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ODD–EVEN DECOMPOSITION OF FUNCTIONS

IOSIF PINELIS

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Abstract

The main result of this note implies that any function from the product of several vector spaces to a vector space can be uniquely decomposed into the sum of mutually orthogonal functions that are odd in some of the arguments and even in the other arguments. Probabilistic notions and facts are employed to simplify statements and proofs.

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For any function $f\colon \mathbb{R}\to\mathbb{R}$, we have the known unique decomposition

$$f = f_1 + f_{-1},$$

where the functions f_1 and f_{-1} are even and odd, respectively, and they are given by the formulas

$$f_1(x) := \frac{f(x)+f(-x)}{2} = \mathsf{E}\, f(\varepsilon x) \quad \text{and} \quad f_{-1}(x) := \frac{f(x)-f(-x)}{2} = \mathsf{E}\, \varepsilon f(\varepsilon x)$$

for real x, where ε is a Rademacher random variable, uniformly distributed on the set $\{-1, 1\}$. Moreover, if $f \in L^2(\mathbb{R})$, then

$$\int_{\mathbb{R}} f_1(x) f_{-1}(x)\, dx = \frac{1}{4}\int_{\mathbb{R}} f(x)^2\, dx - \frac{1}{4}\int_{\mathbb{R}} f(-x)^2\, dx = 0; \tag{1}$$

that is, the even and odd parts of f are mutually orthogonal.

In this note, we shall extend these observations in several ways. First of all, we shall allow f to take several arguments and also vector values. Moreover, we shall use general involutions instead of the particular map $x \mapsto -x$. Finally, instead of the Lebesgue measure in the integrals in (1), we shall consider a general class of measures.

Let X be a set. Recall that an involution of X is a bijection $\iota\colon X \to X$ which coincides with its inverse ι^{-1}. For instance, the product (12)(34)(56)(78) of transpositions is an involution of the set $\{1, \dots, 9\}$. Another example of an involution is the inversion

$\overline{\mathbb{C}} \ni z \mapsto 1/\bar{z} \in \overline{\mathbb{C}}$ of the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, with $1/0 := \infty$ and $1/\infty := 0$.

Take now any natural number n. Suppose that we have mutually commuting involutions $\iota_1, \dots, \iota_n$ of the set X, so that $\iota_j\iota_k = \iota_k\iota_j$ for all j and k in the set $[n] := \{1, \dots, n\}$. We write the composition of involutions, as well as their action, simply as the corresponding concatenation: for any involutions ι and $\tilde{\iota}$ of X and any $x \in X$, $\iota\tilde{\iota} := \iota \circ \tilde{\iota}$ and $\iota x := \iota(x)$.

Example 1. Suppose that $X = X_1 \times \cdots \times X_n$, where $X_1, \dots, X_n$ are any sets with involutions $\tilde{\iota}_j : X_j \to X_j$ for $j \in [n]$. For each $j \in [n]$, let ι_j be the map

$$X \ni x = (x_1, \dots, x_n) \mapsto \iota_j x := (x_1, \dots, x_{j-1}, \tilde{\iota}_j x_j, x_{j+1}, \dots, x_n).$$

Then $\iota_1, \dots, \iota_n$ are mutually commuting involutions of the product set X.

For any $x \in X$ and any $\omega = (\omega_1, \dots, \omega_n) \in \{-1, 1\}^n$, let

$$\omega x := \Big(\prod_{j \in [n]:\ \omega_j = -1} \iota_j \Big) x;$$

in particular, if $\omega = (1, \dots, 1) \in \{-1, 1\}^n$ (so that the set $\{j \in [n] : \omega_j = -1\}$ is empty), then the above definition is understood as $\omega x := x$, in accordance with the convention that the product (that is, composition) of an empty family of transformations of a set X is the identity map of X.

For $j \in [n]$, let us say that a function $g : X \to V$ is j-odd if $g(\iota_j x) = -g(x)$ for all $x \in X$. Replacing here $-g(x)$ by $g(x)$, we get the definition of a j-even function. For any $J \subseteq [n]$, we say that g is J-odd-even if g is j-odd for each $j \in J$ and j-even for each $j \in J^c := [n] \setminus J$. Let us say that g is even if it is $\emptyset$-odd-even.

Let us say that a σ-algebra Σ over the set X is even if for any $A \in \Sigma$ and any $j \in [n]$ the set $\iota_j A := \{\iota_j x : x \in A\}$ is in Σ. A measure μ will be called even if it is defined on an even σ-algebra Σ over X and $\mu(\iota_j A) = \mu(A)$ for all $A \in \Sigma$ and $j \in [n]$.

Theorem 2. *Take any function $f : X \to V$. Then we have a decomposition of the form*

$$f = \sum_{J \subseteq [n]} f_J, \tag{2}$$

where the function $f_J : X \to V$ is J-odd-even for each $J \subseteq [n]$.

This decomposition is uniquely determined by the function f. More specifically, necessarily

$$f_J(x) = g_J(x) := \mathsf{E}\, \varepsilon_J f(\varepsilon x) \tag{3}$$

for all $J \subseteq [n]$ and $x \in X$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ is a uniformly distributed random element of the set $\{-1, 1\}^n$ and

$$\varepsilon_J := \prod_{j \in J} \varepsilon_j.$$

Moreover, the decomposition (2) is orthogonal in the following sense: if the vector space V is endowed with an inner product $\langle \cdot, \cdot \rangle$ and $f \in L^2(X, \mu, V)$ for some even measure μ, then for any distinct subsets J and K of the set $[n]$ the orthogonality condition $\int_X \langle f_J(x), f_K(x) \rangle\, \mu(dx) = 0$ holds.

PROOF. Note that

$$\begin{aligned}
\sum_{J\subseteq[n]} \mathsf{E}\, \varepsilon_J f(\varepsilon x) &= \mathsf{E}\, f(\varepsilon x) \sum_{J\subseteq[n]} \varepsilon_J \\
&= \mathsf{E}\, f(\varepsilon x) \prod_{j\in[n]} (1+\varepsilon_j) \\
&= \mathsf{E}\, f(\varepsilon x) 2^n\, \mathrm{I}\{\varepsilon_1 = \cdots = \varepsilon_n = 1\} \\
&= \mathsf{E}\, f(x) 2^n\, \mathrm{I}\{\varepsilon_1 = \cdots = \varepsilon_n = 1\} \\
&= f(x) 2^n\, \mathsf{P}(\varepsilon_1 = \cdots = \varepsilon_n = 1) = f(x)
\end{aligned}$$

for all $x \in X$, where $\mathrm{I}\{\cdot\}$ denotes the indicator function. So, (2) holds with f_J as in (3).

Next, take any $J \subseteq [n]$ and any $j \in [n]$, and let

$$\varepsilon^{(j)} := (\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_n).$$

Then $\varepsilon^{(j)}$ is uniformly distributed on the set $\{-1, 1\}^n$, so that, for each $x \in X$,

$$\mathsf{E}\, \varepsilon_J f(\varepsilon \iota_j x) = \mathsf{E}(\varepsilon^{(j)})_J f(\varepsilon^{(j)} \iota_j x) = \mathsf{E}(\varepsilon^{(j)})_J f(\varepsilon x) = \begin{cases} -\mathsf{E}\, \varepsilon_J f(\varepsilon x) & \text{if } j \in J, \\ \mathsf{E}\, \varepsilon_J f(\varepsilon x) & \text{if } j \in J^c; \end{cases}$$

the second of the equalities in the line above holds because $\varepsilon^{(j)} \iota_j x = \varepsilon x$, which in turn is true because the involutions $\iota_1, \ldots, \iota_n$ are mutually commuting (it is only here that this commutativity condition is used). This shows that the function f_J in (3) is indeed J-odd-even.

Further, take any distinct subsets J and K of the set $[n]$. Take any j that belongs to exactly one of the sets J and K. Then one of the functions f_J and f_K defined according to (3) is j-odd and the other one is j-even. So, the function $X \ni x \mapsto \langle f_J(x), f_K(x)\rangle \in \mathbb{R}$ is odd. Since the measure μ is even, it follows that indeed $\int_X \langle f_J(x), f_K(x)\rangle\, \mu(dx) = 0$.

Finally, take any $x \in X$ and suppose that (2) holds with some J-odd-even functions $f_J \colon X \to V$. Then, for any $J \subseteq [n]$,

$$f_J(\varepsilon x) = \varepsilon_J f_J(x)$$

and hence, for any $K \subseteq [n]$,

$$\mathsf{E}\, \varepsilon_K f_J(\varepsilon x) = \mathsf{E}\, \varepsilon_K \varepsilon_J f_J(x) = \mathrm{I}\{J = K\} f_J(x),$$

because the ε_j are independent zero-mean random variables. So, by the definition of $g_J(x)$ in (3) and the assumed decomposition (2),

$$g_K(x) = \mathsf{E}\, \varepsilon_K f(\varepsilon x) = \sum_{J\subseteq[n]} \mathsf{E}\, \varepsilon_K f_J(\varepsilon x) = \sum_{J\subseteq[n]} \mathrm{I}\{J = K\} f_J(x) = f_K(x),$$

which proves the uniqueness of the decomposition (2), that is, the first equality in (3).

This concludes the proof of the theorem. □

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REMARK 3. Another way to prove the uniqueness statement in Theorem 2 is to use the orthogonality statement. Indeed, without loss of generality we may assume that $V = \mathbb{R}$; otherwise, replace f by $\ell \circ f$, where ℓ is an arbitrary linear functional on the vector space V. Therefore, and because $\mathbb{R}$ is naturally endowed with an inner product, we may indeed use the orthogonality statement in Theorem 2 for any even measure μ. To choose such a measure most conveniently, fix any $x \in X$ and let μ be the uniform probability distribution on the set $\Omega x := \{\omega x \colon \omega \in \{-1, 1\}^n\}$. The mutual orthogonality in $L^2(X, \mu)$ of the summands f_J in the decomposition (2) implies that the f_J are uniquely determined (by f) μ-almost everywhere. Since μ is the uniform probability distribution on the finite set Ωx and $\Omega x \ni x$, we conclude that the values of $f_J(x)$ are uniquely determined by f for all $J \subseteq [n]$ and for each $x \in X$.

To illustrate Theorem 2, consider Example 1 with $n = 2$, $X_1 = X_2 = \mathbb{R}$ and $\tilde{\imath}_j u := -u$ for $j = 1, 2$ and $u \in \mathbb{R}$. Then, for any function f from $X = \mathbb{R}^2$ to any vector space V, we have the decomposition (2) with

$$\begin{aligned}
f_\emptyset(u, v) &= \tfrac{1}{4}(f(-u, -v) + f(-u, v) + f(u, -v) + f(u, v)),\\
f_{\{1\}}(u, v) &= \tfrac{1}{4}(-f(-u, -v) - f(-u, v) + f(u, -v) + f(u, v)),\\
f_{\{2\}}(u, v) &= \tfrac{1}{4}(-f(-u, -v) + f(-u, v) - f(u, -v) + f(u, v)),\\
f_{\{1,2\}}(u, v) &= \tfrac{1}{4}(f(-u, -v) - f(-u, v) - f(u, -v) + f(u, v))
\end{aligned}$$

for all $(u, v) \in X = \mathbb{R}^2$. Here, $f_\emptyset(u, v)$ is even in u and in v; $f_{\{1\}}(u, v)$ is odd in u and even in v; $f_{\{2\}}(u, v)$ is even in u and odd in v; and $f_{\{1,2\}}(u, v)$ is odd in u and in v.

The orthogonality statement in Theorem 2 means that, for any even measure μ and any $J \subseteq [n]$, the summand f_J in the decomposition (2) equals $P_J f$, where P_J is the orthogonal projector of $H_\mu := L^2(X, \mu, V)$ onto the linear subspace (say $H_{\mu;J}$) of H_μ consisting of all J-odd-even functions in H_μ. This fact can be naturally used to study the distance from a given function $f \in H_\mu$ to any such subspace $H_{\mu;J}$, or to the direct sum of some of these mutually orthogonal subspaces, possibly in contexts with the presence of stochastic noise.

In fact, this note was sparked by the paper [1], which is devoted to extraction of signals from noisy data. Of particular interest to us is decomposition (14) in [1], for $X = \mathbb{R}^2$, with involutions ρ, ρ_1, ρ_2 of $\mathbb{R}^2$ given by the formulas $\rho(u, v) := (v, u)$, $\rho_1(u, v) := (-u, v)$ and $\rho_2(u, v) := (u, -v) = -\rho_1(u, v)$ for $(u, v) \in \mathbb{R}^2$. More specifically, formula (14) in [1] provides a decomposition of an arbitrary function $q \colon \mathbb{R}^2 \to \mathbb{R}$ into the sum of five functions, denoted by $q^{(D_4)}, q^{(D_2)}, q_x^{(C_1)}, q_y^{(C_1)}, q^{(R_2)}$ in [1], where:

$q^{(D_4)}$ is even with respect to ρ, ρ_1 and ρ_2;
$q^{(D_2)}$ is odd with respect to ρ, and even with respect to ρ_1 and ρ_2;
$q_x^{(C_1)}$ is odd with respect to ρ_1 and even with respect to ρ_2;
$q_y^{(C_1)}$ is even with respect to ρ_1 and odd with respect to ρ_2;
$q^{(R_2)}$ is odd with respect to ρ_1 and ρ_2.

One may note that here the role of the involution ρ differs from the roles of each of the involutions ρ_1 and ρ_2.

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The set $\{\rho, \rho_1, \rho_2\}$ of involutions generates the dihedral group D_4 (of order eight) of all symmetries of a square [3]; in fact, each of the sets $\{\rho, \rho_1\}$ and $\{\rho, \rho_2\}$ already generates D_4. However, the group-generating involutions ρ and ρ_1 do not commute with each other. Moreover, since the dihedral group D_4 is not commutative, it cannot be generated by any set of mutually commuting involutions. On the other hand, the commutativity of the involutions $\iota_1, \ldots, \iota_n$ was needed in the proof of Theorem 2 to show that the summands f_J in the decomposition (2) are J-odd-even. So, it is unclear whether analogues of the decomposition (2) can exist for nonabelian groups.

In [2], a decomposition of functions f of several variables $x_1, \ldots, x_n$ into the sum of functions $f^{(J)}$ with $J \subseteq [n]$ was presented, where each function $f^{(J)}$ depends only on the subset $\{x_j : j \in J\}$ of variables. Certain sufficient conditions were given in [2] for the $f^{(J)}$ to be mutually orthogonal.

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IOSIF PINELIS, Department of Mathematical Sciences,
Michigan Technological University, Houghton, Michigan 49931, USA
e-mail: ipinelis@mtu.edu