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The spectral boundary of complemented invariant subspaces in $L^p(\mathbb{R})$

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In this paper we construct a compact set K of zero Hausdorff dimension that satisfies certain 'arithmetic-type' thickness properties. The concept of 'arithmetic thickness' has its origins in applications to harmonic analysis, introduced in a paper by Lebedev and Olevskiĭ. For example, there are no spectral sets whose 'essential boundary' can contain the above set K.

1. Introduction

We say that a Borel set $\Lambda \subseteq \mathbb{R}$ (more precisely the corresponding equivalence class) is a *spectral set* for $L^p(\mathbb{R})$, $1 < p < \infty$, if its indicator function belongs to the multiplier algebra

$$\mathbf{1}_\Lambda \in \mathcal{M}_p(\mathbb{R}). \tag{1.1}$$

If Λ satisfies (1.1), then it yields a *translation invariant complemented* subspace E_Λ in $L^p(\mathbb{R})$,

$$E_\Lambda \stackrel{\text{def}}{=} \text{Clos}\{f \in L^p(\mathbb{R}) \cap L^2(\mathbb{R}) : \hat{f}|_{^c\Lambda} = 0\},$$

where $\hat{f}$ denotes the Fourier transform of f.

Conversely, every invariant complemented subspace can be obtained in this way; in a natural sense, Λ is called the *spectrum* of E_Λ.

For $p = 2$, every Λ is a spectral set. For $p \neq 2$, the situation is much more difficult. Some classical examples are known, like $\Lambda = \mathbb{R}^+$, which is spectral for any $p \in]1, \infty[$, or more general ones, coming from the Littlewood–Paley decomposition.

Everything mentioned above is well known; see [2] or [4] for the precise definitions and references.

It was recently discovered in [2] that, for $p \neq 2$, spectral sets *cannot have a complicated structure.*

For example, a Cantor set of positive Lebesgue measure is never spectral.

Theorem 1.1 (see [2]). *If Λ is spectral for some $p \neq 2$,*

$$\textit{the essential boundary, } \partial\Lambda\textit{, has Lebesgue measure zero,} \tag{1.2}$$

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or, equivalently

$$\varLambda \text{ and } {}^{c}\varLambda \text{ are both (equivalent to) open sets.} \tag{1.3}$$

In the above theorem we used the term *essential boundary* in the sense that $\partial\varLambda = \overline{\varLambda^{\mathrm{d}}} \cap \overline{({}^{c}\varLambda)^{\mathrm{d}}}$, where $\varLambda^{\mathrm{d}}$ denotes the Lebesgue density points of $\varLambda$ and by $\bar{A}$ we denote the closure of A. In the sequel, we also use the (common) essential boundary of two disjoint measurable sets $\varLambda$ and $\varLambda'$, denoted by $\partial(\varLambda, \varLambda')$, and $x \in \mathbb{R}$ belongs to it if and only if any $U(x)$ neighbourhood of x contains portions of positive measure of both sets.

This result means that every spectral set $\varLambda$ can be viewed as a union of some family of disjoint open intervals accumulated to a *closed set F of measure zero*—its essential boundary.

Effective characterization of such families is probably an extremely difficult problem.

In this paper we are interested in what can be said about the essential boundary of a spectral set.

DEFINITION 1.2. For a given $p \neq 2$, a compact set $K \subseteq \mathbb{R}$ is called a *spectral-subboundary* ($K \in SB$) if and only if there exists a spectral set $\varLambda$ such that $\partial\varLambda \supset K$.

Obviously, this notion means some kind of 'thinness'. It is natural to compare it to metrical 'thinness'. Theorem 1.1 says that $K \in SB$ implies that $mK = 0$, where m denotes the Lebesgue measure.

Is the converse implication true?

We prove that the answer is *no*, even if one replaces the Lebesgue measure with the much more sensitive Hausdorff one.

We follow the general approach of [2, 3] which involves an analysis of possible distributions of equidistant nets through $\varLambda$.

DEFINITION 1.3. We say that a given pair of disjoint sets $\varLambda$, $\varLambda'$ has the *universal colouring* property,

$$(\varLambda, \varLambda') \in UC, \tag{1.4}$$

if, for any $N \in \mathbb{N}$ and for any colouring of the set $\{1, \dots, N\}$ by two colours (say, by *red* and *blue*), there exists x_0, $h \in \mathbb{R}$, such that

$$x_k \stackrel{\text{def}}{=} x_0 + kh \in \varLambda \text{ or } \varLambda' \text{ if, } k \text{ is } \textit{blue} \text{ or } \textit{red}\text{, respectively.} \tag{1.5}$$

This notion plays a crucial role in [2,3]. It appeared there in a slightly stronger form: x_k was required to be a density point of the corresponding set; below we will use the UC property for open sets and hence this stronger assumption will also be satisfied.

Actually, theorem 1.1 was obtained by the following implications:

$$m(\partial(\varLambda, \varLambda')) > 0 \Rightarrow (\varLambda, \varLambda') \in UC \Rightarrow \varLambda \text{ is not a spectral set.} \tag{1.6}$$

Next we introduce the property of 'arithmetical thickness' ($\mathcal{A}$-thickness).

DEFINITION 1.4. We say that a compact set K is *$\mathcal{A}$-thick*, that is, $K \in \mathcal{A}$, if and only if, for all pairs of disjoint open sets $\varLambda$, $\varLambda'$,

$$K \subseteq \partial(\varLambda, \varLambda') \Rightarrow (\varLambda, \varLambda') \in UC. \tag{1.7}$$

Our main result says that $\mathcal{A}$-thick compact sets may be of small Hausdorff dimension. We remark that in [1] several different concepts of thin sets in harmonic analysis were considered.

2. Main results

THEOREM 2.1. *There exists a compact perfect set* $K \subseteq \mathbb{R}$ *such that*

(i) $\dim_{\mathrm{H}} K = 0$, *and*

(ii) $K \in \mathcal{A}$.

We remark that $\dim_{\mathrm{H}}$ denotes the usual Hausdorff dimension, and by a suitable refinement of our proof one can obtain compact sets with $\mathcal{H}_h(K) = 0$, where $\mathcal{H}_h$ is the Hausdorff measure defined by using a function h ($h(0) = 0$, $h(x) > 0$, h is monotone increasing and continuous from the right for all $t \geqslant 0$). In the special case when $h(x) = x^s$, that is, we use the s-dimensional Hausdorff measure, we will use the notation $\mathcal{H}^s$.

THEOREM 2.2. *There exists a compact perfect set* K*, satisfying property (i) and*

(iii) $K \notin SB$.

Finally, our last result gives some additional information about $\mathcal{A}$-thickness.

THEOREM 2.3. *If* $K \in \mathcal{A}$*, then it contains arbitrarily long arithmetical progressions.*

Proof of theorem 2.2. We reduce this result to theorem 2.1, which will be proved later. It is enough to show that property (ii) implies property (iii). Let $K \in \mathcal{A}$. Suppose that for some $p \neq 2$, $K \in SB$, that is, there exists a spectral set Λ such that $\partial\Lambda \supset K$. Due to theorem 1.1, we can assume that Λ is open and ${}^{\mathrm{c}}\Lambda$ is equivalent to an open set Λ'. It is clear that Λ' is disjoint from Λ. Since Λ' and ${}^{\mathrm{c}}\Lambda$ are equivalent, $(\Lambda')^{\mathrm{d}} = ({}^{\mathrm{c}}\Lambda)^{\mathrm{d}}$, and hence $K \subseteq \partial\Lambda = \partial(\Lambda, \Lambda')$. Then $K \in \mathcal{A}$ implies that the pair (Λ, Λ') satisfies the UC-property. According to (1.6), Λ is not a spectral set. □

The construction used in the proof of theorem 2.1. We define K as the intersection of the nested closed sets E_k. Each E_k consists of finitely, say n_k, many closed intervals $I_{j,k}$, $j = 1, \dots, n_k$.

We assume that, for a fixed k, the intervals $I_{j,k}$ are indexed from left to right and they are disjoint.

Denote the length of $I_{j,k}$ by $\ell_{j,k}$. For notational convenience, we set $\ell'_k = \frac{1}{2}\ell_k$ for $k = 1, 2, \dots$.

We choose the intervals $I_{j,k}$ in a way that $\ell_{j,k}$ is non-increasing in j, that is, $\ell_{j_1,k} \geqslant \ell_{j_2,k}$ for $j_1 \leqslant j_2$ when k is fixed. We denote by ℓ_k the length of the shortest interval $I_{j,k}$. Clearly, $\ell_k = \ell_{n_k,k}$. For a given $j' \leqslant n_{k-1}$, by our choice $I_{j',k-1} \cap E_k$ will consist of $\nu_{j',k-1}$ many equally spaced intervals of the form $I_{j,k}$ and all these $I_{j,k}$ will be of the same length, denoted by $\eta_{j',k-1}$.

If $j' < j''$ holds, then *the distance between the consecutive intervals* $I_{j,k}$ belonging to $I_{j'',k-1} \cap E_k$ will be *much less than the length of the* $I_{j,k}$ belonging to $I_{j',k-1} \cap E_k$.

Next, we give a formal inductive definition of the sets E_k.

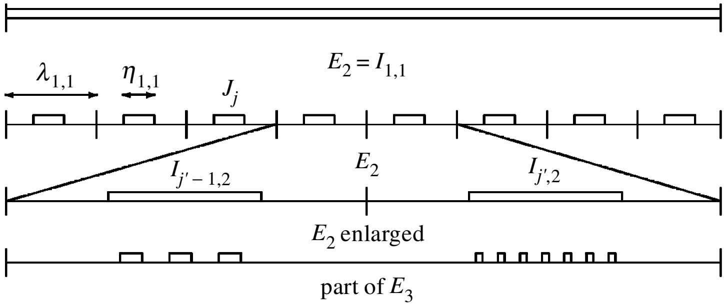

Figure 1.

Put $E_1 = I_{1,1} = [0, 1]$.

Assume that $k \geqslant 2$, $E_{k-1} = \bigcup_{j=1}^{n_{k-1}} I_{j,k-1}$ is defined and

$$\ell_{1,k-1} \geqslant \ell_{2,k-1} \geqslant \cdots \geqslant \ell_{n_{k-1},k-1} = \ell_{k-1}.$$

We also assume that $\ell_{j,k-1}$ is an integer multiple of ℓ_{k-1} for $j = 1, \dots, n_{k-1}$. Set

$$\lambda_{1,k-1} = \frac{\ell_{k-1}}{2k^k} \quad \text{and} \quad \nu_{1,k-1} = \frac{\ell_{1,k-1}}{\lambda_{1,k-1}}.$$

Since $\ell_{1,k-1}$ is an integer multiple of ℓ_{k-1}, it is easy to see that $\nu_{1,k-1}$ is an integer. Divide $I_{1,k-1}$ into $\nu_{1,k-1}$ many intervals of length $\lambda_{1,k-1}$ and denote these intervals by $H_1, \dots, H_{\nu_{1,k-1}}$. Finally, in each interval H_j, choose a small closed interval J_j, which has the same midpoint as H_j, and such that the length of J_j equals

$$\eta_{1,k-1} = \frac{\lambda_{1,k-1}}{\nu_{1,k-1}^{k-1}}.$$

We will define $E_k \cap I_{1,k-1}$ such that it will equal the union of the intervals J_j, that is, $J_j = I_{j,k}$ for $j = 1, \dots, \nu_{1,k-1}$. Observe that ℓ_{k-1} is an integer multiple of $\eta_{1,k-1}$ and $\ell_{j,k} = \eta_{1,k-1}$ for $j = 1, \dots, \nu_{1,k-1}$.

Now assume that $j' \geqslant 2$, the $I_{j,k}$ are defined in the intervals $I_{1,k-1}, \dots, I_{j'-1,k-1}$ and the ones in $I_{j'-1,k-1}$ are of length $\eta_{j'-1,k-1}$. We also assume that ℓ_{k-1} is an integer multiple of $\eta_{j'-1,k-1}$. Since $\ell_{j',k-1}$ is an integer multiple of ℓ_{k-1}, it will also be an integer multiple of $\eta_{j'-1,k-1}$. Set

$$\lambda_{j',k-1} = \frac{\eta_{j'-1,k-1}}{k^k} \quad \text{and} \quad \nu_{j',k-1} = \frac{\ell_{j',k-1}}{\lambda_{j',k-1}}.$$

Observe that $\nu_{j',k-1}$ is an integer. Now we subdivide $I_{j',k-1}$ like we divided $I_{1,k-1}$, that is, divide $I_{j',k-1}$ into $\nu_{j',k-1}$ many intervals of length $\lambda_{j',k-1}$ and denote them by $H_1, \dots, H_{\nu_{j',k-1}}$. In each interval H_j, choose a small closed interval J_j, which has the same midpoint as H_j, and such that the length of J_j equals

$$\eta_{j',k-1} = \frac{\lambda_{j',k-1}}{\nu_{j',k-1}^{k-1}}.$$

Observe that $\lambda_{j',k-1}$ and $\eta_{j'-1,k-1}$ are both integer multiples of $\eta_{j',k-1}$. Finally, let $E_k \cap I_{j',k-1}$ equal the union of the intervals J_j.

Repeating the above steps for a fixed k we can define $E_k \cap I_{j',k-1}$ for all $j' = 1, \dots, n_{k-1}$. Continuing this process, define E_k for all k.

Put $K = \bigcap_{k=1}^{\infty} E_k$.

It is easy to see that K is a nowhere dense non-empty compact perfect set and, for any k and $j \leqslant n_k$, we have $K \cap I_{j,k} \neq \emptyset$. It is also clear that

$$\frac{\ell_{k-1}}{2k^k} = \lambda_{1,k-1} > \eta_{1,k-1} > \cdots > \eta_{n_{k-1},k-1} = \ell_k.$$

This implies that for any $k \geqslant 1$ we have

$$\sum_{j=k+1}^{\infty} \ell_j < \ell_k.$$

Since for $k = 2, 3, \dots$ the choice of $\lambda_{j',k-1}$ implies that each interval $I_{j',k-1}$ contains more than four intervals belonging to E_k, it is always possible to find an $x \in K$ that is in the first quarter of $I_{j',k-1}$.

First we compute the Hausdorff dimension of K. Denote by $d(U)$ the diameter of the set U. Assume that the integer m is fixed. Note that the intervals belonging to E_k cover K for any integer k. Let

$$S_k = \sum_{j=1}^{n_k} d(I_{j,k})^{1/m}.$$

Lemma 2.4. *For any $k \geqslant m+1$, we have $S_k \leqslant S_{k-1}$.*

Proof of lemma 2.4. Assume that $j' \leqslant n_{k-1}$ and $I_{j',k-1} \cap E_k$ consists of the intervals J_j, $j = 1, \dots, \nu_{j',k-1}$. Then $d(J_j) = \eta_{j',k-1}$ for all j and

$$\sum d(J_j)^{1/m} = \nu_{j',k-1} \cdot \eta_{j',k-1}^{1/m} = \nu_{j',k-1} \left(\frac{\lambda_{j',k-1}}{\nu_{j',k-1}^{k-1}} \right)^{1/m} = A.$$

Using $k \geqslant m+1$, we can continue the estimation by

$$A < \nu_{j',k-1} \left(\frac{\lambda_{j',k-1}}{\nu_{j',k-1}^{m-1}} \right)^{1/m} = (\nu_{j',k-1}\lambda_{j',k-1})^{1/m} = (\ell_{j',k-1})^{1/m} = d(I_{j',k-1})^{1/m}.$$

Recalling that the intervals J_j are those intervals $I_{j,k}$ that are in $I_{j',k-1}$, the above result implies that

$$\sum_{\{j : I_{j,k} \subseteq I_{j',k-1}\}} d(I_{j,k})^{1/m} < d(I_{j',k-1})^{1/m}.$$

Summing this for all $I_{j',k-1}$, we obtain $S_k \leqslant S_{k-1}$. This proves lemma 2.4. □

Lemma 2.5. *The set K is of zero Hausdorff dimension.*

Proof of lemma 2.5. We need to verify that, given any $s > 0$ and $\delta > 0$, one can find a cover of K by sets U_i such that $d(U_i) < \delta$ and $\sum d(U_i)^s < \infty$.

It is enough to verify that for each $s_m = 1/m$ a suitable cover can be found; hence assume that m is fixed.

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The intervals belonging to E_k cover K for any integer k. From lemma 2.4, it follows that $S_k \leqslant S_m$ for all $k \geqslant m$. Using $d(I_{j,k}) = \ell_{j,k} < \ell_{k-1}$ and $\ell_k \to 0$ as $k \to \infty$, we obtain that, for any given $\delta > 0$, we can find k such that $d(I_{j,k}) < \delta$ for $j = 1, \dots, n_k$ and $S_k = \sum_j d(I_{j,k})^{1/m} < S_m$. This implies $\mathcal{H}^{1/m}(K) < \infty$. This holds for all m; hence K is of zero Hausdorff dimension. □

We turn to the proof of $K \in \mathcal{A}$.

To state lemma 2.6, and later lemma 2.8, we need more notation.

Denote by Ψ the union of the intervals contiguous to K and by s_n the set $\{1, \dots, n\}$ where n is a positive integer. Assume that Λ and Λ' are disjoint open sets such that $K \subseteq \partial(\Lambda, \Lambda')$. Set $G = \Lambda \cup \Lambda'$. Then $G \subseteq \Psi$ is open and we can introduce the colouring function $\phi_1 : G \to \{\text{blue}, \text{red}\}$ by letting $\phi_1 = \text{red}$ on Λ and $\phi_1 = \text{blue}$ on Λ'. Since $K \subseteq \partial(\Lambda, \Lambda')$, the colouring ϕ_1 of G is *dense*; by this we mean that both red and blue intervals are 'dense' in K, that is, $K \subseteq \overline{\{x : \phi_1(x) = \text{red}\}}$ and $K \subseteq \overline{\{x : \phi_2(x) = \text{blue}\}}$. To verify that $K \in \mathcal{A}$, we need to show that, for any of the above given open sets Λ and Λ', the universal colouring property is satisfied, that is, we need to find $x \in \mathbb{R}$ and $h^* \in \mathbb{R}$ such that $x + ih^* \in G$ for $i = 1, \dots, n$, and the two colourings ϕ_1 and ϕ_2 are *compatible*, that is, $\phi_1(x + ih^*) = \phi_2(i)$ for $i = 1, \dots, n$.

Next we state a rather technical lemma.

Lemma 2.6. *Assume that $k > 2n^2$, y and t are given such that $y \in G$, the points $y + it$ for $i = 1, \dots, n$ belong to different components of E_k and*

$$[y + it, y + it + \ell'_k] \subseteq E_k \quad \textit{for } i = 1, \dots, n.$$

Then there exists t' such that

$$t \leqslant t' \leqslant t + n\frac{\ell'_k}{k},$$

and the points

$$y + it' \in [y + it, y + it + \ell'_k], \quad i = 1, \dots, n,$$

belong to different components of E_{k+1}. Furthermore, if ρ_n denotes the length of the interval $I_{j,k+1} \subseteq E_{k+1}$ that contains the point $y + nt'$, then

$$[y + it', y + it' + \rho_n] \subseteq E_{k+1} \quad \textit{for } i = 1, \dots, n.$$

Remark 2.7. Observe that $\rho_n \geqslant \ell_{k+1} = 2\ell'_{k+1}$, and hence the conclusion of lemma 2.6 implies that

$$[y + it', y + it' + \ell'_{k+1}] \subseteq E_{k+1}$$

also holds for $i = 1, \dots, n$. This implies that we can repeat the application of lemma 2.6.

Proof of lemma 2.6. We do induction on n.

For $n = 1$, we have $[y + t, y + t + \ell'_k] \subseteq E_k$. Choose j_1 such that $y + t \in I_{j_1,k}$. Then there exists $\lambda_1 \leqslant \ell'_k/(k+1)^{(k+1)}$ such that during the construction of E_{k+1}

the interval $I_{j_1,k}$ is being split into intervals of length λ_1 and each such subinterval contains a component of E_{k+1}. Thus we can choose t' such that

$$t \leqslant t' \leqslant t + \lambda_1 < t + \frac{\ell'_k}{k}$$

and $y + t'$ is the left endpoint of a component of E_{k+1}. Then

$$[y + t', y + t' + \rho_1] \subseteq E_{k+1},$$

where ρ_1 is the length of the interval $I_{j,k+1}$ that contains $y + t'$. This completes the proof when $n = 1$.

Now assume that $n \geqslant 2$, $k > 2n^2$, and lemma 2.6 is true for $n - 1$. Observe that using $y \in G$ and the points $y + it$, $i = 1, \dots, n - 1$, the assumptions of lemma 2.6 (used for $n - 1$) are satisfied. Hence, by the induction hypothesis, there exists t'' such that

$$t \leqslant t'' \leqslant t + (n-1)\frac{\ell'_k}{k},$$

and

$$y + it'' \in [y + it, y + it + \ell'_k] \subset E_k \quad \text{for } i = 1, \dots, n-1, \tag{2.1}$$

and these points belong to different components of E_{k+1}. Furthermore, for $i = 1, \dots, n - 1$, we have

$$[y + it'', y + it'' + \rho_{n-1}] \subseteq E_{k+1}, \tag{2.2}$$

where ρ_{n-1} is the length of the interval $I_{j,k+1}$ that contains $y + (n-1)t''$. Clearly, $\rho_{n-1} < \ell'_k/k^k$.

Now, by our assumptions,

$$y + nt \leqslant y + nt'' \leqslant y + nt + n(n-1)\frac{\ell'_k}{k} < y + nt + \tfrac{1}{2}\ell'_k.$$

Hence

$$[y + nt'', y + nt'' + \tfrac{1}{2}\ell'_k] \subseteq [y + nt, y + nt + \ell'_k] \subseteq E_k. \tag{2.3}$$

By (2.1), $y+(n-1)t''$ and $y+(n-1)t$ belong to the same component of E_k. Denote this component by $I_{j_{n-1},k}$. Similarly, by (2.3), $y + nt''$ and $y + nt$ belong to the same component, $I_{j_n,k}$, of E_k. By (2.3) we have

$$[y + nt'', y + nt'' + \tfrac{1}{2}\ell'_k] \subseteq I_{j_n,k} \subseteq E_k. \tag{2.4}$$

The definition of E_{k+1} implies that there exists λ_n and $\eta_{j_n,k} = \rho_n$ such that $I_{j_n,k}$ is divided into subintervals of length λ_n. Each such subinterval contains a component of E_{k+1}, and this component is of length ρ_n. From $j_{n-1} < j_n$ and from our construction, it follows that $\lambda_n < \rho_{n-1}/(k+1)^{(k+1)}$; hence we can choose t' such that

$$t'' \leqslant t' \leqslant t'' + \lambda_n < t'' + \frac{\rho_{n-1}}{(k+1)^{(k+1)}} < t'' + \frac{\ell'_k}{k} \leqslant t + n\frac{\ell'_k}{k}, \tag{2.5}$$

and, using (2.3), we can also assume that $y+nt'$ is the left endpoint of a component of $E_{k+1} \cap I_{j_n,k}$. Then

$$[y + nt', y + nt' + \rho_n] \subseteq E_{k+1}.$$

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It remains to check that, for $i = 1, \dots, n-1$, the small intervals $[y+it', y+it'+\rho_n]$ are in E_{k+1}. From

$$t'' \leqslant t' \leqslant t'' + \frac{\rho_{n-1}}{(k+1)^{(k+1)}}$$

and

$$\rho_n < \lambda_n < \frac{\rho_{n-1}}{(k+1)^{(k+1)}},$$

it follows that

$$y + it' + \rho_n < y + it'' + \frac{(i+1)\rho_{n-1}}{(k+1)^{(k+1)}} < y + it'' + \rho_{n-1}$$

for $i = 1, \dots, n-1$. Thus $y + it'' \leqslant y + it'$ and (2.2) implies

$$[y + it', y + it' + \rho_n] \subseteq E_{k+1}$$

for $i = 1, \dots, n-1$. Observe that above we have already established the last property for $i = n$. This completes the proof of lemma 2.6. □

In the next lemma we will use the notation introduced before the statement of lemma 2.6.

LEMMA 2.8. *Assume that n, k_0, x_n and h_{k_0} are given such that $k_0 > 2n^2$, $x_n \in G$, the points $x_n + ih_{k_0}$ belong to different components of E_{k_0}, we have two colourings ϕ_1 and ϕ_2, $\phi_1(x_n) = \phi_2(n)$, and*

$$[x_n + ih_{k_0}, x_n + ih_{k_0} + \ell'_{k_0}] \subseteq E_{k_0} \quad \textit{for } i = 1, \dots, n.$$

Then there exists $x \in K$ and $h \in \mathbb{R}$ such that

$$|h - h_{k_0}| < n\ell'_{k_0},$$
$$|x - (x_n + nh_{k_0})| < n\ell'_{k_0}$$

and

$$\phi_1(x - ih) = \phi_2(i) \quad \textit{for } i = 1, \dots, n.$$

Proof of lemma 2.8. We do induction on n.

Assume that $k_0 \geqslant 2$ is given. When $n = 1$, by our assumption, $[x_1 + h_{k_0}, x_1 + h_{k_0} + \ell'_{k_0}] \subseteq E_{k_0}$. Choose j_1 such that $x_1 + h_{k_0} \in I_{j_1,k_0}$. Then there exists

$$\lambda < \frac{\ell_{k_0}}{2k_0^{k_0}} = \frac{\ell'_{k_0}}{k_0^{k_0}} < \tfrac{1}{2}\ell'_{k_0}$$

such that any subinterval of I_{j_1,k_0} with length λ contains points of E_{k_0+1}, and hence points of K as well. Thus we can choose $h \geqslant h_{k_0}$ such that $|h - h_{k_0}| < \ell'_{k_0}$ and $x_1 + h = x \in K$. Then $|x - (x_1 + h_{k_0})| = |h - h_{k_0}| < \ell'_{k_0}$ also holds. Clearly, $\phi_1(x - h) = \phi_1(x_1) = \phi_2(1)$.

Assume that lemma 2.8 is valid for $n - 1 \geqslant 1$ with all $k_0 > 2(n-1)^2$.

For the next step of the induction, assume that $k_0 > 2n^2$ is given and the assumptions of lemma 2.8 are satisfied for n, k_0, x_n and h_{k_0}.

Since G is open and $x_n \in G$, there exists $\epsilon > 0$ such that $]x_n - \epsilon, x_n + \epsilon[\subseteq G$. Choose $k_1 > k_0$ such that $\epsilon/2n^2 > \ell'_{k_1}$.

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Assume $k \in \{k_0, \dots, k_1 - 1\}$ is given and h_k is already defined. Using $y = x_n$ and $t = h_k$ in lemma 2.6, we choose $t' = h_{k+1}$. Repeating this procedure we define h_k for $k = k_0 + 1, \dots, k_1$. Then we have

$$h_{k_0} \leqslant h_{k_1} \leqslant h_{k_0} + n\biggl(\frac{\ell'_{k_0}}{k_0} + \frac{\ell'_{k_0+1}}{k_0+1} + \cdots + \frac{\ell'_{k_1-1}}{k_1-1}\biggr) < h_{k_0} + 2n\frac{\ell'_{k_0}}{k_0}, \tag{2.6}$$

the points $x_n + ih_{k_1}$ belong to different components of E_{k_1}, and

$$[x_n + ih_{k_1}, x_n + ih_{k_1} + \rho_{n,k_1}] \subseteq E_{k_1} \quad \text{for } i = 1, \dots, n,$$

where ρ_{n,k_1} denotes the length of the interval $I_{j,k_1} \subseteq E_{k_1}$ that contains $x_n + nh_{k_1}$. Using the fact that $\rho_{n,k_1} \geqslant 2\ell'_{k_1}$, we also have

$$[x_n + ih_{k_1}, x_n + ih_{k_1} + 2\ell'_{k_1}] \subseteq E_{k_1} \quad \text{for } i = 1, \dots, n.$$

Choose j_1 such that $x_n + h_{k_1} \in I_{j_1,k_1}$. Then there exists $\lambda < \ell'_{k_1}/k_1^{k_1} \leqslant \frac{1}{4}\ell'_{k_1}$ such that any subinterval of length λ in I_{j_1,k_1} contains points of E_{k_1+1} and points of K as well. Since we have a dense colouring of G, we can choose an $x_{n-1} \in G$ such that $\phi_1(x_{n-1}) = \phi_2(n-1)$ and $x_{n-1} \in [x_n + h_{k_1}, x_n + h_{k_1} + \lambda]$. Observe that, using (2.6), we have

$$|x_{n-1} - (x_n + h_{k_0})| \leqslant |x_{n-1} - (x_n + h_{k_1})| + |x_n + h_{k_1} - (x_n + h_{k_0})| \leqslant \lambda + 2n\frac{\ell'_{k_0}}{k_0}. \tag{2.7}$$

An easy computation shows that

$$[x_{n-1} + ih_{k_1}, x_{n-1} + ih_{k_1} + \ell'_{k_1}] \subseteq [x_n + (i+1)h_{k_1}, x_n + (i+1)h_{k_1} + 2\ell'_{k_1}] \subseteq E_{k_1}$$

holds for $i = 1, \dots, n-1$ and, by lemma 2.8 applied for $n-1$, k_1, x_{n-1} and h_{k_1}, there exists $x \in K$ and $h \in \mathbb{R}$ such that

$$\begin{aligned} |h - h_{k_1}| &< (n-1)\ell'_{k_1}, \\ |x - (x_{n-1} + (n-1)h_{k_1})| &< (n-1)\ell'_{k_1} \end{aligned}$$

and

$$\phi_1(x - ih) = \phi_2(i) \quad \text{for } i = 1, \dots, n-1.$$

Then, using (2.6), we infer

$$|h - h_{k_0}| \leqslant |h - h_{k_1}| + |h_{k_1} - h_{k_0}| < (n-1)\ell'_{k_1} + 2n\frac{\ell'_{k_0}}{k_0} < n\ell'_{k_0}.$$

Furthermore,

$$\begin{aligned} |x - (x_n + nh_{k_0})| &\leqslant |x - (x_{n-1} + (n-1)h_{k_1})| + |x_{n-1} + (n-1)h_{k_1} - (x_n + nh_{k_0})| \\ &\leqslant (n-1)\ell'_{k_1} + |x_{n-1} - (x_n + h_{k_0})| + (n-1)|h_{k_1} - h_{k_0}| \\ &< (n-1)\ell'_{k_1} + 2n\frac{\ell'_{k_0}}{k_0} + \lambda + (n-1)2n\frac{\ell'_{k_0}}{k_0} < n\ell'_{k_0}, \end{aligned}$$

where, in the last estimations, we used (2.7), $k_0 > 2n^2$, $\ell'_{k_1} < \ell'_{k_0}/k_0^{k_0}$ and $\lambda < \frac{1}{4}\ell'_{k_1}$.

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Finally, observe that

$$|x-(x_n+nh)| \leqslant |x-(x_{n-1}+(n-1)h_{k_1})| + |x_{n-1}-x_n-h_{k_1}| + n|h_{k_1}-h| < (n-1)\ell'_{k_1} + \lambda + n(n-1)\ell'_{k_1} < \epsilon.$$

Thus $x-nh \in]x_n-\epsilon, x_n+\epsilon[$. By the choice of ϵ, the entire interval $]x_n-\epsilon, x_n+\epsilon[$ belongs to G and hence either $]x_n-\epsilon, x_n+\epsilon[\subseteq \Lambda$ or $]x_n-\epsilon, x_n+\epsilon[\subseteq \Lambda'$. Hence

$$\phi_1(x-nh) = \phi_1(x_n) = \phi_2(n).$$

This, together with the induction hypothesis, implies

$$\phi_1(x-ih) = \phi_2(i) \quad \text{for } i = 1, \dots, n.$$

This completes the proof of lemma 2.8. □

Proof of theorem 2.1. By lemma 2.5, K is of zero Hausdorff dimension. It remains to show that K is in $\mathcal{A}$.

Choose $k_0 > 2n^2$ and a $j_0 \leqslant n_{k_0-1}$. Then it is easy to see that $\nu_{j_0,k_0-1} > k_0^{k_0} > n+1$ and hence $I_{j_0,k_0-1} \cap E_{k_0}$ consists of more than $n+1$ intervals of the form I_{j,k_0}. Choose a j_1 such that

$$I_{j_1,k_0}, I_{j_1+1,k_0}, \dots, I_{j_1+n,k_0} \subseteq I_{j_0,k_0-1}.$$

Let $h_{k_0} = \lambda_{j_0,k_0-1}$. Recall that the intervals I_{j_1+i,k_0} are equally spaced in I_{j_0,k_0-1} and I_{j_1+i,k_0} can be obtained from I_{j_1,k_0} by a translation with ih_{k_0}. It is also clear that $\ell_{j_1,k_0} = \ell_{j_1+1,k_0} = \cdots = \ell_{j_1+n,k_0} = \eta_{j_0,k_0}$.

Since the colouring of G is dense and the first quarter of I_{j_1,k_0} contains points of K, we can choose an $x_n \in G$ such that $\phi_1(x_n) = \phi_2(n)$ and $[x_n, x_n+\frac{1}{2}\ell_{j_1,k_0}] \subseteq I_{j_1,k_0}$. Using $\ell_{k_0} \leqslant \ell_{j_1,k_0}$ and the translation property of the intervals I_{j_1+i,k_0}, we have

$$[x_n+ih_{k_0}, x_n+ih_{k_0}+\tfrac{1}{2}\ell_{k_0}] \subseteq I_{j_1+i,k_0} \subseteq E_{k_0} \quad \text{for } i = 0, \dots, n.$$

The assumptions of lemma 2.8 are satisfied. Hence, using $x \in K$ and h from lemma 2.8, let $h^* = -h$ and observe that $\phi_1(x+ih^*) = \phi_2(i)$, $i = 1, \dots, n$, shows that the colouring ϕ_1 of G and ϕ_2 are compatible. □

Proof of theorem 2.3. Given $\ell \in \mathbb{N}$, we denote by $\phi_{2,\ell}$ the colouring of $s_{4\ell+2} = \{1, \dots, 4\ell+2\}$ for which $\phi_{2,\ell}(2i-1) = \text{red}$, $\phi_{2,\ell}(2i) = \text{blue}$ for $i \in \{1, \dots, 2\ell+1\}$.

We can assume that K is non-empty, compact and nowhere dense.

We also assume that K is perfect and does not contain $\ell \geqslant 3$-long arithmetic progressions. We need to show that $K \notin \mathcal{A}$.

We denote by $\mathcal{CK}$ the set consisting of the intervals contiguous to K. If $x \in \Psi = \mathbb{R} \setminus K$, then $I_K(x)$ denotes the $\mathcal{CK}$ interval containing x.

Since the number ℓ is fixed, for ease of notation, we will just write ϕ_2 instead of $\phi_{2,\ell}$. Our goal is to find open sets Λ and Λ' such that $K \subseteq \partial(\Lambda, \Lambda')$, but $(\Lambda, \Lambda') \notin UC$. We will choose Λ and Λ' by defining a colouring $\phi_1 : \Psi \to \{\text{blue}, \text{red}\}$. This colouring will be constant on each $\mathcal{CK}$ interval. We will let

$$\Lambda = \{x \in \Psi : \phi_1(x) = \text{red}\} \quad \text{and} \quad \Lambda' = \{x \in \Psi : \phi_1(x) = \text{blue}\}.$$

It is clear that if we assume that ϕ_1 is a dense colouring, that is, $K \subseteq \bar{\Lambda} \cap \bar{\Lambda}'$, then $K \subseteq \partial(\Lambda, \Lambda')$. Choose an onto mapping $\psi_K : \Psi \to \mathbb{N}$ that is constant on the intervals contiguous to K and takes different values on different intervals.

We say that a pair $(x; h) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$ is admissible if the terms of the sequence $\{x + ih : i \in s_{4\ell+2}\}$ belong to Ψ and $I_K(x + 2ih) \neq I_K(x + 2jh)$ for $i \neq j$, $i, j \in \{1, \dots, 2\ell + 1\}$. The set of admissible pairs is denoted by $\mathcal{P}$.

REMARK 2.9. Observe that if ℓ and ϕ_2 are fixed as above, ϕ_1 is a colouring of Ψ and $\phi_1(x + ih) = \phi_2(i)$ for all $i \in s_{4\ell+2}$, then $(x; h) \in \mathcal{P}$.

Given $\eta > 0$, we denote by $\mathcal{P}_\eta$ the set of those $(x; h) \in \mathcal{P}$ for which $|h| \geqslant \eta$. Clearly, $\mathcal{P}_{\eta_0} = \emptyset$ for a sufficiently large η_0.

Assume that $\eta' < \eta$, the (possibly empty) open set V is the union of finitely many $\mathcal{CK}$ intervals and, for each $(x; h) \in \mathcal{P}_{\eta'} \setminus \mathcal{P}_\eta$, there is at most one $i \in \{1, \dots, 2\ell + 1\}$ such that $x + 2ih \in V$. Set

$$R_V(x; h) = \min\{\psi_K(x + 2ih) : i = 1, \dots, 2\ell + 1 \text{ and } x + 2ih \notin V\}$$

and $\mathcal{R}_{\eta',\eta,V} = \{R_V(x; h) : (x; h) \in \mathcal{P}_{\eta'} \setminus \mathcal{P}_\eta\}$.

LEMMA 2.10. *With our assumptions about K, for any $\eta' < \eta$, the set $\mathcal{R}_{\eta',\eta,V}$ is bounded from above.*

Proof of lemma 2.10. Assume that $\mathcal{R}_{\eta',\eta,V}$ is not bounded from above. Using the compactness of K and the finiteness of $\{1, \dots, 2\ell + 1\}$, we can choose a sequence $(x_n; h_n) \in \mathcal{P}_{\eta'} \setminus \mathcal{P}_\eta$ and an $i^* \in \{1, \dots, 2\ell + 1\}$ such that x_n converges to an $x \in \mathbb{R}$, $h_n \to h$, with $\eta' \leqslant |h| \leqslant \eta$, $\psi_K(x_n + 2ih_n) \to \infty$ and $x_n + 2ih_n \notin V$ for any $i \in \{1, \dots, 2\ell + 1\} \setminus \{i^*\}$. Hence, if $i \neq i^*$ is fixed and $]a_n, b_n[\, = I_K(x_n + 2ih_n)$, then

$$\lim_{n\to\infty} b_n - a_n = 0.$$

Since $a_n \in K$, we obtain that $x + 2ih \in K$.

Therefore, $\{x + 2ih : i = 1, \dots, 2\ell + 1, \ i \neq i^*\} \subseteq K$, but then K would contain an arithmetic progression of length at least ℓ, contradicting our assumption about K. ☐

Now we continue the proof of theorem 2.3 and define by induction the suitable colouring ϕ_1 on Ψ. In each step, first we colour by red finitely many $\mathcal{CK}$ intervals in order to 'take care' of 'blocking' sequences $\{x + ih\}$ with $(x; h) \in \mathcal{P}_{\eta_k}$ with a suitable $\eta_k > 0$. Then, to obtain a dense colouring, we colour some $\mathcal{CK}$ intervals by blue.

Step 1 of the colouring

Set $V_0 = \emptyset$, $\eta = \eta_0$ and choose $\eta_1 > 0$ such that

$$\eta_1 < \frac{\text{diameter}(K)}{4(4\ell + 2)}.$$

Set $R_0(x; h) = R_{V_0}(x; h)$ and $\mathcal{R}_{\eta_1} = \mathcal{R}_{\eta_1,\eta_0,V_0}$. Using lemma 2.10, we can choose $M_1 \in \mathbb{R}$ such that each element of $\mathcal{R}_{\eta_1}$ is bounded by M_1. There are only finitely

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many intervals on which $\psi_K(x) \leqslant M_1$. Denote the union of these intervals by U_1. Without loss of generality, we can assume that U_1 contains the two unbounded components of Ψ. Then $\psi_K(x) > M_1$ for $x \in \mathbb{R} \setminus (K \cup U_1)$. For $x \in U_1$, let $\phi_1(x) = \text{red}$.

Whenever $(x; h) \in \mathcal{P}_{\eta_1} \setminus \mathcal{P}_{\eta_0} = \mathcal{P}_{\eta_1}$, we have $R(x; h) \leqslant M_1$, and hence there exists an $i \in \{1, \dots, 2\ell + 1\}$ such that $x + 2ih \in U_1$. Thus $\phi_1(x + 2ih) \neq \phi_2(2i)$ and hence the colouring ϕ_2 of $s_{4\ell+2}$ is not compatible with the colouring of $\phi_1(x + 2ih)$.

Next, choose finitely many $\mathcal{CK}$ intervals that are disjoint from U_1 such that, denoting their union by V_1, the following two properties are satisfied.

(α_1) If $I_1, I_2 \subseteq V_1$ are disjoint $\mathcal{CK}$ intervals, then $\operatorname{dist}(I_1, I_2) > 2(4\ell + 2)\eta_1$.

(β_1) If $x \in K$, then $(x - 3(4\ell + 2)\eta_1, x + 3(4\ell + 2)\eta_1) \cap V_1 \neq \emptyset$.

It is easy to see that the choice of V_1 can be made satisfying (α_1) and (β_1). For $x \in V_1$, let $\phi_1(x) = \text{blue}$.

Clearly, property (α_1) says that blue intervals are not too close and (β_1) says that they are not too far either.

The general steps of the colouring

Assume now that we have already accomplished step k of our definition. Thus we have the number $\eta_k > 0$ and the sets U_k, V_k such that $U_k \cup V_k \subseteq \Psi$, $\phi_1(U_k) = \{\text{red}\}$, $\phi_1(V_k) = \{\text{blue}\}$, and the sets U_k, V_k consist of the union of finitely many $\mathcal{CK}$ intervals. If $(x; h) \in \mathcal{P}_{\eta_k}$, then there exists $i \in \{1, \dots, 2\ell+1\}$ such that $x + 2ih \in U_k$. Finally, the following two properties are satisfied.

(α_k) If $I_1, I_2 \subseteq V_k$ are disjoint $\mathcal{CK}$ intervals, then $\operatorname{dist}(I_1, I_2) > 2(4\ell + 2)\eta_k$.

(β_k) If $x \in K$, then $(x - 3(4\ell + 2)\eta_k, x + 3(4\ell + 2)\eta_k) \cap V_k \neq \emptyset$.

Step $(k + 1)$ of the colouring

Let $\eta_{k+1} = \frac{1}{2}\eta_k$.

Set $R_k(x; h) = R_{V_k}(x; h)$ and $\mathcal{R}_{\eta_{k+1}} = \mathcal{R}_{\eta_{k+1}, \eta_k, V_k}$.

Observe that assumption (α_k) implies that, for each $(x; h) \in \mathcal{P}_{\eta_{k+1}} \setminus \mathcal{P}_{\eta_k}$, there is at most one $i \in \{1, \dots, 2\ell + 1\}$ such that $x + 2ih \in V_k$.

By lemma 2.10, for a suitable constant M_{k+1}, each element of $\mathcal{R}_{\eta_{k+1}}$ is bounded by M_{k+1}. We can also assume that M_{k+1} is so large that $M_{k+1} > k + 1$ and, whenever $x \in K$, then there exists an interval $]a, b[$ such that $]a, b[\cap]x - \eta_k, x + \eta_k[\neq \emptyset$, $]a, b[\cap V_k = \emptyset$ and $\psi_K(]a, b[)$ is bounded by M_{k+1}.

Denote by U^0_{k+1} the set of points x for which $x \in \Psi$, $\psi_K(x) \leqslant M_{k+1}$, $x \notin U_k \cup V_k$ (that is, we have not defined ϕ_1 at x). For $x \in U^0_{k+1}$, put $\phi_1(x) = \text{red}$, and let $U_{k+1} = U_k \cup U^0_{k+1}$. Observe that $U_{k+1} \cup V_k$ contains all points x in Ψ for which $\psi_K(x) \leqslant M_{k+1}$.

Now take $(x; h) \in \mathcal{P}_{\eta_{k+1}} \setminus \mathcal{P}_{\eta_k}$. Then $R_k(x; h) \leqslant M_{k+1}$, and hence there exists $i \in \{1, \dots, 2\ell + 1\}$ such that $x + 2ih \notin V_k$ and $\psi_K(x + 2ih) \leqslant M_{k+1}$. This implies $x + 2ih \in U_{k+1}$, and hence $\phi_1(x + 2ih) = \text{red} \neq \phi_2(2i)$, that is, the colouring ϕ_1 of $x + 2ih$ is incompatible with the colouring ϕ_2 of $s_{4\ell+2}$.

Next, choose finitely many $\mathcal{CK}$ intervals that are disjoint from U_{k+1} such that, denoting their union by V_{k+1}, we have $V_k \subseteq V_{k+1}$ and propositions (α_{k+1})

and (β_{k+1}) are satisfied (that is, we have propositions (α_k) and (β_k) satisfied with k replaced by $k+1$). It is an easy exercise to show that we can make a suitable choice of V_{k+1}. For $x \in V_{k+1}$, put $\phi_1(x) = \text{blue}$.

By letting $k \to \infty$, we can define a colouring ϕ_1 of Ψ. If one takes $x \in K$, $\delta > 0$, then it is easy to find k such that $U_k \cap]x-\delta, x+\delta[$ and $V_k \cap]x-\delta, x+\delta[$, and hence ϕ_1 is dense.

Since $\mathcal{P} = \mathcal{P}_{\eta_0} \cup \bigcup_{k=1}^{\infty} (\mathcal{P}_{\eta_{k+1}} \setminus \mathcal{P}_{\eta_k})$, our construction implies that, for any $(x; h) \in \mathcal{P}$, there exists $i \in \{1, \dots, 2\ell+1\}$ such that

$$\phi_1(x + 2ih) = \text{red} \neq \phi_2(2i). \tag{2.8}$$

However, remark 2.9 implies that, for $(x; h) \notin \mathcal{P}$, there always exists $i \in \{1, \dots, 2\ell + 1\}$ such that (2.8) holds. Therefore, the open sets Λ and Λ' defined by ϕ_1 do not have the universal colouring property. Hence $K \notin \mathcal{A}$. This concludes our proof for perfect K.

Denote by $\mathcal{N}_\ell$ the set of those non-empty compact sets that does not contain ℓ-long arithmetic progressions. If K is not perfect, then we can use the following result.

Lemma 2.11. *If the compact set $K \in \mathcal{N}_\ell$, then there exists a perfect compact set $K' \in \mathcal{N}_\ell$ such that $K \subseteq K'$.*

First we show that, using this lemma, we can complete the proof of theorem 2.3. Indeed, if $K \in \mathcal{N}_\ell$, then choose K' according to lemma 2.11. Using our previous argument, we can find open sets Λ and Λ' such that $K \subseteq K' \subseteq \partial(\Lambda, \Lambda')$ and $(\Lambda, \Lambda') \notin UC$. This implies that $K \notin \mathcal{A}$. □

Proof of lemma 2.11. For each isolated point y_n of K, we will choose a suitable neighbourhood $I_n =]y_n - r_n, y_n + r_n[$ and denote by G the union of these neighbourhoods. By choosing the r_n sufficiently small, we can assume that $I_n \cap K = \{y_n\}$. Since $K \in \mathcal{N}_\ell$, using induction on n, and at each step a compactness argument, we can assume that the radii, r_n, are so small that $K \cup G$ does not contain an ℓ-long arithmetic progression such that each component of G contains at most one of its terms.

Choose a compact perfect set K_0 for which $0 \in K_0 \subseteq [0, 1]$ and K_0 does not contain a 3-long arithmetic progression. Set

$$K' = K \cup \bigcup_n (y_n + \tfrac{1}{3} r_n K_0).$$

It is not difficult to see that K' satisfies the conditions of the lemma; we leave the details to the reader. □

3. Open problems

(3.1) Suppose that K is an independent compact set (see [1]). Is it true that $K \in SB$? Is it true that there exists a spectral set Λ such that $\partial\Lambda = K$?

(3.2) Does the classical Cantor ternary set belong to SB?

(3.3) Are the classes SB different for different p? (This question is motivated by § 6.2.5 of [5].)

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