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IMPLICIT EQUILIBRIUM DYNAMICS

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We discuss the problem known in economics as backward dynamics occurring in models of perfect foresight, intertemporal equilibrium described mathematically by *implicit* difference equations. In a previously published paper [*Journal of Economic Dynamics and Control* 31 (2007), 1633–1671], we showed that by means of certain mathematical methods and results known as inverse limits theory it is possible to establish a correspondence between the backward dynamics of a noninvertible map and the forward dynamics of a related invertible map acting on an appropriately defined space of sequences, each of whose elements corresponds to an intertemporal equilibrium. We also proved the existence of different types of topological attractors for one-dimensional models of overlapping generations. In this paper, we provide an extension of those results, constructing a Lebesgue-like probability measure on spaces of infinite sequences that allows us to distinguish typical from exceptional dynamical behaviors in a measure–theoretical sense, thus proving that all the topological attractors considered in MR07 are also metric attractors. We incidentally also prove that the existence of chaos (in the Devaney–Touhey sense) backward in time implies (and is implied by) chaos forward in time.

Keywords: Backward Dynamics, Overlapping Generations, Inverse Limits, Attractors, Lebesgue-Like Measure

1. INTRODUCTION AND MOTIVATION

Dynamical system theory (DST) has been extensively and successfully applied to the analysis of economic problems. In this area of research, however, the use of mathematical notions and results is not always straightforward and many delicate questions of economic interpretation arise. To illustrate this point, let us consider two basic types of (discrete-time) models in economic dynamics: optimal growth and intertemporal economic equilibrium (IEE) models. Under generally assumed conditions, optimal growth models can, at least in principle, be represented mathematically by systems of difference equations with the canonical

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form

$$x_{t+1} = F(x_t), \quad \textbf{(1)}$$

where the map F is derived from the solution of a constrained intertemporal optimization problem. Iterations of F generate sequences of optimal values of the state variable x, e.g., the capital stock, *forward in time* from any arbitrarily given initial conditions to an arbitrarily distant future. The methods of DST can therefore be employed to discuss the properties of the orbits of system (1), such as existence and stability of stationary states, existence of other interesting invariant sets and their interpretation, and dependence of dynamical solutions on the system parameters.

The situation is more complicated for IEE models characterized by an infinite horizon and an infinity of agents and commodities, the most popular family of which is given by the overlapping generations (OLG) models. Assuming perfect foresight, here the typical mathematical formulation is a system of *implicit* difference equations such as

$$H(x_t, x_{t+1}) = 0, \quad \textbf{(2)}$$

where x is again a vector of state variables and the function H depends on the economic fundamentals, typically utility or production functions. It is sometimes (but not generally) possible to invert H with respect to x_{t+1} *globally*, and therefore translate (2) into an explicit difference equation such as (1). Unfortunately, in economic models, this procedure may not be possible, nor are there any *a priori* justifications for assuming that it is.

When a map such as F cannot be properly defined, it may still be possible to invert H with respect to x_t globally and write a "backward dynamical system" such as

$$x_t = G(x_{t+1}). \quad \textbf{(3)}$$

In this case, for each given state x_t at some time t, instead of a function F determining the next state x_{t+1}, we have a noninvertible function G that determines the previous state x_{t-1}. Thus, the model does not actually predict the future state from the present, but gives a (possibly empty) set $G^{-1}(x_t)$ of admissible future states. In what follows, we will label maps such as G "BD maps."

The possibility that the mathematical representation of IEEs may yield backward dynamical systems is mentioned in early discussions of OLG models [see, for example, Gale (1973), Benhabib and Day (1982)] but, on the whole, it has been given surprisingly little attention in economic literature. In the presence of BD maps, the approach most commonly adopted is first to locate a steady state solution (a fixed point of the relevant map), next to invoke the implicit function theorem and invert the map around the fixed point, and finally to perform some local analysis of the system [see, for example, Gale (1973, 24–25) and Grandmont (1989, 51–52)]. Although this strategy has produced a number of useful results, it restricts the investigation severely, implicitly leaving out many, possibly most orbits moving forward in time and compatible with the assumed dynamical rules, i.e., disregarding many interesting cases of IEE.

An obvious feature of systems characterized by BD maps such as (3) is that *there is no one–to–one correspondence between initial conditions (points in the state space) and orbits forward in time.* On the contrary, there may be infinitely many such orbits starting from the same point of the state space and converging forward in time to many different sets. Consequently, in order to provide a complete characterization of those orbits and the corresponding IEEs, we need to construct a "larger" space, each of whose elements corresponds to a unique forward orbit. We must also define a map acting on that space such that its iterations can be interpreted as the dynamics *forward in time* for the original problem. There exists an area of research in mathematics called *inverse limits theory* (ILT) that provides a natural framework for this problem. In the present context, the basic idea behind ILT is straightforward: we move from the original state space, typically a subset of $\mathbf{R}^n$, to a more complex space generated by the solutions of the implicit difference equations of the model. In so doing, we transform a noninvertible BD map, such as G of equation (3), into an *invertible* map, which is sometimes called the "natural extension" of the original map G.

Inverse limit theory is a body of mathematical notions and methods developed in the last seventy years or so, but relatively unknown in economics. Medio (1998) provides a preliminary analysis of some of the problems discussed in this paper, in terms of a "natural extension" of noninvertible maps. A more thorough and rigorous investigation of backward dynamical systems in economics is provided by Medio and Raines (2006, 2007). (The latter paper, of which the present one is an extension, will be labeled henceforth "MR07.") A few other notable recent works applying inverse limit spaces to the problem of backward dynamics in economics include Kennedy, Stockman and Yorke (2007, 2008) and Kennedy and Stockman (2008). A different approach to the problem of backward dynamics can be found in Gardini, Hommes, Tramontana and de Vilder (GHTV, 2009), where forward equilibria are defined as sequences where at each step equilibrium selection is determined by a random sunspot sequence. They then apply the method of iterated function schemes (IFS) to a one- and a two-dimensional version of the OLGs model and show that, if the backward dynamics of such a model are chaotic and have a homoclinic orbit, there exists an appropriately restricted IFS whose orbits converge to a fractal attractor. GHTV's approach to the problem of backward dynamics is different from the one adopted in the present paper, both economically and mathematically. Specifically, in MR07 and here we are concerned with perfect foresight, deterministic equilibrium dynamics, whereas GHTV's main result concerns stochastic (sunspot) equilibria.[1]

In order not to overburden our presentation and to avoid useless repetitions, in what follows we will limit our discussion to those aspects of ILT strictly necessary for the present application. For a more general discussion of ILT and its relevance to economic theory, we refer the reader to our MR07 (specifically to its technical Appendix A) and the extended bibliography therein.

The basic ideas and methods presented and applied in this paper, as well as some general results proved in Section 3, are relevant to a wide class of implicitly defined discrete-time dynamical systems arising in economics. However, the main focus of the paper is on one-dimensional OLG models. For those models, we define a small number of basic cases mathematically and economically and, for each of them, we distinguish between "typical or generic" and "exceptional" dynamical behavior. The rest of the paper is organized as follows. In Section 2 we describe a basic OLG model used as a benchmark in the rest of the paper. Section 3 provides an introductory discussion of the basic notions and methods of ILT used in the sequel. In Section 4, we argue that the (metric) attractors of a certain homeomorphism derived from the original, backward-moving map of the model can be used to identify the "typical or generic" forward-in-time orbits implicitly defined by the model. Section 5 deals with the class of OLG models represented by unimodal maps and defines three basic subclasses. In Section 6 we discuss the existence of metric attractors for a special OLG model. In Section 7, we construct a Lebesgue-like probability measure on the space of intertemporal equilibria, and in Section 8, we use this measure to prove the existence of metric attractors for the main cases of OLG models with backward dynamics. Section 9 sums up the paper.

2. THE BASIC MODEL

We begin our discussion by showing how backward dynamics may arise in a basic, one-dimensional OLG model.

For this purpose, we have chosen a slightly modified version of the "leisure–consumption" model used by Grandmont in his much-quoted (1985) analysis of endogenous business cycles [see also Grandmont (1983, 1989)]. Because the model is exceedingly well known and its use in this paper is only instrumental, we limit its presentation to what is necessary to the understanding of our main argument, omitting many technical details.[2]

Basic hypotheses and notation of the model are as follows:

H1. Demography. A constant population of individuals (identical except for their age), living two periods of time, and divided at each period into two equally numerous classes, respectively labeled "young" and "old."

H2. Consumption. At each period t, the young agent consumes a quantity $c_t \geq 0$ of the unique perishable good, and a quantity of leisure $(\bar{l} - l_t) \geq 0$, where $\bar{l} \geq 0$ denotes the constant labor endowment and $l_t \geq 0$ is labor supply. The corresponding quantities for the old agent are $\kappa_t \geq 0$, $\bar{\iota} \geq 0$, and $\iota_t \geq 0$.

H3. Production. Production takes place by means of current labor only, output is traded at a price p_t, and the wage rate is w_t. Physical units of measure of output and labor are normalized so that one unit of labor yields one unit of output. In this case, profit maximization implies that $w_t = p_t\ \forall t$.

H4. Preferences. For each generation living through the periods $(t, t+1)$ preferences are defined by the following utility functions:

$$v[c_t, (\bar{l} - l_t)];\ u[\kappa_{t+1}, (\bar{\iota} - \iota_{t+1})], \tag{4}$$

where v and u are smooth, strictly increasing in each argument, and concave.

H5. Perfect Foresight. For each t, the young agent's expectations concerning the values of the variables at $t+1$ are perfectly fulfilled.

H6. Maximization. At each t, the young agent chooses the present and future levels of consumption and labor supply as functions of the observed current price (= wage rate) and the perfectly anticipated future price p_{t+1} (= perfectly anticipated wage rate), subject to a two-period budget constraint.

H7. Market Clearing Condition. At each t, supply $(l_t + \iota_t)$ of and demand $(c_t + \kappa_t)$ for the consumption good are equal.

The problem can be described in terms of the young agent's first-period excess demand (= dissaving), $z_t = c_t - l_t$, and the *same agent's* second-period excess demand, $\zeta_{t+1} = \kappa_{t+1} - \iota_{t+1}$. Thus, the market-clearing condition reduces to $z_t = -\zeta_t\ \forall t$. The maximizing problem can be represented formally, as follows:

$$\begin{array}{ll} \max\{V(z_t) + U(\zeta_{t+1})\} & \\ \text{s.t. } p_t z_t + p_{t+1}\zeta_{t+1} \geq 0 & \text{budget constraint} \\ z_t \geq -\bar{l},\ \zeta_t \geq -\bar{\iota} & (\text{equivalently, } z_t \leq \bar{\iota};\ \zeta_t \leq \bar{l}), \end{array} \tag{P1}$$

where the functions V and U are derived, respectively, from the basic utility functions v and u of (4), from which they inherit the fundamental properties.[3]

From the first-order conditions of (P1), we deduce that, for each given pair of labor endowments $(\bar{l}, \bar{\iota})$, the optimal current and future consumption and labor supply must satisfy the equation

$$H(z_t, \zeta_{t+1}) = \mathcal{V}(z_t) + \mathcal{U}(\zeta_{t+1}) = 0, \tag{5}$$

where $\mathcal{V}(z_t) = V'(z_t)z_t$ and $\mathcal{U}(\zeta_{t+1}) = U'(\zeta_{t+1})\zeta_{t+1}$.

Using the market-clearing requirement, we can transform (5) into an *implicit* difference equation such as (2) in a single variable (z or, equivalently, ζ). Whether we can also obtain an *explicit* discrete-time dynamical system, moving forward or backward in time, depends on the properties of the functions $\mathcal{V}$ and $\mathcal{U}$ and the endowments.

There are two basic cases, depending on the (derived) utility functions U, V and the labor endowments.[4]

(i) *Classical case*: The young agent is "impatient" and wants to consume more and/or work less, borrowing from the old agent in the first period and paying back to next generation's young agent in the second period.

(ii) *Samuelson case*: The young agent is thrifty and saves in the first period, lending to the old agent, in order to be able to consume more and/or work less in old age.

From the assumed properties of the utility functions, it follows that, in the classical case, the function $\mathcal{U}$ is monotone and therefore we can invert the implicit function H with respect to ζ_{t+1} and, using the market-clearing condition, obtain the equation

$$z_{t+1} = F(z_t), \tag{6}$$

where $z_t \in [0, \bar{\iota})$ and $F(z_t) = -\mathcal{U}^{-1}[-\mathcal{V}(z_t)]$, whose iterations move forward in time. In the classical case, the function F may or may not be invertible. If it is not, the dynamics of (6) may be very complicated, as shown in Benhabib and Day's (1982) pioneering investigation of endogenous cycles and chaos in OLG models.

Vice versa, in the Samuelson case, for which $z_t = -\zeta_t < 0 \; \forall t$, the implicit function H can be inverted globally with respect to z_{t+1}, obtaining the equation

$$\zeta_t = G(\zeta_{t+1}), \tag{7}$$

where $\zeta_t \in [0, \bar{l})$ and $G(\zeta_{t+1}) = -\mathcal{V}^{-1}[-\mathcal{U}(\zeta_{t+1})]$, whose iterations move *backward* in time. With a slight abuse of wording, henceforth we refer to the maps F and G as "offer curves."

If the agent's second-period utility function U is such that the risk aversion $R_U(\zeta) = -U''(\zeta)\zeta/U'(\zeta) < 1$ (substitution effect prevails) for small values of ζ and the opposite is true for larger values, the offer curve G is noninvertible. This is the case discussed by Grandmont (1985) and it gives rise to the problem of backward dynamics on which we focus in this paper.

3. INVERSE LIMIT SPACE AND ADMISSIBLE ORBITS

As we shall see in what follows, BD maps such as (3) and (7) are commonly characterized by the fact that their implicitly defined forward-in-time dynamics includes many, even uncountably many different types of dynamical behavior (e.g., periodic dynamics of many different periods or chaotic dynamics). In this case, we would like to have rigorous criteria for identifying the behaviors that are "typical, or generic" and therefore likely to be observed, and those that are "exceptional" and therefore negligible.

In this section, we argue that this problem can be thoroughly investigated by characterizing the set of forward admissible orbits (i.e., the set of all IEEs) as an inverse limit space and applying to it certain powerful results of inverse limit theory.

Consider the Samuelson OLG model described by equation (7) and assume that G is not globally invertible. We will start with the following definition:

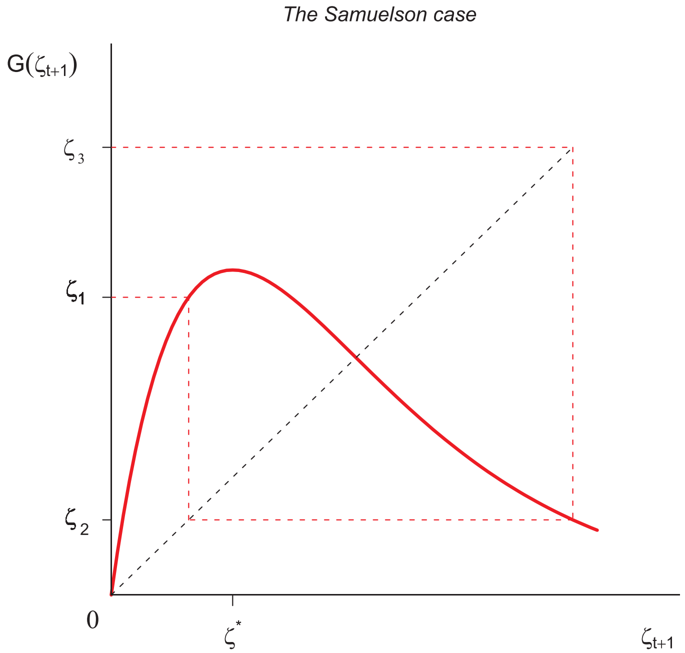

FIGURE 1. A noninvertible offer curve.

DEFINITION 1. *An infinite sequence* $\{\zeta_t\}$ *generated by a BD difference equation such as (7) is said to be* forward admissible *if for each pair* $(t, t+1)$, $\zeta_{t+1} \in G^{-1}(\zeta_t)$, *and for all* $t \in \mathbf{N}$, $0 \leq \zeta_t \leq \bar{l}$.

The admissibility of sequences $\{z_t\}$ can be defined similarly. In economic terms, Definition 1 restricts the sequences of excess demand to those that satisfy the requirement of intertemporal maximization under constraint and market equilibrium; i.e., infinite, forward admissible sequences correspond to intertemporal, perfect foresight competitive equilibria (IEE). Notice that the logic of the model requires that admissible sequences be *infinite.* To see this suppose that, at time T, no value ζ_{T+1} exists such that $\zeta_{T+1} = G^{-1}(\zeta_T)$; i.e., the set $G^{-1}(\zeta_T)$ is empty. Economically, that means that ζ_T will not be realized, since the young agent at time T will not decide to save an amount $-z_T$, because this would only be justified by the (perfectly foreseen) expectation of a positive excess demand in his/her old age equal to $\zeta_{T+1} \in G^{-1}(\zeta_T)$. But if $\zeta_T = -z_T$ are not realized, by the same token neither will be $\zeta_{T-1} = -z_{T-1}$ and so on and so on, back in time all the way to the initial value. In short, no finite sequence can satisfy the requirements of IEE.

Figure 1, depicting a Samuelson OLG model with a noninvertible, unimodal offer curve G, shows an example of an interrupted, nonadmissible forward in time sequence.

As the diagram shows, the set $G^{-1}(\zeta)$ is empty for all $\zeta > \zeta_{\text{MAX}} = G(\zeta^*)$. Thus, for example, no sequence including the subsequence $\zeta_1, \zeta_2, \zeta_3$ can be continued, and therefore it cannot be admissible.

There is a simple way to include only admissible sequences in our construction. Let $X \subset \mathbf{R}^+$ be the domain of G, and let $I = \bigcap_{n\geq 0} G^n(X)$. In our case, it is easy to verify that, if ζ^* is the "critical value" of ζ for which $G'(\zeta) = 0$ and $\zeta_{\text{MAX}} = G(\zeta^*)$, then $I = [0, \zeta_{\text{MAX}}]$ is G-invariant and the restriction $G|_I$ is a surjection, so that, for $\zeta \in I$, $G^{-1}(\zeta)$ is never empty. Clearly, if G is surjective on X to begin with, then $I = X$.

The next step is to provide a proper characterization of the space of all forward admissible sequences. For this purpose, we will make use of ILT. Because we discussed it in great detail in our above-quoted article MR07, and in particular its Appendix A, we will describe here only the essential parts. For a more exhaustive discussion we refer the reader to that article and the references therein.

Consider a sequence $X_1, X_2, \ldots$ of metric spaces (called *factor spaces*) and a sequence $f_1, f_2, \ldots$ of continuous functions (called *bonding maps*) such that, for each $i \in \mathbf{N}$, $f_i : X_{i+1} \to X_i$. The double sequence $\{X_i, f_i\}$ is called an *inverse sequence*.[5] The subset of the product space $\Pi_{i=1}^{\infty} X_i$ to which the point $(x_1, x_2, \ldots)$ belongs if and only if $f_i(x_{i+1}) = x_i \ \forall i \in \mathbf{N}$ is called the *inverse limit* of the inverse sequence $\{X_i, f_i\}$, and is denoted by $\varprojlim\{X_i, f_i\}$. If the factor spaces are metric spaces, so is the inverse limit space derived from them. More specifically, if d_i is a metric on X_i bounded by 1, we can define an induced metric $\hat{d}$ on $\varprojlim\{X_i, f_i\}$ as follows:

$$\hat{d}(\hat{x}, \hat{y}) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^{i-1}}, \tag{8}$$

where $\hat{x} = (x_1, x_2, x_3, \ldots)$ and $\hat{y} = (y_1, y_2, y_3, \ldots)$. Thus, on that space, certain topological notions (such as closed, open, dense set), which we need in our investigation, are well defined.

In the case in which there is a single factor space X and a single bonding map $f : X \to X$, the inverse limit space is simply denoted as $\varprojlim\{X, f\}$. We are mostly concerned with this case, to which we refer as the "simple" inverse limit space.

Unless we state the contrary, in what follows we assume that the bonding map is a surjection on X, or that its domain is restricted to the subset of $X \supset X' = \bigcap_{i\geq 0} f^i(X)$ on which f is a surjection. When the bonding map f is backward-moving, the corresponding forward-in-time dynamics can be described by a map acting on the inverse limit space, as follows:

$$\begin{aligned} &\sigma : \varprojlim\{X, f\} \to \varprojlim\{X, f\}, \\ &\sigma(x_1, x_2, \ldots) = (x_2, x_3, \ldots). \end{aligned} \tag{9}$$

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Notice that the map σ is invertible, with inverse $\sigma^{-1}(x_2, x_3, \ldots) = (f(x_2), f(x_3), \ldots) = (x_1, x_2, \ldots)$. (As a matter of fact, the maps σ/σ^{-1} are homeomorphisms.)

In the following discussion and in the proofs of our results, we often need to use the projection function $\pi_i,\ i = 1, 2, \ldots$[6]:

$$\pi_i \colon \varprojlim\{X, f\} \to X,$$

$$\pi_i(x_1, x_2, \ldots) = x_i.$$

For simplicity's sake, in what follows we put $\pi_1 = \pi$.

We can now draw some interesting conclusions, which we state here for a "simple" inverse limit space, but which could easily be generalized:

- The inverse limit space $\varprojlim\{X, f\}$ is precisely *the space of all the forward-in-time sequences* $\hat{x} = (x_1, x_2, \ldots)$*, implicitly generated by* f*, starting from initial points in* X.
- The map σ acting on $\varprojlim\{X, f\}$ is a (one-sided) shift map that moves a sequence one step to the left and drops the first term. Because, in the present case, f is a BD map, σ is forward-moving in time, in the sense that it moves a sequence one step to the future, discarding its "oldest" element.[7]
- Although the dynamical system defined by σ on the corresponding sequence space cannot be interpreted as a mathematical idealization of any real economic mechanism, as we shall see, the orbit structure of σ reveals many interesting properties of the forward-moving orbits implicitly defined by the bonding map f.

The next step in our discussion is to exploit the knowledge of the (backward) dynamics of f to obtain interesting information on the (forward) dynamics of σ. For this purpose, we shall recall a number of interesting results available in the mathematical literature. They are usually expressed in terms of the relations between the bonding map f and its "induced homeomorphism," which, in our notation, is the map σ^{-1}. However, because the maps σ/σ^{-1} are homeomorphisms, all the results that interest us here can be readily translated into analogous relations between f and σ.

To rigorously classify the dynamical properties of a map, we focus on subsets of the state space that are persistent and dynamically indecomposabile (i.e., they must be studied as a whole). Here is a basic result:

LEMMA 1 [MR07, Lemma 1]. Let $\hat{A} \subseteq \varprojlim\{X, f\}$. Then $\hat{A}$ is closed and σ-invariant [i.e., $\sigma(\hat{A}) = \hat{A}$] if, and only if, $\hat{A} = \varprojlim\{A, f|_A\}$, with $A = \pi(\hat{A})$ and A is f-invariant [i.e., $f(A) = A$].

Indecomposability of a set is commonly characterized by the notion of "topological transitivity." Let $f : X \to X$ be a continuous map of a metric space. There exist two competing definitions of t.t., namely:

DEFINITION 2a. *The map* f *is* topologically transitive (t.t.) *on* A *provided that whenever* U *and* V *are open nonempty subsets of* A *there is an integer* n *such that* $f^n(U) \cap V \neq \emptyset$.

DEFINITION 2b. *The map* f *is t.t. on* A *if there exists a point* $x \in A$ *such that the orbit of* x *under* f *is dense in* A.

The two definitions are equivalent if A is a compact metric space with no isolated points, which is the case we consider here.

LEMMA 2. *The map* f *is topologically transitive on* A *if, and only if, the shift map* σ *is topologically transitive on* $\hat{A} = \underset{\leftarrow}{\lim}\{A, f\}$.

Proof. Li (1992, Theorem C) proved the result for the inverse of σ, σ^{-1}. Because σ/σ^{-1} are homeomorphisms, the extension to σ is immediate.

In the following pages, we will discuss two main types of dynamics, namely simple (periodic) and complex (chaotic). For periodic dynamics, we have the following, entirely intuitive result, which we state without proof:

LEMMA 3. *Let* $A = \{x^1, x^2, \ldots, x^n\}$ *denote a periodic orbit of period* $n \geq 1$ *with* $f(x^1) = x^2, f(x^2) = x^3, \ldots, f(x^n) = x^1$. *Then the set* $\hat{A} = \underset{\leftarrow}{\lim}\{A, f|_A\}$ *is periodic under* σ *with the same period and* $\pi(\hat{A}) = A$[8].

When applied to a BD map such as (7), Lemma 3 simply says that the existence of periodic dynamics backward in time for the map G is a necessary and sufficient condition for the existence of a periodic IEE of the same period.

In the case of chaotic dynamics, the analogous equivalence result is less obvious and, before stating it, we need some definitions and preliminary propositions.

DEFINITION 3. *A continuous map* f *on a metric space* X *is said to be chaotic on* X *if, whenever* U *and* V *are open, nonempty subsets of* X, *there exist a periodic point* $p \in U$ *and a positive integer* k *such that* $f^k(p) \in V$, *that is, every pair of open nonempty sets shares a periodic orbit.*[9]

An immediate consequence of this definition is the following corollary, which we will use in a moment.

COROLLARY 1. *Let* f *be a homeomorphism of a metric space* X. *Then* f *is chaotic on* X *if, and only if, its inverse* f^{-1} *is chaotic on* X.

The proof is straightforward: it is sufficient to interchange the two open sets U and V of Definition 3.

We can now state the following:

THEOREM 1. *Let* A *be a metric compact set. A continuous map* f *on* A *is chaotic if, and only if, the map* σ *is chaotic on the inverse limit space* $\underset{\leftarrow}{\lim}\{A, f|_A\}$.

Proof. Our result is a straightforward consequence of Li (1992, Theorem C), who proved this statement for the inverse of σ, σ^{-1}, using Devaney's definition of chaos. Given the equivalence between Devaney's and Touhey's definitions of chaos and Corollary 1, the required result follows immediately. See also Kennedy and Stockman (2008), where a similar result is proved.

When applied to an OLG model characterized by a BD map such as (7), Theorem 1 states that the existence of chaotic dynamics for the backward-moving map G is a necessary and sufficient condition for the existence of chaotic dynamics for the associated forward-moving map σ and therefore for the existence of chaotic equilibrium dynamics.

The equivalence relations proved so far are interesting but not sufficiently informative. Suppose we are studying an OLG model characterized by a BD map such as G of equation (7). It often happens that the (backward) iterates of a unimodal map yield many (even infinitely many) different types of orbits (e.g., periodic of many different periods, or chaotic). In this case, the results just discussed imply that there will be correspondingly many types of orbits forward in time, i.e., many different kinds of IEEs.

In this situation, the interesting question is: How can we distinguish between orbits that are typical and therefore interesting, and those that are exceptional and therefore negligible?

To answer this question, we take the common view that an event is "typical" ("exceptional") if it belongs to a set that is "large" ("small") *with regards to the set of all possible events*. In mathematics, there exists two main ways of "sizing up" sets: a metric approach based on the natural (Lebesgue) probability measure, and a topological approach based on the notion of (Baire) category. In our article MR07 we have established a number of results based on the topological approach. In this paper, we extend that analysis to the metric approach.

Here are some basic definitions that can be applied both to finite–dimensional state spaces usually encountered in economic models, such as subsets of $\mathbf{R}^n$, and to sequence spaces, such as inverse limit spaces. Once again, for the sake of brevity, we restrict ourselves to the essential points. For greater detail, we refer the reader to MR07.

DEFINITION 4. *Let $f : K \to K$ be a continuous map of a metric space K. Let $x \in K$. Then the* ω-limit set of x *is defined to be* $\omega_f(x) = \bigcap_{i\in\mathbb{Z}^+}(\bigcup_{m\geq i} f^m(x))$. *Let $A \subseteq K$ be closed and forward invariant, i.e., $f[A] = A$, then the* basin of attraction of A *is defined to be* $B(A) = \{x \in K : \omega_f(x) \subseteq A\}$.

Broadly speaking, $\omega_f(x)$ is the set of the limit points of the f–orbit $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ starting from x; the basin of A is the set of points x whose f–orbit converges to A as $n \to \infty$. Then we call A an attractor provided (i) $B(A)$ is "large" and (ii) A is dynamically indecomposable.

More rigorously, these two properties can defined as follows:

DEFINITION 5. *Let $f : K \to K$ be a continuous map of a metric space K. Let $A \subseteq K$ be a closed forward-invariant set. Then A is called a* metric attractor *provided (i) $B(A)$ has positive Lebesgue measure; and (ii) f is topologically transitive on A.*

Thus, from a metric point of view, a set is "large" or "small" if it has, respectively, positive or zero measure.

Attractors, in the sense of Definition 5, are interesting because the orbits that make them up give a likely characterization of the long-run behavior of the system, whereas sets with "small" basins of attraction are made up of orbits unlikely to be observed. Broadly speaking (and ignoring errors and rounding up), attractors are the objects that, transients apart, we expect to observe on the screens of our computers when we perform numerical simulations of dynamical systems, starting from randomly chosen initial conditions.

5. BASIC TYPES OF BACKWARD DYNAMICAL SYSTEMS ARISING FROM OVERLAPPING GENERATIONS MODELS

In order to produce sharp results, we concentrate on a class of noninvertible maps on the interval characterized by a single critical point and called unimodal or, informally, "one-humped" maps. This class of systems concerns virtually all the one-dimensional models of backward dynamics discussed in the economic literature (e.g., OLG models of the "leisure–consumption" or the "pure exchange" type; "cash-in-advance" models). It obviously includes the benchmark OLG model discussed in Section 2. Formally, we have the following definition:

DEFINITION 6. *A continuous map f of an interval $[a, b]$ is called* (strictly) unimodal *if there is a point $x^* \in (a, b)$ such that $f(x^*) \in [a, b]$ and f is strictly increasing on $[a, x^*)$ and strictly decreasing on $(x^*, b]$.*

Notice that if f is a unimodal map, then f is surjective on the interval $I = \bigcap_{n\geq 0} f^n([a, b])$. Recalling our presentation of the inverse limit space, we conclude that $\underleftarrow{\lim}\{[a, b], f\} = \underleftarrow{\lim}\{I, f|_I\}$. In simple terms, this means that, if f is a BD map as we assume here, the space $\underleftarrow{\lim}\{I, f\}$ contains all the forward-admissible orbits associated with it, and only them. For simplicity's sake, and without loss of generality, whenever f is a unimodal map, the factor space will be rescaled to $I = [0, 1]$ so that the corresponding inverse limit space is $\underleftarrow{\lim}\{[0, 1], f|_{[0,1]}\}$. In what follows, all the unimodal maps under consideration will be assumed to be continuous.

To prepare some of our results, we need the following preliminary definitions:

DEFINITION 7. *A C^3 unimodal map f on the interval is said to be* quasi-quadratic (q.q.) *if any sufficiently small perturbation of f in the C^3 topology is topologically conjugate to a quadratic map.*

DEFINITION 8. *For a C^3 unimodal map f on the interval, the* Schwarzian derivative *of f, Sf, is given by*

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

If $Sf(x) < 0$ for all $x \neq x^*$, we say that f is an S-unimodal map.

Remark 1. (i) We can look at the Schwarzian derivative as a property of the curvature of the first derivative of the function f. Considering that

$$\frac{d^2|f'(x)|^{-1/2}}{dx^2} = -(1/2)|f'(x)|^{-1/2}Sf(x),$$

we conclude that $Sf(x) < 0$ implies $|f'(x)|$ being concave over the monotone intervals of f. This assumption has been extensively used to prove certain important results concerning dynamical systems on the interval, but it is restrictive and, in general, it is not even necessary. As a matter of fact, the sign of $Sf(x)$ is not invariant w.r.t. a smooth change of coordinates, and it is perfectly possible to have two maps $f(x)$, $g(x)$ with $Sf(x) > 0$, $Sg(x) < 0$ on the respective domains, which are topologically, or even smoothly conjugate and therefore have the same dynamical properties.

(ii) A more natural class to work with is the family of quasiquadratic unimodal maps, as in Definition 7, of which S-unimodal maps form an open subset. First, the property of being quasiquadratic—differently from having $Sf < 0$—is preserved by smooth conjugacy. Second, in view of the fact that almost all (finitely) renormalizable unimodal maps have a quasiquadratic renormalization, certain important properties of the entire class of unimodal maps can be proved restricting the analysis to q.q. maps. Finally, the class of quasiquadratic unimodal maps defines the most general setting where bifurcations behave as for the quadratic family.[10]

In what follows, we assume q.q. when necessary to obtain sharp results. In these cases, whenever possible, we also explain the consequences of relaxing that assumption. As we shall see, however, in the case of Theorem 3, which covers by far the most common type of BD maps arising in one-dimensional OLG models, our main result can be reached using assumptions that follow directly from the economic hypotheses of the model.

In this paper, we identify three basic subclasses of unimodal maps, labeled Type A, B, and C maps. The identification of each case depends on the basic features of the controlling (backward) map f and therefore on the underlying structural functions. The next step is a description of the economic characterizations of type A, B, and C maps occurring in the OLG models and the corresponding mathematical definitions.

5.1. Type A Maps

This type of map is exemplified by the (Samuelson) OLG models characterized by a noninvertible backward moving offer curve with the following properties: (i) two steady states ("monetary" and "nonmonetary") exist; and (ii) either there is no (second period) utility saturation, or the saturation value of the old agent's excess demand ζ, call it Z, is larger than ζ_{MAX} and therefore irrelevant.[11]

It applies to most one-dimensional OLG models discussed in the economic literature, for example, Grandmont (1985), Boldrin and Woodford (1990), and

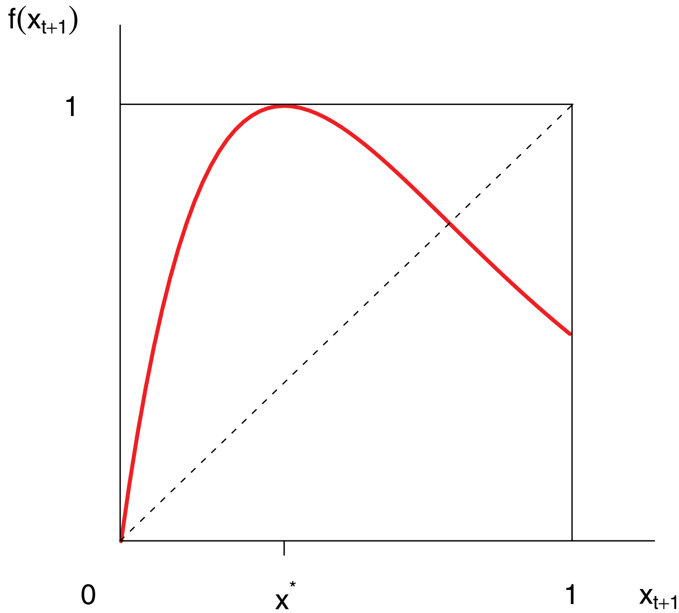

FIGURE 2. A Type A map.

many others. The map characterizing the benchmark model discussed in Section 2 is clearly of Type A.

A precise mathematical definition is the following:

DEFINITION 9. *A Type A map is a unimodal map such that* $f(a) = a$ *and* $f(x^*) < b$.

A Type A map, restricted to $I = [0, f(x^*)]$ so that $f|_I$ is surjective and rescaled to [0, 1], is depicted by Figure 2.

5.2. Type B Maps

The crucial difference between Type A and Type B maps is that for the latter there is no "nonmonetary" stationary equilibrium (the origin is not a fixed point). This case is considered, for example, in Grandmont (1989), in a simplified version of his OLG "leisure–consumption" model in which only young agents work and only old agents consume. In the notation of this paper, this implies that, for all t, $c_t = 0$; $\iota_t = \bar{\iota} = 0$ and consequently $z_t = -l_t$ and $\zeta_t = \kappa_t$. This model is necessarily of the Samuelson type. Assuming that its offer curve is unimodal, there are two possibilities, depending on the properties of the second-period utility function U. If $\lim_{\kappa \to 0} U'(\kappa)\kappa = 0$, we have a type A map as above. If, on the contrary, $\lim_{\kappa \to 0} U'(\kappa)\kappa > 0$, the model has only one steady state equilibrium, namely the "monetary" kind. There is nothing in the economic first principles to

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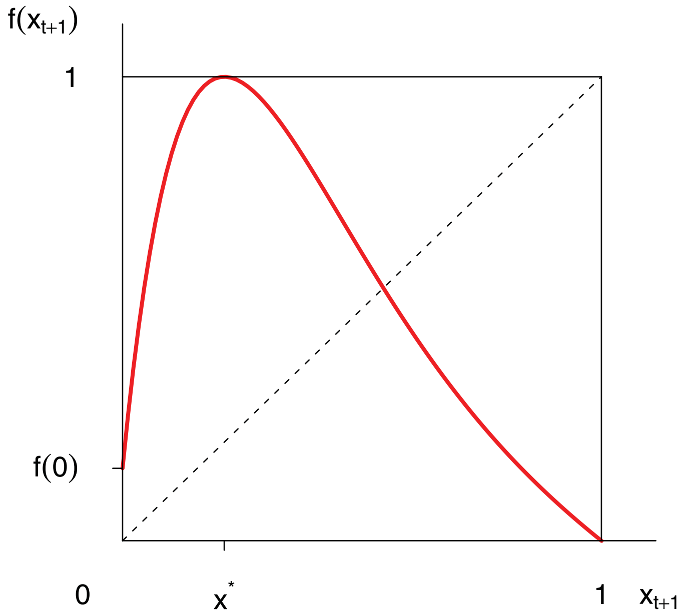

FIGURE 3. A Type B Map.

suggest that either case is exceptional and should be neglected. We call the map describing the second case "Type B."

Formally, we have the following definition:

DEFINITION 10. *A* Type B map *is a unimodal map such that* $f(a) > a$ *and* $f(b) = a$.

In this case, the relevant restriction is $f|_I$ with $I = [f^2(x^*), f(x^*)]$. The inverse limit space $\varprojlim\{[a, b], f\}$ is equal to $\varprojlim\{I, f|_I\}$.[12]

A representation of a restricted Type B map, rescaled to [0, 1], is provided by Figure 3.

5.3. Type C Maps

In the variants of the OLG model discussed so far, it was assumed that either there was no second-period utility saturation, or saturation was irrelevant because it occurred for levels of excess demand outside the admissible interval. We complete the picture, considering a situation characterized economically by the existence of "monetary" and a "nonmonetary" stationary equilibria, both locally unstable backward in time, and by the presence of binding (second-period) consumption saturation. In this case, the controlling map is called "Type C map." Here is a formal definition:

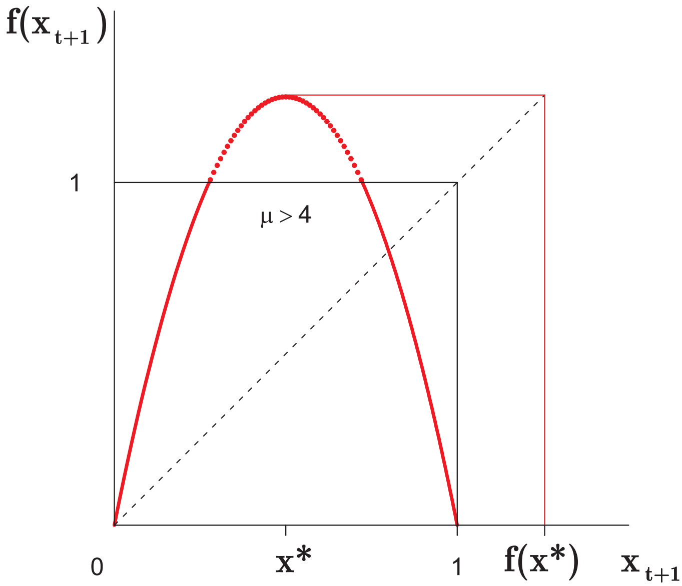

FIGURE 4. A Type C map.

DEFINITION 11. *A map f on an interval $[a, b]$ is called* Type C *if (i) f is not monotone; (ii) there is a point $x^* \in (a, b)$ such that f is monotone on $[a, x^*]$ and $[x^*, b]$; (iii) $f(a) = f(b) = a$; and (iv) $f(x^*) > b$. In this case, $f\colon [a, b] \to [a, f(x^*)]$ is surjective but its range is a superset of $[a, b]$ and therefore f is not a map from $[a, b]$ to $[a, b]$.*

Notice that, contrary to what happens for type A and B maps, in this case the economically relevant interval is not $[a, f(x^*)]$ (i.e., the set of values with a nonempty counterimage under f), but $[a, b] \subsetneq [a, f(x^*)]$, i.e., the (smaller) set of values over which marginal utility is nonnegative. Once again, without loss of generality, the relevant interval $[a, b]$ can be rescaled to [0, 1], so that the saturation point is located at 1. An example of a restricted and rescaled Type C map is provided by Figure 4.

Offer curves of Type C may occur, for example, in the pure exchange OLG models if we assume a (second-period) utility function of the quadratic type, as employed in Gale (1973, 168, Example 3), or Benhabib and Day (1982, 48, Example (ii)), and the "steepness parameter" is sufficiently large.

In MR07, for each of these three basic cases, we provided a classification of *topological* attractors under various more or less restrictive conditions. Unfortunately, the topological and metric notions of "size" are quite distinct and a set may be "small" in one sense and "large" in the other (and vice versa).[13] To avoid this difficulty, it would be desirable to prove that a certain set of interest is "large" ("small") both in a topological and in a metric sense. In MR07, this could be done satisfactorily only in the special case of Type C maps. In general, when the relevant space is not a measurable subset of $\mathbf{R}^n$ or a manifold, there is no obvious way to define a Lebesgue measure, and we are left with the sole topological alternative. In this paper, we strengthen our earlier results, defining a Lebesgue-like measure on $\overleftarrow{\lim}\{I, f\}$ for Type A and B maps as well, which is endowed with the essential properties of the Lebesgue measure. We also prove that all the topological attractors identified in our previous work are also metric attractors with respect to that measure. This implies that the forward-in-time orbits making up those attractors are generic in a measure–theoretic sense, and the same is true for the corresponding IEEs.

Because our present results on metric attractors are an extension and a generalization of the argument used in MR07 for Type C maps, we recall it briefly in the following section.

6. A METRIC ATTRACTOR FOR TYPE C MAPS

Because in this paper the discussion of Type C maps concerns us only as an introduction to the analysis of Type A and B maps, in this section we do not discuss the general case but concentrate on a specific, and particularly transparent example.

Consider again the simplified "leisure–consumption" OLG model in which only young people work and only old people consume, i.e., $z = -l \in (-\bar{l}, 0]$ and $\zeta = \kappa \in [0, \bar{l})$. Putting $V(z) = z$ and $U(\zeta) = \mu\zeta(1 - \zeta/2)$, we obtain the following backward-moving dynamical system in the variable $\zeta = \kappa$ (the old agent's excess demand):

$$\zeta_t = F_\mu(\zeta_{t+1}) = \mu\zeta_{t+1}(1 - \zeta_{t+1}), \qquad \textbf{(10)}$$

where F_μ is a type C map for $\mu > 4$.

The corresponding Type C map is depicted by Figure 4.

For our present purposes, the crucial aspect of this map is that the counterimage of *every* point $\zeta \in [0, 1]$ under F_μ consists exactly of two points belonging to two disjoint subsets of [0, 1], lying respectively on the left and the right side of the critical point $\zeta^* = 1/2$—call them $F^{-1}_{\mu,L}(\zeta)$ ("left inverse") and $F^{-1}_{\mu,R}$ ("right inverse")—and this is also true of the counterimage of the counterimage and so on and so forth up to any arbitrary order.[14] Thus, each forward-in-time sequence generated by the map F_μ starting from any given initial point in $\zeta \in [0, 1]$ can be coded by a unique sequence of two symbols, say {0, 1}.

To apply the machinery of ILT to a Type C map, we need to inject a little additional structure. From now on, we refer to a generic state variable denoted by x.

Let $X_1 = [0, 1]$, $X_2 = X_1 \cap F^{-1}_\mu[X_1]$ and inductively define $X_i = X_{i-1} \cap F^{-1}_\mu[X_{i-1}]$. Also, define $f_i : X_{i+1} \to X_i$ by $f_i = F_\mu|_{X_{i+1}}$. That is to say, in this case the maps f_i are all identical except that each has a different domain. The corresponding inverse limit space $\varprojlim\{X_i, f_i\}$ is the set of all forward admissible sequences, $(x_1, x_2 \ldots)$, permitted by the difference equation (10), with $x_i \in [0, 1]$.

Let $\Lambda = \bigcap_{n\geq 0} F_\mu^{-n}([0, 1])$. This is the (Cantor) set of points $x \in [0, 1]$ such that $F_\mu^n(x) \in [0, 1]\, \forall n \geq 0$. Let $\hat{\Lambda} = \varprojlim\{\Lambda, F_\mu|_\Lambda\}$. The dynamics of F_μ on Λ is well–understood and has been thoroughly discussed in the mathematical literature [see, for example, Katok and Hasseblatt (1995, 80–81)]. In particular, it is known that Λ is an F_μ–invariant, repelling set and the dynamics of F_μ on Λ are chaotic in the sense of Devaney or, equivalently, in the sense of Definition 3. Indeed, $F_\mu|_\Lambda$ is topologically conjugate to the full shift map on the space of one-sided infinite sequences of two symbols,[15] which is "the most chaotic map" in the sense that its iterations can be used as a mathematical representation of an independent stochastic process such as repeated coin tossing. Notice that the periodic orbits of $F_\mu|_\Lambda$ are countably many and they form a "small" set both in a topological and in a metric sense, whereas the typical orbit of $F_\mu|_\Lambda$ is chaotic.

Equipped with this information, we now turn to the analysis of the forward-moving dynamics of σ on $\hat{\Lambda}$. First of all, from Lemma 1 and Lemma 2 we have that $\hat{\Lambda}$ is closed and σ–invariant and that σ is topologically transitive on $\hat{\Lambda}$.

Next, let $\{0, 1\}^{\mathbf{N}}$ denote the (Cantor) set of unilaterally infinite sequences of symbols 0 and 1, $\{(z_1, z_2 \ldots) : z_i \in \{0, 1\} \text{ for each } i \in \mathbf{N}\}$, and consider the standard itinerary mapping $i(x) = 0$ if and only if $x < 1/2$ and $i(x) = 1$ otherwise.

Then we have the following result:

LEMMA 4 [MR07, Lemma 3]. Define h: $\varprojlim\{X_i, f_i\} \to [0, 1] \times \{0, 1\}^{\mathbf{N}}$ by

$$h[(x_1, x_2 \ldots)] = (x_1, (i(x_2), i(x_3), \ldots)) .$$

Then h is a homeomorphism.

Intuitively, this means that each orbit forward in time starting in [0, 1] can be uniquely determined by the initial condition $x \in [0, 1]$ *and* an infinite sequence of two symbols, each of them corresponding, for any x, to a choice between the two elements of the set $F_\mu^{-1}(x)$.

An interesting consequence of this fact is that there exists a well-defined measure on the set $\{0, 1\}^{\mathbf{N}}$ of all infinite sequences of two symbols: it is the product of the measure on the set of two elements, denoted by 0 and 1, that assigns the value 1/2 to each element. Thus, the probability assigned to the subset of $\{0, 1\}^{\mathbf{N}}$ including all the infinite sequences having *given*, finite subsequences of k elements is 2^{-k}. If we call this measure $\hat{\lambda}$ and λ is the ordinary Lebesgue measure on [0, 1], the measurable space $(\{0, 1\}^{\mathbf{N}}, \hat{\lambda})$ is isomorphic (modulo 0) to $([0, 1], \lambda)$.[16] The measure $\hat{\lambda}$ is commonly called the *Lebesgue (uniform) measure* on the space of symbol sequences. Then the product measure $\nu = \lambda \times \hat{\lambda}$ is a meaningful Lebesgue measure on the space $[0, 1] \times \{0, 1\}^{\mathbf{N}}$. It is with respect to the measure ν that, in MR07, Theorem 6, we could prove that the Cantor set $\hat{\Lambda}$ defined above is the unique metric attractor for the shift map σ associated with the Type C map in question (and it is also a topological attractor).

7. A LEBESGUE-LIKE MEASURE ON THE INVERSE LIMIT SPACE

A quick glace at Figures 2 and 3 will suggest that for Type A or B maps we cannot establish a one-to-one correspondence between a forward-in-time orbit starting from a given initial point in [0, 1] and a sequence of two symbols, for the simple reason that there exist subintervals of [0, 1] over which those maps have a single inverse. Consequently we cannot use the product measure ν as defined above.

In search of a viable alternative, we start with some broad methodological considerations.

First, let us recall condition (i) of Definition 5: an attractor is the limit set of orbits originating from a set of initial conditions of positive Lebesgue measure. This condition may be expressed by saying that if we choose an initial condition randomly with regards to the uniform probability density, there is a nonzero probability that the orbit from the chosen initial condition converges to the attractor [see Ott (2006)]. As we have explained elsewhere, for a forward-moving map F such as (1) and (6) choosing initial conditions in the domain of F (usually a subset of $\mathbf{R}^n$) is equivalent to choosing forward-in-time orbits. On the contrary, for a BD map G such as (2) or (7), since to each point in the domain of G there may correspond many forward-in-time orbits, initial conditions must be chosen in the domain of the corresponding shift map σ; i.e., the (inverse limit) space of forward-in-time orbits.

Second, the adoption of Lebesgue measure (uniform probability density) for sizing up the set of initial conditions, which is standard in the discussions of dynamical systems in economics and everywhere else, can be seen as an application of the Bernoulli–Laplace Principle of Insufficient Reason (PIR).[17] In its discrete version—which is relevant here as well as in Section 6—the PIR requires that if there are $n > 1$ mutually exclusive and collectively exhaustive possibilities and if the n possibilities are similar in all discernible relevant respects, then to each possibility should be assigned a probability equal to $1/n$, i.e., the possibilities should have equal probability.

A modern and more sophisticated version of the Principle of Indifference is the Maximum Entropy Principle (MEP). According to the famous formula of Shannon, the information entropy function of n mutually exclusive events E_i, $i = 1, 2, \dots, n$, to each of which a probability p_i is assigned, is defined as

$$H(p_1, p_2, \dots p_n) = -K \sum_{i=1}^{n} p_i \ln p_i,$$

with K an arbitrary, positive constant. The principle states that "in making inferences on the basis of partial information, we must use that probability distribution which has the maximum entropy *subject to whatever is known* [emphasis added]."

Note that, in this case, "entropy" is a synonym of "uncertainty." In the case of absolute ignorance about the events E_i, the maximization of the information uncertainty, with the constraint that $\sum_{i=1}^{n} p_i = 1$, yields $p_i = 1/n$.

In the case of Type C maps, the mutually exclusive possibilities concerned by the PIR/MEP are the two inverses of each point $x \in [0, 1]$ under the map f.

On the other hand, for Type A and B maps "what is known," i.e., the hypotheses of the underlying models and their consequences, imply that the corresponding function f has a single inverse over a certain subset of [0, 1] and a double inverse over another, without any criterion for choosing between the two alternatives. Thus, the application of the PIR/MEP principles requires that whenever there are two possibilities (inverses), we assign equal probabilities (1/2) to each of them and, when a unique possibility (inverse) exists, we assign it probability one.

What follows is a formal definition of a probability measure defined on the inverse limit space that complies with the requirements of the Principles of Insufficient Reason and Maximum Entropy while having some fundamental properties of the Lebesgue measure.

Next, we construct the required measure. Let $\hat{s} = (s_1, s_2, \ldots) \in \{0, 1\}^{\mathbf{N}}$. Let $f\colon [0, 1] \to [0, 1]$ be a Type A or Type B map and let $f_L^{-1}(x)$, $f_R^{-1}(x)$ denote the "left" and "right inverse" of f, respectively.

We now define a function $h_f : [0, 1] \times \{0, 1\} \to [0, 1]$.

First, if f is a rescaled Type A map, put $i \geq 1$ and define h_f by

$$h_f(x_i, s_i) = \begin{cases} f_L^{-1}(x_i), & \text{if } x_i \in [0, f(1)); \\ f_L^{-1}(x_i), & \text{if } x_i \in [f(1), 1] \text{ and } s_i = 0; \\ f_R^{-1}(x_i), & \text{if } x_i \in [f(1), 1] \text{ and } s_i = 1. \end{cases}$$

Next, if f is a rescaled Type B map, then define h_f by

$$h_f(x_i, s_i) = \begin{cases} f_R^{-1}(x_i), & \text{if } x_i \in [0, f(0)); \\ f_L^{-1}(x_i), & \text{if } x_i \in [f(0), 1] \text{ and } s_i = 0; \\ f_R^{-1}(x_i), & \text{if } x_i \in [f(0), 1] \text{ and } s_i = 1. \end{cases}$$

Let $(s_1, s_2, \ldots) \in \{0, 1\}^{\mathbf{N}}$. Then given a point $x_i \in [0, 1]$ let $x_{i+1} = h_f(x_i, s_i)$. Define $H\colon [0, 1] \times \{0, 1\}^{\mathbf{N}} \to \underleftarrow{\lim}\{[0, 1], f\}$ by

$$H(x_1, (s_1, s_2, \ldots)) = (x_1, h_f(x_1, s_1), h_f(x_2, s_2), \ldots).$$

The map H is surjective but, generally, not injective.

Consider now the function

$$\underleftarrow{\lambda} = \nu \circ H^{-1},$$

where $\nu = \lambda \times \hat{\lambda}$ is defined in Section 6.

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From the definition of ν, and considering the fact that $\hat{\lambda}$ is the product of the measure on the set of two elements assigning the value $1/2$ to each of them, it readily follows that $\underset{\leftarrow}{\lambda}$ satisfies the requirements of the PIR/MEP principles.

We now proceed to prove that $\underset{\leftarrow}{\lambda}$ also possesses some basic properties of the Lebesgue measure, according to the following definition:

DEFINITION 12. *Let X be a compact metric space, $\mathcal{B}$ the Borel algebra on X, and $\lambda : \mathcal{B} \to \mathbf{R}$ a measure on $\mathcal{B}$. We say that λ is Lebesgue-like provided (1) λ is a positive Borel measure; (2) if $U \subseteq X$ is open then $\lambda(U) > 0$; (3) if $x \in X$ then $\lambda(\{x\}) = 0$.*

We can state the following theorem:

THEOREM 2. *$\underset{\leftarrow}{\lambda}$ is a Lebesgue-like measure.*

Proof. See Appendix A.

But this is not all. For a class of subsets, $\mathcal{I}$, of the inverse limit space that are the analogues of *subintervals* of the inverse limit space, we have that the $\underset{\leftarrow}{\lambda}$-measure of $J \in \mathcal{I}$ has a fundamental property that the Lebesgue measure has on subintervals of $I \subset \mathbf{R}$, namely it is translation–invariant.

To see that, let us define the set

$$\mathcal{I} = \{\pi^{-1}(I) : I \text{ is a subinterval of } [0, 1]\}.$$

Then if $J \in \mathcal{I}$, $J = \pi^{-1}(I)$ for some subinterval I of $[0, 1]$ and so we can call J a *subinterval* of $\varprojlim\{[0, 1], f\}$. By Lemma A.1, we see that $\underset{\leftarrow}{\lambda}(J) = \lambda(I)$ which is the length of I. For this reason, we can say that $\underset{\leftarrow}{\lambda}(J)$ is the length of J.

For a subinterval $J = \pi^{-1}(I)$ we define *translation* by $t \in \mathbf{R}$ mod 1 by

$$J + t = \pi^{-1}(I + t (\text{mod } 1)).$$

Again, by Lemma A.1, we see that

$$\underset{\leftarrow}{\lambda}(J + t) = \underset{\leftarrow}{\lambda}(J).$$

Therefore the measure $\underset{\leftarrow}{\lambda}$ shares all of the significant properties of Lebesgue measure in this setting.[18]

8. METRIC ATTRACTORS FOR THE MAPS OF TYPE A AND B

Equipped with the results of Section 7, we are now ready to establish the existence of metric attractors for the Type A and B maps as well. Since the proofs of the following theorems are quite technical, we relegate them to the Appendix.

First, we have the following Theorem for Type A maps:

THEOREM 3. *Let* $f : [0, 1] \to [0, 1]$ *be a unimodal Type A map with* $f(1) > 0$ *and* $f'(0) > 1$. *Let* 0 *and* $\bar{x} \in (x^*, 1]$ *be the only fixed points of* f. *Let* $\hat{0} = (0, 0, 0, \ldots) \in \varprojlim\{[0, 1], f\}$. *Then* $\{\hat{0}\}$ *is the only metric attractor for* $\varprojlim\{[0, 1], f\}$ *under* σ.

A rigorous proof of Theorem 3 can be found in Appendix A. In words, this theorem means that, if a OLG model is characterized by a unimodal map of Type A with the specifications of the Theorem, λ-almost all orbits forward-in-time converge asymptotically to the "nonmonetary" stationary state. This is indeed the case for the Grandmont-like basic model discussed in Section 2.

Remark 2. For the class of OLG models described by Type A maps, the standard economic assumptions are sufficient to guarantee that $f'(0) > 1$ and that there exist two stationary equilibrium states, one positive ("monetary") and a second one ("nonmonetary"), located at the origin (cf. Notes 3 and 4). We have excluded the case in which the "monetary" fixed point is located to the left of the critical point (or on it), because in this case the restricted map is monotonically increasing and its dynamics are trivial: all the admissible forward-in-time-orbits converge to the "nonmonetary" fixed point, except one, namely $\hat{x} = (\bar{x}, \bar{x}, \ldots)$. For more general applications of Type A unimodal maps, the assumption that f is q.q. (or $Sf < 0$) is sufficient (though not necessary) to guarantee the uniqueness of the metric attractor $\hat{0}$ for the map σ.

For maps of Type B in which the unique fixed point is located in the subinterval $(x^*, 1)$[19], there are two very different situations. Suppose q is the least point in $[x^*, 1]$ fixed under $f^2 = f \circ f$. Then the first, simpler case occurs when $f(0) > q$. We have the following result:

THEOREM 4. *Let* $f : [0, 1] \to [0, 1]$ *be a quasiquadratic unimodal Type B map with a unique fixed point* $\bar{x} \in (x^*, 1)$, *and let* $|f'(\bar{x})| > 1$ *and* $f(0) > q$. *Then* $\hat{x} = (\bar{x}, \bar{x}, \ldots)$ *is the only metric attractor for* $\varprojlim\{[0, 1], f\}$ *under* σ.

Proof. See Appendix A.

Remark 3. As we mention in the proof, if the map f of Theorem 4 is q.q., $q = \bar{x}$. If f is not q.q., this need not be true and the situation is more complicated. Specifically, the shift map σ associated with f may have multiple attractors. Two main possibilities arise: if $f(0) > f(q)$, depending on the initial conditions, the forward-in-time orbits will be attracted to $\bar{x}$ or to one of the many possible period-two cycles, unstable under f and located in the interval $[q, f(q)]$. If $q \le f(0) \le f(q)$, for appropriate initial conditions, there may also exist a forward-in-time chaotic attractor, formed by two subsets of the "horseshoe" type, each visited by the iterates of σ with periodicity 2.[20]

Next we consider the more interesting and complicated case of $f(0) < q$, where, we recall, q is the first (smallest) fixed point of f^2 in the interval $[x^*, 1]$.

This corresponds to a "chaotic region" of the parameter space, in the sense that the following properties hold [cf. Barge and Diamond (1994) and Ingram (2000b)]: (i) $\varprojlim\{[0,1], f\}$ is indecomposable; (ii) there exists an f-invariant chaotic set (a horseshoe); (iii) the maps f on $[0,1]$ and σ on $\varprojlim\{[0,1], f\}$ have positive topological entropy. We prove below that, in this case, if f satisfies some fairly general assumptions and has an attractor $P \neq [0,1]$ that is either periodic or chaotic, then there is an invariant Cantor set, Λ, which generates a metric attractor for σ, $\hat{\Lambda} = \varprojlim\{\Lambda, f_\Lambda\} \subset \varprojlim\{[0,1], f\}$.

To prepare a rigorous statement of our result, we start with a general result, proved with some variations in a series of recent papers[21]and stating that, for a family of real analytic, unimodal maps, for Lebesgue–almost all parameters, the maps are either "regular" or "stochastic" (chaotic).[22] Here a map f is called "regular" if it is hyperbolic, has a nondegenerate critical point, x^*, $f''(x^*) \neq 0$, which is not periodic or preperiodic. Regular maps have one or more periodic attractors, one of them containing x^* in its immediate basin; stochastic maps have a unique chaotic attractor (transitive cycle of intervals, supporting an absolutely continuous invariant measure), plus perhaps a finite number of periodic attractors, x^* being in the basin of the chaotic attractor. Accordingly, parameters generating attractors of the solenoid type (roughly speaking, attractors that are aperiodic but not chaotic) form sets of measure zero in the parameter space. If, in addition, the map f is quasiquadratic (q.q.) in the sense of Definition 7, it has either a unique periodic attractor or a unique chaotic one, each of them having x^* in its basin (see Avila and co-workers, cited in Note 21).

Next, we make use of a result of Ingram (1995), which we restate using our own notation:

THEOREM [Ingram (1995, Th. 6)]. Let f be a Type B unimodal map on $[0,1]$, with critical point x^* and a unique fixed point $\bar{x} \in (x^*, 1)$. Then f has a periodic point with odd period greater than 1 if, and only if, $f(0) < q$ where q is the first fixed point of f^2 that is in the interval $[x^*, 1]$.

LEMMA 5. *Let f be a type B unimodal q.q. map with $f(0) < q$. Then f has infinitely many periodic orbits.*

Proof. The result follows immediately from Ingram's 1995 theorem and the well-known Sarkovskii theorem.

From the stated results, we conclude that a typical quasiquadratic unimodal map f of Type B with a unique fixed point $\bar{x} \in (x^*, 1)$ and such that $f(0) < q$ satisfies the following properties:

1. f has a unique attractor (either periodic or chaotic) $P \neq [0,1]$;
2. there are finitely many proper subintervals $B_0, \dots B_{n-1}$ (the immediate basin of P) such that
   (a) $P \subseteq \bigcup_{i=0}^{n-1} B_i$;
   (b) $f(B_{i-1}) \subseteq B_i$ for $1 \leq i \leq n-1$, $f(B_{n-1}) \subseteq B_0$;

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(c) ∂B_i is not in the basin of P;
(d) $\partial B_i \cap \partial B_j = \emptyset$ for all $i \neq j$.

Notice that parts (a), (b), and (c) of assumption (2) follow from the definition of the attractor as either periodic (and hence having an immediate basin of subintervals which are cyclically permuted) or chaotic (a transitive cycle of intervals). Lemma 5 establishes that if f is q.q. and $f(0) < q$, then f has infinitely many periodic points. This implies that infinitely many distinct orbits are not attracted to P and hence there must be intervals in the complement of the immediate basin. Assumption 2d will then follow.

Let

$$B = \bigcup_{i=0}^{n-1} B_i$$

and

$$\Lambda = \left\{x \in [0, 1] : f^n(x) \notin B \text{ for all } n \in \mathbf{N}\right\} = [0, 1] \setminus \bigcup_{n\in\mathbf{N}} f^{-n}(B).$$

Because each B_i is a subinterval that does not contain its boundary, we see that each B_i is an open interval. Thus B is an open set and $[0, 1] \setminus B$ contains nondegenerate intervals (by assumption 2d).

LEMMA 6. *Λ is an invariant Cantor set.*

Proof. See Appendix A.

The intervals B_i are cyclically permuted by f and only one of them can contain the critical point x^*. This implies that there is a word in 0 and 1, W, such that $x \in \Lambda$ if and only if the itinerary of x does not contain W or x^*. Because Λ contains none of the preimages of x^* we see that the itinerary mapping, i, is a homeomorphism from Λ onto $i(\Lambda)$[23]. Because the points in Λ are characterized by the property of their itinerary not containing the word W, we see that $i(\Lambda)$ is a subshift of finite type, and hence Λ is conjugate to a subshift of finite type. By Theorem 1 we see then that σ acting on the inverse limit of Λ, which we denote by $\hat{\Lambda} = \varprojlim\{\Lambda, f|_\Lambda\}$, is chaotic. In particular, it is topologically transitive on $\hat{\Lambda}$.

We have then the following theorem:

THEOREM 5. *Let f be a Type B unimodal map of $[0, 1]$ with fixed point $x \in (x^*, 1)$ and such that $f(0) < q$. Suppose that f satisfies assumptions (1) and (2) above. Let $\Lambda = \{x \in [0, 1] : f^n(x) \notin B \text{ for all } n \in \mathbf{N}\}$, and let $\hat{\Lambda} = \{\hat{x} \in \varprojlim\{[0, 1], f\} : x_i \in \Lambda \text{ for all } i \in \mathbf{N}\}$. Then $\hat{\Lambda}$ is the metric attractor for σ in $\varprojlim\{[0, 1], f\}$.*

Proof. See Appendix A.

We now complete the picture, discussing the special case of Type B maps in which the metric attractor of f is the entire interval, i.e., $P = [0, 1]$, and finding out the corresponding metric attractor for the shift map σ.

Before stating our main result, we need a preliminary lemma.

LEMMA 7. *Let f be a Type B map with $0 < f(0) < \bar{x}$, with an attractor $P = [0, 1]$. Then f is topologically exact*[24] *on* $[0, 1]$.

Proof. See Appendix A.

We can now state the following theorem:

THEOREM 6. *Let f be a Type B map with $0 < f(0) < \bar{x}$ and a metric attractor $P = [0, 1]$. Then the only metric attractor for the corresponding map σ on the space $\varprojlim\{[0, 1], f\}$ is the entire space.*

Proof. See Appendix A.

9. CONCLUSIONS

From the formal results proved in the preceding sections, we can draw a number of interesting and sometimes puzzling conclusions concerning the intertemporal economic equilibria (IEEs) generated by overlapping generations models described mathematically by a BD map f on the interval $[0, 1]$ and the corresponding shift map σ on the associated inverse limit space $\varprojlim\{[0, 1], f\}$.[25]

1. If the BD map f is of Type A (no utility saturation, two stationary equilibria, "monetary" and "nonmonetary"), *no matter what the backward-in-time dynamics of f are,* the typical IEE is characterized by an orbit converging forward in time to $\hat{0} = (0, 0, \ldots)$, i.e., an infinite repetition of the "nonmonetary" stationary equilibrium. This result provides a more general and rigorous proof for certain earlier propositions [see Gale (1973, 166, Th. 4)], asserting attractiveness forward in time of the "nonmonetary" steady state equilibrium only on the basis of its local instability backward in time. The discussion of Type B and C maps shows that, contrary to a commonly held view, instability backward in time of a stationary (or a periodic) equilibrium is not a sufficient condition for that equilibrium to be an attractor forward in time *globally*, i.e., when we consider *all* the forward admissible orbits.
2. For Type B maps (no utility saturation, a unique, "monetary" stationary equilibrium), two main situations may occur, with some variations therein.
   (i) The simpler, "periodic case" occurs if the single fixed point of f is unstable and $f(0) > q$, that is, broadly speaking, if there exists a periodic orbit of period 1 or 2 in the subinterval in which the map f has a single inverse. In this case, if f is q.q., the typical IEE is characterized by orbits forward in time converging to the unique stationary equilibrium $\bar{x}$. If the hypothesis of q.q. is dropped, there may be IEEs characterized by (one or more) periodic attractors of period two, plus possibly by "periodic chaos" of period two.
   (ii) The structure of forward-in-time orbits is more complicated if $f(0) < q$—the "chaotic case." Here, if f is q.q. and has a unique metric attractor $P \subsetneq [0, 1]$

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which is periodic or chaotic, the typical IEE is characterized by orbits forward in time converging to a unique metric attractor, a Cantor set on which the dynamics are chaotic.

3. Finally, in the special case in which $P = [0, 1]$ is the unique metric attractor for the BD map f, the typical IEE is characterized by chaotic, forward in time orbits which are dense on the entire interval.

NOTES

1. The difference between the scope and method of the two approaches is best illustrated by the results of their applications to the same model—the leisure–consumption OLG model à la Grandmont (1985), which we label a "Type A" model. In our MR07 and here, we consider the set of *all possible intertemporal perfect foresight equilibria* and prove that, independent of the dynamical properties of the corresponding backward dynamics, the subset of equilibria converging forward in time to the "nonmonetary" steady state is generic in a topological (MR07) as well as in a metric sense (this paper). In contrast, GHTV consider *the subset of equilibria that do not converge to the "nonmonetary" equilibrium* and show that, if the backward dynamics are chaotic and have a homoclinic orbit, the forward orbits generated by a conveniently restricted IFS converge to a fractal attractor.

2. For a more detailed discussion, see also MR07 quoted. Notice that, for our present purpose, the leisure–consumption OLG model à la Grandmont is perfectly equivalent to the model of pure exchange discussed, in various versions, by Samuelson (1958), Gale (1973), Benhabib and Day (1982), and many others. The former model can be transformed into the latter by replacing the hypothesis of variable labor supply and production with the alternative hypothesis of no-production and given endowments of the consumption good. The resulting dynamical equations would be formally identical in the two cases.

3. In particular, we, like Grandmont, assume that $V'(z) > 0, U'(\zeta) > 0; V'' < 0, U''(\zeta) < 0; \lim_{z\to -\bar{l}} V'(z) = \infty; \lim_{\zeta\to -\bar{\iota}} U'(\zeta) = \infty$.

4. In our notation the classical or the Samuelson case obtains if $V'(0) \gtrless U'(0)$, respectively.

5. The term "inverse" refers to the fact that f maps each factor space $X_i, i \geq 2$ to its *antecedent* in the space sequence.

6. In simple words, the maps π_i takes an element of the inverse limit space (an infinite forward-moving sequence) and returns its ith coordinate.

7. The inverse of the shift map, σ^{-1}, acts on a sequence $(x_1, x_2, \ldots)$ by replacing each term with its image under f. Thus, if f is a BD map, the dynamics generated by iterations of σ^{-1} move backward in time.

8. For example, suppose $A = \{x, y, z\}$. Then $\hat{A} = \{\hat{z}, \hat{y}, \hat{x}\}$, with

$$\begin{aligned}\hat{z} &= \{z, y, x, z, y, x, \ldots\},\\ \hat{y} &= \{y, x, z, y, x, z \ldots\},\\ \hat{x} &= \{x, z, y, x, z, y, \ldots\}.\end{aligned}$$

Thus, $\sigma(\hat{z}) = \hat{y}; \sigma(\hat{y}) = \hat{x}; \sigma(\hat{x}) = \hat{z}$, and $\pi(\hat{A}) = A$. The interesting special case $n = 1$ arises when A consists of a fixed point $\bar{x}$ such that $f(\bar{x}) = \bar{x}$. In this case, $\hat{A}$ consists of a single sequence $\hat{x} = (\bar{x}, \bar{x}, \ldots)$.

9. We adopted this characterization of chaos, first suggested by Touhey (1997), because it suits our present purpose very well and immediately leads to the useful Corollary 1. On the other hand, Touhey also proved that his definition is equivalent (in the "iff" sense) to the more common Devaney's definition. A map f is said to be chaotic on a metric space X in the sense of Devaney if it has the following properties: (1) f is t.t. on X; (2) the periodic orbits of f are dense in X; and (3) f has sensitive dependence on initial conditions. It has been proved by Banks *et al.* (1992) that property 3 is redundant because it is implied by the other two. Moreover, Vellekoop and Berglund (1994) proved

that, on intervals, property 1 alone implies the other two. It is also known that, on compact sets, Devaney-chaos implies Li and Yorke-chaos.

10. We are indebted to Artur Avila for discussing with us the properties of q.q. maps. For a thorough analysis of this family of maps, see Avila et al. (2004). Notice that, in specific applications, proving that $Sf < 0$ may be the easiest way to establish that f is q.q.. For example, if in the definition of the map G of equation (7), we put $v(z) = \bar{\iota} + z$ and $u(\zeta) = -r\zeta e^{-(\bar{l}+\zeta)}$, with $r > 0$, we can show that $SG(\zeta) < 0$ for all choices of $\bar{\iota}, \bar{l}, r$. Also, it can be shown that if $v(z) = \overline{(i + z)}^{(1-\alpha_1)}/(1 - \alpha_1)$ and $u(\zeta) = (\bar{l} + \zeta)^{(1-\alpha_2)}/(1 - \alpha_2)$, $SG(\zeta) < 0$ if $\alpha_1 \leq 1$ and $\alpha_2 \geq 2$ [see, with different notation, Grandmont (1985, 1026)].

11. In the Samuelson case for which the young agent's excess demand is at most zero, first-period utility saturation is not relevant.

12. Notice that the inverse limit space of a Type B map can always be constructed from a Type A map f, simply taking the so–called "core map," i.e., the restriction of f to the subinterval $[f^2(x^*), f(x^*)] \subset [0, 1]$. Thus, the space of forward admissible sequences generated by a Type A map always contains a subset R of "Type B sequences," but, as we shall see, for Type A maps, R has zero measure. On the other hand, for a Type B map, the set R include all the forward admissible sequences.

13. Cf. Oxtoby (1971, 4); Katok and Hasselblatt (1995, 287–288).

14. Points $\zeta \in (1, \zeta_{\text{MAX}}]$ have a nonempty counterimage under F_μ too, but they must be discarded since, for those values of ζ the marginal utility is negative.

15. A recent, simpler proof of this fact can be found in Kraft (1999).

16. The probability spaces $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$, are said to be *isomorphic* if there exist $M_1 \in \mathcal{B}_1$ and $M_2 \in \mathcal{B}_2$, with $\mu_1(M_1) = 1 = \mu_2(M_2)$ and an invertible, measure–preserving map $\phi: M_1 \to M_2$, where the space M_i is assumed to be equipped with the σ–algebra $M_i \cap \mathcal{B}_i = \{M_i \cap B | B \in \mathcal{B}_i\}$.

17. The PIR—renamed the "Principle of Indifference" by John Maynard Keynes in his Treatise on Probability (1921)—is a rule for assigning probabilities under ignorance and can be seen as a special case of the Bayesian "prior distribution" in the absence of background information.

18. The $\underset{\leftarrow}{\lambda}$ measure is not the only measure possessing these properties, but what makes it uniquely appealing is that $\underset{\leftarrow}{\lambda}$ is a natural application of certain basic principles of probability theory (Principle of Insufficient Reason/Maximum Entropy) to the situation described by the economic models discussed in the paper.

19. We omit the (trivial) case in which the unique "monetary" stationary state $\bar{x}$ is located on the left of, or on the critical point. In this case, $\hat{x} = (\bar{x}, \bar{x}, \ldots)$ is the only existing forward admissible orbit.

20. See Ingram (1995), in particular Theorems 2 and 3. Moreover, remember that, for any $n > 1$, $\underset{\leftarrow}{\lim}\{X, f\}$ is homeomorphic to $\underset{\leftarrow}{\lim}\{X, f^n\}$.

21. See Avila et al. (2004); Avila and Moreira (2005a, 2005b).

22. In what follows, we will use the term "chaotic" instead of the less common "stochastic." Notice that an attractor that is stochastic in the sense of Avila and co-workers is also chaotic in the sense of Devaney and Touhey.

23. To see this consider the proof contained in Devaney (2003, § 1.7) of the fact that the itinerary mapping is a homeomorphism from his set Λ onto $\{0, 1\}^{\mathbf{N}}$. The only assumptions he uses to prove that the map is continuous and 1–1 are that Λ does not contain the critical point or its preimages and that it has slope larger than 1. Because our set Λ also has these properties, we see that i is continuous and 1–1. It follows that i is a homeomorphism onto the image of Λ, $i(\Lambda)$.

24. A continuous map $f: [0, 1] \to [0, 1]$ is said to be *topologically exact*, or *locally eventually onto (l.e.o.),* if for any open, nonempty set $V \subset [0, 1]$ there exist $n \geq 0$ such that $f^n(V) = [0, 1]$. Topological exactness can be looked at as a strong form of indecomposability that implies (but it is not implied by) transitivity.

25. Because Type C maps has been already covered completely in MR07, here we only deal with Type A and B maps.

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26. We denote by $\bar{\sigma}$ the one-sided shift map on the space of sequences of 2 symbols $\{0, 1\}$, sometimes called (one-sided) "Bernoulli shift." The action of the map consists in deleting the first element and shifting the remaining sequence one step to the left. Notice that $\bar{\sigma}$ is a two-to-one map and therefore not invertible, whereas, as we have already explained, σ is a homeomorphism.

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APPENDIX

A.1. PROOF OF THEOREM 2

Let $\mathcal{M}$ be the σ-algebra of Borel sets in $[0, 1]$. Let $\mathcal{N}$ be the σ-algebra of measurable sets of $\{0, 1\}^{\mathbf{N}}$. Let $\mathcal{A}$ be the algebra of finite, disjoint unions of *rectangles* in $[0, 1] \times \{0, 1\}^{\mathbf{N}}$ such that if $A \in \mathcal{A}$ and $A = \bigcup_{i=1}^{n} A_i$ then each $A_i = B_i \times C_i$, where $B_i \in \mathcal{M}$ and $C_i \in \mathcal{N}$. Often $\mathcal{A}$ is denoted by $\mathcal{M} \otimes \mathcal{N}$. Let $A \in \mathcal{A}$ with $A = \bigcup_{i=1}^{n} A_i$, the A_is disjoint, and each $A_i = B_i \times C_i$, where $B_i \in \mathcal{M}$ and $C_i \in \mathcal{N}$. Then the measure of A is defined by

$$\nu(A) = \sum_{i=1}^{n} \lambda(B_i) \cdot \hat{\lambda}(C_i).$$

It is a standard result that the algebra $\mathcal{A}$ generates a σ-algebra $\mathcal{B}$ and that ν induces a measure (which we also denote by ν) on $\mathcal{B}$.

The set function $\underset{\leftarrow}{\lambda}$ is defined on every set $\hat{K} \subseteq \underset{\leftarrow}{\lim} \{[0, 1], f\}$ with the property that $H^{-1}(\hat{K}) \in \mathcal{B}$. Let $\mathcal{S}$ be the collection of all such sets; i.e., $\mathcal{S}$ is the set of all $\hat{K}$ with the property that $H^{-1}(\hat{K}) \in \mathcal{B}$. We wish to show that $\underset{\leftarrow}{\lambda}$ is a Borel measure. To that end define $\mathcal{R} = \{\hat{M} \subseteq \underset{\leftarrow}{\lim} \{[0, 1], f\} : H^{-1}(\hat{M}) \in \mathcal{A}\}$. It is clear that $\mathcal{R} \subseteq \mathcal{S}$. We will show that $\mathcal{R}$ contains all of the open sets in $\underset{\leftarrow}{\lim} \{[0, 1], f\}$, and that it is an algebra of sets. We will also show that $\mathcal{S}$ is a σ-algebra of sets, and therefore it will contain the Borel σ-algebra.

LEMMA A.1. *Let $K \subseteq [0, 1]$ be Lebesgue measurable. Then $\pi^{-1}(K) \in \mathcal{R}$ and $\underset{\leftarrow}{\lambda}(\pi^{-1}(K)) = \lambda(K)$.*

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Proof. Let $\hat{K} = \pi^{-1}(K)$. We will show that for every $(s_1, \ldots) \in \{0, 1\}^{\mathbf{N}}$ and for every $x_1 \in K$ there is a point $\tilde{x} \in H^{-1}(\pi^{-1}(K))$ with $\tilde{x} = (x_1, (s_1, s_2, \ldots))$.

Let $x_1 \in K$. Let $(s_1, s_2, \ldots) \in \{0, 1\}^{\mathbf{N}}$. By definition, $f^{-1}(x_1) = f_L^{-1}(x_1) \cup f_R^{-1}(x_1)$, and so every point, $\tilde{z}$, in $\pi^{-1}(x_1) \subseteq \hat{K}$ has the property that $z_1 = x_1$ and $z_2 \in f_L^{-1}(x_1) \cup f_R^{-1}(x_1)$. If x_1 is such that both $f_L^{-1}(x_1)$ and $f_R^{-1}(x_1)$ are defined, then clearly there is a point $x_2 \in f^{-1}(x_1)$ such that $x_2 \in f_L^{-1}(x_1)$ or $x_2 \in f_R^{-1}(x_1)$, depending on the value of s_1. If instead, though, one of f_L^{-1} or f_R^{-1} is not defined for x_1, then each $\tilde{z} \in \pi^{-1}(x_1)$ is of the form $\tilde{z} = (x_1, x_2, z_3, z_4 \ldots)$, where x_2 is the unique inverse image of x_1 under f. In this case $h_f(x_1, 0) = x_2$ and $h_f(x_1, 1) = x_2$. Thus we see there are points $(x_1, (0, \ldots))$ and $(x_1, (1, \ldots))$ in $H^{-1}(\hat{K})$. So in this case the specific value of s_1 is not important.

Continuing, suppose that x_j has been chosen for all $1 \leq j < n$ to be compatible with the word $s_1, s_2 \ldots s_{n-1}$. Again we see that $f^{-1}(x_{n-1}) = f_L^{-1}(x_{n-1}) \cup f_R^{-1}(x_{n-1})$, and if both are defined then we can choose x_n to be in $f_L^{-1}(x_{n-1})$ if $s_n = 0$ and $x_n \in f_R^{-1}(x_{n-1})$ if $s_n = 1$. If only one is defined then the value of s_n does not matter and x_n is uniquely defined. This leads to a point $\tilde{x} = (x_1, x_2, \ldots) \in \pi^{-1}(x_1) \subseteq \pi^{-1}(K) = \hat{K}$ with the property that $H^{-1}(\tilde{x}) \ni (x_1, (s_1, s_2, \ldots))$. It follows that $H^{-1}(\hat{K}) = K \times \{0, 1\}^{\mathbf{N}}$. Thus $\underset{\leftarrow}{\lambda}(\hat{K}) = \lambda(K)$.

We shift our focus to verifying that $\underset{\leftarrow}{\lambda}$ is a measure in the case where the bonding map is unimodal of Type B. The case where it is unimodal of Type A is similar.

LEMMA A.2. *Let f be a unimodal Type B map. Let $n \in \mathbf{N}$. Let $K \subseteq [0, 1]$ be Lebesgue measurable such that*

(1) for every $m \leq n$, $f^m(K) \subseteq [0, x^*)$ or $f^m(K) \subseteq (x^*, 1]$, and
(2) if $f^m(K) \cap [0, f(0) \neq \emptyset$ then $f^m(K) \subseteq [0, f(0))$.

Then

$$H^{-1}(\pi_n^{-1}(K)) \in \mathcal{R}$$

and for each $i \leq n$ there is a nonempty set $L_i \subseteq \{0, 1\}$ such that

$$H^{-1}(\pi_n^{-1}(K)) = f^n(K) \times \prod_{i=1}^{n} L_i \times \{0, 1\}^{\mathbf{N}}.$$

Moreover,

$$\underset{\leftarrow}{\lambda}(\pi_n^{-1}(K)) = \lambda(f^n(K)) \cdot \prod_{i=1}^{n} \frac{|L_i|}{2^n}$$

and $|L_i| = 1$ if $f^{n-i}(K) \cap [0, f(0)) = \emptyset$, whereas $|L_i| = 2$ if $f^{n-i}(K) \subseteq [0, f(0))$.

Proof. Let $\hat{K} = \pi_n^{-1}(K)$, and let $\tilde{x} = (x_1, x_2 \ldots) \in \hat{K}$. Then $x_n \in K$ and for every $m < n$, $f^m(x_n) = x_{n-m} \in f^m(K)$. So $x_1 \in f^{n-1}(K)$ and $x_2 \in f^{n-2}(K)$. By assumption (1), either $f^{n-2}(K) \subseteq [0, x^*)$ or $f^{n-2}(K) \subseteq (x^*, 1]$. This implies that x_2 is equal either to $f_R^{-1}(x_1)$ or to $(f_L^{-1}(x_1)$. If $f^{n-1}(K) \cap [0, f(0)) = \emptyset$ then there is a unique $s_1 \in \{0, 1\}$ such that every point in $H^{-1}(\hat{K})$ with first coordinate x_1 has some $(s_1, \ldots)$ as its second coordinate. In this case $L_1 = \{s_1\}$. If instead $f^{n-1}(K) \subseteq [0, f(0))$ then we see that $L_1 = \{0, 1\}$ because both $h_f(x_1, 0)$ and $h_f(x_1, 1)$ equal x_1.

Continuing recursively, suppose that L_m is defined as above and consider $x_m \in f^{n-m}(K)$. Then $x_{m+1} \in f^{-1}(x_m)$ and by assumption (1), this is either in $[0, x^*)$ or in $(x^*, 1]$. Either way there is a unique choice of $f_L^{-1}(x_m)$ or $f_R^{-1}(x_m)$. Again, if $f^{n-m}(K) \cap [0, f(0)) = \emptyset$ then there is a unique $s_m \in \{0, 1\}$ that corresponds to the choice of $f_L^{-1}(x_m)$ or $f_R^{-1}(x_m)$. In this case we take $L_m = \{s_m\}$. In the other case, $f^{n-m}(K) \subseteq [0, f(0))$, we see that $L_m = \{0, 1\}$ because by definition, both $h_f(x_m, 0)$ and $h_f(x_m, 1)$ equal x_{m+1} in this case.

So for each $m < n$ we have defined a set $L_{m+1} \subseteq \{0, 1\}$ that encodes the choice of preimage of a point $x_m \in f^{n-m}(K)$ consistent with the bonding map. Notice that the specific point $\tilde{x}$ did not influence our choice of L_m. In fact the set L_m depends only upon

(1) whether $f^{n-m+1}(K) \subseteq [0, x^*)$ or $f^{n-m+1}(K) \subseteq (x^*, 1]$ and
(2) whether $f^{n-m}(K) \cap [0, f(0)) = \emptyset$ or $f^{n-m}(K) \subseteq [0, f(0))$.

This gives us

$$H^{-1}(\hat{K}) = f^n(K) \times \prod_{i=1}^{n} L_i \times \cdots.$$

The fact that the final factor in $H^{-1}(\hat{K})$ is $\{0, 1\}^{\mathbf{N}}$ follows in a fashion similar to that in the proof of the previous lemma, where instead of starting with $x_1 \in K$ we start with $x_n \in K$.

THEOREM A.3. *Let f be a Type B unimodal map. Let $n \in \mathbf{N}$ and let $K \subseteq [0, 1]$ be measurable. Then $\pi_n^{-1}(K) \in \mathcal{R}$.*

Proof. Let $\hat{K} = \pi_n^{-1}(K)$. We show that $H^{-1}(\hat{K})$ is a finite union of measurable rectangles in $[0, 1] \times \{0, 1\}^{\mathbf{N}}$.

Let $\mathcal{P}_n = \{0 = x_1 < x_2 < \ldots < x_k = 1\}$ be a partition of $[0, 1]$ such that

$$\mathcal{P}_n = \left(\bigcup_{m=0}^{n} f^{-m}(x^*)\right) \cup \left(\bigcup_{i=0}^{n} f^{-i}(f(0))\right).$$

Let $K = \bigcup_{j=1}^{p} K_j$ be the decomposition of K induced by $\mathcal{P}_n$. Then each K_j has the following properties:

(1) either $f^m(K_j) \subseteq [0, x^*)$ or $f^m(K_j) \subseteq (x^*, 1]$, and
(2) either $f^m(K_j) \cap [0, f(0)) = \emptyset$ or $f^m(K_j) \subseteq [0, f(0))$.

So each K_j satisfies the previous lemma. Because $\hat{K} = \pi_n^{-1}(K) = \bigcup_{j=1}^{p} \pi_n^{-1}(K_j)$ and the $\pi_n^{-1}(K_j)$s are disjoint, the result follows.

COROLLARY A.4. *The algebra of sets, $\mathcal{R}$, contains all of the open sets. Thus $\mathcal{S}$ contains the Borel σ-algebra and $\underset{\leftarrow}{\lambda}$ is a Borel measure.*

Proof. It is well known that a basis for the topology of $\underset{\leftarrow}{\lim}\{[0, 1], f\}$ is the collection $\{\pi_n^{-1}(U) | n \in \mathbf{N}$ and U an open set$\}$.

THEOREM A.5. *Let f be a unimodal Type B map. Suppose that for each open set U in $[0, 1]$, $f^k(U)$ has positive Lebesgue measure. Then $\underset{\leftarrow}{\lambda}$ is a Lebesgue-like measure.*

Proof. By the previous theorem we see that $\underset{\leftarrow}{\lambda}(\pi_n^{-1}(U)) > 0$ for each open set U in $[0, 1]$.

LEMMA A.6. *Let $Z \in \mathcal{S}$. For each $i \in \mathbf{N}$ let $L_i \subseteq \{0, 1\}$ be defined by $s_i \in L_i$ if, and only if, there is a point $(x_1, (s_1, s_2, \ldots s_i \ldots)) \in H^{-1}(Z)$. Then*

$$\underset{\leftarrow}{\lambda}(Z) \leq \lambda(\pi(Z)) \cdot \prod_{i=1}^{\infty} \frac{|L_i|}{2^i}.$$

Proof. For each $n \in \mathbf{N}$, $H^{-1}(Z) \subseteq \pi(Z) \times \prod_{i=1}^{n} L_i \times \{0, 1\}^{\mathbf{N}}$.

This lemma gives us the following two useful facts:

(1) Let $Z \in \mathcal{S}$. If $\pi(Z)$ has zero Lebesgue measure then $\underset{\leftarrow}{\lambda}(Z) = 0$.

(2) Let $Z \in \mathcal{S}$. If for infinitely many $i \in \mathbf{N}$ the choice of s_i is unique and $\pi_i(Z) \cap [0, f(0)) = \emptyset$ then $\underset{\leftarrow}{\lambda}(Z) = 0$.

Hence $\underset{\leftarrow}{\lambda}$ is a Lebesgue-like measure.

A.2. PROOF OF THEOREM 3

Let $R = \{\hat{x} \in \underset{\leftarrow}{\lim}\{[0, 1], f\} :$ there exists some $N \in \mathbf{N}$ such that $x_n \in [0, f^2(x^*))$ for all $n \geq N\}$. Because, in the case we are considering, $f(x) > x$ for $x \in (0, x^*)$, it follows that $x_n \to 0$ as $n \to \infty$. Thus for each $\hat{x} \in R$, $\sigma^n(\hat{x}) \to \hat{0}$ as $n \to \infty$. This implies that $R \subseteq B(\{\hat{0}\})$. By the definition of R, we see that $R \supseteq \pi^{-1}[0, f^2(x^*))$, so $\underset{\leftarrow}{\lambda}[R] > 0$. Because $\hat{0}$ is a single point it cannot have a proper subset that is an attractor. Thus, $\{\hat{0}\}$ is a metric attractor for $\underset{\leftarrow}{\lim}\{[0, 1], f\}$ and σ.

We next show that $\underset{\leftarrow}{\lambda}[R] = 1$. It will then follow that $\{\hat{0}\}$ is the only metric attractor for $\lim\{[0, 1], f\}$ and $\overleftarrow{\sigma}$. Let $K = \lim\{[0, 1], f\} \backslash R$ and pick $M \in \mathbf{N}$ large enough so that $f_L^{-M}[0, 1] \subseteq [0, f^2(x^*))$. Then if $\hat{x} \in K$, $\sigma^n(\hat{x}) \in K$ for all n. Thus if $(x_1, (s_1, s_2, \ldots)) \in [0, 1] \times \{0, 1\}^{\mathbf{N}}$ with $H(x_1, (s_1, s_2, \ldots)) = \hat{x}$, then $(s_1, s_2, \ldots)$ cannot have M adjacent 0's. It is not hard then to see that no point of $\Gamma[K]$ has a dense σ-orbit. By Walters (1982, Theorem 1.7), the set of points with a dense orbit, D, in this space has $\hat{\lambda}(D) = 1$. It follows then that $\hat{\lambda}(\Gamma[K]) = 0$. Q.E.D.

A.3. PROOF OF THEOREM 4

First of all, if f is a q.q. map, its second iterate f^2 can have at most three fixed points, namely, the unique fixed point of f, $\bar{x}$, plus, possibly, two fixed points making up a cycle of period two for f. Second, in this paper we are concerned with the case in which the restricted map $f|_{[f^2(x^*), f(x^*)]}$ is unimodal, i.e., when $f^2(x^*) < x^*$. If we now consider the graph of f^2 on $[f^2(x^*), f(x^*)]$ (which, as usual, we rescale to $[0, 1]$), we observe that, if the period-two cycle exists, its smaller element must lie in the subinterval $(0, x^*)$ and the larger one in the subinterval $(\bar{x}, 1)$. Thus, in our case, $q = f(q) = \bar{x}$.

Moreover, if $f(0) > q = \bar{x}$, each of the intervals $[0, \bar{x}]$ and $[\bar{x}, 1]$ is mapped onto itself by the second iterate of f, f^2, and therefore $\underset{\leftarrow}{\lim}\{[0, 1], f^2\} = \underset{\leftarrow}{\lim}\{[0, \bar{x}], f^2_{[0,\bar{x}]}\} \bigcup \underset{\leftarrow}{\lim}\{[\bar{x}, 1], f^2_{[\bar{x},1]}\}$ [cf. Ingram (1995), considering that, due to rescaling, the author's points a, b, c, p correspond to our $0, 1, x^*, \bar{x}$, and his "Type (1) map" corresponds to our "Type B map", respectively].

Consider now that $F_1 = f^2|_{[\bar{x},1]}$ is a Type A map and $F_2 = f^2|_{[0,\bar{x}]}$ is topologically conjugate to a Type A map $G = h \circ F_2 \circ h^{-1}$, with $h(x) = h^{-1}(x) = \bar{x} - x$. Using now the same argument as in the proof of Theorem 3 (and considering that $h^{-1}(0) = \bar{x}$), we can establish that the shift map σ associated with each of the two maps F_1, F_2 has a unique metric attractor $\hat{x} = (\bar{x}, \bar{x}, \ldots)$.

At this point, we invoke the well–known fact that $\varprojlim\{X, f\}$ is homeomorphic to $\{X, f^n\}$ for any $n \geq 1$ [see, Ingram (2000b, Corollary 1.71)] and the result is proved.

A.4. PROOF OF LEMMA 6

Notice that because f is continuous, each $f^{-n}(B)$ is open, and so $\Lambda = [0, 1] \setminus \bigcup_{n \in \mathbf{N}} f^{-n}(B)$ is closed and hence compact. Because the attractor P attracts Lebesgue almost every point in [0, 1] we see that Λ contains no intervals, and is therefore totally disconnected. To see that Λ is perfect, let $x \in \Lambda$ and $\epsilon > 0$. Suppose that every $z \in B_\epsilon(x)$ with $z \neq x$ is in the basin of attraction of P. Let $a = x - \epsilon$ and $b = x + \epsilon$. Each $z \in (a, x) \cup (x, b)$ is mapped into B by some iterate of f. This implies that for each such z there is a B_i and an integer n such that $z \in f^{-n}(B_i)$. Consider (a, x). If there are two different $n, m \in \mathbf{N}$ and i, j with $f^{-n}(B_i) \cap (a, x) \neq \emptyset$ and $f^{-m}(B_j) \cap (a, x) \neq \emptyset$, then because $B_i \cap B_j = \emptyset$, there must be some point $y \in (a, x)$ that is in the boundary of one of $f^{-m}(B_j)$ or $f^{-n}(B_i)$. This point y is then mapped to the boundary of B_i or B_j by f^n or f^m. By assumption 2c this point is not in the basin of P. Hence we see that there cannot be two different $n, m \in \mathbf{N}$ and i, j with $f^{-n}(B_i) \cap (a, x) \neq \emptyset$ and $f^{-m}(B_j) \cap (a, x) \neq \emptyset$. This implies that there is a single $n \in \mathbf{N}$ and i such that every point in (a, x) is in $f^{-n}(B_i)$. Hence x is in the boundary of $f^{-n}(B_i)$. By a similar argument for (x, b) we find some m and j such that x is in the boundary of $f^{-m}(B_j)$. Suppose, without loss of generality, that $m \geq n$. Then $f^m(x)$ is in the boundary of B_j and $f^m(x)$ is in the boundary of $f^m(f^{-n}(B_i)) = f^{m-n}(B_i) \subseteq B_k$ for some k. If $k \neq j$ then $f^m(x)$ is in the intersection $\partial B_k \cap \partial B_j$, which contradicts assumption 2(d). Thus $k = j$. This, though, implies that every point in (x, b) is also in $f^{-m}(B_j)$. So $f^m|_{(a,b)}$ is not monotone and in fact it "folds" at x. Thus x is a preimage of x^*. This is a contradiction because $x^* \in B(P)$ and $x \notin B(P)$.

A.5. PROOF OF THEOREM 5

Before we prove the theorem, we give a few lemmata. Let

$$\hat{B} = \{\hat{x} \in \varprojlim\{[0, 1], f\} : x_i \in B \text{ for all } i \in \mathbf{N}\}.$$

LEMMA A.7. *Let* $\hat{x} \in \varprojlim\{[0, 1], f\}$ *such that* $\hat{x} \notin \hat{B}$*; then* $\hat{x} \in B(\hat{\Lambda})$.

Proof. To show that $\hat{x} \in B(\hat{\Lambda})$, we show that $\omega_\sigma(\hat{x}) \subseteq \hat{\Lambda}$. Let $\hat{z} \in \omega_\sigma(\hat{x})$, and suppose that $\hat{z} \notin \hat{\Lambda}$. Then by the definition of $\hat{\Lambda}$, there is some integer N such that $z_N \notin \Lambda$. Because $z_N \notin \Lambda$, there is some $p \in \mathbf{N}$ such that $f^p(z_N) \in B$. This implies that $z_{p+N} \in B$. Because $\hat{z} \in \omega_\sigma(\hat{x})$, there is an increasing sequence of integers, $m_i \to \infty$, such that $\sigma^{m_i}(\hat{x}) \to \hat{z}$. Because the coordinate maps are continuous, we see that $\pi_N(\sigma^{m_i}(\hat{x})) \to \pi_N(\hat{z})$ as $i \to \infty$. This implies that $x_{N+m_i} \to z_N$ as $i \to \infty$, and because f is continuous, we see that $x_{N+m_i+p} \to z_{N+p} \in B$ as $i \to \infty$. Recall that B is an open set. Hence there is some $R \in \mathbf{N}$ such that $x_{N+m_r+p} \in B$ for all $r \geq R$.

We next show that this implies that $\hat{x} \in \hat{B}$, which will lead to our contradiction. We show this by showing that $x_i \in B$ for all $i \in \mathbf{N}$. Let $i \in \mathbf{N}$ and choose $r \geq R$ so large that

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$N + m_r + p \geq i$. Then by the above and our choice of r we see that $x_{N+m_r+p} \in B$. Because f maps B, into B, we see that

$$x_i = f^{N+m_r+p-i}(x_{N+m_r+p}) \in f^{N+m_r+p-i}(B) \subseteq B.$$

Hence $\hat{x} \in \hat{B}$, a contradiction.

It follows from the lemma and the definition of $\hat{B}$ that

$$B(\hat{\Lambda}) = \{\hat{x} : x_j \notin B \text{ for some } j \in \mathbf{N}\} = \underleftarrow{\lim}\{[0, 1], f\} \setminus \hat{B}.$$

Next we show that $\hat{\Lambda}$ is a metric attractor by showing that

$$\underleftarrow{\lambda}\,[B(\hat{\Lambda})] = 1.$$

LEMMA A.8. $\underleftarrow{\lambda}\,[B(\hat{\Lambda})] = 1$.

Proof. We show that $\underleftarrow{\lambda}[\hat{B}] = 0$; then because $B(\hat{\Lambda}) = \underleftarrow{\lim}\{[0, 1], f\} \setminus \hat{B}$, the result will follow. First, notice that for each $\hat{x} \in \hat{B}$, $x_1 \in B = \bigcup_{i=0}^{n-1} B_i$. So there is some specific interval, say B_i containing x_1. This gives a decomposition of $\hat{B}$ into sets $\hat{B}_i$, $0 \leq i \leq n-1$, by the first coordinate:

$$\hat{B} = \bigcup_{i=0}^{n-1} \hat{B}_i,$$

where $\hat{B}_i = \{\hat{x} \in \hat{B} : x_1 \in B_i\}$ for each $0 \leq i \leq n-1$.

Consider first $\hat{B}_{n-1}$. Each $\hat{x} \in \hat{B}_{n-1}$ has the property that $x_1 \in B_{n-1}$, $x_2 \in B_{n-2}$, $x_3 \in B_{n-3}, \ldots, x_n \in B_0$, $x_{n+1} \in B_{n-1}, \ldots$. This implies that $\hat{x} \in \hat{B}_{n-1}$ if and only if $\hat{x}$ has a backward itinerary of the form

$$\ldots V s_3 V s_2 V s_1 V$$

for some sequence $(\ldots, s_3, s_2, s_1)$ in 0's and 1's and word V that encodes the path of the intervals B_i. So $\hat{B}_{n-1}$ has the property that all of its points have infinitely many restrictions on their backward itinerary. Hence $\underleftarrow{\lambda}\,[\hat{B}_{n-1}] = 0$.

Next notice that $f^j(B_{n-j-1}) = B_{n-1}$, and so $\sigma^j(\hat{B}_{n-1}) = \hat{B}_{n-j-1}$ for all $0 \leq j \leq n-1$. It follows that for all $0 \leq j \leq n-1$, $\hat{B}_{n-j-1}$ has the property that all of its points have infinitely many restrictions on their backward itinerary. Thus $\underleftarrow{\lambda}[\hat{B}_i] = 0$ for all $0 \leq i \leq n-1$. Hence $\underleftarrow{\lambda}[\hat{B}] = 0$ and $\underleftarrow{\lambda}[B(\hat{\Lambda})] = 1$.

The theorem now follows because $\underleftarrow{\lambda}\,[B(\hat{\Lambda})] = 1$, and from the discussion before the theorem we see that $\sigma|_{\hat{\Lambda}}$ is topologically transitive.

A.6. PROOF OF LEMMA 7

By definition of Type B maps and attractors, f is unimodal (piecewise monotonic), surjective, and transitive on [0, 1]. Then, by Shultz's (2007) Theorems 9 and 11, f is topologically conjugate to a "restricted tent map" $T_s : [0, 1] \to [0, 1]$,

$$T_s = \begin{cases} sx - s + 2 & \text{for } 0 \leq x < 1 - 1/s \\ s - sx & \text{for } 1 - 1/s \leq x \leq 1, \end{cases}$$

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with slope s greater or equal to $\sqrt{2}$. This means that there exists a homeomorphism $\vartheta\colon[0,1]\to[0,1]$ such that $\vartheta\circ f = T_s\circ\vartheta$. Next, by Shultz's Theorem 9, a restricted tent map T_s is topologically exact if, and only if, $s>\sqrt{(2)}$, which occurs if, and only if, $T_s(0)<p$ ($p=\vartheta(\bar{x})$ being the unique fixed point of T_s and $\bar{x}$ the only fixed point of f). But $f(0)<\bar{x}$ implies $T_s(0)<p$ and therefore T_s is topologically exact and, because topological exactness is preserved by conjugacy, so is f.

A.7. PROOF OF THEOREM 6

In view of Lemma 7, it will be sufficient to prove the statement of the theorem for a unimodal map $f\colon[0,1]\to[0,1]$ with $f(1)=0$, which is topologically exact on $[0,1]$.

To begin, we mention some well-known properties of the one-sided shift space $\{0,1\}^{\mathbf{N}}$ with its shift map $\bar{\sigma}$ and the inverse limit space and its shift homeomorphism σ.[26] The interested reader should consult Walters (1982, sect. 1.5), for background on ergodic theory and Ingram (2000a, 2000b) for background on inverse limit theory.

First, it is well known that $\bar{\sigma}$ is a measure-preserving transformation and it is ergodic on $\{0,1\}^{\mathbf{N}}$. Every finite word, $w_0w_1\ldots w_k$, in 0 and 1 corresponds to a basic open set, a so-called *cylinder set*, in $\{0,1\}^{\mathbf{N}}$ defined by $\{(s_0s_1\ldots)\in\{0,1\}^{\mathbf{N}} : s_0=w_0, s_1=w_1,\ldots s_k=w_k\}=\{w_0\}\times\{w_1\}\ldots\{w_k\}\times\{0,1\}^{\mathbf{N}}$, which we denote by $\{w_0w_1\ldots w_k\}\times\{0,1\}^{\mathbf{N}}$.

Notice that

$$\bar{\sigma}^{-1}(\{w_0w_1\ldots w_k\}\times\{0,1\}^{\mathbf{N}})=\{0,1\}\times\{w_0w_1\ldots w_k\}\times\{0,1\}^{\mathbf{N}}.$$

In general

$$\bar{\sigma}^{-j}(\{w_0w_1\ldots w_k\}\times\{0,1\}^{\mathbf{N}})=\{0,1\}^j\times\{w_0w_1\ldots w_k\}\times\{0,1\}^{\mathbf{N}}.$$

Notice that the set

$$\mathcal{A}_{w_0w_1\ldots w_k}=\bigcup_{j=0}^{\infty}\bar{\sigma}^{-j}(\{w_0w_1\ldots w_k\}\times\{0,1\}^{\mathbf{N}})$$

is $\bar{\sigma}$-invariant; i.e.,

$$\bar{\sigma}^{-1}(\mathcal{A}_{w_0w_1\ldots w_k})=\mathcal{A}_{w_0w_1\ldots w_k}.$$

Because $\hat{\lambda}$ is ergodic with respect to $\bar{\sigma}$, it must be the case that $\hat{\lambda}(\mathcal{A}_{w_0w_1\ldots w_k})$ is either 0 or 1. Because $\mathcal{A}_{w_0w_1\ldots w_k}$ contains a nonempty open set, it is the case that $\mathcal{A}_{w_0w_1\ldots w_k}$ has $\hat{\lambda}$-measure 1.

Now we consider the inverse limit space and its shift homeomorphism. By definition, $\sigma(x_1,x_2,\ldots)=(x_2,x_3,\ldots)$. So if $(x_1,x_2,\ldots)\in\varprojlim\{[0,1],f\}$ then $\sigma^{-1}(x_1,x_2,\ldots)=(f(x_1),x_1,x_2,\ldots)$. Let $\hat{U}=\pi_n^{-1}(U)$. Then by properties of the inverse limit space, $\pi(\hat{U})=f^{n-1}(U)$, whereas $\sigma^{-1}(\hat{U})=\pi_{n+1}^{-1}(U)$ has π-image $f^n(U)$. In general, $\sigma^{-j}(\hat{U})=\pi_{n+j}^{-1}(U)$ and this set has π-image $f^{n+j-1}(U)$. These facts will be used extensively in the proof of the next lemma.

LEMMA A.9. *Let* $f : [0, 1] \to [0, 1]$ *be a unimodal, exact (l.e.o.) map with* $f(1) = 0$. *Let* $\hat{U} = \pi_n^{-1}(U)$ *be a basic open set in* $\varprojlim\{[0, 1], f\}$. *Then*

$$\bigcup_{j=0}^{\infty} \sigma^{-j}(\hat{U})$$

has $\underleftarrow{\lambda}$*-measure* 1.

Proof. By definition of $\hat{U} = \pi_n^{-1}(U)$ we see that

$$\pi(\hat{U}) = f^{n-1}(U)$$

and U is an open set in $[0, 1]$. Moreover, $\sigma^{-1}(\hat{U}) = \pi_{n+1}^{-1}(U)$, and in general

$$\sigma^{-j}(\hat{U}) = \pi_{n+j}^{-1}(U).$$

Notice also that

$$\pi(\sigma^{-j}(\hat{U})) = f^{n+j-1}(U).$$

Because f is topologically exact, there is an integer J such that $f^J(U) = [0, 1]$. Let $M = \max\{J, n\}$. Then $f^M(U) = [0, 1]$ and $[x^*, 1] \subseteq f^{M-1}(U)$. Hence if we let $j = M - n + 1$ then

$$\pi(\sigma^{-j}(\hat{U})) = f^{n+j-1}(U) = f^M(U) = [0, 1].$$

There is a small subinterval V of U that has the property that $f^{M-1}(V) = [x^*, 1]$ and $f^{M-1}|_V$ is one-to-one. So

$$\hat{V} = \pi_{M+1}^{-1}(V) \subseteq \pi_{M+1}^{-1}(U) = \sigma^{-j}(\hat{U})$$

and

$$\pi(\hat{V}) = f^M(V) = [0, 1],$$

whereas

$$\pi_2(\hat{V}) = f^{M-1}(V) = [x^*, 1].$$

In fact, because $f^{M-1}|_V$ is one-to-one, there is a unique finite word, $w_1 \dots w_{M-1}$, such that $f_{w_1 \dots w_{M-1}}^{-(M-1)}([x^*, 1]) = V$, where by $f_{w_1 \dots w_{M-1}}^{-(M-1)}$ we mean the composition of the branches of the inverse of f, f_L^{-1} and f_R^{-1} (coded as 0 and 1, respectively) in the pattern of the word $w_1 \dots w_{M-1}$. Because $[x^*, 1] = f_R^{-1}([0, 1])$, we see that

$$f_{1 w_1 \dots w_{M-1}}^{-M}([0, 1]) = V.$$

Notice also that $\hat{V}$ satisfies the assumptions of Lemma A.2. Hence

$$H^{-1}(\hat{V}) = f^M(V) \times \prod_{i=1}^{M} L_i \times \{0, 1\}^{\mathbf{N}},$$

where $\prod_{i=1}^{M} L_i = 1 w_1 \dots w_{M-1}$ and $f^M(V) = [0, 1]$. So

$$H^{-1}(\hat{V}) = [0, 1] \times \{1 w_1 w_2 \dots w_{M-1}\} \times \{0, 1\}^{\mathbf{N}}.$$

Now consider $\sigma^{-1}(\hat{V})$. Notice that

$$\pi(\sigma^{-1}(\hat{V})) = [0,1] = \pi_2(\sigma^{-1}(\hat{V})),$$

whereas

$$\pi_3(\sigma^{-1}(\hat{V})) = [x^*, 1]$$

and

$$\pi_{M+2}(\sigma^{-1}(\hat{V})) = V.$$

This implies that

$$H^{-1}(\sigma^{-1}(\hat{V})) = [0,1] \times \{0,1\} \times \{1w_1w_2 \ldots w_{M-1}\} \times \{0,1\}^{\mathbf{N}}$$

and in general we have

$$H^{-1}(\sigma^{-j}(\hat{V})) = [0,1] \times \{0,1\}^j \times \{1w_1 \ldots w_{M-1}\} \times \{0,1\}^{\mathbf{N}}.$$

Because

$$H^{-1}\left(\bigcup_{j=0}^{\infty} \sigma^{-j}(\hat{V})\right) = \bigcup_{j=0}^{\infty} H^{-1}(\sigma^{-j}(\hat{V}))$$

and because

$$\bigcup_{j=0}^{\infty} [0,1] \times \{0,1\}^j \times \{1w_1 \ldots w_{M-1}\} \times \{0,1\}^{\mathbf{N}}$$

$$= [0,1] \times \left(\bigcup_{j=0}^{\infty} \{0,1\}^j \times \{1w_1 \ldots w_{M-1}\} \times \{0,1\}^{\mathbf{N}}\right),$$

we can compute $\underset{\leftarrow}{\lambda}\,[\bigcup_{j=0}^{\infty} \sigma^{-j}(\hat{V})]$ by considering

$$\lambda \times \hat{\lambda}\left[[0,1] \times \left(\bigcup_{j=0}^{\infty} \{0,1\}^j \times \{1w_1 \ldots w_{M-1}\} \times \{0,1\}^{\mathbf{N}}\right)\right].$$

Clearly this is simply

$$1 \cdot \hat{\lambda}\left(\bigcup_{j=0}^{\infty} \{0,1\}^j \times \{1w_1 \ldots w_{M-1}\} \times \{0,1\}^{\mathbf{N}}\right),$$

which is $\hat{\lambda}(\mathcal{A}_{1w_1 \ldots w_{M-1}})$ in the notation of the discussion before the lemma. Again, because $\hat{\lambda}$ is ergodic and $\mathcal{A}_{1w_1 \ldots w_{M-1}}$ is $\bar{\sigma}$-invariant and contains a nonempty open set, $\hat{\lambda}(\mathcal{A}_{1w_1 \ldots w_{M-1}}) = 1$. Hence

$$\underset{\leftarrow}{\lambda}\left(\bigcup_{j=0}^{\infty} \sigma^{-j}(\hat{V})\right) = 1$$

and because $\hat{V} \subseteq \hat{U}$ the lemma follows.

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LEMMA A.10. *The set $\hat{D}$ of points with a dense σ-orbit has*

$$\underset{\leftarrow}{\lambda}(\hat{D}) = 1.$$

Proof. Let $\{U_m\}_{m\in\mathbf{N}}$ be a countable collection of basic open sets of $\underset{\leftarrow}{\lim}\{[0, 1], f\}$ that generates the topology on $\underset{\leftarrow}{\lim}\{[0, 1], f\}$. Then it is easy to see that $\hat{x}$ has a dense σ-orbit if, and only if,

$$\hat{x} \in \bigcap_{m\in\mathbf{N}} \left(\bigcup_{j=0}^{\infty} \sigma^{-j}(U_m) \right).$$

By the previous lemma we have that each set $\bigcup_{j=0}^{\infty} \sigma^{-j}(U_m)$ has $\underset{\leftarrow}{\lambda}$-measure 1. Hence the intersection of all such sets has $\underset{\leftarrow}{\lambda}$-measure 1. Therefore $\hat{D}$ has $\underset{\leftarrow}{\lambda}$-measure 1.

The proof of Theorem 6 can now be completed. Let $K \subsetneq \underset{\leftarrow}{\lim}\{[0, 1], f\}$, and let $B(K)$ be the basin of attraction for K under σ. Notice that $B(K) \subseteq \underset{\leftarrow}{\lim}\{[0, 1], f\} \setminus \hat{D}$. Because $\underset{\leftarrow}{\lambda}(\hat{D}) = 1$, we see that $\underset{\leftarrow}{\lambda}(B(K)) = 0$. Thus K is not a metric attractor under σ.