**, **29**]) and even experimentally [**8**, **9**] that the occurrence of strange non-chaotic attractors (SNAs) in systems with quasiperiodic forcing is a widespread and robust phenomenon, and general methods to rigorously prove their existence have been established in different settings [**4**, **5**, **12**, **22**, **33**]. Thus, SNAs often play a crucial role in the bifurcations of invariant curves and often originate from the collision of these. This pattern for the creation of SNAs has been named *torus collision* or, more specifically, *non-smooth saddle-node bifurcation* [**2**, **17**, **28**].

In contrast to conditions for the existence of SNAs, the structural properties of these objects are far less well understood. From the mathematical viewpoint, much of the relevant information about the geometric and dynamical features of an attractor is encoded in different notions of dimension. Accordingly, the question of computing dimensions of SNAs has been raised already at an early stage. Based on numerical evidence, it has been conjectured in [**7**] that the box (or capacity) dimension of SNAs appearing in different types of qpf systems with one-dimensional base $\mathbb{T}^1$ and one-dimensional fibres equals two, whereas the information dimension equals one. For the simple pinched skew products introduced in [**15**], these findings were confirmed analytically in [**16**, **21**].

The aim here is to perform a similar analysis for SNAs appearing in a more realistic setting. We concentrate on invertible qpf interval maps and focus on such SNAs which are created in non-smooth saddle-node bifurcations. Apart from the dimensions, we obtain the minimality of the dynamics on the attractors and information about their topological structure. On a heuristic level, some inspiration is drawn from the previous work in [**16**, **21**]. Technically, however, the task is considerably more demanding and our approach builds on a detailed multiscale analysis established in the first author's paper [**12**], whose continuation this work presents. Before stating precise results, we need to introduce some general notions and a framework for non-smooth saddle-node bifurcations in qpf interval maps. The latter results from a discrete-time analogue to work of Núñez and Obaya on almost periodically forced scalar differential equations [**25**], which is provided in [**2**].

Given f as in (1.1), an *f-invariant graph* is a measurable function $\phi : \mathbb{T}^d \to X$ that satisfies

$$\tilde{f}(\theta, \phi(\theta)) = \phi(\theta + \omega)$$

for all $\theta \in \mathbb{T}^d$. The associated point set $\Phi = \{(\theta, \phi(\theta)) \mid \theta \in \mathbb{T}^d\}$ is invariant in this case, and in a slight abuse of terminology we will refer to both the function ϕ and the set Φ

† The fact that skew product systems of this type do not allow for positive entropy follows from an old result of Bowen [**6**]; see also [**14**].

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as an invariant graph. As far as functions are concerned, we will not distinguish between invariant graphs that coincide Lebesgue-almost everywhere, and thus implicitly speak of equivalence classes. By saying that an invariant graph has a certain property, such as continuity or semi-continuity, we mean that there exists a representative in the respective equivalence class which has this property. The stability of an invariant graph is determined by its Lyapunov exponent

$$\lambda(\phi) = \int_{\mathbb{T}^d} \log \partial_x \tilde{f}(\theta, \phi(\theta))\, d\theta.$$

If $\lambda(\phi) < 0$, then ϕ is attracting, in the sense that for almost every $\theta \in \mathbb{T}^d$ there is $\varepsilon = \varepsilon(\theta) > 0$ such that

$$|f^n(\theta, x) - (\theta + n\omega, \phi(\theta + n\omega))| \to 0$$

for $n \to \infty$ and $x \in B_\varepsilon(\phi(\theta))$ [**20**]. If ϕ is continuous, then ε can be chosen independent of $\theta \in \mathbb{T}^d$ [**31**]. An SNA, in this setting, is a non-continuous invariant graph with a negative Lyapunov exponent. 'Strange' here simply refers to the lack of continuity. We refer to Milnor [**24**] for a broader discussion of the notion of 'strange attractors'.

In the context of forced systems, the significance of invariant graphs stems from the fact that they are a natural analogue to fixed points of unperturbed maps, and just like the latter they may bifurcate. As mentioned above, we will concentrate on saddle-node bifurcations. In order to keep notation as simple as possible, we may assume without loss of generality that $[0, 1] \subseteq X$ from now on. We denote by $\mathcal{F}_\omega$ the class of $\mathcal{C}^2$-maps of the form (1.1) (with fixed rotation vector $\omega \in \mathbb{T}^d$). Further, by $\mathcal{P}_\omega$ we denote $\mathcal{C}^2$ one-parameter families in $\mathcal{F}_\omega$, that is,

$$\begin{aligned}\mathcal{P}_\omega = \{(f_\beta)_{\beta\in[0,1]} \mid f_\beta \in \mathcal{F}_\omega \text{ for all } \beta \in [0, 1]\\ \text{and } [0, 1] \times \mathbb{T}^d \times X \ni (\beta, \theta, x) \mapsto f_{\beta,\theta}(x) \text{ is } \mathcal{C}^2\}.\end{aligned}$$

Here (and in the following), we adopt the customary notation $f_{\beta,\theta}(\cdot)$ for the fibre map $\tilde{f}_\beta(\theta, \cdot)$ of the family member f_β. Elements of $\mathcal{P}_\omega$ will also be denoted by $\hat{f} = (f_\beta)_{\beta\in[0,1]}$.

We equip $\mathcal{P}_\omega$ with the $\mathcal{C}^2$-metric and simply refer to the induced topology as $\mathcal{C}^2$-topology in all of the following. In order to ensure the occurrence of a saddle-node bifurcation in a prescribed region $\Gamma = \mathbb{T}^d \times [0, 1]$ of the phase space, we need to impose a number of further conditions. The following assumptions are supposed to hold for all $\beta \in [0, 1]$ and all $\theta \in \mathbb{T}^d$ (if applicable):

$$f_{\beta,\theta}(0) \le 0 \quad \text{and} \quad f_{\beta,\theta}(1) \le 1; \tag{1.2}$$

$$f'_{\beta,\theta}(x) > 0 \quad \text{for all } x \in [0, 1]; \tag{1.3}$$

$$f''_{\beta,\theta}(x) < 0 \quad \text{for all } x \in (0, 1); \tag{1.4}$$

$$\frac{\partial}{\partial\beta} f_{\beta,\theta}(x) < 0 \quad \text{for all } x \in [0, 1]; \tag{1.5}$$

$$f_0 \text{ has two continuous invariant graphs in } \Gamma \text{ and } f_1 \text{ has no invariant graph in } \Gamma. \tag{1.6}$$

Here, we say that f has an invariant graph ϕ in $\mathbb{T}^d \times A$ if $\phi(\theta) \in A$ for all $\theta \in \mathbb{T}^d$. We let

$$\mathcal{S}_\omega = \{\hat{f} \in \mathcal{P}_\omega \mid \hat{f} \text{ satisfies (1.2)–(1.6)}\}.$$

THEOREM 1.1. [**2**, Theorem 6.1] *Let* $\hat{f} = (f_\beta)_{\beta\in[0,1]} \in \mathcal{S}_\omega$*. Then there exists a unique critical parameter* $\beta_c \in (0, 1)$ *such that the following hold.*

(i) *If* $\beta < \beta_c$*, then* f_β *has two invariant graphs* $\phi^-_\beta < \phi^+_\beta$ *in* Γ*, both of which are continuous. We have* $\lambda(\phi^-_\beta) > 0$ *and* $\lambda(\phi^+_\beta) < 0$*.*

(ii) *If* $\beta > \beta_c$*, then* f_β *has no invariant graphs in* Γ*.*

(iii) *If* $\beta = \beta_c$*, then one of the following two possibilities holds.*

($\mathcal{S}$*)* Smooth bifurcation: f_{β_c} *has a unique invariant graph* ϕ_{β_c} *in* Γ*, which satisfies* $\lambda(\phi_{\beta_c}) = 0$*. Either* ϕ_{β_c} *is continuous, or it contains both an upper and lower semi-continuous representative in its equivalence class.*

($\mathcal{N}$*)* Non-smooth bifurcation: f_{β_c} *has exactly two invariant graphs* $\phi^-_{\beta_c} < \phi^+_{\beta_c}$ *almost everywhere in* Γ*. The graph* $\phi^-_{\beta_c}$ *is lower semi-continuous, whereas* $\phi^+_{\beta_c}$ *is upper semi-continuous, but none of the graphs is continuous and there exists a residual set* $\Omega \subseteq \mathbb{T}^d$ *such that* $\phi^-_{\beta_c}(\theta) = \phi^+_{\beta_c}(\theta)$ *for all* $\theta \in \Omega$*.*

Remark.

(a) The points in the above set Ω are called *pinched* points. Due to the semi-continuity, it turns out that $\phi^+_{\beta_c}$ and $\phi^-_{\beta_c}$ are actually continuous in the pinched points (cf. [**30**, Lemma 5]).

(b) Note that (1.2) replaces assumption (d2) in [**2**, Theorem 6.1].

(c) In [**2**], the above statement is actually formulated for convex fibre maps (with $f''_{\beta,\theta} > 0$ instead of (1.4) and $(\partial/\partial\beta) f_{\beta,\theta}(x) > 0$ instead of (1.5)). However, by considering the coordinate change $(\theta, x) \mapsto (\theta, -x)$ and the parametrization $\beta \mapsto 1 - \beta$, we obviously get the above formulation; cf. [**2**, Remark 6.2(c)]. Moreover, in [**2**, Theorem 6.1], the strict inequality in (1.4) is actually assumed to hold on the closed interval [0, 1]. Yet, the above statement still holds (and the proof in [**2**] remains exactly the same) when we consider this inequality to hold on (0, 1).

As stated before, the invariant graphs appearing in this statement have to be understood in the sense of equivalence classes. There is, however, an intimate relation to the maximal invariant subset of Γ, given by

$$\Lambda_\beta = \bigcap_{n\in\mathbb{Z}} f^n_\beta(\Gamma),$$

that can be used to obtain well-defined canonical representatives. This will be important in the statement of our main result. We write

$$\Lambda_{\beta,\theta} = \{x \in [0, 1] \mid (\theta, x) \in \Lambda_\beta\}.$$

Due to the invariance of Λ_β and the monotonicity of the fibre map (1.3), the graphs

$$\hat{\phi}^-_\beta(\theta) = \inf \Lambda_{\beta,\theta} \quad \text{and} \quad \hat{\phi}^+_\beta(\theta) = \sup \Lambda_{\beta,\theta} \tag{1.7}$$

are both invariant and thus have to be representatives of the invariant graphs in parts (i) and (iii) of Theorem 1.1. Moreover, if we write $[\hat{\phi}^-_\beta, \hat{\phi}^+_\beta] = \{(\theta, x) \in \Gamma \mid \hat{\phi}^-_\beta(\theta) \le x \le \hat{\phi}^+_\beta(\theta)\}$, then $\Lambda_\beta = [\hat{\phi}^-_\beta, \hat{\phi}^+_\beta]$.

Theorem 1.1 gives a precise meaning to the notion of a saddle-node bifurcation for a family in $\mathcal{S}_\omega$. Moreover, it shows that there are two qualitatively different patterns for

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such a transition, namely the smooth and the non-smooth case. While smooth bifurcations can be realized easily by considering direct products of irrational rotations and suitable interval maps, the existence of non-smooth bifurcations is much more difficult to establish. However, as the following result shows, they are nevertheless a generic case. Recall that $\omega \in \mathbb{T}^d$ is *Diophantine* if there exist $\mathscr{C} > 0$ and $\eta > 1$ such that $d(k\omega, 0) \geq \mathscr{C}|k|^{-\eta}$ for all $k \in \mathbb{Z}\backslash\{0\}$.

THEOREM 1.2. [**12**] *Let*

$$\mathcal{N}_\omega = \{\hat{f} \in \mathcal{S}_\omega \mid f_{\beta_c} \text{ satisfies } (\mathcal{N})\}$$

and suppose that $\omega \in \mathbb{T}^d$ *is Diophantine. Then* $\mathcal{N}_\omega$ *has non-empty interior in the* $\mathcal{C}^2$-*topology on* $\mathcal{P}_\omega$.

This statement is implicitly contained in [**12**]; see [**12**, Theorem 4.18] and [**11**, Theorem 4.2.15] as well as the discussion thereafter. While the assertion may seem rather abstract in the above form, it is important to note that a much more detailed version is given in [**11**, **12**]. It states that $\mathcal{N}_\omega$ contains a $\mathcal{C}^2$-open subset $\mathcal{U}_\omega$ which is completely characterized by a list of $\mathcal{C}^2$-estimates on the respective parameter families. However, since this list consists of 16 different and sometimes rather technical conditions, we refrain from reproducing it here. A partially intrinsic characterization that contains all the information required for our purposes is given in §2.3. In order to fix ideas, readers may restrict their attention to the following explicit example (discussed in [**12**, §3]) which satisfies all the assumptions of our main result below.

PROPOSITION 1.3. (Cf. [**12**]) *Let* $\omega \in \mathbb{T}^d$ *be Diophantine. Then there exists* $a_0 > 0$ *such that, for all* $a > a_0$, *the family* $\hat{f}$ *given by*

$$f_\beta(\theta, x) = (\theta + \omega, 2/\pi \cdot \arctan(ax) - 2\beta - (1 + \cos 2\pi\theta)/4) \tag{1.8}$$

undergoes a non-smooth saddle-node bifurcation, that is, $\hat{f} \in \mathcal{N}_\omega$.

Remark. The above example differs slightly from the one provided in [**12**]. However, apart from minor differences, the discussion in [**12**] also applies to (1.8).

Our main result now provides information on the geometric and topological structure of the SNA and the associated ergodic measure occurring in such non-smooth saddle-node bifurcations. Note that to each invariant graph ϕ an invariant ergodic measure μ_ϕ can be associated by defining

$$\mu_\phi(A) = \text{Leb}_{\mathbb{T}^d}(\pi_{\mathbb{T}^d}(\Phi \cap A)),$$

where $A \subseteq \mathbb{T}^d \times X$ is Borel measurable and $\pi_{\mathbb{T}^d}$ is the canonical projection onto $\mathbb{T}^d$. We denote the box-counting dimension of a set $A \subseteq \mathbb{T}^d \times X$ by $\text{D}_B(A)$ and its Hausdorff dimension by $\text{D}_H(A)$. For the explanation of further dimension-theoretical notions, see §§2.1 and 2.2.

THEOREM 1.4. *Let* $\omega \in \mathbb{T}^d$ *be Diophantine. Then there exists a set* $\widehat{\mathcal{U}}_\omega \subseteq \mathcal{N}_\omega$ *with non-empty* $\mathcal{C}^2$-*interior such that, for all* $\hat{f} \in \widehat{\mathcal{U}}_\omega$, *the SNA* $\hat{\phi}^+_{\beta_c}$ *appearing at the critical bifurcation parameter satisfies the following.*

(i) *$D_B(\hat{\Phi}^+_{\beta_c}) = d + 1$ and $D_H(\hat{\Phi}^+_{\beta_c}) = d$.*
(ii) *The measure $\mu_{\phi^+_{\beta_c}}$ is exact dimensional with pointwise dimension and information dimension equal to d.*
(iii) *The set $\Lambda_{\beta_c} = [\hat{\phi}^-_{\beta_c}, \hat{\phi}^+_{\beta_c}]$ is minimal and we have $\Lambda_{\beta_c} = cl(\hat{\Phi}^-_{\beta_c}) = cl(\hat{\Phi}^+_{\beta_c})$.*
(iv) *The graph $\hat{\phi}^+_{\beta_c}$ is the only semi-continuous representative in the equivalence class $\phi^+_{\beta_c}$.*

Analogous results hold for the repeller $\phi^-_{\beta_c}$. Moreover, for all sufficiently large $a > 0$, the parameter family $\hat{f}$ given by (1.8) is contained in $\widehat{\mathcal{U}}_\omega$.

Property (iii) has already been considered by Herman [**18**]. We observe that it has been proved previously by Bjerklöv for invariant graphs appearing in quasiperiodic Schrödinger cocycles [**3**], which can be considered a special case of our setting. Our proof is inspired by that of Bjerklöv, but puts a stronger focus on the global approximation of the SNA by iterates of continuous curves. This allows us to avoid some technical complications.

We also note that the result on the box-counting dimension is a direct consequence of (iii). Since the box-counting dimension is stable under taking closures, we have $\mathrm{D}_B(\hat{\Phi}^+_{\beta_c}) = \mathrm{D}_B(\Lambda_{\beta_c})$. Since the bounding graphs of Λ_{β_c} are distinct, this set has positive $(d + 1)$-dimensional Lebesgue measure and therefore box-counting dimension $d + 1$.

The strategy of our proof is outlined at the beginning of §3. A crucial ingredient is Proposition 3.1 whose rather technical proof is postponed to the last section. Taking this proposition for granted, the dimension-theoretical results follow straightforwardly (see Theorem 3.2). Points (iii) and (iv) of the above theorem are proven—again by means of Proposition 3.1—in §4.

2. *Preliminaries*

2.1. *Hausdorff and box-counting dimension.*

In the following, we recall the definition of the Hausdorff and box-counting dimension. Further, we state some well-known properties that will be used later on. Suppose that Y is a metric space. We denote the diameter of a subset $A \subseteq Y$ by $|A|$. For $\varepsilon > 0$, we call a finite or countable collection $\{A_i\}$ of subsets of Y an *ε-cover* of A if $|A_i| \leq \varepsilon$ for each i and $A \subseteq \bigcup_i A_i$.

Definition 2.1. For $A \subseteq Y$, $s \geq 0$ and $\varepsilon > 0$, we define

$$\mathcal{H}^s_\varepsilon(A) := \inf\left\{ \sum_i |A_i|^s \;\middle|\; \{A_i\} \text{ is an } \varepsilon\text{-cover of } A \right\}$$

and call

$$\mathcal{H}^s(A) := \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(A)$$

the *s-dimensional Hausdorff measure* of A. The *Hausdorff dimension* of A is defined by

$$D_H(A) := \sup\{s \geq 0 \mid \mathcal{H}^s(A) = \infty\}.$$

The proof of the next lemma is straightforward (cf. [**16**], for example).

LEMMA 2.2. *Let $A \subseteq Y$ be a* lim sup *set, meaning that there exists a sequence $(A_i)_{i\in\mathbb{N}}$ of subsets of Y with*

$$A = \limsup_{i\to\infty} A_i := \bigcap_{i=1}^{\infty}\bigcup_{k=i}^{\infty} A_k.$$

If $\sum_{i=1}^{\infty} |A_i|^s < \infty$ for some $s > 0$, then $\mathcal{H}^s(A) = 0$ and $D_H(A) \le s$.

LEMMA 2.3. [**27**] *Let Y and Z be two metric spaces and assume that $g : A \subseteq Y \to Z$ is a bi-Lipschitz continuous map. Then $D_H(g(A)) = D_H(A)$.*

LEMMA 2.4. [**27**] *The Hausdorff dimension is countably stable, that is, $D_H(\bigcup_i A_i) = \sup_i D_H(A_i)$ for any sequence of subsets $(A_i)_{i\in\mathbb{N}}$ with $A_i \subseteq Y$.*

Definition 2.5. The *lower* and *upper box-counting dimension* of a totally bounded subset $A \subseteq Y$ are defined as

$$\underline{D}_B(A) := \liminf_{\varepsilon\to 0} \frac{\log N(A,\varepsilon)}{-\log\varepsilon},$$
$$\overline{D}_B(A) := \limsup_{\varepsilon\to 0} \frac{\log N(A,\varepsilon)}{-\log\varepsilon},$$

where $N(A, \varepsilon)$ is the smallest number of sets of diameter at most ε needed to cover A. If $\underline{D}_B(A) = \overline{D}_B(A)$, then we call their common value $D_B(A)$ the *box-counting dimension* (or *capacity*) of A.

Remark. In contrast to the previous lemma, we only have that the upper box-counting dimension is finitely stable. Further, $D_B(A) = D_B(\overline{A})$.

THEOREM 2.6. [**19**] *Suppose that Y and Z are two metric spaces and consider the Cartesian product space $Y \times Z$ equipped with the maximum metric. Then for $A \subseteq Y$ and $B \subseteq Z$ totally bounded,*

$$D_H(A \times B) \le D_H(A) + \overline{D}_B(B).$$

2.2. *Exact dimensional and rectifiable measures.* We recall the notions of pointwise and information dimension as well as exact dimensional measures. Further, we provide the definition and some properties of rectifiable measures where we mainly follow [**1**].

Again, let Y be a metric space. For $x \in Y$, $\varepsilon > 0$, let $B_\varepsilon(x)$ be the open ball around x with radius $\varepsilon > 0$.

Definition 2.7. Suppose that μ is a finite Borel measure in Y. For each point x in the support of μ we define the *lower* and *upper pointwise dimension* of μ at x as

$$\underline{d}_\mu(x) := \liminf_{\varepsilon\to 0} \frac{\log \mu(B_\varepsilon(x))}{\log\varepsilon},$$
$$\overline{d}_\mu(x) := \limsup_{\varepsilon\to 0} \frac{\log \mu(B_\varepsilon(x))}{\log\varepsilon}.$$

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If $\underline{d}_\mu(x) = \overline{d}_\mu(x)$, then their common value $d_\mu(x)$ is called the *pointwise dimension* of μ at x. The *information dimension* of μ is defined as

$$\lim_{\varepsilon\to 0} \frac{\int \log \mu(B_\varepsilon(x))\, d\mu(x)}{\log \varepsilon},$$

provided the limit exists. Otherwise, one again defines *upper* and *lower information dimension* via the limit superior and inferior, respectively.

Definition 2.8. We say that the measure μ is *exact dimensional* if the pointwise dimension exists and is constant almost everywhere, that is,

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) =: d_\mu$$

μ-almost everywhere.

Remark. Note that if μ is exact dimensional, then in the setting of separable metric spaces several other dimensions of μ coincide with the pointwise dimension [**34**]. In particular, this is true for the information dimension [**26**, **32**].

Definition 2.9. For $d \in \mathbb{N}$, we call a Borel set $A \subseteq Y$ *countably d-rectifiable* if there exists a sequence of Lipschitz continuous functions $(g_i)_{i\in\mathbb{N}}$ with $g_i : A_i \subseteq \mathbb{R}^d \to Y$ such that $\mathcal{H}^d(A \setminus \bigcup_i g_i(A_i)) = 0$. A finite Borel measure μ is called *d-rectifiable* if $\mu = \Theta\mathcal{H}^d|_A$ for some countably d-rectifiable set A and some Borel measurable density $\Theta : A \to [0, \infty)$.

Observe that, by the Radon–Nikodym theorem, μ is d-rectifiable if and only if μ is absolutely continuous with respect to $\mathcal{H}^d|_A$ where A is a countably d-rectifiable set.

THEOREM 2.10. [**1**, Theorem 5.4] *For a d-rectifiable measure $\mu = \Theta\mathcal{H}^d|_A$, we have*

$$\Theta(x) = \lim_{\varepsilon\to 0} \frac{\mu(B_\varepsilon(x))}{V_d \varepsilon^d},$$

for $\mathcal{H}^d$-almost every $x \in A$, where V_d is the volume of the d-dimensional unit ball. The right-hand side of this equation is called the d-density of μ.

From the last theorem, we can deduce that the d-density exists and is positive μ-almost everywhere for a d-rectifiable measure μ. This directly implies the next corollary.

COROLLARY 2.11. *A d-rectifiable measure μ is exact dimensional with $d_\mu = d$.*

2.3. *Definition of the set $\widehat{\mathcal{U}}_\omega$.* The aim of this section is to define the set $\widehat{\mathcal{U}}_\omega$ in Theorem 1.4. In principle, it would be possible to work directly with the set $\mathcal{U}_\omega$ mentioned after Theorem 1.2, which can be defined in terms of the explicit $\mathcal{C}^2$-estimates used in [**12**]. However, as mentioned, we want to avoid reproducing the somewhat technical characterization. At the same time, we have to state a number of facts concerning the dynamics of the considered parameter families at the bifurcation, which are derived by means of the multiscale analysis carried out in [**12**].

Hence, what we actually do is to omit all those estimates from [**12**] which are only needed to prove the desired dynamical properties—namely, certain slow recurrence conditions for certain critical sets defined in the multiscale analysis. Instead, we define $\widehat{\mathcal{U}}_\omega$

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as the set of parameter families which satisfy those $\mathcal{C}^2$-estimates that are still needed for our purposes and at the same time show the required dynamical behaviour. This means that $\widehat{\mathcal{U}}_\omega$ will be defined in a partially intrinsic and somewhat abstract way. However, the important fact is that it has non-empty $\mathcal{C}^2$-interior (see Proposition 2.15) and contains the example (1.8) for large a.

In the following, let $f \in \mathcal{F}_\omega$ be given. Similarly to the above, we write $f_\theta(\cdot)$ for the fibre map $\tilde{f}(\theta, \cdot)$. We assume the existence of both an *interval of contraction* $C = [c, 1] \subseteq X$ and *expansion* $E = [0, e] \subseteq X$ where $0 < e < c < 1$ (the naming will become clear below) and a closed convex region $\mathcal{I}_0 \subseteq \mathbb{T}^d$, called the *(first) critical region*, such that

$$f_\theta(x) \in C \quad \text{for all } x \in [e, 1] \text{ and } \theta \notin \mathcal{I}_0. \tag{2.1}$$

Furthermore, we suppose that there are $\alpha > 1$, $p \geq \sqrt{2}$ and $S > 0$ such that for arbitrary $\theta, \theta' \in \mathbb{T}^d$,

$$\alpha^{-p}|x - x'| \leq \begin{cases} |f_\theta(x) - f_\theta(x')| \leq \alpha^p |x - x'| & \text{for all } x, x' \in X, \\ |f_\theta(x) - f_{\theta'}(x)| \leq S d(\theta, \theta') & \text{for all } x \in X, \\ |f_\theta(x) - f_\theta(x')| \leq \alpha^{-2/p}|x - x'| & \text{for all } x, x' \in C, \\ |f_\theta(x) - f_\theta(x')| \geq \alpha^{2/p}|x - x'| & \text{for all } x, x' \in E. \end{cases} \tag{2.2}$$

These are the explicit estimates needed to define $\widehat{\mathcal{U}}_\omega$. In order to state the required dynamical properties, let $K_n = K_0\kappa^n$ for some integers $\kappa \geq 2$, $K_0 \in \mathbb{N}$. Set

$$b_0 := 1, \quad b_n := (1 - 1/K_{n-1})b_{n-1} \ (n \in \mathbb{N})$$

and $b := \lim_{n\to\infty} b_n$ and assume K_0 and κ are big enough to ensure that $b > \sqrt{(p^2+1)/(p^2+2)}$. Further, let $(M_n)_{n\in\mathbb{N}_0}$ be a sequence of integers that satisfies $M_n \in [K_{n-1}M_{n-1}, 2K_{n-1}M_{n-1} - 2]$ for all $n \in \mathbb{N}$, where $M_0 \geq 2$.

Definition 2.12. For $n \in \mathbb{N}_0$, we recursively define the $(n+1)$th *critical region* $\mathcal{I}_{n+1}$ in the following way:

- $\mathcal{A}_n := (\mathcal{I}_n - (M_n - 1)\omega) \times C$;
- $\mathcal{B}_n := (\mathcal{I}_n + (M_n + 1)\omega) \times E$;
- $\mathcal{I}_{n+1} := \pi_{\mathbb{T}^d}(f^{M_n-1}(\mathcal{A}_n) \cap f^{-(M_n+1)}(\mathcal{B}_n))$.

Note that we trivially have $\mathcal{I}_{n+1} \subseteq \mathcal{I}_n$. For $n \in \mathbb{N}_0$, set

$$\mathcal{Z}_n^- := \bigcup_{j=0}^{n} \bigcup_{l=-(M_j-2)}^{0} \mathcal{I}_j + l\omega, \quad \mathcal{Z}_n^+ := \bigcup_{j=0}^{n} \bigcup_{l=1}^{M_j} \mathcal{I}_j + l\omega,$$

$$\mathcal{V}_n := \bigcup_{j=0}^{n} \bigcup_{l=1}^{M_j+1} \mathcal{I}_j + l\omega, \quad \mathcal{W}_n := \bigcup_{j=0}^{n} \bigcup_{l=-(M_j-1)}^{0} \mathcal{I}_j + l\omega.$$

Moreover, set $\mathcal{V}_{-1}, \mathcal{W}_{-1} = \emptyset$.

Definition 2.13. Let $n \in \mathbb{N}_0$. For $c_0 > 0$, set $\varepsilon_n := c_0\alpha^{-M_{n-1}\cdot b/(2p)}$, where we put $M_{-1} = 0$ for convenience. We say that f satisfies $(\mathcal{F}1)_n$ and $(\mathcal{F}2)_n$, respectively if $\mathcal{I}_j \neq \emptyset$ and

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$(\mathcal{F}1)_n \quad d(\mathcal{I}_j, \bigcup_{k=1}^{2K_jM_j} \mathcal{I}_j + k\omega) > \varepsilon_j$,
$(\mathcal{F}2)_n \quad (\mathcal{I}_j - (M_j - 1)\omega \cup \mathcal{I}_j + (M_j + 1)\omega) \cap (\mathcal{V}_{j-1} \cup \mathcal{W}_{j-1}) = \emptyset$
for $j = 0, \ldots, n$ and $n \in \mathbb{N}_0$. If f satisfies both $(\mathcal{F}1)_n$ and $(\mathcal{F}2)_n$, we say that f satisfies $(\mathcal{F})_n$. Further, we say f satisfies $(\mathcal{E})_n$ if
$(\mathcal{E})_n \quad |\mathcal{I}_n| < \varepsilon_n$,
where $|\mathcal{I}_n|$ denotes the diameter of $\mathcal{I}_n \subseteq \mathbb{T}^d$.

In the following, we say that f *satisfies (2.1) and (2.2),* $(\mathcal{F})_n$ *and* $(\mathcal{E})_n$ if it satisfies the respective assumptions for some choice of the above constants. With these notions, we are now in a position to define the set $\widehat{\mathcal{U}}_\omega$.

Definition 2.14. For $\omega \in \mathbb{T}^d$, set

$$\widehat{\mathcal{U}}_\omega = \{\hat{f} \in \mathcal{S}_\omega \mid f_{\beta_c} \text{ satisfies (2.1) and (2.2), } (\mathcal{F})_n \text{ and } (\mathcal{E})_n \text{ for all } n \in \mathbb{N}\}.$$

The following result is now contained implicitly in [**12**]; see [**12**, Theorem 4.18] and its proof.

PROPOSITION 2.15. [**12**] *For Diophantine $\omega \in \mathbb{T}^d$, the set $\widehat{\mathcal{U}}_\omega$ has non-empty $\mathcal{C}^2$-interior and we have $\widehat{\mathcal{U}}_\omega \subseteq \mathcal{N}_\omega$. Moreover, for all sufficiently large $a > 0$, the parameter family $\hat{f}$ given by (1.8) is contained in $\widehat{\mathcal{U}}_\omega$.*

Thus, in order to prove Theorem 1.4, our only task is to show that the properties of the parameter families in $\widehat{\mathcal{U}}_\omega$ stated in this section imply the assertions on the dimensions and the topological structure of $\phi^+_{\beta_c}$ and $\phi^-_{\beta_c}$.

3. *Hausdorff, pointwise and information dimension*

Our analysis of the structure of the SNA $\hat{\Phi}^+_{\beta_c}$ appearing in parameter families $\hat{f} \in \widehat{\mathcal{U}}_\omega$ hinges on the fact that the function $\hat{\phi}^+_{\beta_c}$ can be approximated by the images of the curve $\mathbb{T}^d \times \{1\}$ under successive iterates of the map f_{β_c}. Since from now on the critical parameter β_c and thus the map f_{β_c} are fixed, we suppress the parameter from the notation. We hence deal with elements of $\mathcal{F}_\omega$—the class of $\mathcal{C}^2$ qpf monotone interval maps of the form (1.1). More precisely, from now on f will always denote a map that belongs to†

$$\mathcal{V} = \{f \in \mathcal{F}_\omega \mid f \text{ satisfies (1.2)–(1.4), (2.1) and (2.2) as well as } (\mathcal{F})_n \text{ and } (\mathcal{E})_n \text{ for all } n \in \mathbb{N}\}.$$

As before, we let

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(\Gamma)$$

be the maximal f-invariant set inside Γ and denote by ϕ^- and ϕ^+ its bounding graphs, that is, $\phi^-(\theta) = \inf \Lambda_\theta$ and $\phi^+(\theta) = \sup \Lambda_\theta$ (cf. (1.7)). Now given $\theta \in \mathbb{T}^d$, let

$$\phi^+_n(\theta) := f^n_{\theta-n\omega}(1) = f_{\theta-\omega} \circ \cdots \circ f_{\theta-n\omega}(1) \quad \text{and}$$
$$\phi^-_n(\theta) := f^{-n}_{\theta+n\omega}(0) = f^{-1}_{\theta+\omega} \circ \cdots \circ f^{-1}_{\theta+n\omega}(0),$$

† Observe that by definition of $\widehat{\mathcal{U}}_\omega$, the map f_{β_c} corresponding to a one-parameter family $\hat{f} \in \widehat{\mathcal{U}}_\omega$ lies in $\mathcal{V}$.

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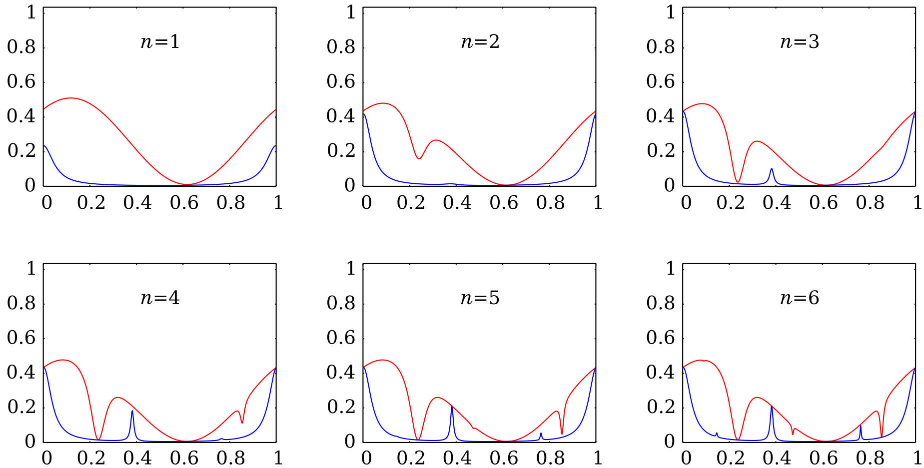

FIGURE 1. The first six iterated upper and lower boundary lines ϕ_n^+ (red) and ϕ_n^- (blue), respectively, of the family (1.8) for $a = 200$ at $\beta = 0.243\,235\,99$ with ω the golden mean.

with $f_\theta^n(x) = \pi_x \circ f^n(\theta, x)$ for all integers† $n \in \mathbb{Z}$ where π_x is the projection to the second coordinate. We call ϕ_n^+ the *nth iterated upper boundary line* and ϕ_n^- the *nth iterated lower boundary line*. Assumption (1.2) and the monotonicity (1.3) yield that $(\phi_n^+)_{n\in\mathbb{N}}$ and $(\phi_n^-)_{n\in\mathbb{N}}$ are monotonously decreasing and increasing, respectively. Moreover, it is easy to see from (1.3) that $[\phi_n^-, \phi_n^+] = \bigcap_{k=-n}^{n} f^n(\Gamma)$. As a consequence, it is immediate that

$$\phi^+(\theta) = \lim_{n\to\infty} \phi_n^+(\theta) \quad \text{and} \quad \phi^-(\theta) = \lim_{n\to\infty} \phi_n^-(\theta).$$

Thus, in order to draw conclusions on the structure of the bounding graphs, it is natural to study the iterated boundary lines first. Figure 1 shows the first six iterated boundary lines for the critical parameter in the example family (1.8) with ω the golden mean and parameters $a = 200$ and $\beta_c \approx 0.243\,235\,99$. These pictures reveal a very characteristic pattern. Let us look carefully at the evolution of ϕ_n^+.

For $n = 1$, we see that a first peak exists in the vicinity of $\theta = \omega$, that is, above the set $\mathcal{I}_0 + \omega$ (cf. (2.1)). After a second iteration, the image of this peak appears as a second peak in the vicinity of 2ω while outside this new peak the graph seems—more or less—unchanged. The second peak is not as pronounced as the first peak yet, since the strong expansion close to the zero line (due to (2.2)) enlarged the tiny gap between $\phi_1^+(\omega)$ and $\phi_1^-(\omega)$. However, after one more iteration, the second peak is *stabilized*, that is, its shape is essentially fixed for higher iterations. It is also important to observe that the graph outside this peak has not changed apart from a small neighbourhood of 3ω in the step from $n = 2$ to $n = 3$. Furthermore, note that the second peak is of much smaller size than the first one.

Although the third peak around 3ω is already hardly visible at $n = 3$, it clearly stabilizes until $n = 6$ and the graph only changes close to 4ω and 5ω along this stabilization. Altogether, this motivates the following qualitative claim.

† Note that the invariant graph $\phi^- \geq 0$ cannot be crossed by any orbit. Hence, due to the monotonicity of f_θ on all X (for each θ) as well as (1.2) and (1.3), $f_\theta^{-n}(0)$ is indeed well defined for all $n \in \mathbb{N}$ and arbitrary $\theta \in \mathbb{T}^d$.

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ϕ_{n+1}^+ differs from ϕ_n^+ only in smaller and smaller neighbourhoods of those peaks around $j\omega$ (for $j = 1, \ldots, n+1$) which are not yet stabilized after n iterations.

The point is that every peak eventually stabilizes in those θ which are not hit by peaks that appear at higher iterations. Moreover, the measure of the these future peaks tends to zero. As ϕ_j^+ is Lipschitz-continuous with a Lipschitz constant L_j, the claim implies that we get essentially the same Lipschitz constant L_j for ϕ_n^+ (with arbitrary $n \geq j$) at all those points at which ϕ_j^+ is already stabilized.

By this means, we are able to establish a decomposition of ϕ^+ into Lipschitz graphs whose Hausdorff dimension equals d (see Lemma 2.3). By the countable stability of the Hausdorff dimension (see Lemma 2.4), this yields that $D_H(\Phi^+) = d$. Parts (iii) and (iv) of Theorem 1.4 are not so easy to illustrate on this qualitative level since we need some understanding of the local densities of those sets which are not hit by future peaks. Still, despite some refinement, the arguments are very much based on the above observations.

To formalize ideas, we introduce

$$\Omega_j^n := \mathbb{T}^d \Big\backslash \bigcup_{k=j}^{\infty} \bigcup_{l=M_{k-1}}^{\min\{n, 2K_k M_k\}} \mathcal{I}_k + l\omega, \quad \Omega_j := \bigcap_{n \in \mathbb{N}} \Omega_j^n = \mathbb{T}^d \Big\backslash \bigcup_{k=j}^{\infty} \bigcup_{l=M_{k-1}}^{2K_k M_k} \mathcal{I}_k + l\omega,$$

where $j, n \in \mathbb{N}$. A way to interpret these definitions in terms of our qualitative discussion is the following: by the recursive definition of $\mathcal{I}_j$ (cf. §2.3), the size of the M_{j-1}th peak is about $|\mathcal{I}_j|$. Hence, Ω_j only contains points which are not hit by any peak that appears after M_{j-1} iterations. Likewise, Ω_j^n contains points at which ϕ_n^+ might stabilize in finite time, but at which new peaks could still appear at future iterations.

Observe that $K_k M_k \leq K_0 \kappa^k \cdot 2K_{k-1} M_{k-1} \leq \cdots \leq K_0^{k+1} \kappa^{\sum_{l=1}^{k} l} 2^k M_0$ while

$$|\mathcal{I}_k| < \varepsilon_k = c_0 \alpha_0^{-M_{k-1}} \leq c_0 \alpha_0^{-K_0^{k-1} \kappa^{\sum_{l=1}^{k-2} l} 2^{k-1} M_0}.$$

Thus, we have $2K_k M_k \varepsilon_k^d < \varepsilon_k^{d/2}$ for large enough k, and hence

$$\mathrm{Leb}_{\mathbb{T}^d}\left(\bigcup_{k=j}^{\infty} \bigcup_{l=M_{k-1}}^{2K_k M_k} \mathcal{I}_k + l\omega\right) < \sum_{k=j}^{\infty} V_d 2K_k M_k \varepsilon_k^d < \sum_{k=j}^{\infty} V_d \varepsilon_k^{d/2}, \tag{3.1}$$

for large enough j, where V_d is the normalizing factor of the d-dimensional Lebesgue measure. Thus, $\mathrm{Leb}_{\mathbb{T}^d}(\Omega_j) > 0$ for large enough $j \in \mathbb{N}$.

There might still be points which get hit by infinitely many peaks so that no eventual stabilization occurs. These are collected within

$$\Omega_\infty := \mathbb{T}^d \Big\backslash \bigcup_{j \in \mathbb{N}} \Omega_j = \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} \bigcup_{l=M_{k-1}}^{2K_k M_k} \mathcal{I}_k + l\omega.$$

In the following, we only consider the upper boundary lines ϕ_n^+ and the upper bounding graph ϕ^+. All of the results and proofs which are only stated in terms of ϕ^+ and ϕ_n^+ hold analogously for the lower boundary lines ϕ_n^- and the lower bounding graph ϕ^-, as can be seen by considering f^{-1} instead of f.

The next proposition is the basis of our geometrical investigation of ϕ^+. Its proof, which is the technical core of this paper, is given in the last section. However, the statement should seem plausible to the reader in the light of the above discussion.

PROPOSITION 3.1. *Let $f \in \mathcal{V}$. There are $\lambda > 0$ and $C > 0$ such that the following is true for sufficiently large j.*

(i) *Suppose $\theta \in \Omega_j^n$ and $n > 2K_{j-1}M_{j-1} - M_{j-1} - 1$. Then $|\phi_n^+(\theta) - \phi_{n-1}^+(\theta)| \leq \alpha^{-\lambda(n-1)}$.*

(ii) *Suppose $\theta, \theta' \in \Omega_j^n$ and $n \in \mathbb{N}$. Then $|\phi_n^+(\theta) - \phi_n^+(\theta')| \leq L_j d(\theta, \theta')$ for some $L_j \leq \varepsilon_j^{-CK_{j-1}}$ independent of n.*

Now, this information on the geometry of the curves ϕ_n^+ allows us to determine the Hausdorff dimension of Φ^+ rather easily (cf. [**16**]).

THEOREM 3.2. *Suppose $f \in \mathcal{V}$. Then the following statements hold:*

(i) $D_H(\Phi^+) = d$;

(ii) *μ_{ϕ^+} is d-rectifiable and exact dimensional with $d_{\mu_{\phi^+}} = d$.*

Proof. For each $j \in \mathbb{N} \cup \{\infty\}$, set $\psi_j := \phi^+|_{\Omega_j}$. First, we want to show that the graph $\Psi_j = \{(\theta, \psi_j(\theta)) : \theta \in \Omega_j\}$ is the image of a bi-Lipschitz continuous function g_j for all $j \in \mathbb{N}$. Define $g_j : \Omega_j \to \Omega_j \times X$ via $\theta \mapsto (\theta, \psi_j(\theta))$ for all $j \in \mathbb{N} \cup \{\infty\}$. We have that $g_j(\Omega_j) = \Psi_j$ and $d_{\mathbb{T}^d \times X}(g_j(\theta), g_j(\theta')) \geq d(\theta, \theta')$ for all $\theta, \theta' \in \Omega_j$. We may assume without loss of generality that j is large enough† so that Proposition 3.1(ii) yields that $\phi_n^+|_{\Omega_j}$ is Lipschitz continuous with Lipschitz constant L_j independent of n. Since $\psi_j = \lim_{n \to \infty} \phi_n^+|_{\Omega_j}$, we also get that ψ_j is Lipschitz continuous with the same constant and therefore

$$d_{\mathbb{T}^d \times X}(g_j(\theta), g_j(\theta')) \leq (1 + L_j)\, d(\theta, \theta'),$$

for all $\theta, \theta' \in \Omega_j$ and $j \in \mathbb{N}$. Hence, g_j is bi-Lipschitz continuous for each $j \in \mathbb{N}$.

(i) We want to make use of the fact that the Hausdorff dimension is countably stable; see Lemma 2.4. Because of the bi-Lipschitz continuity, we get that $D_H(\Psi_j) = D_H(\Omega_j)$. Since $\mathrm{Leb}_{\mathbb{T}^d}(\Omega_j) > 0$ for large enough j, this implies $D_H(\Psi_j) = d$. What is left to show is that $D_H(\Psi_\infty) \leq d$. Observe that Ω_∞ is a lim sup set. With a proper relabelling and doing a similar estimation to that in (3.1), we can use Lemma 2.2 to conclude that $D_H(\Omega_\infty) \leq s$ for all $s > 0$. Therefore, $D_H(\Omega_\infty) = 0$. Furthermore, $\Psi_\infty \subset \Omega_\infty \times X$ and hence $D_H(\Psi_\infty) \leq D_H(\Omega_\infty) + D_B(X) = 1 \leq d$, applying Theorem 2.6.

(ii) Note that by definition, μ_{ϕ^+} is absolutely continuous with respect to $\mathcal{H}^d|_{\Phi^+}$. We have that $\mu_{\phi^+}(\Psi_\infty) = 0$ and therefore μ_{ϕ^+} is also absolutely continuous with respect to $\mathcal{H}^d|_{\Phi^+ \setminus \Psi^\infty}$. Since $\Phi^+ \setminus \Psi_\infty = \bigcup_{j \in \mathbb{N}} \Psi_j$ is a countably d-rectifiable set—using the observation from the beginning of the proof—we get that μ_{ϕ^+} is d-rectifiable, too. Now, by applying Corollary 2.11, we obtain that μ_{ϕ^+} is exact dimensional with pointwise dimension $d_{\mu_{\phi^+}} = d$. □

Remark. By the remark in §2.2, we immediately get that the information dimension of μ_{ϕ^+} equals d.

† Observe that for $j \leq J$, we have $\Psi_j \subseteq \Psi_J$ because $\Omega_j \subseteq \Omega_J$.

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4. *Minimality and box-counting dimension*

For $n \in \mathbb{N}_0$, we denote by $\tilde{\mathcal{I}}_n$ the $\varepsilon_n/2$-neighbourhood of $\mathcal{I}_n$, that is, $\tilde{\mathcal{I}}_n := \bigcup_{\theta\in\mathcal{I}_n} B_{\varepsilon_n/2}(\theta)$, where $B_r(\theta)$ denotes the open ball of radius r centred at θ. Set

$$\tilde{\Omega}_\infty := \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \bigcup_{l=M_{k-1}}^{2K_kM_k} \tilde{\mathcal{I}}_k + l\omega.$$

LEMMA 4.1. *Suppose $\theta \notin \tilde{\Omega}_\infty$. Then there exists $j_0 \in \mathbb{N}$ such that, for all integers $j \geq j_0$, we have $\theta \in \Omega_j$ and*

$$\mathrm{Leb}_{\mathbb{T}^d}(B_{\varepsilon_n/2}(\theta) \cap \Omega_j)/\ \mathrm{Leb}_{\mathbb{T}^d}(B_{\varepsilon_n/2}(\theta)) \to 1, \tag{4.1}$$

for $n \to \infty$.

Proof. By the assumptions, there is $j_0 \in \mathbb{N}$ such that $\theta \notin \bigcup_{k=j_0}^{\infty} \bigcup_{l=M_{k-1}}^{2K_kM_k} \tilde{\mathcal{I}}_k + l\omega$. Fix an arbitrary $j \geq j_0$ and observe that

$$B_{\varepsilon_n/2}(\theta) \cap \left(\bigcup_{k=j}^{n} \bigcup_{l=M_{k-1}}^{2K_kM_k} \mathcal{I}_k + l\omega \right) = \emptyset$$

for $n \geq j$ by definition of $\tilde{\mathcal{I}}_k$. Thus,

$$B_{\varepsilon_n/2}(\theta) \cap \Omega_j = B_{\varepsilon_n/2}(\theta) \Big\backslash \bigcup_{k=j}^{\infty} \bigcup_{l=M_{k-1}}^{2K_kM_k} \mathcal{I}_k + l\omega = B_{\varepsilon_n/2}(\theta) \Big\backslash \bigcup_{k=n+1}^{\infty} \bigcup_{l=M_{k-1}}^{2K_kM_k} \mathcal{I}_k + l\omega.$$

Similarly as in (3.1), we get $\mathrm{Leb}_{\mathbb{T}^d}(\bigcup_{k=n+1}^{\infty} \bigcup_{l=M_{k-1}}^{2K_kM_k} \mathcal{I}_k + l\omega) < \sum_{k=n+1}^{\infty} V_d \varepsilon_k^{d/2}$ for large enough n, where V_d normalizes the Lebesgue measure. □

LEMMA 4.2. *Suppose $\theta \in \tilde{\Omega}_\infty$. For each $\ell \in \mathbb{N}$, there are arbitrarily large j such that*

$$B_{\varepsilon_j/2}(\theta) \subseteq \Omega_{j+1}^{2K_{j+\ell}M_{j+\ell}} \tag{4.2}$$

and

$$\mathrm{Leb}_{\mathbb{T}^d}(B_{\varepsilon_j/2}(\theta)) - \mathrm{Leb}_{\mathbb{T}^d}(B_{\varepsilon_j/2}(\theta) \cap \Omega_{j+1}) < \varepsilon_{j+\ell}. \tag{4.3}$$

Proof. For $n \in \mathbb{N}$, we define

$$j_n := \max\{p \in \mathbb{N}_0 \colon \exists l \in [M_{p-1}, \min\{n, 2K_pM_p\}] \text{ such that } \theta \in \tilde{\mathcal{I}}_p + l\omega\}$$

and let $l_n \in [M_{j_n-1}, 2K_{j_n}M_{j_n}]$ be the corresponding time such that $\theta \in \tilde{\mathcal{I}}_{j_n} + l_n\omega$, where uniqueness is guaranteed by $(\mathcal{F}1)_{j_n}$. Note that j_n and l_n are well defined for sufficiently large n and $j_n \overset{n\to\infty}{\longrightarrow} \infty$ because $\theta \in \tilde{\Omega}_\infty$.

Further, let $\theta_* \in \bigcap_{n=0}^{\infty} \mathcal{I}_n$. Note that $d(\theta_* + l\omega, \theta) < \frac{3}{2}\varepsilon_{j_n}$ for all l for which $\theta \in \tilde{\mathcal{I}}_{j_n} + l\omega$. Now, suppose there is $k \in \mathbb{N}$ such that $\theta \in \tilde{\mathcal{I}}_{j_n} + l_n\omega + k\omega$. Then

$$d(k\omega, 0) = d(\theta_* + (l_n + k)\omega, \theta_* + l_n\omega) \leq d(\theta_* + (l_n + k)\omega, \theta) + d(\theta, \theta_* + l_n\omega) < 3\varepsilon_{j_n}.$$

As ω is Diophantine, this means $\mathscr{C}|k|^{-\eta} < d(k\omega, 0) < 3\varepsilon_{j_n}$ and hence

$$|k| > \tilde{c}\varepsilon_{j_n}^{-1/\eta}, \tag{4.4}$$

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where $\tilde{c} > 0$. Define

$$J_n := \max\{N : 2K_N M_N < \tilde{c}\varepsilon_{j_n}^{-1/\eta}\}.$$

By (4.4), we have

$$B_{\varepsilon_{j_n}/2}(\theta) \subseteq \Omega_{j_n+1}^{2K_{J_n}M_{J_n}}.$$

Since $j_n/J_n \stackrel{n\to\infty}{\longrightarrow} 0$, we have thus shown that, for any $\ell \in \mathbb{N}$, there is arbitrarily large j such that $B_{\varepsilon_j/2}(\theta) \subseteq \Omega_{j+1}^{2K_{j+\ell}M_{j+\ell}}$.

Given $\ell \in \mathbb{N}$, assume that j is such that (4.2) holds. Then

$$\begin{aligned} B_{\varepsilon_j/2}(\theta) \cap \Omega_{j+1} &= B_{\varepsilon_j/2}(\theta) \Big\backslash \bigcup_{k=j+1}^{\infty} \bigcup_{l=M_{k-1}}^{2K_k M_k} \mathcal{I}_k + l\omega \\ &= B_{\varepsilon_j/2}(\theta) \Big\backslash \bigcup_{k=j+\ell+1}^{\infty} \bigcup_{l=M_{k-1}}^{2K_k M_k} \mathcal{I}_k + l\omega. \end{aligned}$$

Finally, $\mathrm{Leb}_{\mathbb{T}^d}(\bigcup_{k=j+\ell+1}^{\infty} \bigcup_{l=M_{k-1}}^{2K_k M_k} \mathcal{I}_k + l\omega) < \sum_{k=j+\ell+1}^{\infty} V_d \varepsilon_k^{d/2} < \varepsilon_{j+\ell}$ for large enough j. □

COROLLARY 4.3. *Let $f \in \mathcal{V}$. If $\phi = \phi^+$ almost everywhere and ϕ is an upper semi-continuous invariant graph, then $\phi = \phi^+$. In other words, ϕ^+ is the unique upper semi-continuous invariant graph in its equivalence class. Further,*

$$\phi^+(\overline{B_r(\theta)}) \subseteq \overline{\phi^+(B_r(\theta))}, \tag{4.5}$$

for all $\theta \in \mathbb{T}^d$ and all $r > 0$.

Proof. We first show (4.5). Let $\theta \in \mathbb{T}^d$ and $r > 0$ be given and let $\theta_0 \in \partial B_r(\theta) = \overline{B_r(\theta)} \backslash B_r(\theta)$.

Consider the case where $\theta_0 \notin \tilde{\Omega}_\infty$ and let j be as in Lemma 4.1. Equation (4.1) yields that for every $\rho > 0$ there is $\theta' \in B_r(\theta) \cap B_\rho(\theta_0)$ such that $\theta' \in \Omega_j$. Without loss of generality we may assume that j is large enough so that Proposition 3.1(ii) gives

$$|\phi_n^+(\theta_0) - \phi_n^+(\theta')| \leq L_j d(\theta_0, \theta')$$

for arbitrary n and thus $|\phi^+(\theta_0) - \phi^+(\theta')| \leq L_j d(\theta_0, \theta') \leq L_j \rho$ as $\phi_n^+ \to \phi^+$ pointwise. Sending ρ to zero proves the statement in the case $\theta_0 \notin \tilde{\Omega}_\infty$.

Now suppose $\theta_0 \in \tilde{\Omega}_\infty$ and let $\delta > 0$. Lemma 4.2 yields that there is arbitrarily large $j \in \mathbb{N}$ such that $\theta_0 \in \Omega_j^{2K_{j+2}M_{j+2}}$. For sufficiently large j, equation (4.3) gives $B_r(\theta) \cap B_{\delta\varepsilon_j^{CK_{j-1}}}(\theta_0) \cap \Omega_j \neq \emptyset$, where we may choose C such that $L_j \leq \varepsilon_j^{-CK_{j-1}}$ (see Proposition 3.1(ii)). Let $\theta' \in B_r(\theta) \cap B_{\delta\varepsilon_j^{CK_{j-1}}}(\theta_0) \cap \Omega_j$. Then $|\phi_{2K_jM_j}^+(\theta_0) - \phi_{2K_jM_j}^+(\theta')| \leq \delta$ by Proposition 3.1(ii). Without loss of generality we may further assume that j is large enough to ensure $|\phi^+(\theta_0) - \phi_{2K_jM_j}^+(\theta_0)| \leq \delta$ and $\sum_{k=2K_jM_j}^{\infty} \alpha^{-\lambda k} \leq \delta$, for λ as in Proposition 3.1(ii). This eventually gives

$$\begin{aligned} |\phi^+(\theta_0) - \phi^+(\theta')| \leq |\phi^+(\theta_0) - \phi_{2K_jM_j}^+(\theta_0)| + |\phi_{2K_jM_j}^+(\theta_0) - \phi_{2K_jM_j}^+(\theta')| \\ + |\phi_{2K_jM_j}^+(\theta') - \phi^+(\theta')| \leq 3\delta, \end{aligned}$$

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where we used Proposition 3.1(ii) (again, assuming large enough j) to estimate the last term.

Given arbitrary $\theta \in \mathbb{T}^d$ and $r > 0$, we have thus shown that for each $\theta_0 \in \partial B_r(\theta)$ there is a sequence $\theta_n \overset{n\to\infty}{\longrightarrow} \theta_0$ within $B_r(\theta)$ such that $\phi^+(\theta_0) = \lim_{n\to\infty} \phi^+(\theta_n)$. Hence, (4.5) holds. In fact, the construction shows that even if $\phi = \phi^+$ only *almost* everywhere, we still find a sequence $\tilde{\theta}_n \overset{n\to\infty}{\longrightarrow} \theta_0$ within $B_r(\theta)$ such that $\phi(\tilde{\theta}_n) = \phi^+(\tilde{\theta}_n) \overset{n\to\infty}{\longrightarrow} \phi^+(\theta_0)$. Thus, if ϕ is upper semi-continuous, this necessarily yields $\phi \geq \phi^+$. On the other hand, if ϕ is invariant, its graph is contained entirely within the maximal invariant set Λ so that $\phi \leq \phi^+$. Thus, $\phi = \phi^+$. □

Given an f-invariant and closed set $B \subseteq \mathbb{T}^d \times X$, the associated *upper* and *lower bounding graphs*

$$\phi_B^+(\theta) := \sup\{x : (\theta, x) \in B\} \quad \text{and} \quad \phi_B^-(\theta) := \inf\{x : (\theta, x) \in B\}$$

are invariant graphs, where ϕ_B^+ is upper semi-continuous and ϕ_B^- is lower semi-continuous. Conversely, continuity of f straightforwardly gives that the topological closure of an invariant graph Φ is a closed invariant set. Further, if ϕ is upper (lower) semi-continuous, then it equals the corresponding upper (lower) bounding graph: $\phi = \phi_{\overline{\Phi}}^+$ ($\phi = \phi_{\overline{\Phi}}^-$) (see [**30**, Corollaries 1 and 2]).

Remark. For the proof of the next statement, it is important to note that due to the non-zero Lyapunov exponents there is no lower and upper semi-continuous invariant graph that coincides almost everywhere with ϕ^+ and ϕ^-, respectively (cf. [**20**, Lemma 3.2]).

THEOREM 4.4. *Let $f \in \mathcal{V}$. Then $[\Phi^-, \Phi^+]$ is minimal. As a consequence, $D_B(\Phi^-) = D_B(\Phi^+) = d + 1$.*

Proof. As ϕ^- and ϕ^+ are lower and upper semi-continuous invariant graphs, respectively, $[\phi^-, \phi^+]$ is a compact invariant set.

For a contradiction, assume that $[\phi^-, \phi^+]$ is not minimal. Then there is a proper subset $M \subset [\phi^-, \phi^+]$ which is compact and invariant. Theorem 1.1 ($\mathcal{N}$) and Corollary 4.3 as well as the above remark yield that $\phi_M^{\pm} = \phi^{\pm}$. Hence, there have to be $\theta \in \mathbb{T}^d$ and $x \in (\phi^-(\theta), \phi^+(\theta))$ with $(\theta, x) \notin M$. Since M is compact, there is an open strip $S := B_{\varepsilon_1}(\theta_0) \times B_{\varepsilon_2}(x_0)$ with $\varepsilon_1, \varepsilon_2 > 0$ centred at some $(\theta_0, x_0) \in \mathbb{T}^d \times X$ such that $(\theta, x) \in S$ and $S \cap M = \emptyset$.

By Theorem 1.1, we may assume without loss of generality that there is a pinched point $\theta' \in B_{\varepsilon_1}(\theta_0)$ with $\phi^-(\theta') = \phi^+(\theta') \leq x_0 - \varepsilon_2$. In other words, Φ^- and Φ^+ have a common point below S. By continuity of ϕ^+ at the pinched points (see the remark below Theorem 1.1), we have that $\Phi^+|_{B_r(\theta')} := \Phi^+ \cap B_r(\theta') \times [0, 1]$ is below S for all small enough $r > 0$. Denote by R the supremum of all such r and suppose without loss of generality that $B_R(\theta') \subseteq B_{\varepsilon_1}(\theta')$. Then, $\Phi^+|_{B_R(\theta')}$ is below S, while $\Phi^+|_{B_{R+\delta}(\theta')}$ necessarily contains points above S for each $\delta > 0$. Hence, there is $\theta'' \in \partial B_R(\theta')$ such that $(\theta'', \phi^+(\theta''))$ is above S, contradicting Corollary 4.3 (cf. Figure 2). This proves the desired minimality.

As an immediate consequence, we have $\overline{\phi^-} = \overline{\phi^+} = [\phi^-, \phi^+]$ and so, by the remark in §2.1, $D_B(\phi^-) = D_B(\phi^+) = D_B([\phi^-, \phi^+])$. Since $\phi^- < \phi^+$ almost everywhere, we further have $D_B([\phi^-, \phi^+]) = d + 1$. □

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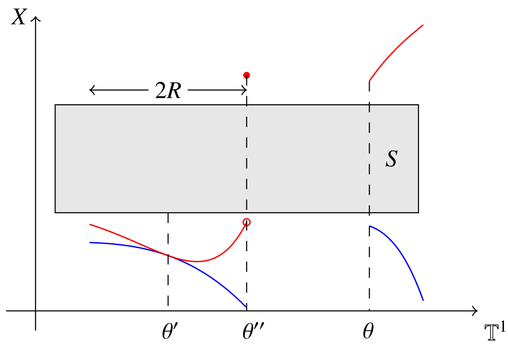

FIGURE 2. The one-dimensional case: assuming a gap within the minimal set implies the existence of a point $(\theta'', \phi^+(\theta''))$ which is isolated from one side (here, from the left). This contradicts Corollary 4.3.

5. *Proof of Proposition 3.1*

We now turn to the proof of Proposition 3.1. It is based on both the $\mathcal{C}^2$-estimates and the dynamical assumptions that define the set $\widehat{U}_\omega$ (see §2.3).

A crucial point is to control the number of times a forward orbit occurs in the contracting and a backward orbit occurs in the expanding region, respectively. For $n, N \in \mathbb{N}$, set

$$\mathcal{P}_n^N(\theta, x) := \#\{l \in [n, N-1] \cap \mathbb{N}_0 : f_\theta^l(x) \in C \text{ and } \theta + l\omega \notin \mathcal{I}_0\},$$

$$\mathcal{Q}_n^N(\theta, x) := \#\{l \in [n, N-1] \cap \mathbb{N}_0 : f_\theta^{-l}(x) \in E \text{ and } \theta - l\omega \notin \mathcal{I}_0 + \omega\}.$$

The following combinatorial lemmas are important ingredients for this control. Their proofs can be found in [**12**]. In the following, it is convenient to set $M_{-1} := 0$ (as before) and $\mathcal{I}_{-1} := \mathcal{I}_0$ as well as $\mathcal{Z}_{-1}^-, \mathcal{Z}_{-1}^+ := \emptyset$.

Definition 5.1. (θ, x) satisfies $(\mathcal{B}1)_n$ and $(\mathcal{B}2)_n$, respectively, if
$(\mathcal{B}1)_n$ $x \in C$ and $\theta \notin \mathcal{Z}_{n-1}^-$,
$(\mathcal{B}2)_n$ $x \in E$ and $\theta \notin \mathcal{Z}_{n-1}^+$.

LEMMA 5.2. (Cf. [**12**, Lemma 4.4]) *Let $f \in \mathcal{V}$, $n \in \mathbb{N}_0$ and assume that (θ, x) satisfies $(\mathcal{B}1)_n$. Let $\mathcal{L}$ be the first time l such that $\theta + l\omega \in \mathcal{I}_n$ and let $0 < \mathcal{L}_1 < \cdots < \mathcal{L}_N = \mathcal{L}$ be all those times $m \leq \mathcal{L}$ for which $\theta + m\omega \in \mathcal{I}_{n-1}$. Then $f^{\mathcal{L}_i + M_{n-1}+2}(\theta, x)$ satisfies $(\mathcal{B}1)_n$ for each $i = 1, \ldots, N-1$ and the following implication holds:*

$$f_\theta^k(x) \notin C \Rightarrow \theta + k\omega \in \mathcal{V}_{n-1} \quad \textit{and} \quad f_\theta^k(x) \in [0, 1] \quad (k = 1, \ldots, \mathcal{L}).$$

Analogously for backward iteration: instead of $(\mathcal{B}1)_n$, assume that (θ, x) satisfies $(\mathcal{B}2)_n$. Let $\mathcal{R}$ be the first time r such that $\theta - r\omega \in \mathcal{I}_n + \omega$ and let $0 < \mathcal{R}_1 < \cdots < \mathcal{R}_N = \mathcal{R}$ be all those times $m \leq \mathcal{R}$ for which $\theta - m\omega \in \mathcal{I}_{n-1}$. Then $f^{-\mathcal{R}_i - M_{n-1}}(\theta, x)$ satisfies $(\mathcal{B}2)_n$ for each $i = 1, \ldots, N-1$ and the following implication holds:

$$f_\theta^{-k}(x) \notin E \Rightarrow \theta - k\omega \in \mathcal{W}_{n-1} \quad \textit{and} \quad f_\theta^{-k}(x) \in [0, 1] \quad (k = 1, \ldots, \mathcal{R}).$$

LEMMA 5.3. (Cf. [**12**, Lemma 4.8]) *Let $f \in \mathcal{V}$ and assume that (θ, x) satisfies $(\mathcal{B}1)_n$ for $n \in \mathbb{N}$. Let $0 < \mathcal{L}_1 < \cdots < \mathcal{L}_N = \mathcal{L}$ be as in Lemma 5.2. Then, for each $i = 1, \ldots, N$, we have*

$$\mathcal{P}_k^{\mathcal{L}_i}(\theta, x) \geq b_n(\mathcal{L}_i - k) \quad (k = 0, \ldots, \mathcal{L}_i - 1). \tag{5.1}$$

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Analogously, assume that (θ, x) *satisfies* $(\mathcal{B}2)_n$ *for* $n \in \mathbb{N}$. *Let* $0 < \mathcal{R}_1 < \cdots < \mathcal{R}_N = \mathcal{R}$ *be as in Lemma 5.2. Then, for each* $i = 1, \ldots, N$, *we have*

$$\mathcal{Q}_k^{\mathcal{R}_i}(\theta, x) \geq b_n(\mathcal{R}_i - k) \quad (k = 1, \ldots, \mathcal{R}_i - 1).$$

As before, we consider the iterated upper boundary lines only. Given fixed $n \in \mathbb{N}$ and $\theta \in \mathbb{T}^d$, we set

$$\theta_k := \theta - (n-k)\omega \quad \text{and} \quad x_k := f_{\theta_0}^k(1)$$

such that $\phi_k^+(\theta_k) = x_k$.

Let $p \in \mathbb{N}$ and consider a finite orbit $\{(\theta_0, x), \ldots, f^n(\theta_0, x)\}$ which initially satisfies $(\mathcal{B}1)_p$ and hits $\mathcal{I}_p$ only at $\theta_0 + n\omega$. Lemma 5.3 provides us with a lower bound on the times spent in the contracting region between any time k and only such following times at which the orbit hits $\mathcal{I}_{p-1}$. If we want a lower bound on the times in the contracting region between any two consecutive moments $k < l$, we have to deal with the fact that Lemma 5.2 might allow the orbit to stay in the expanding region $M_{p-1} + 1$ times after hitting $\mathcal{I}_{p-1}$. This is taken care of in the following corollary of Lemmas 5.2 and 5.3.

For $\theta \in \mathbb{T}^d$ and $0 \leq k \leq n$, set

$$p_k^n(\theta) = \max\{p \in \mathbb{N}_0 \colon \exists l \in [M_{p-1}, \min\{n, n-k+M_p+1\}] \text{ such that } \theta - l\omega \in \mathcal{I}_p\}$$

with $\max \emptyset := -1$. At times, the following (and obviously equivalent) characterization of $p_k^n(\theta)$ is useful:

$$p_k^n(\theta) = \max\{p \in \mathbb{N}_0 \colon \exists l \in [\max\{0, k - M_p - 1\}, n - M_{p-1}] \text{ such that } \theta_l \in \mathcal{I}_p\}.$$

Observe that $p_\ell^n(\theta)$ and $p_{k-\ell}^{n-\ell}(\theta)$ are non-increasing in ℓ.

COROLLARY 5.4. *Let* $f \in \mathcal{V}$ *and suppose* $(\theta_0, x) = (\theta - n\omega, x)$ *satisfies* $(\mathcal{B}1)_{p_0^n(\theta)+1}$. *Then*

$$\mathcal{P}_k^n(\theta_0, x) \geq b_{p_k^n(\theta)+1}\left(n - k - \sum_{j=0}^{p_k^n(\theta)} (M_j + 2)\right) \quad \textit{for each } k = 0, \ldots, n-1. \tag{5.2}$$

Proof. For integers $p \geq -1$, set

$$\Theta_p := \{(\theta, x, n) \in \mathbb{T}^d \times [c, 1] \times \mathbb{N} \colon p_0^n(\theta) \leq p \text{ and } (\theta - n\omega, x) \text{ satisfy } (\mathcal{B}1)_{p_0^n(\theta)+1}\}.$$

We say (5.2) holds within Θ_p if (5.2) is true for all $(\theta, x, n) \in \Theta_p$. We show by induction on p that (5.2) holds within Θ_p for all p. Note that within Θ_{-1} inequality (5.2) follows directly from (2.1).

Suppose that there is an integer $p_0 \geq -1$ so that (5.2) holds within Θ_{p_0}. Set $p = p_0 + 1$ and fix $(\theta, x, n) \in \Theta_p \backslash \Theta_{p_0}$ which is assumed to be non-empty without loss of generality. Let $\mathcal{L}$ be the largest positive integer not bigger than $n - M_{p-1}$ such that $\theta_{\mathcal{L}} \in \mathcal{I}_p$ and assume without loss of generality that $\mathcal{L} < n$. Note that $p_{\mathcal{L}}^n(\theta) = p$. First, let $k \in [\mathcal{L}, n-1]$. There are two cases to consider. (a) Suppose that $\mathcal{L} \geq n - M_p - 2$. Then $\mathcal{L} \in [\max\{0, k - M_p - 1\}, n - M_{p-1}]$ for all $k \leq n-1$, by definition of $\mathcal{L}$. Hence, $p_k^n(\theta) = p$ for all $k \in [\mathcal{L}, n-1]$ since $\theta_{\mathcal{L}} \in \mathcal{I}_p$. Thus, $k \geq \mathcal{L} \geq n - M_{p_k^n(\theta)} - 2$ and so $M_{p_k^n(\theta)} \geq n - k - 2$ so that (5.2) holds trivially.

(b) Suppose that $\mathcal{L} < n - M_p - 2$. First, consider $k \geq \mathcal{L} + M_p + 2$. Then $p_k^n(\theta) < p$ and hence $p_{k-(\mathcal{L}+M_p+2)}^{n-(\mathcal{L}+M_p+2)}(\theta) \leq p_k^n(\theta) < p$. Furthermore, by Lemma 5.2, $f^{\mathcal{L}+M_p+2}(\theta_0, x)$ satisfies $(\mathcal{B}1)_{p+1}$ and thus $(\mathcal{B}1)_{p_0+1}$. Hence, we get

$$\begin{aligned}\mathcal{P}_k^n(\theta_0, x) &= \mathcal{P}_{k-(\mathcal{L}+M_p+2)}^{n-(\mathcal{L}+M_p+2)}(f^{\mathcal{L}+M_p+2}(\theta_0, x)) \\ &\geq b_{p_{k-(\mathcal{L}+M_p+2)}^{n-(\mathcal{L}+M_p+2)}(\theta)+1}\Bigg(n - k - \sum_{j=0}^{p_{k-(\mathcal{L}+M_p+2)}^{n-(\mathcal{L}+M_p+2)}(\theta)} (M_j + 2)\Bigg) \\ &\geq b_{p_k^n(\theta)+1}\Bigg(n - k - \sum_{j=0}^{p_k^n(\theta)} (M_j + 2)\Bigg),\end{aligned}$$

where the first estimate follows by the induction hypothesis and the last estimate from the fact that b_q is decreasing in q. Now, if $k \in [\mathcal{L}, \mathcal{L} + M_p + 1]$, then

$$\begin{aligned}\mathcal{P}_k^n(\theta_0, x) &= \mathcal{P}_k^{\mathcal{L}+M_p+2}(\theta_0, x) + \mathcal{P}_{\mathcal{L}+M_p+2}^n(\theta_0, x) \geq \mathcal{P}_{\mathcal{L}+M_p+2}^n(\theta_0, x) \\ &\geq b_{p_{\mathcal{L}+M_p+2}^n(\theta)+1}\Bigg(n - \mathcal{L} - M_p - 2 - \sum_{j=0}^{p_{\mathcal{L}+M_p+2}^n(\theta)} (M_j + 2)\Bigg) \\ &\geq b_{p_k^n(\theta)+1}\Bigg(n - k - M_p - 2 - \sum_{j=0}^{p_{\mathcal{L}+M_p+2}^n(\theta)} (M_j + 2)\Bigg) \\ &\geq b_{p_k^n(\theta)+1}\Bigg(n - k - \sum_{j=0}^{p_k^n(\theta)} (M_j + 2)\Bigg),\end{aligned}$$

where the last estimate holds since $p_k^n(\theta) = p$ for $k \leq \mathcal{L} + M_p + 1$.

We have thus shown that

$$\mathcal{P}_k^n(\theta_0, x) \geq b_{p_k^n(\theta)+1}\Bigg(n - k - \sum_{j=0}^{p_k^n(\theta)} (M_j + 2)\Bigg) \tag{5.3}$$

for $k \in [\mathcal{L}, n - 1]$.

It remains to consider $k < \mathcal{L}$. Since $p_k^n(\theta) \geq p_{\mathcal{L}}^n(\theta) = p$, we obtain

$$\begin{aligned}\mathcal{P}_k^n(\theta_0, x) &= \mathcal{P}_k^{\mathcal{L}}(\theta_0, x) + \mathcal{P}_{\mathcal{L}}^n(\theta_0, x) \geq b_{p+1}(\mathcal{L} - k) + b_{p_{\mathcal{L}}^n(\theta)+1}\Bigg(n - \mathcal{L} - \sum_{j=0}^{p_{\mathcal{L}}^n(\theta)} M_j + 2\Bigg) \\ &\geq b_{p+1}\Bigg(n - k - \sum_{j=0}^{p} M_j + 2\Bigg),\end{aligned}$$

where we used equations (5.1) and (5.3) in the first estimate. As (θ, x, n) was arbitrary in $\Theta_p \backslash \Theta_{p_0}$, this shows that (5.2) holds within Θ_p. □

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For $k, n \in \mathbb{N}$, set $i_k^n := \max\{l : n - k \geq 2K_l M_l - M_l - 1\}$.

PROPOSITION 5.5. *Suppose $\theta \in \Omega_j^n$ for some $j \in \mathbb{N}$. Then $i_k^n \geq p_k^n(\theta)$ for all $0 \leq k \leq n - (2K_{j-1}M_{j-1} - M_{j-1} - 1)$.*

Proof. Note that by the assumptions $i_k^n \geq j - 1$. Thus, without loss of generality we may assume $p_k^n(\theta) > j - 1$. By definition of $p_k^n(\theta)$, there is $l \in [M_{p_k^n(\theta)-1}, n - k + M_{p_k^n(\theta)} + 1]$ such that $\theta - l\omega \in \mathcal{I}_{p_k^n(\theta)}$. Since $\theta \in \Omega_j^n$, this implies $l > 2K_{p_k^n(\theta)} M_{p_k^n(\theta)}$ and thus, $n - k > 2K_{p_k^n(\theta)} M_{p_k^n(\theta)} - M_{p_k^n(\theta)} - 1$, which means $i_k^n \geq p_k^n(\theta)$. □

Proof of Proposition 3.1. Let $\theta \in \Omega_j^n$ and let $\mathcal{L}$ be the smallest positive integer such that $\theta_0 - \mathcal{L}\omega = \theta - (\mathcal{L} + n)\omega \in \mathcal{I}_{p_0^n(\theta)}$. Then $(\theta_0 - (\mathcal{L} - 1)\omega, 1)$ satisfies $(\mathcal{B}1)_{p_0^n(\theta)+1}$ because of $(\mathcal{F}1)_{p_0^n(\theta)}$. By (1.2) and by the monotonicity (1.3) of the fibre maps, we have the implication

$$f_{\theta_0-(\mathcal{L}-1)\omega}^{\mathcal{L}-1+k}(1) \in C \implies f_{\theta_0}^k(1) \in C,$$

for all $k \geq 0$. Further, we observe that $p_0^n(\theta) = p_{\mathcal{L}-1}^{\mathcal{L}-1+n}(\theta)$ and actually $p_k^n(\theta) = p_{\mathcal{L}-1+k}^{\mathcal{L}-1+n}(\theta)$ for all $k = 0, \ldots, n$. By Corollary 5.4, we thus get

$$\begin{aligned} \mathcal{P}_k^n(\theta_0, 1) &\geq \mathcal{P}_{\mathcal{L}-1+k}^{\mathcal{L}-1+n}(\theta_0 - (\mathcal{L} - 1)\omega, 1) \geq b_{p_k^n(\theta)+1}\left(n - k - \sum_{\ell=0}^{p_k^n(\theta)} (M_\ell + 2)\right) \\ &\overset{\text{Proposition 5.5}}{\geq} b_{i_k^n+1}\left(n - k - \sum_{\ell=0}^{i_k^n} (M_\ell + 2)\right), \end{aligned} \tag{5.4}$$

for $0 \leq k \leq n - (2K_{j-1}M_{j-1} - M_{j-1} - 1)$. Now note that $\sum_{\ell=0}^{i_k^n}(M_\ell + 2) \leq \frac{3}{2}M_{i_k^n}$ for large enough i_k^n (and hence, for large enough j since $i_k^n \geq j - 1$). Further, $(n - k)/K_{i_k^n} \geq 2M_{i_k^n} - M_{i_k^n}/K_{i_k^n} - 1/K_{i_k^n}$ by definition of i_k^n. Thus for large enough j, we have $\sum_{\ell=0}^{i_k^n}(M_\ell + 2) \leq (n - k)/K_{i_k^n}$ and so, by (5.4),

$$\mathcal{P}_k^n(\theta_0, 1) \geq b_{i_k^n+1}(1 - 1/K_{i_k^n})(n - k) \geq b^2(n - k). \tag{5.5}$$

Hence, we have

$$\begin{aligned} &|\phi_n^+(\theta) - \phi_{n-1}^+(\theta)| \\ &\quad = \phi_{n-1}^+(\theta) - \phi_n^+(\theta) = (\phi_0^+(\theta_1) - \phi_1^+(\theta_1)) \cdot \prod_{k=1}^{n-1} \frac{\phi_k^+(\theta_{k+1}) - \phi_{k+1}^+(\theta_{k+1})}{\phi_{k-1}^+(\theta_k) - \phi_k^+(\theta_k)} \\ &\quad \leq \prod_{k=1}^{n-1} \frac{f_{\theta_k}(\phi_{k-1}^+(\theta_k)) - f_{\theta_k}(\phi_k^+(\theta_k))}{\phi_{k-1}^+(\theta_k) - \phi_k^+(\theta_k)} \leq \alpha^{p((n-1)-\mathcal{P}_1^n(\theta_0,1))-2\mathcal{P}_1^n(\theta_0,1)/p} \\ &\quad \overset{(5.5)}{\leq} \alpha^{(p(1-b^2)-2b^2/p)(n-1)}, \end{aligned}$$

where we assumed—without loss of generality—that $\phi_{k-1}^+(\theta_k) - \phi_k^+(\theta_k) > 0$ for all $k \in \{1, \ldots, n\}$. This proves the first part with $\lambda = 2b^2/p - p(1 - b^2) > 0$.

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Let $\wp_k^n(\theta, \theta') := \#\{\ell \in [k, n-1] \cap \mathbb{N}_0 \colon x_\ell, x'_\ell \in C\}$ for $\theta, \theta' \in \mathbb{T}^d$. By induction on n, we first show that, for all $n \in \mathbb{N}$,

$$|\phi_n^+(\theta) - \phi_n^+(\theta')| \leq Sd(\theta, \theta') \sum_{k=1}^{n} \alpha^{p(n-k-\wp_k^n(\theta,\theta'))-2\wp_k^n(\theta,\theta')/p}. \tag{5.6}$$

For $n = 1$, this is equation (2.2). Suppose that (5.6) holds for some $n \in \mathbb{N}$. Since $\wp_k^n(\theta - \omega, \theta' - \omega) + \wp_n^{n+1}(\theta, \theta') = \wp_k^{n+1}(\theta, \theta')$, this yields

$$\begin{aligned}
|\phi_{n+1}^+(\theta) - \phi_{n+1}^+(\theta')| &= |f_{\theta-\omega}(\phi_n^+(\theta - \omega)) - f_{\theta'-\omega}(\phi_n^+(\theta' - \omega))| \\
&\leq \alpha^{p(1-\wp_n^{n+1}(\theta,\theta'))-(2/p)\wp_n^{n+1}(\theta,\theta')} |\phi_n^+(\theta - \omega) - \phi_n^+(\theta' - \omega)| \\
&\quad + Sd(\theta - \omega, \theta' - \omega) \\
&\leq Sd(\theta, \theta') \sum_{k=1}^{n+1} \alpha^{p(n+1-k-\wp_k^{n+1}(\theta,\theta'))-2\wp_k^{n+1}(\theta,\theta')/p},
\end{aligned}$$

where we used (2.2) in the first estimate and the induction hypothesis in the last step. Hence, equation (5.6) holds.

Now consider sufficiently large j and suppose $\theta, \theta' \in \Omega_j^n$. Suppose $n > 2K_{j-1}M_{j-1} - M_{j-1} - 1$ and observe that equation (5.5) gives

$$\begin{aligned}
\wp_k^n(\theta, \theta') &\geq n - k - (2(n-k) - \mathcal{P}_k^n(\theta) - \mathcal{P}_k^n(\theta')) \geq n - k - 2(1 - b^2)(n - k) \\
&= (2b^2 - 1)(n - k)
\end{aligned}$$

for all $k = 0, \ldots, n - (2K_{j-1}M_{j-1} - M_{j-1} - 1)$. Plugging this into (5.6) yields

$$\begin{aligned}
&|\phi_n^+(\theta) - \phi_n^+(\theta')| \\
&\leq Sd(\theta, \theta') \Bigg(\sum_{k=1}^{n-2K_{j-1}M_{j-1}-M_{j-1}-1} \alpha^{(2p(1-b^2)-2(2b^2-1)/p)(n-k)} \\
&\quad + \sum_{k=n-2K_{j-1}M_{j-1}-M_{j-1}}^{n} \alpha^{p(n-k-\wp_k^n(\theta,\theta'))-2\wp_k^n(\theta,\theta')/p} \Bigg) \\
&\leq L_j d(\theta, \theta'),
\end{aligned}$$

where

$$L_j := S \cdot \Bigg(\sum_{l=2K_{j-1}M_{j-1}-M_{j-1}-1}^{\infty} \alpha^{(2p(1-b^2)-2(2b^2-1)/p)l} + \sum_{l=0}^{2K_{j-1}M_{j-1}-M_{j-1}} \alpha^{pl} \Bigg).$$

It is immediate that $|\phi_n^+(\theta) - \phi_n^+(\theta')| \leq L_j d(\theta, \theta')$ holds for $n \leq 2K_{j-1}M_{j-1} - M_{j-1} - 1$, too. Finally, observe that there is $C > 0$ (independent of j) such that $L_j \leq \varepsilon_j^{-CK_{j-1}}$ for large enough j. □

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