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SHORT MATURITY ASIAN OPTIONS FOR THE CEV MODEL

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We present a rigorous study of the short maturity asymptotics for Asian options with continuous-time averaging, under the assumption that the underlying asset follows the constant elasticity of variance (CEV) model. The leading order short maturity limit of the Asian option prices under the CEV model is obtained in closed form. We propose an analytical approximation for the Asian options prices which reproduces the exact short maturity asymptotics, and demonstrate good numerical agreement of the asymptotic results with Monte Carlo simulations and benchmark test cases for option parameters relevant for practical applications.

Keywords: Asian options, CEV model, large deviations, short maturity, variational problem

1. INTRODUCTION

Asymptotics for option prices and implied volatility of European options for the short maturity regime have been extensively studied in the literature, see e.g. [8,16,39,40,43] for local volatility models, [4,6,30,53] for the exponential Lévy models [2,9,28,29,31–33,45] for stochastic volatility models, and [38,49] for model-free approaches.

Recently, this asymptotic regime was also investigated for Asian options in [50] under the assumption that the asset price follows a local volatility model. More precisely, the paper [50] considered arithmetic averaging Asian options in continuous time under the assumption that the asset price follows a local volatility model

$$dS_t = (r-q)S_t dt + \sigma(S_t)S_t dW_t, \quad S_0 > 0, \tag{1}$$

where W_t is a standard Brownian motion, $r \geq 0$ is the risk-free rate, $q \geq 0$ is the continuous dividend yield, and $\sigma(\cdot)$ is the local volatility function.

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It was shown by Varadhan [54] that, under certain boundedness and Lipschitz conditions on the local volatility function $\sigma(\cdot)$, the log-stock price $X_t := \log S_t$ satisfies a sample path large deviation principle on an appropriate functional space.

This result was used in [50] together with the contraction principle, to derive large deviations for the time average of the diffusion $(1/T)\int_0^T S_t dt$, and short maturity asymptotics for out-of-the-money (OTM) Asian options. On the other hand, the short maturity asymptotics for at-the-money (ATM) Asian options are dominated by the fluctuations of the time average around the mean, and have power-law form $C(T) \sim T^{1/2}$. The results of [50] cover in particular the Black–Scholes model, and more explicit formulas are derived for this case.

The boundedness and Lipschitz conditions assumed in [50] are not satisfied by some of the models that are popular in financial practice. One important model of this type is the constant elasticity of variance (CEV) model [17], which is defined by the diffusion

$$dS_t = (r-q)S_t dt + \sigma S_t^\beta dW_t, \quad S_0 > 0. \tag{2}$$

This model is used for modeling the skew in equities and FX markets, and allows the flexibility of calibrating to the ATM slope of the implied volatility by choosing appropriately the exponent β. For $\beta < 1$, the model reproduces the leverage effect observed in many financial markets, which is manifested as a decreasing volatility as the asset price increases. The result of this inverse relationship between the price and volatility is the implied volatility skew. See [47] for a survey of the mathematical properties of the CEV model and also the pricing of vanilla options under the CEV model.

In practical applications, the exponent β is usually chosen in the range $0 < \beta \leq 1$. The case $\beta = 1/2$ corresponds to the square-root model of Cox and Ross [19], and is obtained as a particular case of the Feller process [18,27]

$$dx_t = (bx_t + c)dt + \sqrt{2ax_t} dW_t, \tag{3}$$

with $a = (1/2)\sigma^2$, $b = r - q$, $c = 0$. The case of general β can also be mapped to the diffusion process (3) by a change of variable. See [48] for a detailed study of the properties of this process.

The model (2) is a local volatility model of type (1) with a volatility function $\sigma(S_t) := \sigma S_t^{\beta-1}$. For $0 < \beta < 1$ this is not a bounded function. This implies that the results of [54] cannot be directly applied to this case.

The pricing of Asian options has been widely studied in the mathematical finance literature. The pricing under the Black–Scholes model has been studied in [13,24,41,46], using a relation between the distributional property of the time-integral of the geometric Brownian motion and Bessel processes. See [26] for an overview, and [35] for a comparison with alternative simulation methods, such as the Monte Carlo (MC) approach.

An approach based on partial differential equations (PDE) has been proposed in [51, 56,57] for pricing Asian options. This can be used either as a numerical approach [3,56,57], or can be combined with asymptotic expansion methods to derive analytical approximation formulae. Foschi, Pagliarani, and Pascucci [34] used heat kernel expansion methods and developed approximate formulae expressed in terms of elementary functions for the density, the price and the Greeks of path-dependent options of Asian style. Asymptotic expansion leading to analytical approximation with error bounds for Asian options have been obtained also using Malliavin calculus in [42,52]. A small-time expansion for Asian options has been proposed in [11], and a general framework for pricing Asian options under Markov processes was given in [12]. We also note the optimized upper and lower bounds on Asian options given in [37].

Asian options pricing under the CEV model with $\beta = 1/2$ has been studied in [20,25]. A detailed study under the $\beta = 1/2$ model both with discrete and continuous time averaging was presented in [36]. The general case of the CEV model was studied in [34] using heat kernel expansion methods in the PDE approach [51,56,57]. The paper [34] presented detailed numerical tests of their method under the CEV model, which show good convergence and stability of the expansion.

The short maturity asymptotics for vanilla options under the CEV model has been studied in the literature using several approaches; see [16,43,45]. In this paper, we study the short maturity asymptotics for the Asian options in this model. We consider both the fixed strike and floating strike Asian options. Our main tool is the large deviations theory from probability theory. For the theory and applications of Large Deviations theory, we refer to the books [21] and [55]. Some basic definitions and results needed in this paper will be provided in Appendix A.

The case of the square-root model $\beta = 1/2$ is special as the model is affine, and the moment generating function of the time integral $\int_0^T S_t dt$ can be found in closed form. Then the application of the Gärtner–Ellis theorem gives the large deviations for the time average of the asset price. Large deviations for the square-root process $\beta = 1/2$ were studied in [22].

For $1/2 < \beta < 1$ we use a recent large deviations result due to Baldi and Caramelino [7] for the CEV model to derive a variational problem for the rate function determining the short maturity asymptotics of the Asian options. The variational problem is solved completely. We derive large and small-strike asymptotics for the rate function.

Some of the methods proposed in the literature for pricing Asian options are less efficient in the small maturity/volatility limit. This is a well-known problem in several of the methods proposed for the Black–Scholes model [26], but a similar phenomenon appears also for the method of [20] in the square-root model, where the convergence of the expansion is slower for small maturity/volatility. The short maturity asymptotic expansion proposed in this paper complements the use of these methods in a regime where their numerical efficiency is less than optimal. A recent paper [5] obtains short-maturity asymptotics for Asian options in local volatility models, using similar large deviations methods.

The paper is organized as follows. In Section 2, we present asymptotics for OTM Asian options in the square-root model $\beta = 1/2$. Section 3 considers the case of the general CEV model with $1/2 \leq \beta < 1$. The asymptotics for OTM Asian options is given by the solution of a variational problem, which is solved in closed form. We also obtain the asymptotics for ATM Asian options. Section 4 considers the asymptotics of Asian options with floating strike. In Section 5, we present an analytical approximation for the Asian options prices which has the same short maturity asymptotics as that obtained in Sections 2 and 3. This approximation is compared against benchmark results in the literature, and good agreement is demonstrated for model and option parameters relevant for practical applications. Finally, the background of large deviations theory and the proofs of the main results are given in Appendix A.

Notations and preliminaries

The price of the Asian call and put options with maturity T and strike K with continuous time averaging are given by expectations in the risk-neutral measure

$$C(T) := e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right], \tag{4}$$

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$$P(T) := e^{-rT}\mathbb{E}\left[\left(K - \frac{1}{T}\int_0^T S_t dt\right)^+\right], \tag{5}$$

where $C(T)$ and $P(T)$ emphasize the dependence on the maturity T.

We denote the expectation of the averaged asset price in the risk-neutral measure as

$$A(T) := \frac{1}{T}\int_0^T \mathbb{E}[S_t]dt = S_0 \frac{1}{(r-q)T}\left(e^{(r-q)T} - 1\right), \tag{6}$$

for $r - q \neq 0$ and $A(T) := S_0$ for $r - q = 0$. When $K > A(T)$, the call Asian option is OTM and $C(T) \to 0$ as $T \to 0$. When $A(T) > K$, the put Asian option is OTM and $P(T) \to 0$ as $T \to 0$.

The prices of call and put Asian options are related by put–call parity as

$$C(K,T) - P(K,T) = e^{-rT}(A(T) - K). \tag{7}$$

As $T \to 0$, we have $A(T) = S_0 + O(T)$. Therefore, for the small maturity regime, the call Asian option is OTM if and only if $K > S_0$, etc. For the purposes of the short maturity limit, the call Asian option is said to be OTM (resp. in-the-money) if $K > S_0$ (resp. $K < S_0$), and the put Asian option is said to be OTM (resp. in-the-money) if $K < S_0$ (resp. $K > S_0$), and finally they are said to be ATM if $K = S_0$.

2. SHORT MATURITY ASIAN OPTIONS IN THE SQUARE-ROOT MODEL

We assume in this section that the asset value S_t follows a square-root process:

$$dS_t = (r-q)S_t dt + \sigma\sqrt{S_t}dW_t, \tag{8}$$

with $S_0 > 0$ and W_t is a standard Brownian motion starting at zero at time zero $W_0 = 0$. r, q, σ are positive real parameters.

We have the following result.

THEOREM 2.1: *$\mathbb{P}((1/T)\int_0^T S_t dt \in \cdot)$ satisfies a large deviation principle with rate function*

$$\mathcal{I}(x, S_0) = \sup_{\theta\in\mathbb{R}}\{\theta x - \Lambda(\theta)\}, \tag{9}$$

where

$$\Lambda(\theta) := \lim_{T\to 0} T\log\mathbb{E}\left[e^{(\theta/T^2)\int_0^T S_t dt}\right] = \begin{cases} \frac{\sqrt{2\theta}}{\sigma}\tan\left(\frac{\sigma}{2}\sqrt{2\theta}\right)S_0, & \text{if } 0 \le \theta < \frac{\pi^2}{2\sigma^2} \\ \frac{-\sqrt{-2\theta}}{\sigma}\tanh\left(\frac{\sigma}{2}\sqrt{-2\theta}\right)S_0, & \text{if } \theta \le 0 \\ +\infty & \text{otherwise} \end{cases}. \tag{10}$$

Indeed, the rate function in Theorem 2.1 has a more explicit expression. Together with Theorem 2.1 and Lemma 3.1 that we prove in Section 3, we have the following result.

PROPOSITION 2.2: *Assume the square-root model: $\beta = 1/2$.*

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(i) *For* $K \le S_0$*, the put option is OTM, and* $P(T) = e^{-(1/T)\mathcal{I}(K,S_0)+o(1/T)}$*, as* $T \to 0$*, where*

$$\mathcal{I}(K,S_0) = \frac{S_0}{\sigma^2}\frac{x^2}{\cosh^2(x)}\left(\frac{\sinh(2x)}{2x} - 1\right), \tag{11}$$

where x *is the solution of the equation*

$$\frac{1}{2\cosh^2(x)}\left(1 + \frac{\sinh(2x)}{2x}\right) = \frac{K}{S_0}. \tag{12}$$

(ii) *For* $K \ge S_0$*, the call option is OTM, and* $C(T) = e^{-(1/T)\mathcal{I}(K,S_0)+o(1/T)}$*, as* $T \to 0$*, where*

$$\mathcal{I}(K,S_0) = \frac{S_0}{\sigma^2}\frac{x^2}{\cos^2(x)}\left(1 - \frac{\sin(2x)}{2x}\right), \tag{13}$$

where $0 \le x \le \pi/2$ *is given by the solution of the equation*

$$\frac{1}{2\cos^2(x)}\left(1 + \frac{\sin(2x)}{2x}\right) = \frac{K}{S_0}. \tag{14}$$

We can study also the small/large strike asymptotics of the rate function.

Proposition 2.3:

(i) *The large strike asymptotics for the rate function of OTM Asian call options* $K > S_0$ *in the square-root model* $\beta = 1/2$ *is*

$$\lim_{K\to\infty}\frac{\mathcal{I}(K,S_0)}{K} = \frac{\pi^2}{2\sigma^2}. \tag{15}$$

(ii) *The small strike* $K \to 0$ *asymptotics of the rate function for OTM Asian put options* $K < S_0$ *in the square-root model* $\beta = 1/2$ *is*

$$\mathcal{I}(K,S_0) \sim \frac{S_0^2}{2\sigma^2 K}, \quad \text{as } K \to 0. \tag{16}$$

2.1. Expansion of the Rate Function around the ATM Point

We give also the expansion of the rate function for Asian options in the square-root model ($\beta = 1/2$) in power series of $x = \log(K/S_0)$. The first few terms are

$$\mathcal{I}(K,S_0) = \frac{S_0}{\sigma^2}\left\{\frac{3}{2}x^2 + \frac{3}{5}x^3 + \frac{271}{1400}x^4 + O(x^5)\right\}. \tag{17}$$

The rate function $\mathcal{I}(K,S_0)$ in the square-root model was evaluated numerically using Proposition 2.2. Figure 1 shows the plot of this function vs. K/S_0 (left) and vs. $x = \log(K/S_0)$ (right). The right plot shows also the approximation of the rate function obtained by keeping the first three terms in the series expansion (17).

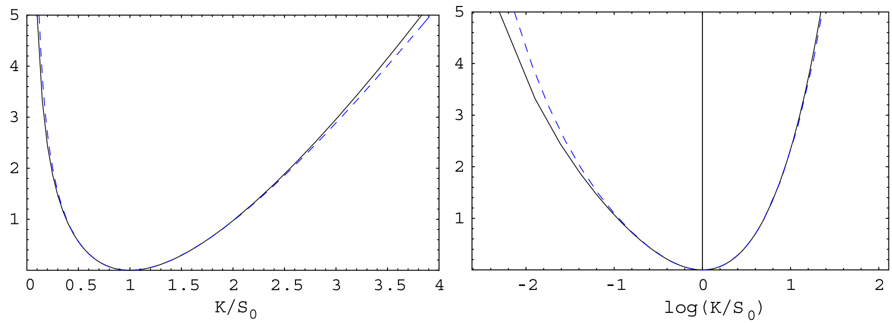

FIGURE 1. The rate function $\mathcal{I}(K, S_0)$ for $\beta = 1/2$ in units of S_0/σ^2 vs. K (left) and vs. $\log(K/S_0)$ (right) (solid black curve) and the Taylor expansion (39) keeping the first three terms (dashed blue).

3. ASIAN OPTIONS IN THE CEV MODEL

The CEV model is defined by the one-dimensional diffusion under the risk-neutral measure

$$dS_t = (r - q)S_t dt + \sigma S_t^\beta dW_t, \tag{18}$$

with $S_0 > 0$. r, q, σ are real positive parameters.

It is easy to check that the following Lemma holds.

LEMMA 3.1: *For an Asian OTM call option, that is, $K > S_0$, we have for $1/2 \leq \beta < 1$*

$$\lim_{T\to 0} T \log C(T) = \lim_{T\to 0} T \log \mathbb{P}\left(\frac{1}{T}\int_0^T S_t dt \geq K\right). \tag{19}$$

For an Asian OTM put option, that is, $K \leq S_0$, we have for $1/2 \leq \beta < 1$

$$\lim_{T\to 0} T \log P(T) = \lim_{T\to 0} T \log \mathbb{P}\left(\frac{1}{T}\int_0^T S_t dt \leq K\right). \tag{20}$$

Using this result we can prove the short maturity asymptotics for OTM Asian options in the CEV model (2).

THEOREM 3.2: *The short maturity asymptotics for OTM Asian options in the CEV model (2) with $1/2 \leq \beta < 1$ is given by*

$$\lim_{T\to 0} T \log C(T) = -\mathcal{I}(K, S_0), \tag{21}$$

where the rate function is given by the solution of a variational problem specified as follows.

(i) *For OTM Asian call options $K > S_0$ we have*

$$\mathcal{I}(K, S_0) = \inf_{\int_0^1 g(t)dt \geq K, g(0)=S_0, g(t)\geq 0, 0\leq t\leq 1} \frac{1}{2}\int_0^1 \frac{(g'(t))^2}{\sigma^2 g(t)^{2\beta}} dt, \quad K > S_0. \tag{22}$$

(ii) *For OTM Asian put options $K < S_0$ we have*

$$\mathcal{I}(K, S_0) = \inf_{\int_0^1 g(t)dt \leq K, g(0)=S_0, g(t)\geq 0, 0\leq t\leq 1} \frac{1}{2}\int_0^1 \frac{(g'(t))^2}{\sigma^2 g(t)^{2\beta}} dt, \quad K < S_0. \tag{23}$$

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3.1. ATM Asian Options

For the ATM case, that is, $K = S_0 > 0$, we have the following short maturity asymptotics.

THEOREM 3.3: *As $T \to 0$, we have in the CEV model with $1/2 \leq \beta < 1$*

$$C(T) = \sigma S_0^\beta \frac{\sqrt{T}}{\sqrt{6\pi}} + O(T), \quad P(T) = \sigma S_0^\beta \frac{\sqrt{T}}{\sqrt{6\pi}} + O(T). \tag{24}$$

3.2. Variational Problem for Short-Maturity Asymptotics for Asian Options in the CEV Model

Theorem 3.2 gives the rate function $\mathcal{I}(K, S_0)$ of an Asian option in the CEV model as a variational problem. For OTM Asian call option $K > S_0$ this variational problem reads

$$\mathcal{I}(K, S_0) = \inf_g \frac{1}{2\sigma^2} \int_0^1 \frac{(g'(t))^2}{g(t)^{2\beta}} dt, \tag{25}$$

where the function $g(t)$ is differentiable and satisfies $g(0) = S_0$, $g(t) > 0, 0 \leq t \leq 1$ and the infimum is taken under the constraint

$$\int_0^1 g(t) dt \geq K. \tag{26}$$

Similarly, for OTM Asian put option with $K < S_0$, the rate function $\mathcal{I}(K, S_0)$ is given by the variational problem (25) with inequality constraint $\int_0^1 g(t)dt \leq K$.

Define $\mathcal{I}_K(K, S_0)$ as the solution of the variational problem (25), obtained by replacing the inequality (26) with equality. The strategy of the proof will be to show that $\mathcal{I}_K(K, S_0)$ is an increasing function for $K > S_0$ and thus the solution of the variational inequality is given by $\mathcal{I}(K, S_0) = \mathcal{I}_K(K, S_0)$. For $K < S_0$ we will show that $\mathcal{I}_K(K, S_0)$ is a decreasing function for $K < S_0$, and thus the solution of the variational inequality is given by $\mathcal{I}(K, S_0) = \mathcal{I}_K(K, S_0)$.

We give next the solution of the variational problem (25) with the equality constraint $\int_0^1 g(t)dt = K$. This is given by the following result.

PROPOSITION 3.4: *The solution of the variational problem (25) with the equality constraint $\int_0^1 g(t)dt = K$ is given by*

$$\mathcal{I}_K(K, S_0) = \begin{cases} \dfrac{S_0^{2(1-\beta)}}{2\sigma^2} a^{(+)}(x) b^{(+)}(x) & K \leq S_0, \\ \dfrac{S_0^{2(1-\beta)}}{2\sigma^2} a^{(-)}(x) b^{(-)}(x) & K \geq S_0. \end{cases} \tag{27}$$

The two cases are as follows:

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(i) $K \leq S_0$. $0 < x \leq 1$ *is the solution of the equation*

$$\frac{K}{S_0} = x + \frac{b^{(+)}(x)}{a^{(+)}(x)}, \tag{28}$$

with

$$a^{(+)}(x) = 2x^{-\beta}(1-x)^{1/2}\,{}_2F_1\left(\beta, \frac{1}{2}; \frac{3}{2}; 1-\frac{1}{x}\right), \tag{29}$$

$$b^{(+)}(x) = \frac{2}{3}x^{-\beta}(1-x)^{3/2}\,{}_2F_1\left(\beta, \frac{3}{2}; \frac{5}{2}; 1-\frac{1}{x}\right). \tag{30}$$

The argument $z = 1-(1/x)$ *of the hypergeometric function* ${}_2F_1(a,b;c;z)$ *is negative.*

(ii) $K \geq S_0$. $x \geq 1$ *is the solution of the equation*

$$\frac{K}{S_0} = x - \frac{b^{(-)}(x)}{a^{(-)}(x)}, \tag{31}$$

with

$$a^{(-)}(x) = 2x^{-\beta}(x-1)^{1/2}{}_2F_1\left(\beta, \frac{1}{2}; \frac{3}{2}; 1-\frac{1}{x}\right), \tag{32}$$

$$b^{(-)}(x) = \frac{2}{3}x^{-\beta}(x-1)^{3/2}{}_2F_1\left(\beta, \frac{3}{2}; \frac{5}{2}; 1-\frac{1}{x}\right). \tag{33}$$

The argument $z = 1-(1/x)$ *of the hypergeometric function* ${}_2F_1(a,b;c;z)$ *is positive.*

An alternative form of the solution for $\mathcal{I}_K(K, S_0)$ which gives additional information on the continuity and monotonicity properties of this function in K is given by the following result.

Proposition 3.5:

(i) *The function* $\mathcal{I}_K(K, S_0)$ *is given for* $K > S_0$ *by*

$$\mathcal{I}_K(K, S_0) = \inf_{\varphi > K/S_0} \frac{1}{2}\frac{[\mathcal{G}^{(-)}(\varphi)]^2}{\varphi - (K/S_0)}, \tag{34}$$

with

$$\begin{aligned}\mathcal{G}^{(-)}(\varphi) &= \frac{S_0^{1-\beta}}{\sigma}\int_1^{\varphi} z^{-\beta}\sqrt{\varphi - z}\,dz \\ &= \frac{S_0^{1-\beta}}{\sigma}\frac{2}{3}\varphi^{-\beta}(\varphi-1)^{3/2}{}_2F_1\left(\frac{3}{2}, \beta; \frac{5}{2}; 1-\frac{1}{\varphi}\right).\end{aligned} \tag{35}$$

(ii) *The function* $\mathcal{I}_K(K, S_0)$ *is given for* $K < S_0$ *by*

$$\mathcal{I}_K(K, S_0) = \inf_{0<\chi<K/S_0} \frac{1}{2}\frac{[\mathcal{G}^{(+)}(\chi)]^2}{(K/S_0) - \chi}, \tag{36}$$

with

$$\mathcal{G}^{(+)}(\chi) = \frac{S_0^{1-\beta}}{\sigma}\int_\chi^1 z^{-\beta}\sqrt{z-\chi}dz$$
$$= \frac{S_0^{1-\beta}}{\sigma}\frac{2}{3}\chi^{-\beta}(1-\chi)^{3/2}{}_2F_1\left(\frac{3}{2},\beta;\frac{5}{2};1-\frac{1}{\chi}\right). \quad \textbf{(37)}$$

From the representation of Proposition 3.5 it follows that $\mathcal{I}_K(K,S_0)$ is a continuous function of K. We also obtain the monotonicity properties of this function, which imply the relation to the rate function $\mathcal{I}(K,S_0)$ given by Theorem 3.2.

Corollary 3.6: *We have the following monotonicity properties of the function $\mathcal{I}_K(K,S_0)$ with respect to strike K:*

(i) *For $K > S_0$ the function $\mathcal{I}_K(K,S_0)$ is an increasing function of K.*
(ii) *For $K < S_0$ the function $\mathcal{I}_K(K,S_0)$ is a decreasing function of K.*
(iii) *The rate function $\mathcal{I}(K,S_0)$ is given by*

$$\mathcal{I}(K,S_0) = \mathcal{I}_K(K,S_0). \quad \textbf{(38)}$$

Remark 3.7: It is easy to check that the results of Proposition 3.4 recover the result of Proposition 2.2 for $\beta = 1/2$. For this case, the hypergeometric functions can be expressed in terms of elementary functions. The rate function $\mathcal{I}(K,S_0)$ in the CEV model was evaluated numerically using Proposition 3.4 and Corollary 3.6. Figure 2 shows the result for $\beta = 1/2, 2/3, 5/6$.

Remark 3.8: For $\beta \to 1$, the results of Proposition 3.4 recover the rate function for the Black–Scholes model in Proposition 12 of [50].

3.3. Expansion of the Rate Function Around the ATM Point

Using the same approach as in the proof of Proposition 14 in [50] one can expand the rate function in power series of $x = \log(K/S_0)$ for arbitrary β. The first few terms are

$$\mathcal{I}(K,S_0) = \frac{S_0^{2(1-\beta)}}{\sigma^2}\left\{\frac{3}{2}x^2 + \left(-\frac{3}{10}+\frac{9}{5}(1-\beta)\right)x^3 + \left(\frac{109}{1400}-\frac{117}{350}(1-\beta)+\frac{198}{175}(1-\beta)^2\right)x^4 + O(x^5)\right\}. \quad \textbf{(39)}$$

For $\beta = 1/2$ this reduces to the expansion of the rate function in the square-root model given in Eq. (17).

3.4. Asymptotics of the Rate Function

We discuss next the asymptotics of the rate function $\mathcal{I}(K,S_0)$ in the CEV model for very small/large strike K. This is given by the following result, which generalizes the results of Proposition 2.3 to general $1/2 \le \beta < 1$.

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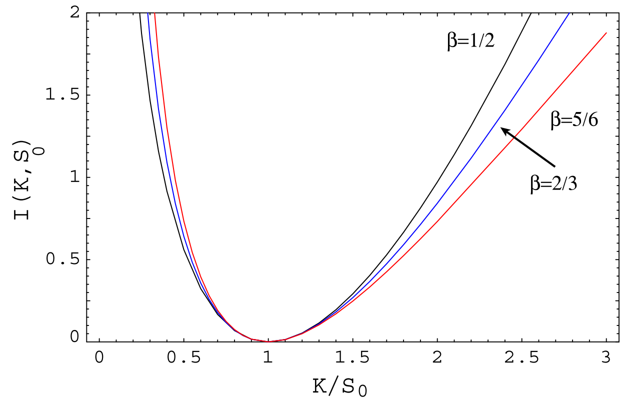

FIGURE 2. The rate function $\mathcal{I}(K,S_0)/(S_0^{2(1-\beta)}/\sigma^2)$ vs. K/S_0 for Asian options in the CEV model with $\beta=1/2$ (black), $\beta=2/3$ (blue) and $\beta=5/6$ (red).

PROPOSITION 3.9 Large strike asymptotics: *We have, for* $\beta\in[1/2,1)$,

$$\mathcal{I}(K,S_0)\sim\frac{S_0^{2(1-\beta)}}{2\sigma^2}\frac{\pi\Gamma^2(1-\beta)}{(3-2\beta)\Gamma^2(3/2-\beta)}\left(\frac{3-2\beta}{2(1-\beta)}\frac{K}{S_0}\right)^{2(1-\beta)},\quad \text{as } K\to\infty, \tag{40}$$

where $\Gamma(\cdot)$ *is the Gamma function.*

For $\beta=1/2$ *this reproduces the result* (i) *of Proposition* 2.3.

$$\mathcal{I}(K,S_0)\sim\frac{\pi^2K}{2\sigma^2},\quad \text{as } K\to\infty. \tag{41}$$

PROPOSITION 3.10 Small strike asymptotics: *The* $K\to 0$ *asymptotics of the rate function for* $\beta\in 1/2,1$ *is given by*

$$\lim_{K\to 0}\frac{K}{S_0}\mathcal{I}(K,S_0)=\frac{2S_0^{2(1-\beta)}}{\sigma^2(3-2\beta)^2}. \tag{42}$$

For $\beta\to 1/2$, *this reproduces the result* (ii) *of Proposition* 2.3

$$\lim_{K\to 0}\frac{K}{S_0}\mathcal{I}(K,S_0)=\frac{2S_0^{2(1-\beta)}}{\sigma^2(3-2\beta)^2}\to\frac{S_0}{2\sigma^2}, \tag{43}$$

4. FLOATING STRIKE ASIAN OPTIONS

We consider in this section the short maturity asymptotics for floating strike Asian options. The prices of the floating strike Asian call/put options are given by risk-neutral expectations

$$C_f(T):=e^{-rT}\mathbb{E}\left[\left(\kappa S_T-\frac{1}{T}\int_0^T S_t dt\right)^+\right], \tag{44}$$

$$P_f(T):=e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt-\kappa S_T\right)^+\right]. \tag{45}$$

First of all, similar to Lemma 3.1, we have:

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(i) For an Asian OTM call option, that is, $\kappa < 1$, we have for $1/2 \leq \beta < 1$

$$\lim_{T\to 0} T\log C_f(T) = \lim_{T\to 0} T\log \mathbb{P}\left(\frac{1}{T}\int_0^T S_t dt \leq \kappa S_T\right). \tag{46}$$

(ii) For an Asian OTM put option, that is, $\kappa > 1$, we have for $1/2 \leq \beta < 1$

$$\lim_{T\to 0} T\log P_f(T) = \lim_{T\to 0} T\log \mathbb{P}\left(\frac{1}{T}\int_0^T S_t dt \geq \kappa S_T\right). \tag{47}$$

We start by considering the square-root model:

$$dS_t = (r-q)S_t dt + \sigma\sqrt{S_t}dW_t, \tag{48}$$

with $S_0 > 0$ and W_t is a standard Brownian motion starting at zero at time zero.

THEOREM 4.1: *For $\beta = 1/2$, $\mathbb{P}((1/T)\int_0^T S_t dt - \kappa S_T \in \cdot)$ satisfies a large deviation principle with the rate function*

$$I_f(x) := \sup_{\theta\in\mathbb{R}}\{\theta x - \Lambda_f(\theta)\}, \tag{49}$$

where $\Lambda_f(\theta)$ is given by

$$\Lambda_f(\theta) := \lim_{T\to 0} T\log \mathbb{E}\left[e^{(\theta/T^2)\int_0^T S_t dt}\right]$$
$$= \begin{cases} \dfrac{\sqrt{2\theta}}{\sigma}\tan\left(\dfrac{\sigma}{2}\sqrt{2\theta} + \tan^{-1}\left(-\sigma\kappa\sqrt{\dfrac{\theta}{2}}\right)\right)S_0, & \text{if } 0 \leq \theta < \theta_c \\ -\dfrac{\sqrt{-2\theta}}{\sigma}\tanh\left(\dfrac{\sigma}{2}\sqrt{-2\theta} + \tanh^{-1}\left(-\sigma\kappa\sqrt{\dfrac{-\theta}{2}}\right)\right)S_0, & \text{if } \theta \leq 0 \\ +\infty, & \text{otherwise} \end{cases}. \tag{50}$$

where θ_c is the unique positive solution of the equation

$$\sqrt{\frac{\sigma^2\theta_c}{2}} + \tan^{-1}\left(-\sigma\kappa\sqrt{\frac{\theta_c}{2}}\right) = \frac{\pi}{2}. \tag{51}$$

It follows from (46) and (47) that for $\kappa < 1$, the call option is OTM and $C_f(T) = e^{-(1/T)\mathcal{I}_f(\kappa,S_0)+o(1/T)}$, as $T\to 0$, and for $\kappa > 1$, the put option is OTM and $P_f(T) = e^{-(1/T)\mathcal{I}_f(\kappa,S_0)+o(1/T)}$, as $T\to 0$, where

$$\mathcal{I}_f(\kappa, S_0) = I_f(0) = \sup_{\theta\in\mathbb{R}}\{-\Lambda_f(\theta)\}. \tag{52}$$

The result of Theorem 4.1 for $\mathcal{I}_f(\kappa, S_0)$ for the square-root model can be put into a more explicit form, as

$$\mathcal{I}_f(\kappa, S_0) = \frac{S_0}{\sigma^2}\mathcal{J}_f(\kappa), \tag{53}$$

where $\mathcal{J}_f(\kappa)$ is given by:

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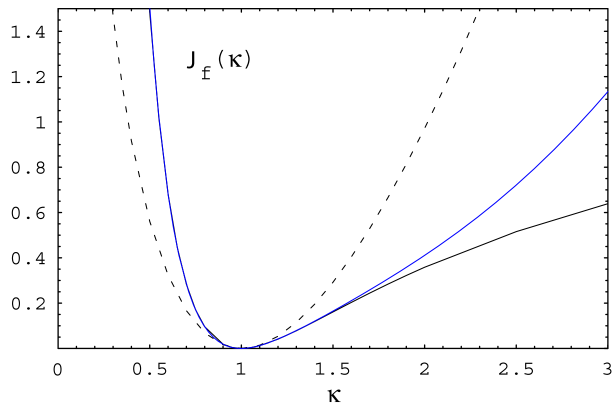

FIGURE 3. The rate function $\mathcal{J}_f(\kappa)$ vs. κ for floating strike Asian options in the square-root model with $\beta = 1/2$ (black solid curve). The solid blue curve shows the approximation of this function obtained by keeping the first three terms in the expansion (58). This is compared against the fixed strike rate function $\mathcal{I}(S_0\kappa, S_0)$ (in units of S_0/σ^2) for the same model (dashed curve), given by Proposition 2.2.

(i) For $\kappa \geq 1$,

$$\mathcal{J}_f(\kappa) = 2z\frac{\kappa z - \tan z}{1 + \kappa z \tan z}, \tag{54}$$

where z is the solution of the equation

$$1 + \kappa^2 z^2 + (1 - \kappa^2 z^2)\frac{\sin 2z}{2z} = 2\kappa \cos^2 z. \tag{55}$$

The solution is defined up to a sign, but this ambiguity is not relevant for computing $\mathcal{J}_f(\kappa)$.

(ii) For $\kappa \leq 1$,

$$\mathcal{J}_f(\kappa) = 2z\frac{\kappa z - \tanh z}{1 - \kappa z \tanh z}, \tag{56}$$

where z is the solution of the equation

$$1 - \kappa^2 z^2 + (1 + \kappa^2 z^2)\frac{\sinh 2z}{2z} = 2\kappa \cosh^2 z. \tag{57}$$

The rate function $\mathcal{J}_f(\kappa, S_0)$ for the square-root model $\beta = 1/2$ is shown in Figure 3 (solid black curve). This is compared against the rate function $\mathcal{I}(\kappa)$ for fixed strike Asian options given by Proposition 2.2 (dashed curve). In the Black–Scholes model they are equal [50], which follows from the equivalence relations for fixed/floating strike Asian options [44]. These relations do not hold beyond the Black–Scholes model, and as a consequence the corresponding rate functions are different.

The floating strike rate function has the expansion around the ATM point $\kappa = 1$

$$\mathcal{J}_f(\kappa) = \frac{3}{2}\log^2\kappa - \frac{33}{20}\log^3\kappa + \frac{5809}{5600}\log^4\kappa + O(\log^5\kappa). \tag{58}$$

This is obtained by expanding the solution of Eqs. (55) and (57) in powers of z, and inserting the result into (54) and (56). The approximation for the rate function $\mathcal{J}_f(\kappa)$ obtained by keeping the first three terms in this expansion is shown in Figure 3 as the solid blue curve.

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For the general CEV model with $1/2 \leq \beta < 1$, following the proof of Theorem 3.2, we get the following result:

Theorem 4.2: *The short maturity asymptotics for OTM floating strike Asian options in the CEV model (2) with $1/2 \leq \beta < 1$ is given by*

(i) *For $\kappa < 1$, corresponding to an OTM floating strike Asian call option,*

$$\lim_{T\to 0} T \log C_f(T) = -\mathcal{I}_f(\kappa, S_0), \tag{59}$$

where

$$\mathcal{I}_f(\kappa, S_0) = \inf_{\int_0^1 g(t)dt \leq \kappa g(1), g(0)=S_0, g(t)\geq 0, 0\leq t\leq 1} \frac{1}{2}\int_0^1 \frac{(g'(t))^2}{\sigma^2 g(t)^{2\beta}} dt. \tag{60}$$

(ii) *For $\kappa > 1$, corresponding to a OTM floating strike Asian put option,*

$$\lim_{T\to 0} T \log P_f(T) = -\mathcal{I}_f(\kappa, S_0), \tag{61}$$

where

$$\mathcal{I}_f(\kappa, S_0) = \inf_{\int_0^1 g(t)dt \geq \kappa g(1), g(0)=S_0, g(t)\geq 0, 0\leq t\leq 1} \frac{1}{2}\int_0^1 \frac{(g'(t))^2}{\sigma^2 g(t)^{2\beta}} dt. \tag{62}$$

Let us consider the ATM case, that is, $\kappa = 1$. For this case, we have the following result. The proof is very similar to the proof of Theorem 3.3, and is hence omitted here.

Theorem 4.3: *As $T \to 0$, we have in the CEV model with $1/2 \leq \beta < 1$,*

$$C_f(T) = \sigma S_0^\beta \frac{\sqrt{T}}{\sqrt{6\pi}} + O(T), \quad P_f(T) = \sigma S_0^\beta \frac{\sqrt{T}}{\sqrt{6\pi}} + O(T). \tag{63}$$

5. NUMERICAL TESTS

We present in this section a few numerical tests of the short-maturity asymptotic results for Asian options in the CEV model obtained in this paper. Following [50] we will use the Asian option pricing formulas

$$C_{\text{asympt}}(K,T) = e^{-rT}(A(T)N(d_1) - KN(d_2)), \tag{64}$$

$$P_{\text{asympt}}(K,T) = e^{-rT}(KN(-d_2) - A(T)N(-d_1)), \tag{65}$$

where $A(T)$ is the expectation of the averaged asset price,

$$A(T) = S_0 \frac{1}{(r-q)T}(e^{(r-q)T} - 1), \tag{66}$$

and

$$d_{1,2} = \frac{1}{\Sigma_{LN}\sqrt{T}}\left(\log\frac{A(T)}{K} \pm \frac{1}{2}\Sigma_{LN}^2 T\right). \tag{67}$$

The equivalent log-normal volatility of the Asian option is defined by

$$\Sigma_{LN}^2(K, S_0) = \frac{\log^2(K/S_0)}{2\mathcal{I}(K, S_0)}, \tag{68}$$

where $\mathcal{I}(K, S_0)$ is the rate function, given for the general CEV model in (27), and for the square-root model $\beta = 1/2$ in Proposition 2.2. As shown in Proposition 18 of [50],

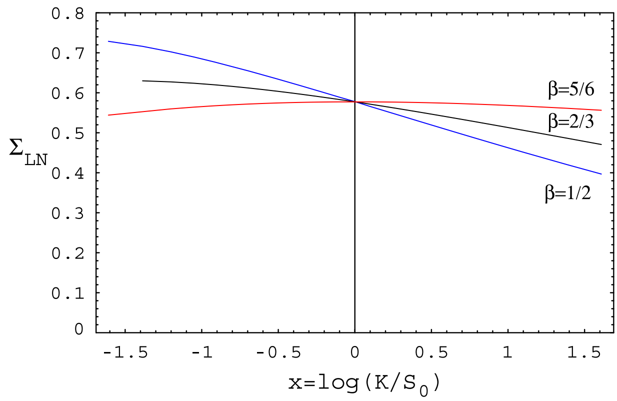

FIGURE 4. Small-maturity equivalent log-normal volatility $\Sigma_{\mathrm{LN}}(K,S_0)/(\sigma S_0^{\beta-1})$ vs. $x=\log(K/S_0)$ for an Asian option in the CEV model with $\beta=1/2$ (blue), $\beta=2/3$ (black) and $\beta=5/6$ (red).

the approximation (64),(65) has the same short maturity asymptotics as that given by Proposition 2.2 for the square-root model $\beta=1/2$, and by Theorem 3.2 for the general CEV model.

5.1. Equivalent Log-normal Volatility of Asian Options in the CEV Model

The series expansion of the equivalent log-normal volatility $\Sigma_{\mathrm{LN}}(K,S_0)$ in powers of log-strike $x=\log(K/S_0)$ can be obtained by substituting (39) into the definition (68). This is

$$\Sigma_{LN}(K,S_0)=\sigma\frac{1}{\sqrt{3}}S_0^{\beta-1}\left\{1+\left(\frac{1}{10}+\frac{3}{5}(\beta-1)\right)x\right.$$
$$\left.+\left(-\frac{23}{2100}+\frac{12}{175}(\beta-1)+\frac{57}{350}(\beta-1)^2\right)x^2+O(x^3)\right\}. \quad (69)$$

For ATM Asian options $K=S_0$ the equivalent log-normal volatility is

$$\Sigma_{LN}(S_0,S_0)=\sigma\frac{1}{\sqrt{3}}S_0^{\beta-1}. \quad (70)$$

For the square-root model $\beta=1/2$ the ATM skew and convexity of the equivalent log-normal volatility are $-1/5$ and $-19/4200$ of the ATM equivalent volatility, respectively. We show in Figure 4 the plot of $\Sigma_{\mathrm{LN}}/\sigma$ vs $x=\log(K/S_0)$ obtained using Eq. (68) for the square-root model $\beta=1/2$.

The plot in Figure 4 is in general qualitative agreement with the shape of the implied volatility for options on realized variance in the Heston model, which are mathematically equivalent to Asian options in the square-root model. See Figure 5 (right) in [23]. As noted in the literature [23], the down-sloping shape of the implied volatility is a deficiency of the Heston model, as the observed smile for variance options in equity markets is up-sloping. The reference [23] proposes as an alternative model which has up-sloping smiles for variance options, the 3/2 stochastic volatility model [14,15].

From the expansion (69) one can obtain the dependence of the ATM skew and convexity on the β parameter. For $\beta=1/2$ the ATM skew is negative; as β is increased, the ATM skew increases, crosses zero at $\beta=5/6$ and becomes positive. The ATM convexity is always negative for $1/2\le\beta<1$, so the equivalent log-normal volatility smile is slightly concave.

TABLE 1. Comparison of the short-maturity asymptotic formulas for Asian options in the square-root model $\beta = 1/2$ for the seven scenarios considered by Dassios and Nagardjasarma [20]. The results are compared against those of [20] (DN) and those of Foschi et al. [34] (denoted as FPP3)

Case	S_0	K	r	σ	T	$C_{\mathrm{asympt}}(K,T)$	DN	FPP3
1	2	2	0.02	0.14	1	0.055474	0.0197	0.055562
2	2	2	0.18	0.42	1	0.216013	0.2189	0.217874
3	2	2	0.0125	0.35	2	0.170568	0.1725	0.170926
4	1.9	2	0.05	0.69	1	0.189863	0.1902	0.190834
5	2	2	0.05	0.72	1	0.250113	NA	0.251121
6	2.1	2	0.05	0.72	1	0.307731	0.3098	0.308715
7	2	2	0.05	0.71	2	0.350516	0.3339	0.353197

TABLE 2. Numerical tests for the scenarios proposed in [20] and [34]

Case	σ	T	$C_{\mathrm{asympt}}(K,T)$	DN	FPP3
1	0.71	0.1	0.075354	0.0751	0.075387
2	0.71	0.5	0.172813	0.1725	0.173175
3	0.71	1.0	0.247020	0.2468	0.248016
4	0.71	2.0	0.350516	0.3339	0.353197
5	0.71	5.0	0.536611	0.3733	0.545714
6	0.1	1.0	0.061310	0.0484	0.061439
7	0.3	1.0	0.120226	0.1207	0.120680
8	0.5	1.0	0.181983	0.1827	0.182723
9	0.7	1.0	0.243926	0.2446	0.244913

5.2. Numerical Scenarios

We present next numerical tests for Asian option pricing in the square-root model $\beta = 1/2$, for the seven scenarios proposed in Dassios and Nagardjasarma [20]. We also compare with the third-order approximation of Foschi et al. [34] (denoted as FPP3), listed in Table 5 of [34]. The results are shown in Table 1.

We note that the agreement of the asymptotic result with FPP3 is always better than 1% in relative value. The differences becomes larger for cases where rT is larger, since the asymptotic result for $\Sigma_{\mathrm{LN}}(K, S_0)$ does not take into account dependence on this parameter, and for larger maturity.

A second set of scenarios proposed by DN [20] is shown in Table 2. There are nine scenarios with $S_0 = K = 2, r = 0.05, q = 0, \beta = 1/2$. The asymptotic results are shown in Table 2, comparing with the results of [20,34] (Table 6 in this reference). Since they are all ATM scenarios, the use of the asymptotic formulas is very simple, and reduces to the use of Eq. (70).

The agreement of the asymptotic result with FPP3 is again very good, except for the $T = 5Y$ case. In all these cases (except $T = 5Y$) the difference between them is <1% in relative value. The deviations from the short maturity asymptotic results are expected to increase with the maturity, as the expectations giving the Asian option prices receive also contributions from typical paths, in addition to the rare events paths which dominate in the

TABLE 3. Tests of the short maturity asymptotic formulas for Asian options for $\beta \in [0.5, 1]$. The model parameters correspond to cases 1,2,3 in Table 1, except for σ which is scaled as shown such that the ATM value is independent of β. The benchmark evaluation is obtained using MC simulation as discussed in text (the MC error corresponding to one standard deviation is shown in brackets)

	Case 1		Case 2		Case 3	
β	C_{asympt}	C_{MC}	C_{asympt}	C_{MC}	C_{asympt}	C_{MC}
0.5	0.055474	0.055401(0.000074)	0.216013	0.217260(0.000244)	0.170568	0.170386(0.000257)
0.6	0.058651	0.058568(0.000079)	0.223504	0.224746(0.000262)	0.181843	0.181561(0.000280)
0.7	0.062063	0.061967(0.000085)	0.231666	0.232881(0.000282)	0.193920	0.193516(0.000307)
0.8	0.065724	0.065614(0.000091)	0.240540	0.241704(0.000307)	0.206855	0.206301(0.000337)
0.9	0.069654	0.069528(0.000098)	0.250167	0.251253(0.000334)	0.220705	0.219973(0.000372)
1.0	0.073871	0.073726(0.000105)	0.260594	0.261570(0.000365)	0.235534	0.234589(0.000414)

large deviations limit. For maturities <1Y, the difference is always below 0.5% in relative value.

We present in Table 3 also numerical tests of β dependence of the short maturity asymptotic results for several values of $\beta \in [0.5, 1.0]$. The model parameters correspond to Cases 1,2, and 3 in Table 1. The results are compared against those obtained from a MC simulation using Euler discretization with $n = 10^3$ time steps and $N_{\text{MC}} = 10^6$ MC paths. We observe the same pattern as for the $\beta = 1/2$ results in Table 1. The agreement is best for case 1 and the differences with the MC simulation are larger for case 2 (due to larger value of $rT = 0.18$) and case 3 (due to larger maturity).

5.3. Floating-strike Asian Options

We discuss in this section the pricing of floating-strike Asian options. They can be considered as call and put options on the underlying $B_T := \kappa S_T - A_T$. The forward price of this asset is

$$F_f(T) := \mathbb{E}[B_T] = S_0 \left(\kappa e^{(r-q)T} - \frac{e^{(r-q)T} - 1}{(r-q)T} \right). \tag{71}$$

For $\kappa \geq 0$, the underlying B_T takes values on the entire real axis. For this reason a Black–Scholes representation of this asset is not appropriate.

We propose to approximate the prices of floating-strike Asian options using a Bachelier (normal) approximation. These options are approximated as zero strike put and call options on the asset B_T, and their prices are

$$C_f(\kappa, T) = e^{-rT} \left[F_f(T)\Phi(d) + \frac{1}{\sqrt{2\pi}} \Sigma_N \sqrt{T} e^{-(1/2)d^2} \right],$$
$$P_f(\kappa, T) = e^{-rT} \left[-F_f(T)\Phi(-d) + \frac{1}{\sqrt{2\pi}} \Sigma_N \sqrt{T} e^{-(1/2)d^2} \right], \tag{72}$$

with $d = (F_f(T)/\Sigma_N \sqrt{T})$.

The equivalent normal volatility $\Sigma_N(\kappa, T)$ is specified by requiring that the small-maturity asymptotics of the floating-strike Asian options matches that of the Bachelier expression. This is given by the following result.

PROPOSITION 5.1: *The short-maturity limit of the equivalent normal volatility in the square-root model $\beta = 1/2$ is given by:*

(i) *for OTM floating strike Asian options $\kappa \neq 1$*

$$\lim_{T\to 0} \Sigma_N(\kappa, T) = \frac{\sigma^2}{2S_0} \frac{(\kappa-1)^2}{\mathcal{J}_f(\kappa)}, \tag{73}$$

where $\mathcal{J}_f(\kappa)$ is given by (53).

(ii) *for ATM floating strike Asian options $\kappa = 1$*

$$\lim_{T\to 0} \Sigma_N(\kappa, T) = \sigma\sqrt{\frac{S_0}{3}}. \tag{74}$$

PROOF: The proof is similar to that of Proposition 18 in [50] and will be omitted. ■

The pricing of floating-strike Asian options in the square-root model has been considered in [36]. This paper studied the pricing of options with payoff $(-S_T + A_T - K)^+$ with K both positive and negative, using both discrete and continuous time monitoring. We will compare the result for $K = 0$ with continuous time averaging, which corresponds in our notations to a floating strike Asian put option with $\kappa = 1$.

The model parameters used in [36] are $S_0 = 1, r = 0.04, \sigma = 0.7$, and the option maturity is $T = 1$. The price quoted in Table 3 of this paper with $K = 0$ is $C_f(1, T) = 0.14376$. The asymptotic formula (72) gives $C_f(1, T) = 0.14524$, which is in reasonably good agreement with the result of [36] (1% relative difference).

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APPENDIX A. PROOFS

Appendix A.1. Background of Large Deviations Theory

We start by giving a formal definition of the large deviation principle. We refer to Dembo and Zeitouni [21] and Varadhan [55] for general background of large deviations theory and its applications.

Definition A.1 (Large Deviation Principle): *A sequence $(P_\epsilon)_{\epsilon\in\mathbb{R}^+}$ of probability measures on a topological space X satisfies the large deviation principle with rate function $I : X \to \mathbb{R}$ if I is non-negative, lower semicontinuous and for any measurable set A, we have*

$$-\inf_{x\in A^o} I(x) \le \liminf_{\epsilon\to 0} \epsilon \log P_\epsilon(A) \le \limsup_{\epsilon\to 0} \epsilon \log P_\epsilon(A) \le -\inf_{x\in \overline{A}} I(x). \tag{A.1}$$

Here, A^o is the interior of A and $\overline{A}$ is its closure.

The contraction principle plays a key role in our proofs. For the convenience of the readers, we state the result as follows:

Theorem A.2 (Contraction Principle, e.g. Theorem 4.2.1. [21]): *If P_ϵ satisfies a large deviation principle on X with rate function $I(x)$ and $F : X \to Y$ is a continuous map, then the probability measures $Q_\epsilon := P_\epsilon F^{-1}$ satisfies a large deviation principle on Y with rate function*

$$J(y) = \inf_{x:F(x)=y} I(x). \tag{A.2}$$

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We will use the following version of the Gärtner–Ellis Theorem in the proofs in this paper.

THEOREM A.3 (Gärtner–Ellis Theorem, e.g. Theorem 2.3.6 in [21]): *Let* Z_ϵ *be a sequence of random variables on* $\mathbb{R}$. *Assume the limit* $\Lambda(\theta) := \lim_\epsilon \epsilon \log \mathbb{E}[e^{(\theta/\epsilon)Z_\epsilon}]$ *exists on the extended real line and the interior of the set* $\mathcal{D} := \{\theta : \Lambda(\theta) < \infty\}$ *contains* 0, *and* $\Lambda(\theta)$ *is differentiable for any* θ *in the interior of* $\mathcal{D}$ *and* $|\Lambda'(\theta)| \to \infty$ *as* θ *approaches to the boundary of* $\mathcal{D}$. *Then* $\mathbb{P}(Z_\epsilon \in \cdot)$ *satisfies a large deviation principle with the rate function* $I(x) := \sup_{\theta\in\mathbb{R}}\{\theta x - \Lambda(\theta)\}$.

Appendix A.2. Proofs of the Results in Section 2

Proof of Theorem 2.1: For any $\theta \in \mathbb{R}$, $u(t,x) = \mathbb{E}[e^{\theta\int_0^t S_s ds}|S_0 = x]$ satisfies the PDE:

$$\frac{\partial u}{\partial t} = (r-q)x\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 x\frac{\partial^2 u}{\partial x^2} + \theta x u(t,x), \tag{A.3}$$

with $u(0,x) \equiv 1$. This affine PDE has the solution $u(t,x) = e^{A(t)x+B(t)}$, where

$$A'(t) = (r-q)A(t) + \frac{1}{2}\sigma^2 A(t)^2 + \theta, \tag{A.4}$$

$$B'(t) = 0, \tag{A.5}$$

with $A(0) = B(0) = 0$ and hence $B(t) = 0$ and for $\theta > 0$ sufficiently large,

$$\frac{2}{\sqrt{2\sigma^2\theta - (r-q)^2}}\tan^{-1}\left(\frac{r-q+\sigma^2 A}{\sqrt{2\sigma^2\theta-(r-q)^2}}\right)\Bigg|_{A=0}^{A=A(t)} = t, \tag{A.6}$$

and thus

$$A(t;\theta) = \frac{\sqrt{2\sigma^2\theta-(r-q)^2}}{\sigma^2}\tan\left[\frac{\sqrt{2\sigma^2\theta-(r-q)^2}}{2}t + \tan^{-1}\left(\frac{r-q}{\sqrt{2\sigma^2\theta-(r-q)^2}}\right)\right] - \frac{r-q}{\sigma^2}. \tag{A.7}$$

We denoted $A(t;\theta) = A(t)$ by making explicit the dependence on θ.

For $\theta < 0$ sufficiently negative,

$$\frac{2}{\sigma^2}\frac{1}{2\sqrt{((r-q)^2/\sigma^4) - (2\theta/\sigma^2)}}\log\left(\frac{(r-q/\sigma^2) - \sqrt{((r-q)^2/\sigma^4)-(2\theta/\sigma^2)} + A}{(r-q/\sigma^2) + \sqrt{((r-q)^2/\sigma^4)-(2\theta/\sigma^2)} + A}\right)\Bigg|_{A=0}^{A=A(t)} = t, \tag{A.8}$$

and thus

$$A(t;\theta) = \frac{e^{t\delta}-1}{\frac{\sigma^2}{r-q-\delta} - \frac{\sigma^2 e^{t\delta}}{r-q+\delta}} = \frac{2\theta(e^{t\delta}-1)}{(r-q)(1-e^{t\delta}) + \delta(e^{t\delta}+1)} \tag{A.9}$$

with $\delta := \sqrt{(r-q)^2 - 2\theta\sigma^2}$.

Let us study now the $T \to 0$ limit. We note that for any $T > 0$ sufficiently small, we have

$$\mathbb{E}\left[e^{(\theta/T^2)\int_0^T S_t dt}\right] = e^{A(T;(\theta/T^2))S_0}. \tag{A.10}$$

For $0 \le \theta < \pi^2/(2\sigma^2)$,

$$\lim_{T\to 0} TA\left(T;\frac{\theta}{T^2}\right) = \sqrt{\frac{2\theta}{\sigma^2}}\tan\sqrt{\frac{\sigma^2\theta}{2}}, \tag{A.11}$$

and this limit is ∞ if $\theta \ge \pi^2/(2\sigma^2)$.

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For $\theta < 0$,

$$\lim_{T\to 0} TA\left(T; \frac{\theta}{T^2}\right) = \frac{-\sqrt{-2\theta}}{\sigma}\frac{e^{\sigma\sqrt{-2\theta}}-1}{e^{\sigma\sqrt{-2\theta}}+1} = \frac{-\sqrt{-2\theta}}{\sigma}\tanh\left(\frac{\sigma}{2}\sqrt{-2\theta}\right). \tag{A.12}$$

Therefore,

$$\Lambda(\theta) := \lim_{T\to 0} T\log\mathbb{E}\left[e^{\frac{\theta}{T^2}\int_0^T S_t dt}\right] = \begin{cases} \frac{\sqrt{2\theta}}{\sigma}\tan\left(\frac{\sigma}{2}\sqrt{2\theta}\right)S_0 & \text{if } 0 \le \theta < \frac{\pi^2}{2\sigma^2} \\ \frac{-\sqrt{-2\theta}}{\sigma}\tanh\left(\frac{\sigma}{2}\sqrt{-2\theta}\right)S_0 & \text{if } \theta \le 0 \\ +\infty & \text{otherwise} \end{cases}. \tag{A.13}$$

For $0 < \theta < \pi^2/(2\sigma^2)$ and $\theta < 0$, $\Lambda(\theta)$ is differentiable and it is also easy to check that $\Lambda(\theta)$ is differentiable at $\theta = 0$. Finally, for $0 < \theta < \pi^2/(2\sigma^2)$, we can compute that

$$\frac{\partial\Lambda(\theta)}{\partial\theta} = \frac{\sqrt{2}}{\sigma 2\sqrt{\theta}}\tan\left(\frac{\sigma}{2}\sqrt{2\theta}\right)S_0 + \frac{\sqrt{2\theta}}{\sigma}\frac{\sigma\sqrt{2}}{4\sqrt{\theta}}\sec^2\left(\frac{\sigma}{2}\sqrt{2\theta}\right)S_0 \to +\infty, \tag{A.14}$$

as $\theta \uparrow \pi^2/(2\sigma^2)$. Hence, we proved the essential smoothness condition. The conclusion follows from the Gärtner–Ellis theorem, see Theorem A.3 in Appendix A. ■

Proof for Proposition 2.2: The result follows from Lemma 3.1, Theorem 2.1 and the Gärtner–Ellis theorem. According to this result the rate function is given by the Legendre transform of the cumulant function

$$\mathcal{I}(K, S_0) = \sup_{\theta\in\mathbb{R}}\{\theta K - \Lambda(\theta)\}, \tag{A.15}$$

where the cumulant function $\Lambda(\theta)$ is given by Theorem 2.1.

(i) $K \ge S_0$. This case corresponds to $0 \le \theta \le (\pi^2/2\sigma^2)$. The cumulant function $\Lambda(\theta)$ is given by

$$\Lambda(\theta) = \frac{S_0}{\sigma^2}\sqrt{2\theta\sigma^2}\tan\sqrt{\frac{1}{2}\sigma^2\theta} = \frac{S_0}{\sigma^2}F_+(\theta\sigma^2). \tag{A.16}$$

where we defined $F_+(y) := \sqrt{2y}\tan\sqrt{(1/2)y}$.
The optimal value of θ in (A.15) is given by the solution of the equation

$$K = S_0 F'_+(\theta_*\sigma^2), \quad F'_+(y) = \frac{1}{2\cos^2\sqrt{y/2}}\left(1 + \frac{\sin\sqrt{2y}}{\sqrt{2y}}\right). \tag{A.17}$$

Numerical evaluation shows that $F'_+(y) : [0,\infty) \to [1,\infty)$ is a bijective map, such that this equation will have a solution for $K > S_0$. Identifying $x = \sqrt{(1/2)\theta_*\sigma^2}$, it is easy to see that the equation for θ_* is the same as (14). The result for the rate function is

$$\begin{aligned}\mathcal{I}(K, S_0) = \theta_* K - \Lambda(\theta_*) &= \frac{S_0}{\sigma^2}\left(\theta_*\sigma^2\frac{K}{S_0} - F_+(\theta_*\sigma^2)\right) \\ &= \frac{S_0}{\sigma^2}\left(2x^2\frac{1}{2\cos^2 x}\left(1 + \frac{\sin 2x}{2x}\right) - 2x\tan x\right) \\ &= \frac{S_0}{\sigma^2}\frac{x^2}{\cos^2 x}\left(1 - \frac{\sin(2x)}{2x}\right),\end{aligned} \tag{A.18}$$

which yields Eq. (13).

(ii) $K \leq S_0$. This case corresponds to $\theta \leq 0$. The cumulant function $\Lambda(\theta)$ is

$$\Lambda(\theta) = -\frac{S_0}{\sigma^2}\sqrt{-2\theta\sigma^2}\tanh\sqrt{-\frac{1}{2}\theta\sigma^2} = \frac{S_0}{\sigma^2}F_-(\theta\sigma^2), \tag{A.19}$$

where we introduced $F_-(y) := -\sqrt{-2y}\tanh\sqrt{-(1/2)y}$. This is related to the function appearing for the previous case as $F_-(iy) = F_+(y)$.

The optimal θ is given by the solution of the equation

$$\frac{K}{S_0} = F_-'(\theta_*\sigma^2), \quad F_-'(y) = \frac{1}{2\cosh^2\sqrt{-(1/2)y}}\left(1 + \frac{\sinh\sqrt{-2y}}{\sqrt{-2y}}\right). \tag{A.20}$$

Numerical evaluation gives that $F_-'(y) : (-\infty, 0] \to (0, 1]$ is a bijective function, so this equation will have a solution for $K < S_0$. Identifying $x = \sqrt{-(1/2)\theta_*\sigma^2}$ we see that Eq. (A.20) reproduces (12). The result for the rate function is

$$\begin{aligned}
\mathcal{I}(K, S_0) = \theta_* K - \Lambda(\theta_*) &= \frac{S_0}{\sigma^2}\left(\theta_*\sigma^2\frac{K}{S_0} - F_-(\theta_*\sigma^2)\right) \\
&= \frac{S_0}{\sigma^2}\left(-2x^2\frac{1}{2\cosh^2 x}\left(1 + \frac{\sinh 2x}{2x}\right) + 2x\tanh x\right) \\
&= -\frac{S_0}{\sigma^2}\frac{x^2}{\cosh^2 x}\left(1 - \frac{\sinh(2x)}{2x}\right),
\end{aligned} \tag{A.21}$$

which gives the result of Eq. (11). ■

Proof of Proposition 2.3:

(i) This is obtained starting with the relation

$$\mathcal{I}(K, S_0) = \sup_{0 \leq \theta < (\pi^2/2\sigma^2)}\left\{\theta K - \frac{\sqrt{2\theta}}{\sigma}\tan\left(\frac{\sigma}{2}\sqrt{2\theta}\right)S_0\right\}. \tag{A.22}$$

On the one hand, $\mathcal{I}(K, S_0) \leq \sup_{0 \leq \theta < (\pi^2/2\sigma^2)} \theta K = (\pi^2/2\sigma^2)K$. On the other hand, for any $\epsilon > 0$, for sufficiently large K,

$$\begin{aligned}
\mathcal{I}(K, S_0) &= \sup_{(\pi^2/2\sigma^2) - \epsilon \leq \theta < (\pi^2/2\sigma^2)}\left\{\theta K - \frac{\sqrt{2\theta}}{\sigma}\tan\left(\frac{\sigma}{2}\sqrt{2\theta}\right)S_0\right\} \\
&\geq \left(\frac{\pi^2}{2\sigma^2} - \epsilon\right)K - \Lambda\left(\frac{\pi^2}{2\sigma^2} - \epsilon\right).
\end{aligned} \tag{A.23}$$

Thus, $\liminf_{K\to\infty}(\mathcal{I}(K, S_0)/K) \geq ((\pi^2/2\sigma^2) - \epsilon)$. Since it holds for any $\epsilon > 0$, we conclude that the relation (15) holds.

(ii) This is obtained starting from the relation

$$\mathcal{I}(K, S_0) = \sup_{\theta \leq 0}\left\{K\theta + \frac{\sqrt{-2\theta}}{\sigma}\tanh\left(\frac{\sigma}{2}\sqrt{-2\theta}\right)S_0\right\}. \tag{A.24}$$

At optimality we have

$$K = \frac{\sqrt{2}}{2\sigma\sqrt{-\theta}}\tanh\left(\frac{\sigma}{2}\sqrt{-2\theta}\right)S_0 + \frac{1}{2}\left[1 - \tanh^2\left(\frac{\sigma}{2}\sqrt{-2\theta}\right)\right]S_0. \tag{A.25}$$

Note that the function $\tanh x$ approaches 1 exponentially fast as $x \to \infty$. Therefore, $\theta \sim -(S_0^2/2\sigma^2 K^2)$ as $K \to 0$ and the result (16) follows. ■

Appendix A.3. Proofs of the Results in Section 3

Proof of Lemma 3.1: We will prove the result for the case of the Asian call option. The case of the Asian put option is very similar.

Note that by Hölder's inequality, for any $(1/p) + (1/p') = 1$, $p, p' > 1$ and $p \geq 2$,

$$\begin{aligned} C(T) &= e^{-rT}\mathbb{E}\left[\left|\frac{1}{T}\int_0^T S_t dt - K\right| 1_{(1/T)\int_0^T S_t dt \geq K}\right] \\ &\leq e^{-rT}\left(\mathbb{E}\left[\left|\frac{1}{T}\int_0^T S_t dt - K\right|^p\right]\right)^{1/p} \mathbb{P}\left(\frac{1}{T}\int_0^T S_t dt \geq K\right)^{1/p'} \\ &\leq e^{-rT}2^{\frac{p-1}{p}}\left(K^p + \left(\mathbb{E}\left[\frac{1}{T}\int_0^T S_t^p dt\right]\right)\right)^{1/p} \mathbb{P}\left(\frac{1}{T}\int_0^T S_t dt \geq K\right)^{1/p'}, \end{aligned} \tag{A.26}$$

where in the last step we used Jensen's inequality to write

$$\begin{aligned} \mathbb{E}\left[\left|\frac{1}{T}\int_0^T S_t dt - K\right|^p\right] &\leq \mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt + K\right)^p\right] \\ &\leq 2^{p-1}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt\right)^p + K^p\right] \leq 2^{p-1}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t^p dt\right) + K^p\right]. \end{aligned} \tag{A.27}$$

The second inequality follows by noting that for $p \geq 2$, $x \to x^p$ is a convex function for $x \geq 0$, which gives by Jensen's inequality $((x + y/2))^p \leq (x^p + y^p/2)$ for any $x, y \geq 0$. This gives

$$\begin{aligned} \mathbb{E}\left[\left|\frac{1}{T}\int_0^T S_t dt - K\right|^p\right] &\leq \mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt + K\right)^p\right] \\ &\leq 2^{p-1}\left[\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt\right)^p\right] + K^p\right]. \end{aligned} \tag{A.28}$$

The last inequality follows again from the Jensen's inequality which gives for $p \geq 2$ $\mathbb{E}[((1/T)\int_0^T S_t dt)^p] \leq \mathbb{E}[(1/T)\int_0^T S_t^p dt]$.

For any $p \geq 2$,

$$\frac{1}{T}\int_0^T \mathbb{E}[S_t^p]dt = O(1), \tag{A.29}$$

since for the CEV process, all these moments are finite and well-behaved as $T \to 0$. The marginal distribution of S_t in this model is known [47] and the above expression can be computed explicitly.

Therefore, we have

$$\limsup_{T\to 0} T \log C(T) \leq \limsup_{T\to 0} \frac{1}{p'} T \log \mathbb{P}\left(\frac{1}{T}\int_0^T S_t dt \geq K\right). \tag{A.30}$$

Since it holds for any $2 > p' > 1$, we have the upper bound.

Next we derive a matching lower bound on $C(T)$. For any $\epsilon > 0$,

$$C(T) \geq e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right) 1_{(1/T)\int_0^T S_t dt \geq K+\epsilon}\right]$$
$$\geq e^{-rT}\epsilon\mathbb{P}\left(\frac{1}{T}\int_0^T S_t dt \geq K+\epsilon\right), \qquad \textbf{(A.31)}$$

which implies that

$$\liminf_{T\to 0} T\log C(T) \geq \liminf_{T\to 0} T\log \mathbb{P}\left(\frac{1}{T}\int_0^T S_t dt \geq K+\epsilon\right). \qquad \textbf{(A.32)}$$

Since it holds for any $\epsilon > 0$, we get the lower bound by letting $\epsilon \to 0$, provided that the limit $\mathcal{I}(K, S_0) := -\lim_{T\to 0} T\log \mathbb{P}((1/T)\int_0^T S_t dt \geq K)$ exists and is continuous in K. The continuity in K can be seen from the expression in Proposition 3.5. ■

Proof of Theorem 3.2: We split the proof into several steps.

Step 1. We need to prove that

$$\lim_{T\to 0} T\log \mathbb{P}\left(\frac{1}{T}\int_0^T S_t dt \geq K\right) = \lim_{T\to 0} T\log \mathbb{P}\left(\frac{1}{T}\int_0^T \hat{S}_t dt \geq K\right), \qquad \textbf{(A.33)}$$

where

$$d\hat{S}_t = \sigma \hat{S}_t^{\beta} dW_t, \qquad \textbf{(A.34)}$$

with $\hat{S}_0 = S_0$. That is, the drift term is negligible for small time large deviations. Let us now prove (A.33). Note that

$$S_t = S_0 e^{(r-q)t+\int_0^t \sigma S_s^{\beta} dW_s-(1/2)\sigma^2\int_0^t S_s^{2\beta} ds} = e^{(r-q)t}\tilde{S}_t, \qquad \textbf{(A.35)}$$

where

$$d\tilde{S}_t = \sigma \tilde{S}_t^{\beta} e^{-(r-q)\beta t} dW_t, \quad \tilde{S}_0 = S_0 > 0. \qquad \textbf{(A.36)}$$

By the time change $d\tau(t) = e^{-2(r-q)\beta t}dt$, $\tau(0) = 0$, $\tilde{S}_t = \hat{S}_{\tau(t)}$, where $\hat{S}$ is defined in (A.34).

Hence,

$$\lim_{T\to 0} T\log \mathbb{P}\left(\frac{1}{T}\int_0^T S_t dt \geq K\right)$$
$$= \lim_{T\to 0} T\log \mathbb{P}\left(\frac{1}{T}\int_0^T e^{(r-q)t}\hat{S}_{\tau(t)} dt \geq K\right)$$
$$= \lim_{T\to 0} T\log \mathbb{P}\left(\frac{1}{T}\int_0^{\tau(T)} e^{(r-q)(1-2\beta)\tau^{-1}(t)}\hat{S}_t dt \geq K\right). \qquad \textbf{(A.37)}$$

It is easy to check that $(\tau(T)/T) \to 1$ as $T \to 0$ and $\lim_{T\to 0}\inf_{0\leq t\leq T} e^{(r-q)(1-2\beta)\tau^{-1}(t)} = \lim_{T\to 0}\sup_{0\leq t\leq T} e^{(r-q)(1-2\beta)\tau^{-1}(t)} = 1$. Hence, (A.33) follows.

Step 2. Now assume that $r = q = 0$ so that

$$dS_t = \sigma S_t^\beta dW_t, \tag{A.38}$$

with $S_0 > 0$. Therefore, for $0 \le t \le 1$,

$$dS_{tT} = \sigma S_{tT}^\beta dW_{tT} = \sqrt{T}\sigma S_{tT}^\beta d(W_{tT}/\sqrt{T}) = \sqrt{T}\sigma S_{tT}^\beta dB_t, \tag{A.39}$$

where $B_t := W_{tT}/\sqrt{T}$ is a standard Brownian motion by the scaling property of the Brownian motion. Therefore, by letting $T = \epsilon$,

$$\lim_{T\to 0} T \log \mathbb{P}\left(\int_0^1 S_{tT} dt \ge K\right) = \lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\int_0^1 S_t^\epsilon dt \ge K\right), \tag{A.40}$$

where

$$dS_t^\epsilon = \sqrt{\epsilon}\sigma (S_t^\epsilon)^\beta dB_t, \tag{A.41}$$

with $S_0^\epsilon = S_0 > 0$.

Step 3. We need to show that

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\int_0^1 S_t^\epsilon dt \ge K\right) = \lim_{\delta\to 0}\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\int_0^1 S_t^\epsilon dt \ge K, S_t^\epsilon \ge \delta, 0 \le t \le 1\right). \tag{A.42}$$

Note that conditional of $\int_0^1 S_t^\epsilon dt \ge K$, the event that $S_t^\epsilon \ge \delta, 0 \le t \le 1$ is a typical event, while the event that $S_t^\epsilon \le \delta$ for some $0 \le t \le 1$ is a rare event. Therefore, for sufficiently small $\delta > 0$,

$$\mathbb{P}\left(\int_0^1 S_t^\epsilon dt \ge K\right) \le 2\mathbb{P}\left(\int_0^1 S_t^\epsilon dt \ge K, S_t^\epsilon \ge \delta, 0 \le t \le 1\right). \tag{A.43}$$

On the other hand, for any $\delta > 0$,

$$\mathbb{P}\left(\int_0^1 S_t^\epsilon dt \ge K\right) \ge \mathbb{P}\left(\int_0^1 S_t^\epsilon dt \ge K, S_t^\epsilon \ge \delta, 0 \le t \le 1\right), \tag{A.44}$$

which implies that, for any $\delta > 0$.

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\int_0^1 S_t^\epsilon dt \ge K\right) \ge \lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\int_0^1 S_t^\epsilon dt \ge K, S_t^\epsilon \ge \delta, 0 \le t \le 1\right). \tag{A.45}$$

Hence, (A.42) follows from (A.43) and (A.45).

Step 4. Define

$$dS_t^{\epsilon,\delta} = b^\delta(S_t^{\epsilon,\delta})dt + \sqrt{\epsilon}\sigma(S_t^{\epsilon,\delta})^\beta dB_t, \quad S_0^{\epsilon,\delta} = S_0, \tag{A.46}$$

where $b^\delta(x) = 0$ for any $x > \delta$ and also is locally Lipschitz continuous and $b^\delta(0) > 0$. Morever, $S \mapsto S^\beta$ is Hölder continuous with exponent $\ge 1/2$ and for $\beta < 1$, it has sublinear growth at ∞. The dynamics (A.46) satisfies the assumption A1.1. in Baldi and Caramellino [7]. It is easy to see that

$$\mathbb{P}\left(\int_0^1 S_t^\epsilon dt \ge K, S_t^\epsilon \ge \delta, 0 \le t \le 1\right) = \mathbb{P}\left(\int_0^1 S_t^{\epsilon,\delta} dt \ge K, S_t^{\epsilon,\delta} \ge \delta, 0 \le t \le 1\right). \tag{A.47}$$

By Theorem 1.2 in Baldi and Caramellino [7] it follows that $\mathbb{P}(S^{\epsilon,\delta} \in \cdot)$ satisfies a large deviation principle on $C_{S_0}([0,1])$, the space of continuous functions starting at S_0 equipped with uniform topology, with the rate function $(1/2)\int_0^1((g'(t) - b^\delta(g(t)))^2/\sigma^2 g(t)^{2\beta})dt$, with the understanding

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that the rate function is $+\infty$ if g is not differentiable. Moreover, the map $g \mapsto (\int_0^1 g(t)dt, g)$ is continuous from $C_{S_0}[0,1]$ to $\mathbb{R}_+ \times C_{S_0}[0,1]$.

By the contraction principle, see Theorem A.2 in Appendix A, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\int_0^1 S_t^{\epsilon} dt \geq K, S_t^{\epsilon,\delta} \geq \delta, 0 \leq t \leq 1\right) \\
&= -\inf_{\int_0^1 g(t)dt \geq K, g(0)=S_0, g(t)\geq\delta, 0\leq t\leq 1} \frac{1}{2}\int_0^1 \frac{(g'(t) - b^{\delta}(g(t)))^2}{\sigma^2 g(t)^{2\beta}} dt \\
&= -\inf_{\int_0^1 g(t)dt \geq K, g(0)=S_0, g(t)\geq\delta, 0\leq t\leq 1} \frac{1}{2}\int_0^1 \frac{(g'(t))^2}{\sigma^2 g(t)^{2\beta}} dt. \qquad \textbf{(A.48)}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\int_0^1 S_t^{\epsilon} dt \geq K\right) \\
&= \lim_{\delta\to 0}\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\int_0^1 S_t^{\epsilon} dt \geq K, S_t^{\epsilon} \geq \delta, 0 \leq t \leq 1\right) \\
&= -\inf_{\int_0^1 g(t)dt \geq K, g(0)=S_0, g(t)\geq 0, 0\leq t\leq 1} \frac{1}{2}\int_0^1 \frac{(g'(t))^2}{\sigma^2 g(t)^{2\beta}} dt. \qquad \textbf{(A.49)}
\end{aligned}$$

■

Proof of Theorem 3.3: We consider only the Asian call option case. The proof for the put option is very similar and is omitted. As $T \to 0$,

$$C(T) = e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] = \mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] + O(T), \qquad \textbf{(A.50)}$$

and we showed that

$$\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] = \mathbb{E}\left[\left(\frac{1}{T}\int_0^{\tau(T)} e^{(r-q)(1-2\beta)\tau^{-1}(t)}\hat{S}_t dt - K\right)^+\right], \qquad \textbf{(A.51)}$$

where $d\hat{S}_t = \sigma \hat{S}_t^{\beta} dW_t$ and $\hat{S}_0 = S_0$.

It is easy to show that

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\frac{1}{T}\int_0^{\tau(T)} e^{(r-q)(1-2\beta)\tau^{-1}(t)}\hat{S}_t dt - K\right)^+\right] - \mathbb{E}\left[\left(\frac{1}{T}\int_0^{\tau(T)} \hat{S}_t dt - K\right)^+\right]\right| \\
&\leq \mathbb{E}\left[\frac{1}{T}\int_0^{\tau(T)} |e^{(r-q)(1-2\beta)\tau^{-1}(t)} - 1|\hat{S}_t dt\right] \\
&= S_0 \frac{1}{T}\int_0^{\tau(T)} |e^{(r-q)(1-2\beta)\tau^{-1}(t)} - 1| dt = O(T). \qquad \textbf{(A.52)}
\end{aligned}$$

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Moreover, we can show that

$$\left|\mathbb{E}\left[\left(\frac{1}{T}\int_0^T \hat{S}_t dt - K\right)^+\right] - \mathbb{E}\left[\left(\frac{1}{T}\int_0^{\tau(T)} \hat{S}_t dt - K\right)^+\right]\right|$$

$$\leq \mathbb{E}\left|\frac{1}{T}\int_{\tau(T)}^T \hat{S}_t dt\right| = S_0\frac{1}{T}|T-\tau(T)| = O(T). \quad \textbf{(A.53)}$$

Next, let $dX_t = \sigma S_0^\beta dW_t$ and $X_0 = S_0$, that is $X_t = S_0 + \sigma S_0^\beta W_t$. By Itô's formula and taking the expectations, we get

$$\mathbb{E}(\hat{S}_t - X_t)^2 = \sigma^2 \int_0^t \mathbb{E}(\hat{S}_s^\beta - S_0^\beta)^2 ds$$

$$\leq 2\sigma^2 \int_0^t \mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^2]ds + 2\sigma^2 \int_0^t \mathbb{E}[(X_s^\beta - S_0^\beta)^2]ds. \quad \textbf{(A.54)}$$

For any $x > 0$, $y \geq 0$ and $1/2 \leq \beta < 1$, we have $|x^\beta - y^\beta| \leq |x-y|x^{\beta-1}$, see e.g. Lemma 2.2. in Cai and Wang [10]. Hence,

$$2\sigma^2 \int_0^t \mathbb{E}[(X_s^\beta - S_0^\beta)^2]ds \leq 2\sigma^2 S_0^{2(\beta-1)} \int_0^t \mathbb{E}[(X_s - S_0)^2]ds = \sigma^2 S_0^{2(\beta-1)}\sigma^2 S_0^{2\beta} t^2. \quad \textbf{(A.55)}$$

Moreover, for $S_0 > \delta > 0$,

$$2\sigma^2 \int_0^t \mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^2]ds$$

$$= 2\sigma^2 \int_0^t \mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^2 1_{X_s \geq \delta}]ds + 2\sigma^2 \int_0^t \mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^2 1_{X_s < \delta}]ds. \quad \textbf{(A.56)}$$

On the one hand,

$$2\sigma^2 \int_0^t \mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^2 1_{X_s \geq \delta}]ds \leq 2\sigma^2 \delta^{2(\beta-1)} \int_0^t \mathbb{E}[(\hat{S}_s - X_s)^2]ds. \quad \textbf{(A.57)}$$

On the other hand,

$$2\sigma^2 \int_0^t \mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^2 1_{X_s < \delta}]ds$$

$$\leq 2\sigma^2 \int_0^t \sqrt{\mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^4]}\sqrt{\mathbb{P}(X_s < \delta)}ds$$

$$\leq 2\sigma^2 \max_{0\leq s\leq t} \sqrt{\mathbb{P}(X_s < \delta)} \int_0^t \sqrt{\mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^4]}ds. \quad \textbf{(A.58)}$$

Note that

$$\int_0^t \sqrt{\mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^4]}ds \leq \int_0^t \sqrt{4\mathbb{E}[\hat{S}_s^{4\beta} + X_s^{4\beta}]}ds, \quad \textbf{(A.59)}$$

and we can compute $\mathbb{E}[\hat{S}_s^{4\beta}]$ and $\mathbb{E}[X_s^{4\beta}]$ explicitly since $\hat{S}_t$ is a CEV process and X_t is a Brownian motion. It is therefore easy to check that $\int_0^T \sqrt{\mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^4]}ds = O(T)$. Furthermore,

$$2\sigma^2 \max_{0\leq s\leq t} \sqrt{\mathbb{P}(X_s < \delta)} = 2\sigma^2 \Phi\left(\frac{\delta - S_0}{\sigma S_0^\beta \sqrt{t}}\right), \quad \textbf{(A.60)}$$

where $\Phi(x) := (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-(y^2/2)} dy$. Hence, by Gronwall's inequality, we conclude that

$$\mathbb{E}[(\hat{S}_T - X_T)^2] = O(T^2). \quad \textbf{(A.61)}$$

Note that $\hat{S}_t - X_t$ is a martingale. By Doob's martingale inequality,

$$\mathbb{E}\left[\max_{0\le t\le T} |\hat{S}_t - X_t|\right] \le C\sqrt{\mathbb{E}[(\hat{S}_T - X_T)^2]} = O(T). \quad \textbf{(A.62)}$$

Therefore, we conclude that

$$\begin{aligned} C(T) &= \mathbb{E}\left[\left(\frac{1}{T}\int_0^T X_t dt - S_0\right)^+\right] + O(T) \\ &= \mathbb{E}\left[\left(\sigma S_0^\beta \frac{1}{T}\int_0^T W_t dt\right)^+\right] + O(T) \\ &= \sigma S_0^\beta \frac{\sqrt{T}}{\sqrt{3}} \mathbb{E}[Z 1_{Z>0}] + O(T), \end{aligned} \quad \textbf{(A.63)}$$

where $Z \sim N(0,1)$. Finally, we can compute that

$$\mathbb{E}[Z 1_{Z>0}] = \frac{1}{\sqrt{2\pi}} \int_0^\infty x e^{-(x^2/2)} dx = \frac{1}{\sqrt{2\pi}}. \quad \textbf{(A.64)}$$

Hence, we proved the desired result. ■

Proof of Proposition 3.4: We will define $\mathcal{I}_K(K, S_0)$ as the solution of the variational problem (25), obtained by replacing the inequality (26) with the equality constraint $\int_0^1 g(t)dt = K$. This is solved by considering the variational problem for the auxiliary functional

$$\Lambda[g] := \frac{1}{2\sigma^2} \int_0^1 \frac{(g'(t))^2}{g(t)^{2\beta}} dt - \lambda \left(\int_0^1 g(t)dt - K \right), \quad \textbf{(A.65)}$$

where λ is a Lagrange multiplier.

The solution of this variational problem satisfies the Euler-Lagrange equation

$$g''(t) = \beta \frac{[g'(t)]^2}{g(t)} - \lambda\sigma^2 (g(t))^{2\beta}, \quad \textbf{(A.66)}$$

with initial condition $g(0) = S_0$ and transversality condition $g'(1) = 0$.

This equation can be simplified by the change of variable

$$g(t) = S_0 (y(t))^{1/(1-\beta)}. \quad \textbf{(A.67)}$$

Expressed in terms of $y(t)$, the Euler–Lagrange Eq. (A.66) becomes

$$y''(t) = C(y(t))^{\beta/(1-\beta)}, \quad \textbf{(A.68)}$$

with $C := -\lambda\sigma^2(1-\beta)S_0^{2\beta-1}$. The solution $y(t)$ satisfies the initial condition $y(0) = 1$ and transversality condition $y'(1) = 0$. The rate function and the constraint $\int_0^1 g(t)dt = K$ take the

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form

$$\mathcal{I}_K(K,S_0)=\frac{S_0^{2-2\beta}}{2\sigma^2(1-\beta)^2}\int_0^1[y'(t)]^2dt,\quad \int_0^1(y(t))^{1/(1-\beta)}dt=\frac{K}{S_0}. \qquad \textbf{(A.69)}$$

The differential Eq. (A.68) is known as the Emden–Fowler equation. The exponent $\gamma := \beta/(1-\beta)$ satisfies $\gamma \geq 1$ for the cases considered here $\beta \in [1/2, 1)$. This equation can be reduced to a first order ordinary differential equation (ODE) by noting the conservation of the quantity

$$E := \frac{1}{2}[y'(t)]^2 - C(1-\beta)(y(t))^{\gamma+1}. \qquad \textbf{(A.70)}$$

Taking into account the boundary condition $y'(1)=0$ this allows us to express the first derivative as

$$[y'(t)]^2 = 2C(1-\beta)\left([y(t)]^{\gamma+1}-y_1^{\gamma+1}\right), \qquad \textbf{(A.71)}$$

where we denoted $y_1 := y(1)$.

We distinguish the two cases:

1. $C>0$. This case has $y'(t)<0$ and $y(1)<y(0)=1$, and corresponds to $K<S_0$.
2. $C<0$. This case has $y'(t)>0$ and $y(1)>y(0)=1$, and corresponds to $K>S_0$.

We consider the two cases separately.

Case 1. $C>0$. One first relation between y_1 and C follows from

$$1=\int_0^1 dt = \int_{y_1}^{y(0)}\frac{dy}{y'} = \frac{1}{\sqrt{2C(1-\beta)}}\int_{y_1}^1\frac{dy}{\sqrt{y^{\gamma+1}-y_1^{\gamma+1}}}. \qquad \textbf{(A.72)}$$

This relation can be used to eliminate C in terms of y_1 as $C=(1/2(1-\beta))[A^{(+)}(y_1)]^2$, where we defined the function

$$\begin{aligned}A^{(+)}(x) &:= \int_x^1\frac{dy}{\sqrt{y^{\gamma+1}-x^{\gamma+1}}}\\ &= \frac{2x}{\gamma+1}\frac{\sqrt{1-x^{\gamma+1}}}{x^{\gamma+1}}\,{}_2F_1\left(\frac{\gamma}{\gamma+1},\frac{1}{2};\frac{3}{2};1-\frac{1}{x^{\gamma+1}}\right),\quad 0<x\leq 1. \end{aligned}\qquad \textbf{(A.73)}$$

The integral constraint on $y(t)$ can be written equivalently using (A.71) as

$$\frac{K}{S_0}=\int_0^1[y(t)]^{\gamma+1}dt = y_1^{\gamma+1}+\frac{1}{2C(1-\beta)}\int_0^1[y'(t)]^2dy. \qquad \textbf{(A.74)}$$

The integral can be expressed by a change of variable as

$$\begin{aligned}\int_0^1 dy[y'(t)]^2 &= \int_{y(0)}^{y_1}y'dy = \sqrt{2C(1-\beta)}\int_{y_1}^1\sqrt{y^{\gamma+1}-y_1^{\gamma+1}}dy\\ &= A^{(+)}(y_1)B^{(+)}(y_1),\end{aligned} \qquad \textbf{(A.75)}$$

where we defined

$$\begin{aligned}B^{(+)}(x) &:= \int_x^1\sqrt{y^{\gamma+1}-x^{\gamma+1}}dy\\ &= \frac{2x}{3(\gamma+1)}\frac{(1-x^{\gamma+1})^{3/2}}{x^{\gamma+1}}\,{}_2F_1\left(\frac{\gamma}{\gamma+1},\frac{3}{2};\frac{5}{2};1-\frac{1}{x^{\gamma+1}}\right),\quad 0<x\leq 1.\end{aligned} \qquad \textbf{(A.76)}$$

The integral (A.75) is the same as the integral appearing in the expression for the rate function (A.69).

In conclusion, the rate function $\mathcal{I}_K(K, S_0)$ for $K < S_0$ is given by

$$\mathcal{I}_K(K, S_0) = \frac{S_0^{2(1-\beta)}}{2\sigma^2(1-\beta)^2} A^{(+)}(y_1) B^{(+)}(y_1), \tag{A.77}$$

where $y_1 < 1$ is the solution of the equation

$$\frac{K}{S_0} = y_1^{\gamma+1} + \frac{B^{(+)}(y_1)}{A^{(+)}(y_1)}. \tag{A.78}$$

These expressions can be simplified by defining $a^{(+)}(x), b^{(+)}(x)$ as $a^{(+)}(x) = (1/1-\beta)A^{(+)}(x^{1-\beta})$ and $b^{(+)}(x) = (1/1-\beta)B^{(+)}(x^{1-\beta})$.

Case 2. $C < 0$. It is similar to the $C > 0$ case and the proof is omitted here. ■

Proof of Proposition 3.5: The proof is similar to that of Proposition 9 in [50]. ■

Proof of Corollary 3.6:

(i) follows from Lemma 29 in [50]. The technical conditions of this Lemma require that $\mathcal{G}^{(-)}(\varphi)$ is an increasing function and that $[\mathcal{G}^{(-)}(\varphi)]^2$ has superlinear growth as $\varphi \to \infty$. The first condition is satisfied as the derivative of $\mathcal{G}^{(-)}(\varphi)$ is given in the equation:

$$\mathcal{F}^{(-)}(\varphi) = 2\frac{d}{d\varphi}\mathcal{G}^{(-)}(\varphi) = \frac{S_0^{1-\beta}}{\sigma}\int_1^{\varphi} \frac{dz}{z^\beta\sqrt{\varphi - z}} \tag{A.79}$$

which is a positive function.
The second technical condition is also satisfied, as follows. Using the asymptotics of the hypergeometric function

$${}_2F_1\left(\frac{3}{2}, \beta; \frac{5}{2}; 1 - \frac{1}{\varphi}\right) = \frac{\Gamma(5/2)\Gamma(1-\beta)}{\Gamma((5/2)-\beta)} + O(\varphi^{-1}), \quad \text{as } \varphi \to \infty \tag{A.80}$$

we get that

$$[\mathcal{G}^{(-)}(\varphi)]^2 \sim \frac{(\varphi-1)^3}{\varphi^{2\beta}}, \quad \text{as } \varphi \to \infty. \tag{A.81}$$

This has indeed superlinear growth provided that $\beta < 1$.

(ii) follows from Lemma 30 in [50]. This requires the following two technical conditions: $\mathcal{G}^{(+)}(\chi)$ is a decreasing function, and the infimum in (36) is not reached at the lower boundary $\chi = 0$. The first condition follows indeed from

$$\mathcal{F}^{(+)}(\chi) = -2\frac{d}{d\chi}\mathcal{G}^{(+)}(\chi) = \frac{S_0^{1-\beta}}{\sigma}\int_\chi^1 \frac{dz}{z^\beta\sqrt{z-\chi}} \tag{A.82}$$

as the integral in this expression is positive.

The second condition follows by noting that we have, for $\beta \geq 1/2$

$$\lim_{\chi \to 0} \frac{d}{d\chi} \left(\frac{(1/2)[\mathcal{G}^{(+)}(\chi)]^2}{(K/S_0) - \chi} \right) = -\infty. \tag{A.83}$$

This is obtained by writing the derivative explicitly

$$\frac{d}{d\chi} \left(\frac{(1/2)[\mathcal{G}^{(+)}(\chi)]^2}{(K/S_0) - \chi} \right) = \frac{\mathcal{G}^{(+)}(\chi)(d/\chi)\mathcal{G}^{(+)}(\chi)}{(K/S_0) - \chi} + \frac{1}{2} \frac{[\mathcal{G}^{(+)}(\chi)]^2}{((K/S_0) - \chi)^2}. \tag{A.84}$$

Furthermore, the functions appearing here have the $\chi \to 0$ limits, for $\beta \geq 1/2$,

$$\mathcal{G}^{(+)}(\chi) = 1 + O\left(\chi^{(3/2)-\beta}\right), \frac{d}{d\chi}\mathcal{G}^{(+)}(\chi) = -\infty \quad \text{as } \chi \to 0. \tag{A.85}$$

The first relation (A.85) follows from the $\chi \to 0$ asymptotics of the hypergeometric function, which can be extracted from Eq. (A.93)

$$_2F_1\left(\frac{3}{2}, \beta; \frac{5}{2}; 1 - \frac{1}{\chi}\right) = \frac{3}{3 - 2\beta}\chi^\beta + \frac{\Gamma(5/2)\Gamma(\beta - (3/2))}{\Gamma(\beta)}\chi^{3/2}. \tag{A.86}$$

The second relation (A.85) is obtained from (A.82) by noting that the integral on the RHS is bounded from below as

$$\int_\chi^1 \frac{dz}{z^\beta\sqrt{z - \chi}} \geq \int_\chi^1 dz z^{-(1/2)-\beta} = \frac{1}{(1/2) - \beta}(1 - \chi^{(1/2)-\beta}) \to +\infty, \quad \chi \to 0_+. \tag{A.87}$$

In the last step, we used $\beta > 1/2$. The conclusion holds also for $\beta = 1/2$, using the relation

$$\int_\chi^1 \frac{dz}{\sqrt{z(z - \chi)}} = 2\log(\sqrt{1 - \chi} + 1) - \log\chi \to \infty, \quad \chi \to 0_+. \tag{A.88}$$

This shows that the infimum in (36) is not reached at the lower boundary $\chi = 0$. This justifies the application of Lemma 30 in [50].

(iii) The conclusion follows immediately from the result for the rate function $\mathcal{I}(K, S_0)$ given by Theorem 3.2 and the monotonicity properties of $\mathcal{I}_K(K, S_0)$ proven above in (i) and (ii).

■

Proof of Proposition 3.9 (Large strike asymptotics): For this case we are interested in the $x \to \infty$ asymptotics of the functions $a^{(-)}(x), b^{(-)}(x)$. For this purpose it is useful to transform the argument $z = 1 - (1/x)$ of the hypergeometric functions appearing in the expressions of these functions as $z \to 1 - z = (1/x)$ using the identity 15.3.6 in Abramowitz and Stegun [1].

We get, for $\beta \in [1/2, 1)$,

$$_2F_1\left(\beta, \frac{1}{2}; \frac{3}{2}; 1 - \frac{1}{x}\right) = \frac{\Gamma(3/2)\Gamma(1-\beta)}{\Gamma((3/2) - \beta)} + O(x^{\beta-1}), \tag{A.89}$$

$$_2F_1\left(\beta, \frac{3}{2}; \frac{5}{2}; 1 - \frac{1}{x}\right) = \frac{\Gamma(5/2)\Gamma(1-\beta)}{\Gamma((5/2) - \beta)} + O(x^{\beta-1}), \tag{A.90}$$

as $x \to \infty$.

The solution of Eq. (31) for x for $K/S_0 \gg 1$ is

$$x = \frac{3 - 2\beta}{2(1-\beta)}\left(\frac{K}{S_0}\right) + O(K/S_0). \tag{A.91}$$

Substituting x into the expression for the rate function of Proposition 3.4 we obtain the large-strike asymptotics of $\mathcal{I}(K, S_0)$ given in Proposition 3.9. ■

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Proof of Proposition 3.10 (Small strike asymptotics): We require the $x \to 0_+$ asymptotics for $a^{(+)}(x), b^{(+)}(x)$. This is obtained by changing the $z = 1 - (1/x)$ argument of the hypergeometric functions appearing in the expressions for these functions as $z \to (1/z - 1) = -x$, using the identity 15.3.8 in Abramowitz and Stegun [1].

We get

$$
\begin{aligned}
{}_2F_1\left(\beta, \frac{1}{2}; \frac{3}{2}; 1 - \frac{1}{x}\right) &= x^{\beta} \frac{\Gamma(3/2)\Gamma(1/2-\beta)}{\Gamma(1/2)\Gamma(3/2-\beta)}(1 + O(x)) + x^{1/2} \frac{\Gamma(3/2)\Gamma(\beta - 1/2)}{\Gamma(\beta)}(1 + O(x)) \\
&= x^{\beta} \frac{1}{1-2\beta}(1 + O(x)) + x^{1/2} \frac{\Gamma(3/2)\Gamma(\beta - 1/2)}{\Gamma(\beta)}(1 + O(x)), \qquad \textbf{(A.92)}
\end{aligned}
$$

and

$$
\begin{aligned}
{}_2F_1\left(\beta, \frac{3}{2}; \frac{5}{2}; 1 - \frac{1}{x}\right) &= x^{\beta} \frac{\Gamma(5/2)\Gamma(3/2-\beta)}{\Gamma(3/2)\Gamma(5/2-\beta)}(1 + O(x)) + x^{3/2} \frac{\Gamma(5/2)\Gamma(\beta - 3/2)}{\Gamma(\beta)}(1 + O(x)) \\
&= x^{\beta} \frac{3}{3-2\beta}(1 + O(x)) + x^{3/2} \frac{\Gamma(5/2)\Gamma(\beta - 3/2)}{\Gamma(\beta)}(1 + O(x)). \qquad \textbf{(A.93)}
\end{aligned}
$$

For $1/2 < \beta < 1$, the dominant term in these expansions as $x \to 0_+$ is the second term in (A.92), and the first term in (A.93).

The equation for x as $K \to 0$ becomes approximately

$$
\frac{K}{S_0} = x^{\beta-(1/2)} \frac{\Gamma(\beta)}{\sqrt{\pi}((3/2) - \beta)\Gamma(\beta - (1/2))} + O(x). \qquad \textbf{(A.94)}
$$

Substituting x into the expression for the rate function of Proposition 3.4 we obtain the small-strike asymptotics of $\mathcal{I}(K, S_0)$ given in Proposition 3.10. ■

Appendix A.4. Proof of the Results in Section 4

Proof of Theorem 4.1: For any $\theta \in \mathbb{R}$, $\mathbb{E}[e^{(\theta/T^2)\int_0^t S_s ds - (\theta\kappa/T)S_T} | S_0] = e^{A(T;(\theta/T^2), -(\theta\kappa/T))S_0}$, where $A(t;\theta;\phi)$ satisfies the ODE:

$$
A'(t;\theta,\phi) = (r-q)A(t;\theta,\phi) + \frac{1}{2}\sigma^2 A(t;\theta,\phi)^2 + \theta, \qquad \textbf{(A.95)}
$$

with $A(0;\theta,\phi) = \phi$.

For $\theta > 0$,

$$
\begin{aligned}
A(t;\theta,\phi) = {} & \frac{\sqrt{2\sigma^2\theta - (r-q)^2}}{\sigma^2} \tan\left[\frac{\sqrt{2\sigma^2\theta - (r-q)^2}}{2} t + \tan^{-1}\left(\frac{r - q + \sigma^2\phi}{\sqrt{2\sigma^2\theta - (r-q)^2}}\right)\right] \\
& - \frac{r-q}{\sigma^2}, \qquad \textbf{(A.96)}
\end{aligned}
$$

and for $\theta < 0$,

$$
\begin{aligned}
A(t;\theta,\phi) = {} & \frac{(\frac{r-q}{\sigma^2} - \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + \phi)(\frac{r-q}{\sigma^2} + \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}})e^{t\sqrt{(r-q)^2 - 2\theta\sigma^2}}}{(\frac{r-q}{\sigma^2} + \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + \phi) - e^{t\sqrt{(r-q)^2 - 2\theta\sigma^2}}(\frac{r-q}{\sigma^2} - \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + \phi)} \\
& - \frac{(\frac{r-q}{\sigma^2} - \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}})(\frac{r-q}{\sigma^2} + \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + \phi)}{(\frac{r-q}{\sigma^2} + \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + \phi) - e^{t\sqrt{(r-q)^2 - 2\theta\sigma^2}}(\frac{r-q}{\sigma^2} - \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + \phi)}. \qquad \textbf{(A.97)}
\end{aligned}
$$

For $0 \le \theta < \theta_c$,

$$\lim_{T\to 0} TA\left(T; \frac{\theta}{T^2}, \frac{-\kappa\theta}{T}\right) = \sqrt{\frac{2\theta}{\sigma^2}} \tan\left(\sqrt{\frac{\sigma^2\theta}{2}} + \tan^{-1}\left(-\sigma\kappa\sqrt{\frac{\theta}{2}}\right)\right), \tag{A.98}$$

and this limit is ∞ if $\theta \ge \theta_c$, where θ_c is the unique positive solution to the equation:

$$\sqrt{\frac{\sigma^2\theta_c}{2}} + \tan^{-1}\left(-\sigma\kappa\sqrt{\frac{\theta_c}{2}}\right) = \frac{\pi}{2}. \tag{A.99}$$

To see that (A.99) has a unique positive solution, let us define:

$$F(x) := \sqrt{\frac{\sigma^2}{2}}x + \tan^{-1}\left(-\sigma\kappa\frac{1}{\sqrt{2}}x\right) - \frac{\pi}{2}. \tag{A.100}$$

Then, $F(0) = -\pi/2$ and $F(\infty) = \infty$. On the other hand, we can compute that

$$F'(x) = \sqrt{\frac{\sigma^2}{2}} - \frac{\sigma\kappa}{\sqrt{2}}\frac{1}{(1/2)\sigma^2\kappa^2x^2 + 1}, \quad F''(x) = \frac{\sigma\kappa}{\sqrt{2}}\frac{\sigma^2\kappa^2 x}{((1/2)\sigma^2\kappa^2x^2+1)^2}. \tag{A.101}$$

Since $F''(x) > 0$ for any $x > 0$, and $F(0) = -\pi/2 < 0$ and $F(\infty) = \infty$, it follows that $F(x) = 0$ has a unique positive solution.

For $\theta < 0$,

$$\begin{aligned}\lim_{T\to 0} TA\left(T; \frac{\theta}{T^2}, \frac{-\kappa\theta}{T}\right) &= -\frac{\sqrt{-2\theta}}{\sigma}\frac{(\sqrt{-2\theta}/\sigma)(e^{\sigma\sqrt{-2\theta}} - 1) + \theta\kappa(1 + e^{\sigma\sqrt{-2\theta}})}{(\sqrt{-2\theta}/\sigma)(1 + e^{\sigma\sqrt{-2\theta}}) - \theta\kappa(1 - e^{\sigma\sqrt{-2\theta}})} \\ &= -\frac{\sqrt{-2\theta}}{\sigma}\frac{((\sqrt{-2\theta}/\sigma) + \theta\kappa/(\sqrt{-2\theta}/\sigma) - \theta\kappa e^{\sigma\sqrt{-2\theta}} - 1}{((\sqrt{-2\theta}/\sigma) + \theta\kappa/(\sqrt{-2\theta}/\sigma) - \theta\kappa)e^{\sigma\sqrt{-2\theta}} + 1} \\ &= -\frac{\sqrt{-2\theta}}{\sigma}\tanh\left(\frac{\sigma}{2}\sqrt{-2\theta} + \tan h^{-1}\left(-\sigma\kappa\sqrt{\frac{-\theta}{2}}\right)\right).\end{aligned} \tag{A.102}$$

Therefore,

$$\Lambda(\theta) := \lim_{T\to 0} T\log\mathbb{E}\left[e^{(\theta/T^2)\int_0^T S_t dt - (\theta/T)\kappa S_T}\right]$$

$$= \begin{cases} \dfrac{\sqrt{2\theta}}{\sigma}\tan\left(\dfrac{\sigma}{2}\sqrt{2\theta} + \tan^{-1}\left(-\sigma\kappa\sqrt{\dfrac{\theta}{2}}\right)\right)S_0, & \text{if } 0 \le \theta < \theta_c \\ -\dfrac{\sqrt{-2\theta}}{\sigma}\tanh\left(\dfrac{\sigma}{2}\sqrt{-2\theta} + \tan h^{-1}\left(-\sigma\kappa\sqrt{\dfrac{-\theta}{2}}\right)\right)S_0, & \text{if } \theta \le 0 \\ +\infty, & \text{otherwise} \end{cases}. \tag{A.103}$$

It is easy to show that $\Lambda_f(\theta)$ is differentiable for any $\theta < \theta_c$ and $\Lambda_f'(\theta) \to \infty$ as $\theta \uparrow \theta_c$. Hence, $\mathbb{P}(\frac{1}{T}\int_0^T S_t dt - \kappa S_T \in \cdot)$ satisfies a large deviation principle with the rate function $\mathcal{I}_f(\kappa, S_0)$ given in (49) by applying the Gärtner–Ellis theorem, see Theorem A.3 in Appendix A. ■