Canad. Math. Bull. Vol. 65 (1), 2022, pp. 225–233
http://dx.doi.org/10.4153/S0008439521000230

Uniconnected solutions to the Yang–Baxter equation arising from self-maps of groups

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Abstract. Set-theoretic solutions to the Yang–Baxter equation can be classified by their universal coverings and their fundamental groupoids. Extending previous results, universal coverings of irreducible involutive solutions are classified in the degenerate case. These solutions are described in terms of a group with a distinguished self-map. The classification in the nondegenerate case is simplified and compared with the description in the degenerate case.

1 Introduction

Very soon after Drinfeld's suggestion [9] to study set-theoretic solutions $S: X^2 \to X^2$ to the Yang–Baxter equation

$$(S \times 1_X)(1_X \times S)(S \times 1_X) = (1_X \times S)(S \times 1_X)(1_X \times S)$$

in X^3, these solutions found increasing attention during the past 30 years (see, e.g., [1, 4–8, 10–12, 14, 15, 18, 19, 21, 23, 25–30]. Due to their close connection with braidings, they arise in various topics, including noncommutative regular rings [13, 14], regular affine groups [2–4], Hopf–Galois theory [1, 6, 11, 16], and Garside groups [7, 8, 23].

A first systematic study of involutive solutions was undertaken by Etingof et al. [10] who introduced the *structure group* $(G_X;\circ)$ of a solution $S(x,y) = ({}^x y, x^y)$, generated by X, with relations $x \circ y = {}^x y \circ x^y$. A solution S is said to be *nondegenerate* if the component maps $y \mapsto {}^x y$ and $y \mapsto y^x$ are invertible for all $x \in X$. In [21], it was shown that left nondegenerate involutive solutions S are equivalent to *cycle sets*, that is, sets X with a single binary operation $\cdot$ such that the self-maps $\sigma(x): X \to X$ with $\sigma(x)(y) := x \cdot y$ are bijective, and X satisfies the equation

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z). \tag{1.1}$$

Furthermore, it was shown that finite cycle sets are (two-sided) nondegenerate.

The structure group G_X of a nondegenerate cycle set X is obtained by a process of discrete integration where equation (1.1) plays the role of an integrability condition. More precisely, X embeds into G_X, and the operation on X extends to G_X and makes G_X into a cycle set. The action $G_X \times X \to X$ given by $(a,x) \mapsto a \cdot x$ leads to a factor group $G(X)$ of G_X, acting faithfully on X, such that $G(X)$ is generated by the

Received by the editors November 25, 2020, accepted March 27, 2021.
Published online on Cambridge Core April 20, 2021.
AMS subject classification: 81R50, 16T25, 20L05.
Keywords: Braces, coverings, Yang–Baxter equation, structure group, cycle set.

permutations $\sigma(x)$ with $x \in X$. Therefore, $G(X)$ is called the *permutation group* of X. If X is nondegenerate, $G(X)$ is also a cycle set, induced by the cycle set structure of G_X.

Until now, several types of cycle sets with an underlying group structure have been found. The cycle sets G_X and $G(X)$ are two examples. Both are *braces* [22], that is, additive abelian groups with a ring-like multiplication where $(G_X; \circ)$ plays the role of an *adjoint group*, similar to that of a radical ring: $a \circ b = ab + a + b$. Any brace satisfies

$$(a+b)\cdot c = (a\cdot b)\cdot(a\cdot c),$$

so that equation (1.1) follows by the commutativity of addition. Etingof et al. [10] already considered *affine solutions* to the Yang–Baxter equation. These also can be described as cycle sets with an underlying abelian group. In [24], we studied two other types of cycle set structures on an abelian group. Given the close connection between group structures and symmetries, the cumulative occurrence of groups in relation with cycle sets indicates a high level of inner balance of this structure. Surprisingly, there is another connection with groups arising in the context of coverings.

In this paper, we introduce a class of self-maps of a group G, leading to a cycle set structure on G, such that the universal covering of any cycle set is of that type. In [26], we developed a general covering theory which applies to noninvolutive and even degenerate solutions to the Yang–Baxter equation. Here, we focus upon the involutive case. To highlight the analogy with topological coverings of connected spaces, special attention is payed to *irreducible* cycle sets, consisting of a single orbit under the permutation group.

By [26, Theorem 6], every cycle set X has a universal *covering* $\widetilde{p}\colon \widetilde{X} \twoheadrightarrow X$, a cycle set morphism which does not change the permutation group $G(X)$, and does not increase the set of $G(X)$-orbits. Like in topology, it is *universal*, hence essentially unique, so that it factors through each covering of X. The relationship between X and $\widetilde{X}$ is determined by the *fundamental groupoid* $\pi_1(X)$, which can be replaced by a group if X is irreducible. Thus, in order to classify all irreducible cycle sets, the main step consists in the knowledge of cycle sets arising as universal coverings, i.e., those with a trivial fundamental group. In [27], these *uniconnected* cycle sets, analogous to simply connected spaces, are classified in the nondegenerate case. Here, we remove the restriction by characterizing arbitrary uniconnected cycle sets as groups with a specific self-map (Theorem 3.2).

Using a result of [28], we show that nondegenerate uniconnected cycle sets are equivalent to braces with a distinguished generator (Theorem 3.3). Here, the uniconnected cycle set structure differs from the cycle set structure as a brace! As a corollary, it follows that a uniconnected cycle set is nondegenerate if and only if its underlying group is the adjoint group of a brace. It turns out that a great many of universal coverings are degenerate: For example, each group with a right-invariant lattice structure (e.g., Artin's braid group) gives rise to a degenerate uniconnected cycle set (Example 1). A class of nondegenerate uniconnected cycle sets is obtained from differential groups (Example 2).

2 Preliminaries

In this section, we briefly recall some facts on cycle sets needed in what follows. A set X with a binary operation $\cdot$ is said to be a *cycle set* [21] if the left multiplication $\sigma(x)$: $y \mapsto x \cdot y$ is invertible, and the equation (1.1) holds for all $x, y, z \in X$. So σ defines a map

$$\sigma : X \to \mathfrak{S}(X) \tag{2.1}$$

into the permutation group $\mathfrak{S}(X)$. The subgroup $G(X)$ of $\mathfrak{S}(X)$ generated by the image of σ is said to be the *permutation group* of X. The image $\sigma(X)$ admits a well-defined operation

$$\sigma(x) \cdot \sigma(y) := \sigma(x \cdot y),$$

so that equation (1.1) is retained for $\sigma(X)$, the *retraction* [10, 21] of X. In general, $\sigma(X)$ is not a cycle set, unless X is *nondegenerate*, that is, the square map $x \mapsto x \cdot x$ is bijective. The retraction $\sigma(X)$ is then again nondegenerate, so that the retraction process can be iterated. Finite cycle sets are always nondegenerate [21, Theorem 2].

A *brace* A is a cycle set with an abelian group structure satisfying

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c), \tag{2.2}$$

$$(a + b) \cdot c = (a \cdot b) \cdot (a \cdot c), \tag{2.3}$$

for all $a, b, c \in A$. Note that equation (1.1) follows by equation (2.3) and the commutativity of $(A; +)$. Equation (2.2) says that $\sigma(a)$ is a group automorphism for all $a \in A$. As a cycle set, any brace A is nondegenerate. Moreover, it is a group with respect to the operation

$$a \circ b := a^b + b, \tag{2.4}$$

where $a^b := \sigma(b)^{-1}(a)$. The group $A^\circ := (A; \circ)$ is called the *adjoint group* of A. Typical examples of braces are radical rings R (=Jacobson radicals of rings, viewed as pseudorings) with the operation $a \cdot b := b(1 + a)^{-1}$. Then, the adjoint group is given by Jacobson's circle operation [17]

$$a \circ b = a + ab + b.$$

In accordance with Jacobson's notation [17], we write a' for the inverse in the adjoint group of a brace. Note that the unit element of A° is 0.

For any nondegenerate cycle set X, the permutation group $G(X)$ is the adjoint group of a unique brace $A(X)$ such that $\sigma: X \to A(X)$ is a cycle set morphism. A subset of a brace A is said to be a *cycle base* [22, Definition 4] if it is invariant under the adjoint group A° and generates the additive group of A. By [28, Proposition 10], a brace A is generated by an element $e \in A$ if and only if $X := \{e^a \mid a \in A\}$ is a cycle base of A. Thus, each *transitive* cycle base (i.e., such that A° acts transitively on it) is given by a generator $e \in A$.

A *right ideal* [22] of a brace A is an additive subgroup which is invariant under the action of A°. By equation (2.4), any right ideal is a subbrace. For radical rings, this concept coincides with the usual one. We say that a brace A is a *torsion brace* if

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each $a \in A$ is of finite order in $(A;+)$. For a torsion brace A, equation (2.2) implies that the primary decomposition of the additive group is a decomposition into right ideals.

The *fundamental groupoid* [26] $\pi_1(X)$ of a cycle set X has X as set of objects, and pairs $(a,x) \in G(X) \times X$ as morphisms from x to $a(x)$. Composition is the obvious one:

$$x \longrightarrow b(x) \longrightarrow ab(x),$$

given by $(a, b(x))(b,x) := (ab, x)$. Inverses are $(a,x)^{-1} := (a^{-1}, a(x))$. The isomorphism classes of objects in $\pi_1(X)$ form a set $C(X)$, consisting of the $G(X)$-orbits of X. Any morphism $f\colon X \to Y$ of cycle sets induces a map $C(f)\colon C(X) \to C(Y)$. If f is surjective, it gives rise to a commutative diagram

$$\begin{tikzcd} X \arrow[r, two heads, "f"] \arrow[d, "\sigma"] & Y \arrow[d, "\sigma"] \\ G(X) \arrow[r, two heads, "G(f)"] & G(Y) \end{tikzcd}$$

with a surjective group homomorphism $G(f)$. We call $f\colon X \twoheadrightarrow Y$ a *covering* [26] if $C(f)$ and $G(f)$ are bijective. By [26, Theorem 6], any cycle set X admits a covering $\widetilde{X} \twoheadrightarrow X$ which is *universal* in the sense that it factors uniquely though each covering $Y \twoheadrightarrow X$, so that $\widetilde{X} \to Y$ is again a covering. Cycle sets X with no proper coverings $Y \twoheadrightarrow X$ are characterized by their fundamental groupoid which is skeletal.

A cycle set X is said to be *indecomposable* if $G(X)$ acts transitively on X, that is, $|C(X)| = 1$. Then, $\widetilde{X}$ is indecomposable, too. Thus, $G(X) = G(\widetilde{X})$ acts freely and transitively on $\widetilde{X}$. Being quite analogous to simply connected topological spaces, such cycle sets are called *uniconnected* [27].

3 Bracial self-maps of groups

Let $\varepsilon\colon G \to G$ be a self-map of a group G. We define the kernel of ε to be the subgroup

$$\operatorname{Ker}\varepsilon := \{h \in G \mid \forall\, g \in G\colon \varepsilon(hg) = \varepsilon(g)\}.$$

For a group homomorphism ε, this concept coincides with the usual one. Note, however, that $\operatorname{Ker}\varepsilon$ need not be normal for an arbitrary self-map ε.

Definition 3.1 We call a cycle set structure $(B;\odot)$ on a group B *bracial* if it satisfies $a \odot b = b(a \odot 1)$ for all $a, b \in B$.

Thus, B is uniquely determined by its group structure and the self-map

$$\varepsilon(a) := (a \odot 1)^{-1}.$$

We call ε the *structure map* and $\varepsilon(B)$ the *image* of B. The kernel $\operatorname{Ker} B := \operatorname{Ker}\varepsilon$ of ε will be called the *kernel* of B. In terms of ε, we have

Proposition 1 *A self-map $\varepsilon: B \to B$ of a group B defines a bracial cycle set if and only if it satisfies the equation*

$$\varepsilon\big(a\varepsilon(b)^{-1}\big)\varepsilon(b) = \varepsilon\big(b\varepsilon(a)^{-1}\big)\varepsilon(a). \tag{3.1}$$

Proof For $a, b, c \in B$, we have

$$\begin{aligned}(a \odot b) \odot (a \odot c) &= b\varepsilon(a)^{-1} \odot c\varepsilon(a)^{-1} = c\varepsilon(a)^{-1}\varepsilon(b\varepsilon(a)^{-1})^{-1} \\ &= c(\varepsilon\big(b\varepsilon(a)^{-1}\big)\varepsilon(a))^{-1}.\end{aligned}$$

Equation (3.1) says that this expression is symmetric in a and b. ∎

Consider the subgroup $\langle\varepsilon(B)\rangle$ of B generated by the image of B. For $a, b \in B$, we have $a \odot b = b(a \odot 1) = b\varepsilon(a)^{-1}$. Therefore, the right regular representation of B induces a group isomorphism

$$\pi: \langle\varepsilon(B)\rangle \xrightarrow{\sim} G(B) \tag{3.2}$$

onto the permutation group of B, which maps $a \in \langle\varepsilon(B)\rangle$ to the right multiplication $b \mapsto ba^{-1}$. On the other hand, we have a map

$$\sigma: B \to G(B)$$

with $\sigma(a)(b) := a \odot b$. So the isomorphism (3.2) yields a factorization

$$\sigma: B \xrightarrow{\varepsilon} \varepsilon(B) \xrightarrow{\iota} \langle\varepsilon(B)\rangle \underset{\sim}{\xrightarrow{\pi}} G(B). \tag{3.3}$$

Theorem 3.1 *Let B be a bracial cycle set. Then, $G(B)$ acts freely on B. The subset $\langle\varepsilon(B)\rangle$ of B is a uniconnected subcycle set. The left cosets of $\langle\varepsilon(B)\rangle$ are the $G(B)$-orbits of B.*

Proof Assume that $\alpha(b) = b$ for some $\alpha \in G(B)$ and $b \in B$. Then, $b = b\pi^{-1}(\alpha)^{-1}$, which yields $\pi^{-1}(\alpha) = 1$, that is, $\alpha = 1_B$. For $a \in \langle\varepsilon(B)\rangle$ and $b \in B$, we have $b \odot a = a\varepsilon(b)^{-1} \in \langle\varepsilon(B)\rangle$. Thus, $\langle\varepsilon(B)\rangle$ is a subcycle set. By induction, the equation $b \odot a = a\varepsilon(b)^{-1}$ shows that the cycle set $\langle\varepsilon(B)\rangle$ is indecomposable, hence uniconnected. Moreover, the equation implies that the left cosets of $\langle\varepsilon(B)\rangle$ are the $G(B)$-orbits of B. ∎

As a consequence, we get a classification of uniconnected cycle sets.

Theorem 3.2 *A bracial cycle set B is uniconnected if and only if $\langle\varepsilon(B)\rangle = B$. Conversely, every uniconnected cycle set B is bracial and satisfies $\langle\varepsilon(B)\rangle = B$.*

Proof The first statement follows by Theorem 3.1. Conversely, let $(B; \odot)$ be any uniconnected cycle set. Choose an element $1 \in B$. Then, there is a bijection $\gamma: G(B) \xrightarrow{\sim} B$ with $\gamma(\alpha) := \alpha(1)$. So the map $\sigma: B \to G(B)$ with $\sigma(a)(b) = a \odot b$ satisfies $\sigma(a) = \gamma^{-1}(a \odot 1)$. Define a multiplication in B as follows:

$$ab := \gamma^{-1}(b)(a). \tag{3.4}$$

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For given $a, b \in B$, consider the automorphisms $\alpha := \gamma^{-1}(a)$ and $\beta := \gamma^{-1}(b)$ in $G(B)$. Because $\gamma(\alpha\beta) = \alpha\gamma(\beta)$, we have $\gamma\big(\gamma^{-1}(b)\gamma^{-1}(a)\big) = \gamma^{-1}(b)(a) = ab$, which yields $\gamma^{-1}(ab) = \gamma^{-1}(b)\gamma^{-1}(a)$. Thus, equation (3.4) defines a group structure on B with an isomorphism

$$\gamma \colon G(B)^{\mathrm{op}} \xrightarrow{\sim} B.$$

In particular, we obtain $\langle \varepsilon(B) \rangle = B$. For $a, b \in B$, equation (3.4) gives $a \odot b = \sigma(a)(b) = \gamma^{-1}(a \odot 1)(b) = b(a \odot 1)$. Whence B is a bracial cycle set. ∎

In [27, Theorem 2], nondegenerate uniconnected cycle sets are classified in terms of braces with a transitive cycle base. The following example shows that uniconnected cycle sets need not be nondegenerate.

Example 1 Let G be a *right ℓ-group* [23], that is, a group with a lattice order, satisfying

$$a \leqslant b \implies ac \leqslant bc$$

for all $a, b, c \in G$. Consider the self-map $\varepsilon \colon G \to G$ given by $\varepsilon(a) := a \vee 1$. Then, $\varepsilon(a\varepsilon(b)^{-1})\varepsilon(b) = \big(a(b \vee 1)^{-1} \vee 1\big)(b \vee 1) = a \vee b \vee 1$, which is symmetric in a and b. Hence, ε makes G into a bracial cycle set. The kernel of ε is $\{1\}$, while the image $\varepsilon(G)$ is the positive cone of G. Therefore, the corresponding cycle set $(G; \odot)$ with $a \odot b := b(a \vee 1)^{-1}$ is uniconnected. However, $(G; \odot)$ is degenerate: For $a \geqslant 1$, we have $a \odot a = a(a \vee 1)^{-1} = 1$. So the square map $a \mapsto a \odot a$ is not injective if $|G| \neq 1$.

The diagram (3.3) shows that the structure map ε of a bracial cycle set B gives the retraction $\varepsilon(B)$ of B. By [21, Section 6], this implies that the cycle set structure of B induces a well-defined binary operation on $\varepsilon(B)$:

$$\varepsilon(a) \cdot \varepsilon(b) := \varepsilon(b\varepsilon(a)^{-1}). \tag{3.5}$$

Indeed, $\varepsilon(b) = \varepsilon(c)$ implies that $\varepsilon(b\varepsilon(a)^{-1})\varepsilon(a) = \varepsilon(a\varepsilon(b)^{-1})\varepsilon(b) = \varepsilon(a\varepsilon(c)^{-1})\varepsilon(c) = \varepsilon(c\varepsilon(a)^{-1})\varepsilon(a)$, which yields $\varepsilon(b\varepsilon(a)^{-1}) = \varepsilon(c\varepsilon(a)^{-1})$. Thus, $\varepsilon(a) \cdot \varepsilon(b)$ does not depend on the choice of b. So the operation (3.5) satisfies equation (1.1), which makes $\varepsilon(B)$ into a cycle set if B is nondegenerate. In general, however, $(\varepsilon(B); \cdot)$ is not a subcycle set of $(B; \odot)$.

Example 2 Let C be a *differential group* [20], that is, an abelian group with an endomorphism $d \colon C \to C$ satisfying $d^2 = 0$. For an element $b \in C$ with $db \neq 0$, we define $\varepsilon(x) := dx + b$. Then, $\varepsilon(x - \varepsilon(y)) + \varepsilon(y) = d(x + y) - db + 2b$. By symmetry, this gives a bracial cycle set C with $x \odot y = y - \varepsilon(x) = y - dx - b$. Because $1 - d$ is invertible, the cycle set is nondegenerate. However, $\varepsilon(C)$ is not a subcycle set. Indeed, $\varepsilon(x) \odot \varepsilon(y) = d(y - b) \notin \varepsilon(C)$, because $db \neq 0$. The subcycle set $\langle \varepsilon(C) \rangle = dC + \mathbb{Z}b$ is uniconnected.

Now, we turn our attention to the nondegenerate case. Note that a bracial cycle set B is nondegenerate if and only if the map $a \mapsto a\varepsilon(a)^{-1}$ is bijective.

Definition 3.2 We call a self-map $\varepsilon: G \to G$ of a group G *right invariant* if the implication

$$\varepsilon(a) = \varepsilon(b) \implies \varepsilon(ac) = \varepsilon(bc) \tag{3.6}$$

holds for $a, b, c \in G$.

Let $\varepsilon: G \to G$ be a self-map with kernel H. Because $ac = (ab^{-1})bc$, condition (3.6) says that $\varepsilon(a) = \varepsilon(b)$ if and only if $ab^{-1} \in H$. So the fibers of ε are the right cosets of H. Therefore, ε induces a bijection $\overline{\varepsilon}: G\backslash H \xrightarrow{\sim} \varepsilon(G)$ from the set $G\backslash H$ of right cosets onto the image of ε. In Example 2, but not in Example 1, the map ε is right invariant.

Proposition 2 *Let G be a group with a right invariant self-map $\varepsilon: G \to G$. Then,*

$$g \cdot \varepsilon(a) := \varepsilon(ag^{-1}) \tag{3.7}$$

(with $g, a \in G$) defines a transitive action of G on the image of ε.

Proof Because ε is right invariant, equation (3.7) gives a well-defined map $G \times \varepsilon(G) \to \varepsilon(G)$. Furthermore, $gh \cdot \varepsilon(a) = \varepsilon(ah^{-1}g^{-1}) = g \cdot \varepsilon(ah^{-1}) = g \cdot (h \cdot \varepsilon(a))$ and $1 \cdot \varepsilon(a) = \varepsilon(a)$. Thus, equation (3.7) defines an action on $\varepsilon(G)$. Because $aG = G$, this action is transitive. ∎

Proposition 3 *The structure map $\varepsilon: B \to B$ of a nondegenerate uniconnected cycle set B is right invariant.*

Proof The diagram (3.3) shows that $\varepsilon(a) = \varepsilon(b)$ is equivalent to $\sigma(a) = \sigma(b)$. Assuming this, we have to show that $\sigma(ac) = \sigma(bc)$ holds for all $c \in B$. Because B is uniconnected, it is enough to verify the equivalence

$$\sigma(a) = \sigma(b) \iff \sigma(c \odot a) = \sigma(c \odot b).$$

Indeed, $\sigma(a) = \sigma(b)$ implies $(c \odot a) \odot (c \odot d) = (a \odot c) \odot (a \odot d) = (b \odot c) \odot (b \odot d) = (c \odot b) \odot (c \odot d)$ for all $d \in B$. Thus, $\sigma(c \odot a) = \sigma(c \odot b)$. Conversely, assume that $\sigma(c \odot a) = \sigma(c \odot b)$. Then, $(a \odot c) \odot (a \odot c) = (c \odot a) \odot (c \odot c) = (c \odot b) \odot (c \odot c) = (b \odot c) \odot (b \odot c)$. Because B is nondegenerate, we obtain $a \odot c = b \odot c$. As B is uniconnected, this implies that $\sigma(a) = \sigma(b)$. ∎

For a nondegenerate bracial cycle set B, the group $G(B)$ is the adjoint group of a brace. So the group isomorphism (3.2) also makes $\langle \varepsilon(B) \rangle$ into a brace $A(B)$. Taking π as an identification, we get a factorization

$$\sigma: B \overset{\varepsilon}{\twoheadrightarrow} \varepsilon(B) \overset{\iota}{\hookrightarrow} A(B) \tag{3.8}$$

which identifies $\varepsilon(B)$ with the image of σ.

Theorem 3.3 *Let B be a brace, generated by $e \in B$. Then, there is a unique self-map $\varepsilon: B \to B$ with $\varepsilon(0) = e$ such that ε is the structure map of a nondegenerate uniconnected*

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cycle set with group B°. Conversely, every nondegenerate uniconnected cycle set arises in this way.

Proof By [28, Proposition 10], $X := \{e^a \mid a \in B\}$ is a transitive cycle base of B. So [27, Theorem 2] shows that the operation

$$a \odot b := b \circ (e^a)'$$

makes B into a nondegenerate uniconnected cycle set. In particular, $(B; \odot)$ is a bracial cycle set with $\varepsilon(a) = (a \odot 0)' = e^a$. Thus, $\varepsilon(B) = X$ and $\varepsilon(0) = e$. By Propositions 2 and 3, the structure map ε is unique: $\varepsilon(a) = \varepsilon(0 \circ a) = \varepsilon(0)^a = e^a$.

Conversely, let B be a nondegenerate uniconnected cycle set. Because $\varepsilon(B)$ is the retraction of B, the map $\varepsilon: B \twoheadrightarrow \varepsilon(B)$ is a cycle set morphism. So the cycle set morphism (3.8) shows that ι is a morphism of cycle sets. Thus, $\varepsilon(B)$ is a cycle base of $A(X)$. Equation (3.5) shows that it is transitive.

By equation (3.5), the action (3.7) coincides with the adjoint action in $A(B)$. Hence, $\varepsilon(a) = a^{-1} \cdot \varepsilon(1) = \varepsilon(1)^a$ for all $a \in A(B)$. Thus, $a \odot b = b\varepsilon(a)^{-1} = b(\varepsilon(1)^a)^{-1}$. By [27, Theorem 2], the proof is complete. ∎

Corollary *A uniconnected cycle set B is nondegenerate if and only if B is the adjoint group of a brace.*

Proof By [27, Proposition 2], a cycle set is nondegenerate if and only if it is a cycle base of a brace. ∎

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