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CLOSED FORMS FOR DEGENERATE BERNOULLI POLYNOMIALS

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Abstract

Qi and Chapman [‘Two closed forms for the Bernoulli polynomials’, *J. Number Theory* **159** (2016), 89–100] gave a closed form expression for the Bernoulli polynomials as polynomials with coefficients involving Stirling numbers of the second kind. We extend the formula to the degenerate Bernoulli polynomials, replacing the Stirling numbers by degenerate Stirling numbers of the second kind.

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1. Introduction

The nth Bernoulli polynomial $B_n(x)$ is given by

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \frac{te^{xt}}{e^t - 1}.$$

In particular, $B_n := B_n(0)$ is the nth Bernoulli number. The Bernoulli polynomials and the Bernoulli numbers have many important applications in number theory (see [8, 11, 13]). A basic symmetric formula for Bernoulli polynomials is

$$B_n(1 - x) = (-1)^n B_n(x) \tag{1.1}$$

for each $n \geq 0$.

Recently, with the help of Faá di Bruno’s formula, Qi and Chapman [12, Theorem 1.1] obtained a curious closed form for the Bernoulli polynomials:

$$B_n(x) = n! \sum_{k=1}^{n} k! \sum_{\substack{i+j=k \\ i,j\geq 0}} \sum_{\substack{u+v=n \\ u,v\geq 0}} \frac{(1-x)^{u+i}}{(u+i)!} \frac{x^{v+j}}{(v+j)!} (-1)^v \cdot \Big(\sum_{s=0}^{i} \sum_{t=0}^{j} (-1)^{s+t} \binom{u+i}{i-s} \binom{v+j}{j-t} S(u+s, s) S(v+t, t) \Big), \tag{1.2}$$

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where $S(n,k)$ is the Stirling number of the second kind defined by

$$\sum_{n=k}^{\infty}\frac{S(n,k)}{n!}t^n=\frac{(e^t-1)^k}{k!}.$$

The formula (1.2) implies (1.1). In fact, replacing x by $1-x$ in (1.2) gives

$$B_n(1-x)=n!\sum_{k=1}^{n}k!\sum_{\substack{i+j=k\\ i,j\geq 0}}\sum_{\substack{u+v=n\\ u,v\geq 0}}\frac{x^{u+i}}{(u+i)!}\frac{(1-x)^{v+j}}{(v+j)!}(-1)^{n-u}$$
$$\cdot\Big(\sum_{s=0}^{i}\sum_{t=0}^{j}(-1)^{s+t}\binom{u+i}{i-s}\binom{v+j}{j-t}S(u+s,s)S(v+t,t)\Big)=(-1)^nB_n(x).$$

It is well known that $S(n,k)$ counts the ways of partitioning $\{1,\ldots,n\}$ into k nonempty subsets. A natural generalisation of $S(n,k)$ is the r-associated Stirling number of the second kind: $S_r(n,k)$ counts the ways of partitioning $\{1,\ldots,n\}$ into k subsets such that each subset contains at least r elements, with $S_r(n,k)=0$ if $n<rk$. Clearly, $S_1(n,k)=S(n,k)$. From [4, page 222], $S_r(n,k)$ has the generating function

$$\sum_{n,k\geq 0}^{\infty}\frac{S_r(n,k)}{n!}u^kt^n=\exp\Big(u\sum_{j=r}^{\infty}\frac{t^j}{j!}\Big),$$

that is,

$$\sum_{n=rk}^{\infty}\frac{S_r(n,k)}{n!}t^n=\frac{1}{k!}\Big(e^t-1-t-\frac{t^2}{2!}-\cdots-\frac{t^{r-1}}{(r-1)!}\Big)^k.$$

In particular,

$$\sum_{n=2k}^{\infty}\frac{S_2(n,k)}{n!}t^n=\frac{(e^t-1-t)^k}{k!}=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(e^t-1)^j(-t)^{k-j}$$
$$=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-t)^{k-j}\cdot j!\sum_{i=j}^{\infty}\frac{S(i,j)}{i!}t^i.$$

Comparing the coefficients of t^n on both sides of this identity,

$$S_2(n,k)=\sum_{j=0}^{k}(-1)^{k-j}\binom{n}{k-j}S(n-k+j,j). \tag{1.3}$$

It is not difficult to verify that (1.2) can be rewritten as

$$B_n(x)=n!\sum_{k=1}^{n}(-1)^kk!\sum_{\substack{i+j=k\\ i,j\geq 0}}\sum_{\substack{u+v=n\\ u,v\geq 0}}(-1)^v\frac{(1-x)^{u+i}}{(u+i)!}\frac{x^{v+j}}{(v+j)!}S_2(u+i,i)S_2(v+j,j). \tag{1.4}$$

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Carlitz [2] considered a degenerate extension of the Bernoulli polynomials. The nth degenerate Bernoulli polynomial $\beta_n(\lambda, x)$ is defined by

$$\sum_{n=0}^{\infty} \frac{\beta_n(\lambda, x)}{n!} t^n = \frac{t}{(1+\lambda t)^{1/\lambda} - 1} (1+\lambda t)^{x/\lambda}.$$

Since

$$\lim_{\lambda \to 0} (1+\lambda t)^{1/\lambda} = \Big(\lim_{\lambda \to 0} (1+\lambda t)^{1/\lambda t} \Big)^t = e^t,$$

$$\lim_{\lambda \to 0} \beta_n(\lambda, x) = B_n(x).$$

The degenerate Bernoulli number $\beta_n(\lambda)$ is given by $\beta_n(\lambda) := \beta_n(\lambda, 0)$. It is not difficult to verify that

$$\beta_n(\lambda, x) = \sum_{k=0}^{n} \binom{n}{k} \beta_k(\lambda) \cdot x(x-1)(x-2) \cdots (x-(n-k-1)\lambda).$$

In [2], Carlitz extended the Staudt–Clausen congruence to the degenerate Bernoulli numbers and, in [7], Howard obtained an explicit form for $\beta_n(\lambda)$ by using the Stirling numbers of the first kind.

Carlitz [3] defined the degenerate extensions of $S(n, k)$ by

$$\sum_{n=k}^{\infty} \frac{S(n, k|\lambda)}{n!} t^n = \frac{1}{k!} ((1+\lambda t)^{1/\lambda} - 1)^k.$$

Howard [6, (2.8)] considered the degenerate 2-associated Stirling number of the second kind, which is given by

$$\sum_{n=2k}^{\infty} \frac{S_2(n, k|\lambda)}{n!} t^n = \frac{1}{k!} ((1+\lambda t)^{1/\lambda} - 1 - t)^k.$$

In particular, $S(n, k|\lambda) = 0$ if $0 \le n < k$ and $S_2(n, k|\lambda) = 0$ if $0 \le n < 2k$. Clearly,

$$\lim_{\lambda \to 0} S(n, k|\lambda) = S(n, k), \qquad \lim_{\lambda \to 0} S_2(n, k|\lambda) = S_2(n, k).$$

For more results on degenerate Bernoulli polynomials and degenerate Stirling numbers, see [1, 5, 9, 10, 14, 15].

In this paper, we shall extend (1.4) to the degenerate Bernoulli polynomials.

THEOREM 1.1. *We have*

$$\beta_n(\lambda, x) = n! \sum_{k=1}^{n} (-1)^k k! \sum_{\substack{i+j=k \\ i,j \ge 0}} \sum_{\substack{u+v=n \\ u,v \ge 0}} \frac{(1-x)^{u+i}}{(u+i)!} \frac{x^{v+j}}{(v+j)!} \cdot (-1)^v S_2\Big(u+i, i \Big| \frac{\lambda}{1-x}\Big) S_2\Big(v+j, j \Big| -\frac{\lambda}{x}\Big). \tag{1.5}$$

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Similarly to (1.3),

$$S_2(n,k|\lambda) = \sum_{j=0}^{k}(-1)^{k-j}\binom{n}{k-j}S(n-k+j, j|\lambda).$$

So, (1.5) is equivalent to

$$\beta_n(\lambda,x) = n!\sum_{k=1}^{n}k!\sum_{\substack{i+j=k\\ i,j\geq 0}}\sum_{\substack{u+v=n\\ u,v\geq 0}}\frac{(1-x)^{u+i}}{(u+i)!}\frac{x^{v+j}}{(v+j)!}(-1)^v \cdot\bigg(\sum_{s=0}^{i}\sum_{t=0}^{j}(-1)^{s+t}\binom{u+i}{i-s}\binom{v+j}{j-t}S\Big(u+s,s\Big|\frac{\lambda}{1-x}\Big)S\Big(v+t,t\Big|-\frac{\lambda}{x}\Big)\bigg). \tag{1.6}$$

Furthermore, replacing x by $1-x$ and λ by $-\lambda$ in (1.5),

$$\beta_n(-\lambda,1-x) = n!\sum_{k=1}^{n}(-1)^k k!\sum_{\substack{i+j=k\\ i,j\geq 0}}\sum_{\substack{u+v=n\\ u,v\geq 0}}\frac{x^{u+i}}{(u+i)!}\cdot\frac{(1-x)^{v+j}}{(v+j)!} \cdot(-1)^{n-u}S_2\Big(u+i,i\Big|-\frac{\lambda}{x}\Big)S_2\Big(v+j,j\Big|\frac{\lambda}{1-x}\Big),$$

that is, $\beta_n(\lambda,x)$ satisfies the symmetric relation

$$\beta_n(-\lambda,1-x) = (-1)^n\beta_n(\lambda,x), \tag{1.7}$$

which is a degenerate extension of (1.1).

Note that the right-hand side of (1.6) involves five summations. We propose another similar explicit expression of the degenerate Bernoulli polynomials, which involves fewer summations.

THEOREM 1.2. *We have*

$$\beta_n(\lambda,x) = n!\sum_{k=1}^{n}k!\sum_{\substack{i+j\leq k\\ i,j\geq 0}}\frac{(-1)^{i+j}}{(k-i-j)!}\sum_{\substack{u+v=n\\ u,v\geq 0}}\frac{(1-x)^{u+i}}{(u+i)!}\frac{x^{v+j}}{(v+j)!} \cdot(-1)^v S\Big(u+i,i\Big|\frac{\lambda}{1-x}\Big)S\Big(v+j,j\Big|-\frac{\lambda}{x}\Big). \tag{1.8}$$

Letting λ tend to 0 in (1.8) gives a new closed form for the Bernoulli polynomials.

COROLLARY 1.3. *We have*

$$B_n(x) = n!\sum_{k=1}^{n}k!\sum_{\substack{i+j\leq k\\ i,j\geq 0}}\frac{(-1)^{i+j}}{(k-i-j)!}\sum_{\substack{u+v=n\\ u,v\geq 0}}\frac{(1-x)^{u+i}}{(u+i)!}\frac{x^{v+j}}{(v+j)!} \cdot(-1)^v S(u+i,i)S(v+j,j). \tag{1.9}$$

Evidently, (1.8) also implies (1.7) by replacing x and λ by $1-x$ and $-\lambda$, respectively.

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2. Proofs of Theorems 1.1 and 1.2

In this section, we shall prove Theorems 1.1 and 1.2. We introduce the Bell polynomial $\mathcal{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ given by

$$\mathcal{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) := n! \sum_{\substack{j_1, j_2, \ldots, j_{n-k+1} \geq 0 \\ j_1 + j_2 + \cdots + j_{n-k+1} = k \\ j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n}} \prod_{i=1}^{n-k+1} \frac{1}{j_i!} \cdot \left(\frac{x_i}{i!}\right)^{j_i}.$$

The classical Faá di Bruno formula (see [4, Theorem A, page 137]) is

$$\frac{d^n}{dz^n} f(g(z)) = \sum_{k=1}^{n} f^{(k)}(g(z)) \cdot \mathcal{B}_{n,k}(g'(z), g''(z), \ldots, g^{(n-k+1)}(z)) \tag{2.1}$$

for any n-times differentiable functions $f(z)$ and $g(z)$. The next lemma gives some special values of Bell polynomials.

Lemma 2.1. *Let $g(z)$ be an analytic function of z. Then*

$$\sum_{n=k}^{\infty} \mathcal{B}_{n,k}(g'(0), g''(0), \ldots, g^{(n-k+1)}(0)) \frac{t^n}{n!} = \frac{1}{k!}(g(t) - g(0))^k. \tag{2.2}$$

Proof. From [3, page 136, formula (30)],

$$\sum_{n=k}^{\infty} \mathcal{B}_{n,k}(u_1, u_2, \ldots, u_{n-k+1}) \frac{t^n}{n!} = \frac{1}{k!}\left(\sum_{h=1}^{\infty} u_h \frac{t^h}{h!}\right)^k$$

for any sequence $u_0, u_1, u_2, \ldots \in \mathbb{C}$. The desired result (2.2) follows because

$$\sum_{h=1}^{\infty} g^{(h)}(0) \frac{t^h}{h!} = \sum_{h=0}^{\infty} g^{(h)}(0) \frac{t^h}{h!} - g(0) = g(t) - g(0).$$

□

Proof of Theorem 1.1. Write

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_n(\lambda, x) \frac{t^n}{n!} &= \frac{t}{(1+\lambda t)^{1/\lambda} - 1}(1+\lambda t)^{x/\lambda} \\
&= \frac{t}{(1+\lambda t)^{(1-x)/\lambda} - (1+\lambda t)^{-x/\lambda}} = f(g(t)),
\end{aligned}$$

where $f(z) := 1/z$ and

$$g(t) := \frac{(1+\lambda t)^{(1-x)/\lambda} - (1+\lambda t)^{-x/\lambda}}{t}.$$

In particular, we set

$$g(0) = \lim_{t \to 0} g(t) = \lim_{t \to 0}\left(\frac{(1+\lambda t)^{(1-x)/\lambda} - 1}{t} - \frac{(1+\lambda t)^{-x/\lambda} - 1}{t}\right) = 1.$$

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By Faá di Bruno's formula (2.1), since $f^{(k)}(z) = (-1)^k k!/z^k$,

$$\begin{aligned}\beta_n(\lambda, x) &= \lim_{z\to 0} \frac{d^n}{dz^n} f(g(z)) = \lim_{z\to 0} \sum_{k=1}^{n} f^{(k)}(g(z)) \mathcal{B}_{n,k}(g'(z), g''(z), \dots, g^{(n-k+1)}(z)) \\ &= \sum_{k=1}^{n} \frac{(-1)^k k!}{g(0)^{k+1}} \mathcal{B}_{n,k}(g'(0), g''(0), \dots, g^{(n-k+1)}(0)). \qquad (2.3)\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}\sum_{n=k}^{\infty} \mathcal{B}_{n,k}(g'(0), \dots, g^{(n-k+1)}(0)) \frac{t^n}{n!} &= \frac{1}{k!} \Big(\frac{(1+\lambda t)^{(1-x)/\lambda} - (1+\lambda t)^{-x/\lambda}}{t} - 1 \Big)^k \\ &= \frac{1}{k!} \Big(\frac{(1+\lambda t)^{(1-x)/\lambda} - 1 - (1-x)t}{t} - \frac{(1+\lambda t)^{-x/\lambda} - 1 - (-x)t}{t} \Big)^k. \qquad (2.4)\end{aligned}$$

From the definition of $S_2(n, k|\lambda)$,

$$\Big(\frac{(1+\lambda t)^{z/\lambda} - 1 - zt}{t} \Big)^k = \frac{1}{t^k} \Big(\Big(1 + \frac{\lambda}{z} \cdot zt \Big)^{z/\lambda} - 1 - zt \Big)^k = \frac{k!}{t^k} \sum_{n=k}^{\infty} S_2\Big(n, k \Big| \frac{\lambda}{z}\Big) \cdot \frac{z^n t^n}{n!},$$

so

$$\begin{aligned}&\sum_{n=k}^{\infty} \mathcal{B}_{n,k}(g'(0), \dots, g^{(n-k+1)}(0)) \frac{t^n}{n!} \\ &\quad= \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \Big(\frac{(1+\lambda t)^{(1-x)/\lambda} - 1 - (1-x)t}{t} \Big)^i \Big(\frac{(1+\lambda t)^{-x/\lambda} - 1 - (-x)t}{t} \Big)^{k-i} \\ &\quad= \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \Big(i! \sum_{u=0}^{\infty} S_2\Big(u+i, i \Big| \frac{\lambda}{1-x}\Big) \frac{(1-x)^{u+i} t^u}{(u+i)!} \Big) \\ &\qquad\quad \cdot \Big((k-i)! \sum_{v=0}^{\infty} S_2\Big(v+k-i, k-i \Big| -\frac{\lambda}{x}\Big) \frac{(-x)^{v+k-i} t^v}{(v+k-i)!} \Big).\end{aligned}$$

Comparing the coefficients of t^n on both sides of this identity,

$$\begin{aligned}&\frac{\mathcal{B}_{n,k}(g'(0), \dots, g^{(n-k+1)}(0))}{n!} \\ &\quad= \sum_{\substack{i+j=k \\ i,j\ge 0}} (-1)^j \sum_{\substack{u+v=n \\ u,v\ge 0}} S_2\Big(u+i, i \Big| \frac{\lambda}{1-x}\Big) S_2\Big(v+j, j \Big| -\frac{\lambda}{x}\Big) \frac{(1-x)^{u+i}}{(u+i)!} \frac{(-x)^{v+j}}{(v+j)!}. \qquad (2.5)\end{aligned}$$

It is not difficult to check that (1.5) follows from (2.3) and (2.5). □

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PROOF OF THEOREM 1.2 AND COROLLARY 1.3. From (2.4),

$$\sum_{n=k}^{\infty}\mathcal{B}_{n,k}(g'(0),\dots,g^{(n-k+1)}(0))\frac{t^n}{n!}=\frac{1}{k!}\Big(\frac{(1+\lambda t)^{(1-x)/\lambda}-1}{t}-\frac{(1+\lambda t)^{-x/\lambda}-1}{t}-1\Big)^k$$
$$=\frac{1}{k!}\sum_{\substack{i+j\le k\\ i,j\ge 0}}\frac{(-1)^{k-i}}{i!\,j!\,(k-i-j)!}\Big(\frac{(1+\lambda t)^{(1-x)/\lambda}-1}{t}\Big)^i\Big(\frac{(1+\lambda t)^{-x/\lambda}-1}{t}\Big)^j.$$

Since

$$\Big(\frac{(1+\lambda t)^{z/\lambda}-1}{t}\Big)^j=\frac{1}{t^j}\Big(\Big(1+\frac{\lambda}{z}\cdot zt\Big)^{z/\lambda}-1\Big)^j=\frac{j!}{t^j}\sum_{n=j}^{\infty}S\Big(n,j\Big|\frac{\lambda}{z}\Big)\frac{z^nt^n}{n!}, \tag{2.6}$$

$$\sum_{n=k}^{\infty}\mathcal{B}_{n,k}(g'(0),\dots,g^{(n-k+1)}(0))\frac{t^n}{n!}=\frac{1}{k!}\Big(\frac{(1+\lambda t)^{(1-x)/\lambda}-1}{t}-\frac{(1+\lambda t)^{-(x/\lambda)}-1}{t}-1\Big)^k$$
$$=\sum_{\substack{i+j\le k\\ i,j\ge 0}}\frac{(-1)^{k-i}}{i!\,j!\,(k-i-j)!}\Big(i!\sum_{u=0}^{\infty}S\Big(u+i,i\Big|\frac{\lambda}{1-x}\Big)\frac{(1-x)^{u+i}t^u}{(u+i)!}\Big)$$
$$\cdot\Big(j!\sum_{v=0}^{\infty}S\Big(v+j,j\Big|-\frac{\lambda}{x}\Big)\frac{(-x)^{v+j}t^v}{(v+j)!}\Big).$$

Comparing the coefficients of t^n on both sides of this identity,

$$\frac{\mathcal{B}_{n,k}(g'(0),\dots,g^{(n-k+1)}(0))}{n!}$$
$$=\sum_{\substack{i+j\le k\\ i,j\ge 0}}\frac{(-1)^{k-i}}{(k-i-j)!}\sum_{\substack{u+v=n\\ u,v\ge 0}}S\Big(u+i,i\Big|\frac{\lambda}{1-x}\Big)S\Big(v+j,j\Big|-\frac{\lambda}{x}\Big)\frac{(1-x)^{u+i}}{(u+i)!}\frac{(-x)^{v+j}}{(v+j)!}.$$

This gives (1.8) and also (1.9). □

3. Closed forms for further degenerate sequences

From (2.4),

$$\sum_{n=1}^{\infty}\beta_n(\lambda,x)\frac{t^n}{n!}=\sum_{n=1}^{\infty}\frac{t^n}{n!}\Big(\sum_{k=1}^{n}(-1)^kk!\mathcal{B}_{n,k}(g'(0),\dots,g^{(n-k+1)}(0))\Big)$$
$$=\sum_{k=1}^{\infty}(-1)^kk!\sum_{n=k}^{\infty}\mathcal{B}_{n,k}(g'(0),\dots,g^{(n-k+1)}(0))\frac{t^n}{n!}$$
$$=\sum_{k=1}^{\infty}(-1)^k\Big(\frac{(1+\lambda t)^{(1-x)/\lambda}-(1+\lambda t)^{-x/\lambda}}{t}-1\Big)^k. \tag{3.1}$$

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According to our discussions in Section 2, both (1.5) and (1.8) can be proved by comparing the coefficients of t^n on both sides of (3.1). In fact, (3.1) also can be directly deduced from the observation

$$1+\sum_{k=1}^{\infty}(-1)^k\Big(\frac{(1+\lambda t)^{(1-x)/\lambda}-(1+\lambda t)^{-x/\lambda}}{t}-1\Big)^k=\frac{t}{(1+\lambda t)^{(1-x)/\lambda}-(1+\lambda t)^{-x/\lambda}}. \quad (3.2)$$

Motivated by (3.2), we may study closed forms for other degenerate sequences. Define the degenerate Euler polynomial $\epsilon_n(\lambda, x)$ by

$$\sum_{n=0}^{\infty}\epsilon_n(\lambda,x)\frac{t^n}{n!}=\frac{2}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda}.$$

In particular,

$$\lim_{\lambda\to 0}\epsilon_n(\lambda,x)=E_n(x),$$

where $E_n(x)$ is the classical Euler polynomial given by

$$\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}=\frac{2e^{xt}}{e^t+1}.$$

Clearly,

$$\begin{aligned}\frac{2(1+\lambda t)^{x/\lambda}}{(1+\lambda t)^{1/\lambda}+1}&=\frac{2}{(1+\lambda t)^{(1-x)/\lambda}+(1+\lambda t)^{-x/\lambda}}\\&=\frac{1}{1+\frac{1}{2}((1+\lambda t)^{(1-x)/\lambda}+(1+\lambda t)^{-x/\lambda}-2)}\\&=\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k}\cdot((1+\lambda t)^{(1-x)/\lambda}+(1+\lambda t)^{-x/\lambda}-2)^k. \quad (3.3)\end{aligned}$$

In view of (2.6),

$$\begin{aligned}((1+\lambda t)^{(1-x)/\lambda}+(1+\lambda t)^{-x/\lambda}-2)^k&=\sum_{i=0}^{k}\binom{k}{i}((1+\lambda t)^{(1-x)/\lambda}-1)^i((1+\lambda t)^{-x/\lambda}-1)^{k-i}\\&=\sum_{i=0}^{k}\binom{k}{i}\Big(i!\sum_{u=i}^{\infty}S\Big(u,i\Big|\frac{\lambda}{1-x}\Big)\frac{(1-x)^u t^u}{u!}\Big)\\&\quad\cdot\Big((k-i)!\sum_{v=k-i}^{\infty}S\Big(v,k-i\Big|-\frac{\lambda}{x}\Big)\frac{(-x)^v t^v}{v!}\Big).\end{aligned}$$

Comparing the coefficients of t^n on both sides of this identity gives the following result.

THEOREM 3.1. *We have*

$$\epsilon_n(\lambda,x)=n!\sum_{k=0}^{n}\frac{(-1)^k k!}{2^k}\sum_{\substack{i+j=k\\ i,j\ge 0}}\sum_{\substack{u+v=n\\ u,v\ge 0}}\frac{(1-x)^u}{u!}\frac{x^v}{v!}(-1)^v S\Big(u,i\Big|\frac{\lambda}{1-x}\Big)S\Big(v,j\Big|-\frac{\lambda}{x}\Big). \quad (3.4)$$

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In particular, letting $\lambda \to 0$ in (3.4),

$$E_n(x) = n! \sum_{k=0}^{n} \frac{(-1)^k k!}{2^k} \sum_{\substack{i+j=k \\ i,j\geq 0}} \sum_{\substack{u+v=n \\ u,v\geq 0}} \frac{(1-x)^u}{u!} \frac{x^v}{v!} (-1)^v S(u,i) S(v,j).$$

Of course, we can rewrite (3.3) as

$$\begin{aligned}\frac{2}{(1+\lambda t)^{(1-x)/\lambda} + (1+\lambda t)^{-x/\lambda}} &= 2\sum_{k=0}^{\infty} (-1)^k ((1+\lambda t)^{(1-x)/\lambda} + (1+\lambda t)^{-x/\lambda} - 1)^k \\ &= 2\sum_{k=0}^{\infty} (-1)^k \sum_{i=0}^{k} \binom{k}{i} ((1+\lambda t)^{(1-x)/\lambda} - 1)^i (1+\lambda t)^{-(k-i)x/\lambda}.\end{aligned}$$

Note that

$$(1+\lambda t)^{z/\lambda} = \sum_{u=0}^{\infty} \binom{z/\lambda}{u} \lambda^u t^u = \sum_{u=0}^{\infty} (z|\lambda)_u \frac{t^u}{u!},$$

where

$$(z|\lambda)_u = \begin{cases} z(z-\lambda)\cdots(z-(u-1)\lambda) & \text{if } u \geq 1, \\ 1 & \text{if } u = 0. \end{cases}$$

From this,

$$\epsilon_n(\lambda, x) = 2n! \sum_{k=0}^{n} (-1)^k k! \sum_{\substack{i+j=k \\ i,j\geq 0}} \frac{1}{j!} \sum_{\substack{u+v=n \\ u,v\geq 0}} \frac{(1-x)^u}{u!} \frac{(-jx|\lambda)_v}{v!} S\left(u, i \,\middle|\, \frac{\lambda}{1-x}\right). \tag{3.5}$$

Symmetrically,

$$\epsilon_n(\lambda, x) = 2n! \sum_{k=0}^{n} (-1)^k k! \sum_{\substack{i+j=k \\ i,j\geq 0}} \frac{1}{i!} \sum_{\substack{u+v=n \\ u,v\geq 0}} \frac{(i-ix|\lambda)_u}{u!} \frac{(-x)^v}{v!} S\left(v, j \,\middle|\, -\frac{\lambda}{x}\right). \tag{3.6}$$

Since $(z|\lambda)_k$ tends to z^k as $\lambda \to 0$,

$$\begin{aligned} E_n(x) &= 2n! \sum_{k=0}^{n} (-1)^k k! \sum_{\substack{i+j=k \\ i,j\geq 0}} \frac{1}{j!} \sum_{\substack{u+v=n \\ u,v\geq 0}} \frac{(1-x)^u}{u!} \frac{j^v x^v}{v!} (-1)^v S(u,i) \\ &= 2n! \sum_{k=0}^{n} (-1)^k k! \sum_{\substack{i+j=k \\ i,j\geq 0}} \frac{1}{i!} \sum_{\substack{u+v=n \\ u,v\geq 0}} \frac{i^u (1-x)^u}{u!} \frac{x^v}{v!} (-1)^v S(v,j). \end{aligned}$$

For each integer $r \geq 1$, define the degenerate Bernoulli polynomial of order r by

$$\sum_{n=0}^{\infty} \frac{\beta_n^{(r)}(\lambda, x)}{n!} t^n = \left(\frac{t}{(1+\lambda t)^{1/\lambda} - 1}\right)^r (1+\lambda t)^{x/\lambda}$$

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and the degenerate Euler polynomial of order r by

$$\sum_{n=0}^{\infty} \frac{\epsilon_n^{(r)}(\lambda, x)}{n!} t^n = \Big(\frac{2}{(1+\lambda t)^{1/\lambda}+1}\Big)^r (1+\lambda t)^{x/\lambda}.$$

Clearly,

$$\begin{aligned}\Big(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\Big)^r (1+\lambda t)^{rx/\lambda} &= \frac{1}{(1+\frac{1}{t}\{(1+\lambda t)^{(1-x)/\lambda}-(1+\lambda t)^{-x/\lambda}\}-1)^r} \\ &= \sum_{k=0}^{\infty}(-1)^k \binom{k+r-1}{r-1}\Big(\frac{(1+\lambda t)^{(1-x)/\lambda}-1-(1-x)t}{t} - \frac{(1+\lambda t)^{-x/\lambda}-1-(-x)t}{t}\Big)^k \\ &= \sum_{k=0}^{\infty}(-1)^k \binom{k+r-1}{r-1}\Big(\frac{(1+\lambda t)^{(1-x)/\lambda}-1}{t} - \frac{(1+\lambda t)^{-x/\lambda}-1}{t} - 1\Big)^k.\end{aligned}$$

Hence,

$$\begin{aligned}\beta_n^{(r)}(\lambda, rx) = n! \sum_{k=1}^{n} (-1)^k k! \binom{k+r-1}{r-1} \sum_{\substack{i+j=k\\ i,j\geq 0}} \sum_{\substack{u+v=n\\ u,v\geq 0}} \frac{(1-x)^{u+i}}{(u+i)!}\frac{x^{v+j}}{(v+j)!} \\ \cdot (-1)^v S_2\Big(u+i, i \Big| \frac{\lambda}{1-x}\Big) S_2\Big(v+j, j\Big| -\frac{\lambda}{x}\Big)\end{aligned}$$

and

$$\begin{aligned}\beta_n(\lambda, rx) = n! \sum_{k=1}^{n} k! \binom{k+r-1}{r-1} \sum_{\substack{i+j\leq k\\ i,j\geq 0}} \frac{(-1)^{i+j}}{(k-i-j)!} \sum_{\substack{u+v=n\\ u,v\geq 0}} \frac{(1-x)^{u+i}}{(u+i)!}\frac{x^{v+j}}{(v+j)!} \\ \cdot (-1)^v S\Big(u+i, i \Big| \frac{\lambda}{1-x}\Big) S\Big(v+j, j\Big| -\frac{\lambda}{x}\Big).\end{aligned}$$

Similarly, (3.4), (3.5) and (3.6) can be generalised, giving

$$\begin{aligned}\epsilon_n^{(r)}(\lambda, rx) = n! \sum_{k=0}^{n} \frac{(-1)^k k!}{2^k} \binom{k+r-1}{r-1} \sum_{\substack{i+j=k\\ i,j\geq 0}} \sum_{\substack{u+v=n\\ u,v\geq 0}} \frac{(1-x)^{u}}{u!}\frac{x^{v}}{v!} \\ \cdot (-1)^v S\Big(u, i \Big| \frac{\lambda}{1-x}\Big) S\Big(v, j\Big| -\frac{\lambda}{x}\Big)\end{aligned}$$

and

$$\begin{aligned}\epsilon_n^{(r)}(\lambda, x) &= 2n! \sum_{k=0}^{n} (-1)^k k! \binom{k+r-1}{r-1} \sum_{\substack{i+j=k\\ i,j\geq 0}} \frac{1}{i!} \sum_{\substack{u+v=n\\ u,v\geq 0}} \frac{(i-ix|\lambda)_u}{u!}\frac{(-x)^v}{v!} S\Big(v, j\Big| -\frac{\lambda}{x}\Big) \\ &= 2n! \sum_{k=0}^{n} (-1)^k k! \binom{k+r-1}{r-1} \sum_{\substack{i+j=k\\ i,j\geq 0}} \frac{1}{i!} \sum_{\substack{u+v=n\\ u,v\geq 0}} \frac{(i-ix|\lambda)_u}{u!}\frac{(-x)^v}{v!} S\Big(v, j\Big| -\frac{\lambda}{x}\Big).\end{aligned}$$

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