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# On the entropy of actions of nilpotent Lie groups and their lattice subgroups

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*Dedicated to the memory of Daniel Rudolph*

*Abstract*. We consider a natural class $\mathcal{ULG}$ of connected, simply connected nilpotent Lie groups which contains $\mathbb{R}^n$, the group $\mathcal{UT}_n(\mathbb{R})$ of all triangular unipotent matrices over $\mathbb{R}$ and many of its subgroups, and is closed under direct products. If $G \in \mathcal{ULG}$, then $\Gamma_1 = G \cap \mathcal{UT}_n(\mathbb{Z})$ is a lattice subgroup of $G$. We prove that if $G \in \mathcal{ULG}$ and $\Gamma$ is a lattice subgroup of $G$, then a free ergodic measure-preserving action $T$ of $G$ on a probability space $(X, \mathcal{B}, \mu)$ has completely positive entropy (CPE) if and only if the restriction $T^\Gamma$ of $T$ to $\Gamma$ has CPE. We can deduce from this the following version of a well-known conjecture in this case: the action $T$ has CPE if and only if $T$ is uniformly mixing. Moreover, such $T$ has a Lebesgue spectrum with infinite multiplicity. We further consider an ergodic free action $T$ with positive entropy and suppose $T^\Gamma$ is ergodic for any lattice subgroup $\Gamma$ of $G$. This holds, in particular, if the spectrum of $T$ does not contain a discrete component. Then we show the Pinsker algebra $\Pi(T)$ of $T$ exists and coincides with the Pinsker algebras $\Pi(T^\Gamma)$ of $T^\Gamma$ for any lattice subgroup $\Gamma$ of $G$. In this case, $T$ always has Lebesgue spectrum with infinite multiplicity on the space $\mathcal{L}_0^2(X, \mu) \ominus \mathcal{L}_0^2(\Pi(T))$, where $\mathcal{L}_0^2(\Pi(T))$ contains all $\Pi(T)$-measurable functions from $\mathcal{L}_0^2(X, \mu)$. To prove these results, we use the following formula: $h(T) = |G(\Gamma)|^{-1} h_K(T^\Gamma)$, where $h(T)$ is the Ornstein–Weiss entropy of $T$, $h_K(T^\Gamma)$ is a Kolmogorov–Sinai entropy of $T^\Gamma$, and the number $|G(T^\Gamma)|$ is the Haar measure of the compact subset $G(\Gamma)$ of $G$. In particular, $h(T) = h_K(T^{\Gamma_1})$, and $h_K(T^{\Gamma_1}) = |G(\Gamma)|^{-1} h_K(T^\Gamma)$. The last relation is an analogue of the Abramov formula for flows.

## 1. *Introduction*

The theory of entropy has a long history in the theory of ergodic theory and dynamical systems. First introduced for an action of $\mathbb{Z}$ by Kolmogorov [**26**] in 1958, and refined by Sinai in 1959, it is still finding important recent applications in wider settings [**6, 15, 22, 39**]. In particular, Kolmogorov discussed the class of actions of $\mathbb{Z}$ with completely positive entropy (CPE), and Rokhlin and Sinai [**37**] described mixing and spectral properties of these systems. Ornstein [**29**] proved that if two Bernoulli actions of $\mathbb{Z}$ have the same Kolmogorov–Sinai entropy, then they are isomorphic. Furthermore, Pinsker [**31**] proved that if a $\mathbb{Z}$-action on a probability space $(X, \mathcal{B}, \mu)$ has positive Kolmogorov–Sinai entropy $h$, then there exists a maximal $\mathbb{Z}$-invariant factor-space $\Pi$ of $X$ such that the restriction of $\mathbb{Z}$ to $\Pi$ has entropy zero: in this situation a theorem of Sinai [**43**] tells us further that there exists a Bernoulli $\mathbb{Z}$-invariant subfactor of $X$ with entropy $h_1$, for any $0 < h_1 \leq h$.

It is natural to consider generalizations of these results to other locally compact amenable groups. In particular, Feldman [**13**] found an analogue of Kolmgorov–Sinai entropy $h(T)$ for an action $T$ of $\mathbb{R}^n$, $1 \leq n < \infty$ and he proved that two Bernoulli actions $T_i$, $i = 1, 2$ of $\mathbb{R}^n$ with $h(T_1) = h(T_2)$ are isomorphic. We will describe his results in more detail below in 2.1.

Feldman's approach was extended by Ornstein and Weiss [**30**] to the class $\mathcal{G}$ of amenable locally compact unimodular groups with zero self-entropy: $\mathcal{G}$ contains all discrete and all nilpotent Lie groups. If $G \in \mathcal{G}$, $T$ is a measure-preserving action of $G$ on a probability space $(X, \mathcal{B}, \mu)$, and $\rho$ is a finite partition of $X$, then Ornstein and Weiss define the *spatial entropy* $\mathrm{sh}(T, \rho)$ of the process $(T, \rho)$. Then $h(T) = \sup_\rho \mathrm{sh}(T, \rho)$ is defined as the entropy of the action $T$ of $G$. However, the definition of spatial entropy takes a different approach from Kolmogorov's in [**26**]: we will discuss the relationships between them below. It was further proved in [**30**] that $h(T)$ is a full invariant of the isomorphism of Bernoulli actions of $G$. Ornstein and Weiss also proved an analogue of the Sinai theorem on the existence of the Bernoulli subfactor, mentioned above, for an action $T$ of $G \in \mathcal{G}$ with $h(T) > 0$, and an analogue of the classical Rudolph theorem on Bernoulli actions of a group $G \in \mathcal{G}$ and its closed cocompact subgroups. For further details, see §§2.1–2.3 below.

If $G$ is a discrete amenable group, and $X$, $T$ and $\rho$ are as above, then Kieffer [**25**] proved an analogue of the Shannon–McMillan theorem for the action $T$ using the Kolmogorov entropy $h_K(T, \rho)$ of the process $(T, \rho)$. Furthermore, it was shown in [**13, 30**] (see also Lemma 2.3 below) that $\mathrm{sh}(T, \rho) = h_K(T, \rho)$. It follows that $h(T) = h_K(T)$, where $h_K(T)$ is the classical Kolmogorov–Sinai entropy of $T$. These results suggest that the entropy theory of actions of countable infinite amenable groups is very similar to the entropy theory of $\mathbb{Z}$-actions, and this intuition is supported by the study of CPE actions.

The initial results on CPE actions were obtained by Rokhlin and Sinai [**37**], who showed that an action of $\mathbb{Z}$ has CPE if and only if it is uniformly mixing. Furthermore, they showed that a CPE action of $\mathbb{Z}$ has Lebesgue spectrum with infinite multiplicity. To prove this, they developed the method of perfect partitions. Later, this was extended by Kamiński [**21**] to actions of $\mathbb{Z}^d$, $d < \infty$, and then to actions of the group $\mathcal{UT}_d(\mathbb{Z})$ of upper unipotent triangular $d \times d$-matrices over $\mathbb{Z}$ and its subgroups. Unfortunately, as is well known, these methods cannot be applied to arbitrary discrete amenable groups.

A new approach to this problem, which can be applied to any countable infinite amenable group, was introduced by Rudolph and Weiss [**41**], who applied results of Connes *et al* [**4**] on the properties of orbits of actions of amenable countable groups. Rudolph and Weiss [**41**] proved that any free CPE action of an amenable countable group is uniformly mixing. In fact, it is not difficult to use the Rudolph–Weiss theory to show that an action of a countable infinite amenable group is CPE if and only if it is uniformly mixing, see [**12, 18, 47**]. Dooley and Golodets [**11**] proved that any CPE action of a countable amenable group has Lebesgue spectrum with infinite multiplicity.

The results of Rudolph and Weiss have led to an increased interest in the entropy of actions of countable amenable groups. Glasner *et al* [**16**] studied Pinsker algebras in this setting: Danilenko and Park [**7, 8**] developed a new approach to these problems using cocycles, and Weiss [**47**] proved several versions of Shannon–McMillan theorems for actions of monotileable amenable groups which we will use in this paper. Dooley *et al* [**12**] studied non-Bernoulli CPE actions of countable amenable groups constructed as coinduced actions: a version of this construction also appeared in [**17**] in connection with a question of Thouvenot on CPE actions for discrete nilpotent groups.

New important results in the study of CPE actions of continuous amenable groups were recently obtained by Avni [**2**], who developed the theory to include a notion of entropy for cross-sections, and applied it to CPE actions of amenable groups in $\mathcal{G}$. The theory of cross-sections of locally compact groups actions was initially worked out by Feldman *et al* [**14**]: Avni proved the following version of the Thouvenot conjecture.

THEOREM 1.1. *Let $G$ be an amenable group with zero self-entropy and let $T$ be a free CPE action of $G$ on a probability space. Then $T$ is uniformly mixing and has Lebesgue spectrum with infinite multiplicity.*

In this paper, we consider a special class of nilpotent Lie groups, which we call *unicommutator* Lie groups, $\mathcal{ULG}$. We prove that if $G$ belongs to this class, then a CPE action $T$ of $G$ on a probability space $(X, \mu)$ is equivalent to a system which is uniformly mixing and has infinite Lebesgue spectrum. If the action $T$ is free ergodic with a positive entropy and the action $T^{\Gamma}$ of any lattice subgroup $\Gamma$ of $G$ is also ergodic then we prove the existence of the Pinsker algebra $\Pi(T)$ for $T$. Moreover, $T$ has Lebesgue spectrum on $\mathcal{L}^2(X, \mu) \ominus \mathcal{L}^2(\Pi(T))$, where $\mathcal{L}^2(\Pi(T))$ contains all functions from $\mathcal{L}^2(X, \mu)$ measurable with respect to $\Pi(T)$. Our approach to this problem differs from that of Avni. It was outlined to us by Benjy Weiss and we develop it below.

Let us describe the class of unicommutator Lie groups $\mathcal{ULG}$.

*Definition 1.2.* We say that $G$ belongs to $\mathcal{ULG}$ if it is connected simply connected and if, furthermore, its Lie algebra $\mathfrak{g}$ has a basis $\{e_i\}_1^N$ whose commutators satisfy, for all $1 \le i < j \le N$, that there is $1 \le k(i, j) \le N$ such that:

(i) $[e_i, e_j] = 0$ or $[e_i, e_j] = e_{k(i,j)}$; and

(ii) $[e_i, e_{k(i,j)}] = [e_j, e_{k(i,j)}] = 0$.

These assumptions on $\mathfrak{g}$ are made for technical reasons: we hope to weaken them later on.

The simplest examples of groups from this class are $\mathbb{R}^n$, $n \ge 1$, the group $\mathcal{UT}_n(\mathbb{R})$ of all triangular unipotent $n \times n$-matrices over $\mathbb{R}$, certain subgroups of $\mathcal{UT}_n(\mathbb{R})$ and their direct products.

First we investigated the case of $\mathbb{R}^n$, and then we introduced the $\mathcal{ULG}$ groups for which it is possible to extend this technique.

Recall that a discrete closed subgroup $\Gamma$ of a locally compact group $G$ is called *a lattice subgroup* if $G/\Gamma$ has a finite $G$-invariant measure [**33**, II]. A lattice subgroup is called *uniform* if $\Gamma$ is cocompact in $G$. Recall also that all lattice subgroups of a simply connected nilpotent Lie group are uniform [**33**]. The existence of a lattice subgroup $\Gamma$ in $G \in \mathcal{ULG}$ follows from the commutations relations above for $\mathfrak{g}$ [**33**].

We will see in §4.2 that $G \in \mathcal{ULG}$ is a subgroup of $\mathcal{UT}_n(\mathbb{R})$ for some integer $n$, where $N < n$ with $N$, as above. Moreover, $\Gamma_1 = G \cap \mathcal{UT}_n(\mathbb{Z})$ is also a lattice subgroup of $G$.

The main idea of this paper is to reduce the study of the entropy of an action of groups in $\mathcal{ULG}$ to the study of the entropy of actions of its lattice subgroups $\Gamma$.

This idea is not new in ergodic theory. Rudolph [**40**] proved that a free action $S_t$, $t \in \mathbb{R}$ of $\mathbb{R}$ (i.e. a flow) with $h_K(S_t) < \infty$ for each $t \in \mathbb{R}$ is CPE if and only if there is some $t$ for which $S_t$ is a CPE action. The general situation, without this assumption, was considered by Blanchard [**3**]. A similar result was obtained by Gurevich [**20**]. Sinai proved that if a flow $S_t$, $t \in \mathbb{R}$ has CPE, then it has Lebesgue spectrum with infinite multiplicity [**6**].

To prove these results, the authors use the methods of perfect partitions and properties of special flows. However, these approaches are difficult to apply in our situation. We use rather the spatial entropy of Ornstein–Weiss [**30**], its special case, Feldman's $r$-entropy, and the connection of this entropy with the classical Kolmogorov–Sinai entropy [**13**]. These matters are discussed in §2.

One of the main results of this paper is the following.

THEOREM 1.3. *Let $T$ be a free Borel ergodic action of a group $G$ from $\mathcal{ULG}$ by measure-preserving automorphisms of a probability space, and let $\Gamma$ be a lattice subgroup of $G$. Then the action $T$ is CPE if and only if the restriction $T^\Gamma$ of $T$ to $\Gamma$ is CPE.*

The proof of this statement is given in Theorem 3.2 for $\mathbb{R}^n$-actions, and in Theorem 4.10 for actions of a group $G \in \mathcal{ULG}$.

We derive two consequences from this statement.

- A free action $T$ of a group $G$ from $\mathcal{ULG}$ has CPE if and only if $T$ is uniformly mixing.
- If $G$ and $T$ are as above, then $T$ has Lebesgue spectrum with infinite multiplicity.

We prove these results in §3 for $\mathbb{R}^n$-actions, and in §4 for $G \in \mathcal{ULG}$.

Let $G$ belong to $\mathcal{ULG}$, and let $\Gamma$ be a lattice subgroup of $G$. Consider the action of $\Gamma$ on $G$ by left shifts. Since $\Gamma$ is cocompact in $G$, then it follows from the definition of the class $\mathcal{ULG}$ that there is a compact subset $G(\Gamma)$ of $G$ such that $G(\Gamma)$ intersects any $\Gamma$-orbit at only one point. This observation is easy to see for the lattice $\Gamma_1$, introduced above. (We check this for the Heisenberg group $\mathcal{H}$ in the proof of Theorem 4.3 below.) Denote by $|G(\Gamma)|$ the Haar measure of $G(\Gamma) \in G$. We choose the Haar measure on $G$ such that $|G(\Gamma_1)| = 1$.

THEOREM 1.4. *Let $G$, $\Gamma$, $\Gamma_1$ be as above, $T$ an ergodic Borel action of $G$ by measure-preserving automorphisms of a probability space $(X, \mathcal{B}, \mu)$, and the spectrum of $T$ does not contain a discrete component. If $T^\Gamma$ is the restriction of $T$ to $\Gamma$, then*

$$h(T) = |G(\Gamma)|^{-1} h_K(T^\Gamma),$$

*where $h(T)$ is the Ornstein–Weiss entropy of $T$ and $h_K(T^\Gamma)$ is the classical entropy of $T^\Gamma$. In particular, we have $h(T) = h_K(T^{\Gamma_1})$.*

One can easily derive from this theorem the following analogue of Abramov's formula for the entropy of a flow.

COROLLARY 1.5. *If the conditions of Theorem 1.4 hold, then*

$$h_K(T^{\Gamma_1}) = |G(\Gamma)|^{-1} h_K(T^{\Gamma}).$$

Recall that if $S_t$, $t \in \mathbb{R}$ is a flow, then $h_K(S_t) = |t| h_K(S_1)$ [**1**], and $S_t$ and $S_1$ are not necessarily ergodic. A similar formula was found by Conze [**5**] for the entropy of an action of a lattice subgroup of $\mathbb{R}^n$.

The proof of Theorem 1.4 for $\mathbb{R}^n$-actions is given in 2.14, and for actions of a non-commutative group from $\mathcal{ULG}$, in 4.7. It seems to us that Theorem 1.4 might hold for a larger class of groups than nilpotent Lie groups, (see [**30**, Appendix B]).

The second subject studied in this paper is the *Pinsker algebra* of actions of $G \in \mathcal{ULG}$.

Suppose that a locally compact amenable group $G$ has a measure-preserving free action $T$ on a probability space $(X, \mathcal{B}, \mu)$. The maximal $T$-invariant sub-$\sigma$-algebra of $\mathcal{B}$, containing all finite partitions $\mathcal{P}$ such that the process $(T, \mathcal{P})$ has entropy $h(T, \mathcal{P}) = 0$, is called the Pinsker algebra of $\Pi(T)$ of the action $T$. This algebra was introduced and studied by Pinsker [**31**] for $\mathbb{Z}$-actions.

Rokhlin and Sinai [**37**] described the Pinsker algebra of $\mathbb{Z}$-actions using perfect partitions and proved that $T$ has infinite Lebesgue spectrum on the space $\mathcal{L}_0^2(X, \mu) \ominus \mathcal{L}_0^2(\Pi(T))$, where $\mathcal{L}_0^2(\Pi(T))$ contains all $\Pi(T)$-measurable functions from $\mathcal{L}_0^2(X, \mu)$. These and other results on the Pinsker algebra can be found in the recent monograph of Glasner [**15**] and the survey of Thouvenot [**45**].

The Pinsker algebra and the Pinsker factor for actions of countable amenable groups were further investigated by Glasner *et al* [**16**] and Danilenko [**7**]. The spectral properties of these actions were studied by the authors in [**11**]. Essentially, all the major results on the Pinsker algebra of $\mathbb{Z}$-actions extend to this setting.

Pinsker algebras for flows were studied by Gurevich [**20**]. He showed that if $S_t$, $t \in \mathbb{R}$ is a flow, and $\Pi(S_t)$ is the Pinsker algebra of the automorphism $S_t$, then the $\sigma$-algebra $\Pi(S_t)$ is independent of $t \neq 0$. To prove this assertion, the authors applied the Abramov formula for the entropy of a flow, mentioned above. Spectral properties of a flow $S_t$, $t \in \mathbb{R}$, with a non-trivial Pinsker algebra $\Pi(S_t)$ for $t \neq 0$, appear not to have been considered.

We will study the Pinsker algebra via the connection between the entropy of an action $T$ of a group $G$ and the entropy of the restriction $T^{\Gamma}$.

Our second major result is the following theorem.

THEOREM 1.6. *Let $G$ be a nilpotent Lie group in $\mathcal{ULG}$, $\Gamma$ a lattice subgroup of $G$, and let $T$ be a free ergodic measure-preserving action of $G$ by automorphisms on a probability space $(X, \mathcal{B}, \mu)$, with positive entropy, and the spectrum of $T$ does not contain a discrete component. If $T^{\Gamma}$, is the restriction of $T$ to $\Gamma$, then the Pinsker algebra $\Pi(T)$ of $T$ exists and $\Pi(T) = \Pi(T^{\Gamma})$, where $\Pi(T^{\Gamma})$ is the Pinsker algebra of $T^{\Gamma}$. Furthermore, the action $T$ has infinite Lebesgue spectrum on the space $\mathcal{L}_0^2(X, \mu) \ominus \mathcal{L}_0^2(\Pi(T))$, where $\mathcal{L}_0^2(\Pi(T))$ contains all $\Pi(T)$-measurable functions from $\mathcal{L}_0^2(X, \mu)$.*

The first part of this theorem follows from Theorem 1.4. Notice also that if the spectrum of $T$ does not contain a discrete component, then $T^{\Gamma}$ is ergodic for any lattice $\Gamma$ of $G$ (see Proposition 2.12).

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The paper is organized as follows. Section 2 contains preliminary results: we define the Feldman $r$-entropy of actions of groups $\mathbb{Z}^n$ and $\mathbb{R}^n$, $n \geq 1$, and its generalization by Ornstein and Weiss [**30**] for a rather large class of locally compact unimodular amenable groups. We also discuss some results on connections of this entropy with the classical Kolmogorov–Sinai entropy. In §3, we study entropy of actions of $\mathbb{R}^n$ and its lattice subgroups. Section 4 is devoted to the study of the entropy of actions of nilpotent Lie groups from $\mathcal{ULG}$.

## 2. *The entropy of actions of* $\mathbb{R}^n$

In this section, we present some preliminaries and new results. In §2.1, we consider the Feldman $r$-entropy [**13**] for $\mathbb{Z}$-actions, and in §2.2 we introduce the spatial entropy [**30**] and discuss some of its properties. In §2.3, we present Feldman's theorem on the connection between the spatial entropy of actions of the group $\mathbb{R}^n$, $n \in \mathbb{N}$, and those of its lattice subgroups. We also give some applications and generalizations of this theorem. In particular, we show that the Ornstein–Weiss entropy of an $\mathbb{R}^n$-action coincides with the classical entropy of the action of its lattice subgroup $\mathbb{Z}^n$ (see Theorem 2.14).

2.1. *Feldman's $r$-entropy in $\mathbb{Z}^n$.* Let us consider $\mathbb{Z}$-actions in more detail: we will return to the case $\mathbb{Z}^n$ at the end of this subsection. Let $T$ be an ergodic measure-preserving automorphism of $(X, \mathcal{B}, \mu)$ and $\rho$ a finite partition of $(X, \mu)$. The classical entropy $h_K(T, \rho)$ of the process $(T, \rho)$ is

$$h_K(T, \rho) = \limsup_{N\to\infty}(1/N)H\bigg(\bigvee_{j=1}^{N} T^{-j}\rho\bigg), \tag{2.1}$$

where $H(Q)$ is defined for a finite partition $Q = \{Q_i\}$ of $(X, \mu)$ by $H(Q) = -\sum \mu(Q_i)\log \mu(Q_i)$.

Now choose a real number $r > 0$ and consider a collection B of disjoint $\bigvee_{j=1}^{N} T^{-j}\rho$-measurable sets each having diameter $\leq r$ with respect to the normalized Hamming metric on $\rho - N$-names of points, that is

$$x, y \in B \in \mathrm{B} \Rightarrow d_N^{\rho}(x, y) = (1/N)|\{j : 1 \leq j \leq N \text{ and } \rho(T^{-j}x) \neq \rho(T^{-j}y)\}| \leq r,$$

where $\rho(x) = \rho_i$ is an element of the partition $\rho = \{\rho_i\}$ such that $x \in \rho_i (= \rho(x))$. This family B is called a $(\rho, N, r)$-family.

*Definition 2.1.* [**13**] We define $h_r(T, \rho)$, the $r$-entropy, as the infimum of the set of real numbers $b$ such that for every $\varepsilon > 0$, there exists $N_0$ such that if $N > N_0$, then there exists a $(\rho, N, r)$-family B with $\mu(\bigcup \mathrm{B}) > 1 - \varepsilon$ and $(1/N)H(\mathrm{B}) \leq b$; in symbols

$$h_r(T, \rho) = \sup_{\varepsilon>0} \liminf_{N\to\infty}\bigg\{\frac{H(\mathrm{B})}{N}\bigg\},$$

where B is a $(\rho, N, r)$-family with $\mu(\bigcup \mathrm{B}) > 1 - \varepsilon$.

Clearly, $h_r(T, \rho) \leq h_K(T, \rho)$.

*Definition 2.2.* [**13**] For $r > 0$, define $k_r(T, \rho)$ to be the same supremum as in equation 2.1, but with $\log |\mathrm{B}|$ replacing $H(\mathrm{B})$. It follows from the Shannon–McMillan theorem that $k_r(T, \rho) \le h_K(T, \rho)$, and also, clearly, $h_r(T, \rho) \le k_r(T, \rho)$. Furthermore, the number of atoms of $\bigvee_{j=1}^{N} T^{-j}\rho$ in a single $B \in \mathrm{B}$ is dominated by $\binom{N}{[Nr]}$, where $\binom{n}{k}$ is the binomial coefficient, and $[x]$ means the greatest integer not exceeding $x$.

The following properties are easy to verify.

- $h_r(T, \rho) = k_r(T, \rho) = 0$ if $r \ge 1$.
- $h_r(T, \rho)$ and $k_r(T, \rho)$ are monotone non-increasing functions of $r$.

It is, furthermore, not difficult to prove that $k_r(T, \rho)$ is convex, and hence continuous and strictly monotonic for $r > 0$ (see [**13**, Proposition 2.3]). Finally, $h_r(T, \rho) = k_r(T, \rho)$ for all $r$ [**13**, Corollary 2.6].

LEMMA 2.3. [**13**] *Let $T$ and $\rho$ be as above, and let $k(T, \rho) = \lim_{r\to 0} k_r(T, \rho)$, then $k(T, \rho) = h_K(T, \rho)$.*

*Proof.* The idea of the proof was given in [**13**]. We present a somewhat more detailed proof because we will use it in the following. Let $\mathrm{B}$ be a $(\rho, N, r)$-family with $\mu(\bigcup \mathrm{B}) > 1 - \varepsilon/2$. Then each $B \in \mathrm{B}$ contains no more than $\binom{N}{[Nr]}|\rho|^{[Nr]}$ atoms. For sufficiently large $N$, the Shannon–McMillan theorem gives a $(\bigvee_{j=1}^{N} T^{-j}\rho)$-measurable set $E$ of measure at least $1 - \varepsilon/2$ such that all atoms in $E$ have measure at most $2^{-(h_K(T,\rho)-\varepsilon/2)N}$. Consider the $(\rho, N, r)$-family $\mathrm{C} = (B \cap E : B \in \mathrm{B})$. Then

$$H(\mathrm{C}) = -\sum_B \mu(B \cap E)(\log \mu(B \cap E)) \tag{2.2}$$

$$= H(\mathrm{B}) - \sum_B \mu(B)\bigg(\log \frac{\mu(B \cap E)}{\mu(B)}\bigg) + \sum_B \mu(B \cap (X \backslash E)) \log(\mu(B)) \tag{2.3}$$

$$\le H(\mathrm{B}) - \sum_B \mu(B)\bigg(\log \frac{\mu(B \cap E)}{\mu(B)}\bigg). \tag{2.4}$$

Let $\mathrm{B}_1 = \{B \in \mathrm{B} : \mu(B \cap E) \ge (1 - \sqrt{\varepsilon})\mu(B)\}$ and $\mathrm{B}_2 = \mathrm{B}\backslash(\mathrm{B}_1)$.
Then

$$- \sum_{B\in \mathrm{B}_1} \mu(B) \log\bigg(\frac{\mu(B \cap E)}{\mu(B)}\bigg) \le -\log(1 - \sqrt{\varepsilon})(1 - \varepsilon/2). \tag{2.5}$$

Notice that

$$-\sum_{B\in \mathrm{B}_2} \log\bigg(\frac{\mu(B \cap E)}{\mu(B)}\bigg) < -\sum_{B\in \mathrm{B}_2} (\log \mu(B \cap E))\mu(B).$$

Since $\mu(B \cap E) < (1 - \sqrt{\varepsilon})\mu(B)$ for $B \in \mathrm{B}_2$, we have $\sqrt{\varepsilon}\mu(B) < \mu(B\backslash(B \cap E))$, for any $B \in \mathrm{B}_2$. Hence

$$\sum_{B\in \mathrm{B}_2} \mu(B) < (1/\sqrt{\varepsilon})\mu\bigg(X\backslash\bigg(\bigcup\bigg(B \bigcap E\bigg)\bigg)\bigg) < \sqrt{\varepsilon}.$$

Let

$$C(N, \rho, r, \varepsilon) = -\log\bigg(\binom{N}{[Nr]}|\rho|^{[Nr]}2^{-Nh_K(T,\rho)-\varepsilon/2}\bigg).$$

Now the following estimate is obvious:

$$-\sum_{B\in\mathrm{B}}\mu(B)\log\left(\frac{\mu(B\cap E)}{\mu(B)}\right) < C(N, \rho, r, \varepsilon)\sqrt{\varepsilon}. \tag{2.6}$$

Let $D(N, \rho, r, \varepsilon) = (1/N)C(N, \rho, r, \varepsilon)$. We may deduce from (2.2)–(2.6) that

$$\frac{H(\mathrm{C})}{N} \leq \frac{H(\mathrm{B})}{N} + D(N, \rho, r, \varepsilon)\sqrt{\varepsilon}.$$

Since $(h_K(T, \rho) - \varepsilon/2)(1-\varepsilon) \leq (1/N)H(\mathrm{C})$ by the Shannon–McMillan theorem, one can use Definition 2.1 to obtain the following estimate:

$$(h_K(T, \rho) - \varepsilon/2)(1-\varepsilon) \leq h_r(T, \rho) + D(N, \rho, r, \varepsilon)\sqrt{\varepsilon}. \tag{2.7}$$

We use Stirling's formula to estimate $D(N, \rho, r, \varepsilon)$ for large enough $N$, obtaining:

$$D(N, \rho, r, \varepsilon) = -r\log r - (1-r)\log(1-r) - r\log|\rho| + h_K(T, \rho) - \varepsilon/2 + O(1).$$

Hence, one can assume that $\lim_{r\to 0} D(N, \rho, r, \varepsilon) < h_K(T, \rho) - \varepsilon/4$, for sufficiently large $N$, and it follows from (2.7) that

$$(h_K(T, \rho) - \varepsilon/2)(1-\varepsilon) \leq k(T, \rho) + (h_K(T, \rho) - \varepsilon/4)\sqrt{\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, it follows from the last inequality that $h_K(T, \rho) \leq k(T, \rho)$. But we have seen above that $k_r(T, \rho) \leq h_K(T, \rho)$, and hence we deduce that $k(T, \rho) = h_K(T, \rho)$. □

Conze [**5**] and Katznelson and Weiss [**23**] showed that the above definitions and theorems all hold for $\mathbb{Z}^n$, $n > 1$. In particular, one can apply the version of Shannon–McMillan from [**47**] to prove an analogue of Lemma 2.3 for $\mathbb{Z}^n$-actions.

2.2. *Ornstein–Weiss spatial entropy.* Suppose that a locally compact unimodular group acts freely and preserves the measure on a probability space $(X, \mathcal{B}, \mu)$ via $T_g : x \mapsto gx$. Ornstein and Weiss [**30**] introduced the notion of spatial entropy to extend the above theory to this setting. We summarize here their major results.

If $\rho$ is a finite partition of $X$, one can consider a measurable mapping, also denoted $\rho$, from $X$ to a compact metric space $(\Omega, d)$, whose level sets form the partition

$$\rho : X \to \Omega.$$

The special case where $d$ is the normalized Hamming metric on the set $\Omega = \{1, 2, \ldots, |\rho|\}$ was considered in §§2.1. For convenience, it is assumed that the diameter of $\Omega$ is 1. Let $F$ be a compact subset of $G$. The basic concept in this approach is a $\rho - r - F$-ball which is a measurable subset $E$ of $X$ such that for all $x, y \in E$,

$$d_F^{\rho}(x, y) < r,$$

where

$$d_F^{\rho}(x, y) = 1/|F| \int_F d(\rho(fx), \rho(fy))\, df,$$

and $|F|$ is the Haar measure of $F \subset G$.

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If $\rho$ is fixed, we write $d_F$ instead of $d_F^\rho$. Let $\{F_n\}$ be an increasing sequence of Følner subsets of $G$ such that $\bigcup_n F_n = G$. For fixed $r > 0$, consider the set $H_r$ of all $h > 0$ such that for all $\delta > 0$, if $n$ is large enough, one can cover $(1-\delta)$ of $X$ by fewer than $2^{h|F_n|}$ $\rho - r - F_n$-balls.

Ornstein and Weiss proved that $H_r$ is not empty for $0 < r < 1$, and defined the *spatial r-entropy* of the process of $(T, \rho)$ by

$$\mathrm{sh}(T, \rho, r) = \inf_h H_r.$$

They further showed that $sp(T, \rho, r)$ is a continuous monotonic increasing function in $r$, and defined the *spatial entropy* $\mathrm{sh}(T, \rho)$ of $(T, \rho)$ by $\mathrm{sh}(T, \rho) = \lim_{r\to 0} \mathrm{sh}(T, \rho, r)$.

In the case of $G = \mathbb{Z}$, we introduced the $r$-entropy $k_r(T, \rho)$ in Definition 2.2. It is clear that $\mathrm{sh}(T, \rho, r) = k_r(T, \rho)$, and Lemma 2.3 shows that $\mathrm{sh}(T, \rho) = \lim_{r\to 0} \mathrm{sh}(T, \rho, r) = h_K(T, \rho)$, where $h_K(T, \rho)$ is the Kolmogorov–Sinai entropy of the process $(T, \rho)$. It turns out that this assertion holds for any countable amenable group.

PROPOSITION 2.4. [**30**] *Let $G$ be an infinite countable amenable group, $T$ an ergodic action of $G$ on a probability space $(X, \mathcal{B}, \mu)$, and $\rho$ a finite partition of $X$. Then* $\mathrm{sh}(T, \rho) = h_K(T, \rho)$.

The proof uses the arguments of the proof of Lemma 2.3 and the version of the Shannon–McMillan theorem for actions of countable amenable groups proved in [**25**] and [**47**, Theorems 4.4, 4.12].

Ornstein and Weiss [**30**] introduced a class $\mathcal{G}$ of groups which they called groups of *zero self-entropy*. This class contains all discrete amenable groups, all nilpotent Lie groups, some solvable Lie groups, and is closed under direct products. We consider groups from this class in this paper.

Let $G$ belong to $\mathcal{G}$ and let $T$ be a measure-preserving, free ergodic action on a probability space. Then the entropy $h(T)$ of the action $T$ is defined by:

$$h(T) = \sup_\rho \mathrm{sh}(T, \rho),$$

where $\rho$ is a finite partition of $X$, $\mathrm{sh}(T, \rho) = \lim_{r\to 0} \mathrm{sh}(T, \rho, r)$, and

$$h(T) = \mathrm{sh}(T, \rho)$$

if a measurable partition $\rho$ is a generator for $T$.

Ornstein and Weiss studied $h(T)$ as an invariant for the action $T$. In particular, they showed that if $T_1$ and $T_2$ are Bernoulli and $h(T_1) = h(T_2)$, then $T_1$ and $T_2$ are metrically isomorphic. They also proved some useful properties of $h(T)$ which we will use later on.

2.3. *Spatial entropy for actions of $\mathbb{R}^n$.* If $f$ and $g$ are measurable functions from a measurable subset $C$ of $\mathbb{R}^n$ to a finite index set, then we denote by $d_C(f, g)$, or $d(f, g)$, the number $(1/|C|)|\{f \neq g\}|$, where $|C|$ denotes the Haar measure of $C$. For a measure-preserving action $\varphi$ of $\mathbb{R}^n$ on $(X, \mathcal{B}, \mu)$ as above, and a finite partition $\rho$ of $X$, one can define a metric $d_C^\rho(x, y) = d(f, g)$, where $f(v) = \rho(\varphi_v x)$ and $g(v) = \rho(\varphi_v y)$, $v \in C$, $x, y \in X$, that is $(1/|C|)|\{v : \rho(\varphi_v x) \neq \rho(\varphi_v y)\}|$.

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Let $N$ be a positive real number and let $C_N$ denote the cube of side $N$, i.e. all of whose vertices have coordinates either 0 or $N$. Denote by $\rho_{C_N}$ the $\sigma$-field spanned by $\{\varphi_v^{-1}\rho;\ v \in C_N\}$. Then the family B of disjoint sets is called a $(\rho, N, r)$-family if:

(i) each $B \in$ B is in $\rho_{C_N}$; and

(ii) each $B \in$ B has $d^{\rho}_{C_N}$-diameter $\leq r$.

The $r$-spatial entropy $\mathrm{sh}(\varphi, \rho, r)$, or $r$-entropy, and the spatial entropy $\mathrm{sh}(\varphi, \rho)$ are defined by analogy with §§2.1 and 2.2, and $\mathrm{sh}(\varphi, \rho) = \lim_{r\to 0} \mathrm{sh}(\varphi, \rho, r)$.

Now let $\varphi^D$ be the $\mathbb{Z}^n$-action obtained from $\varphi$ on the $D\mathbb{Z}^n$-lattice: $\varphi_v^D = \varphi_{Dv}$, where $D$ is a positive real number. Then one can consider the $\mathbb{Z}^n$-process $(\varphi^D, \rho)$, and define the spatial entropy of this process by $\mathrm{sh}(\varphi^D, \rho)$. Feldman [**13**] found an important connection between the spatial entropy $\mathrm{sh}(\varphi, \rho)$ of the process $(\varphi, \rho)$, and the spatial entropy $\mathrm{sh}(\varphi^{D_i}, \rho)$ of the process $(\varphi^{D_i}, \rho)$, where $D_{i-1} \subset D_i$, $i \in \mathbb{N}$, and $D_i \to 0$.

THEOREM 2.5. [**13**] *Let $\varphi$ be an ergodic measure-preserving action of $\mathbb{R}^n$ on a probability space $(X, \mathcal{B}, \mu)$. If $\rho$ is a finite partition of $X$, then*

$$\mathrm{sh}(\varphi, \rho) = \lim_{D\downarrow 0} |C_D|^{-1}\mathrm{sh}(\varphi^D, \rho). \tag{2.8}$$

*Proof.* The proof which we will give is based largely on [**13**]: we give sufficient detail to establish some key estimates which will be used subsequently. To simplify our argument, we consider actions $\varphi$ such that $\varphi^{\Gamma}$ is ergodic for any lattice subgroup $\Gamma$ of $\mathbb{R}^n$. In particular, $\varphi$ has this property if its spectrum does not contain a discrete component (see Proposition 2.12 below).

Fix $D > 0$. The *continuous* $d^{\rho}(x, y)$-distance on $C_N$ between $x$ and $y$ from $X$ may be computed by taking the *discrete* $d^{\rho}$-distance between $\varphi_v x$ and $\varphi_v y$ over $C_D$*-lattice points in* $C_N$ and taking the normalized integral of this as $v$ ranges over $C_D$. More exactly,

$$d_{C_N}(x, y) = \int_{C_D} \frac{dv}{|C_D|}\bigg(\frac{1}{(N/D)^n}\sum_{w\in(N/D)^n} |\{w : \rho(\varphi_{w+v}x) \neq \rho(\varphi_{w+v}y)\}|\bigg),$$

where we suppose that $N/D$ is an integer.

Let B be a $(\rho, N, \delta)$-family of a measure greater than $1-\varepsilon$, i.e. $\mu(\bigcup \mathrm{B}) > (1-\varepsilon)$. If $x$ and $y$ are in the same $B \in$ B, i.e. $d_{C_N}(x, y) < \delta$, then a sequence of estimates based on the Fubini theorem shows us that for any given $\varepsilon > 0$, a sufficiently small choice of $\delta > 0$ guarantees that there exists a set $V \subset C_D$ with $|V|/|C_D| > 1-\varepsilon$ such that for each $v \in V$ there is a set $S_v \subset X$, $\mu(S_v) > 1-\varepsilon$ with the following properties.

For each $B \in$ B, $x \in B \cap S_v$, there is a set $R_x \subset B$, $\mu(R_x) > (1-\varepsilon)\mu(B)$ such that if $w \in V$, $x \in B \cap S_v$ and $y \in R_x$, then the discrete distance from $\varphi_v x$ to $\varphi_v y$ (over the $C_D$ lattice points in $C_N$) is less than $\varepsilon/2$.

Thus, if $y'$, $y''$ are in $R_x$, the discrete distance from $\varphi_v y'$ to $\varphi_v y''$ is less than $\varepsilon$. Choose some fixed $v \in V$ and some $x(B)$ in each non-empty $B \cap S_v$, and let $\mathrm{B}_0 = \{\varphi_{-v}R_{x(B)} : B \cap S_v \neq \emptyset\}$. Then $\mu(\bigcup \mathrm{B}_0) > 1-3\varepsilon$, $|\mathrm{B}_0| \leq |\mathrm{B}|$, and each $B \in \mathrm{B}_0$ has discrete $d^{\rho}$ diameter at most $3\varepsilon$ over the $C_D$ lattice points in $C_N$. Expand each set $B \in \mathrm{B}_0$ to the set $\tilde{B}$ by adding the set $B$ of all points which have the same $\rho$-name over the $C_D$ lattice in $C_N$ as any point of $B$. The family so obtained consists of sets which are measurable with respect to $\bigvee\{\varphi_v\rho : v \in (D\mathbb{Z})^n\}$. However, they may no longer be disjoint. We disjointify them. Thus, we have produced a $(\rho, N/D, 3\varepsilon)$-family for $\varphi^D$, of measure at least $1-3\varepsilon$,

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and of cardinality $\leq |\mathrm{B}|$. This shows that

$$\mathrm{sh}(\varphi, \rho, \delta) \geq |C_D|^{-1}\mathrm{sh}(\varphi^D, \rho, 3\varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we have

$$\mathrm{sh}(\varphi, \rho, \delta) \geq |C_D|^{-1}\mathrm{sh}(\varphi^D, \rho),$$

by the properties of $\mathrm{sh}(\varphi, \rho, \gamma)$, and hence

$$\mathrm{sh}(\varphi, \rho) \geq |C_D|^{-1}\mathrm{sh}(\varphi^D, \rho), \tag{2.9}$$

which gives the result in one direction.

To prove the opposite direction, we require a lemma.

LEMMA 2.6. **[13]** *Fix $\varepsilon > 0$. There exists $D > 0$ and $N_0$ such that if $N > N_0$, and if we let $L_x$ be the set of $C_D$ lattice points $v$ in $C_N$ for which*

$$|\{w \in C_D : \rho(\varphi_{v+w}x) = \rho(\varphi_v x)\}| > (1-\varepsilon)|C_D|,$$

*then $R = \{x : |L_x| > (1-\varepsilon)|N/D|^n\}$ has measure $> 1-\varepsilon$, where $|N/D|^n$ is just the number of $C_D$ lattice points in $C_N$.*

*Proof.* We present a sketch of the proof of this lemma. By a straightforward argument involving Fubini's theorem, we get, for sufficiently small $D > 0$, that

$$|\{w \in C_D : \rho(\varphi_w x) = \rho(x)\}| > (1-\varepsilon)|C_D|$$

for all $x$ in a set of measure greater than $1-\varepsilon^2$. Since $\varphi^D$ is ergodic by the assumptions of the theorem, one can apply the mean ergodic theorem to $\varphi^D$. □

*Proof of Theorem 2.5.* Let $R$ be as in the statement of Lemma 2.6. Then $\mu(R) > (1-\varepsilon)$, where $R$ is a $\rho_{C_N}$-measurable set, and if $B$ is a $\{\varphi_v\rho : v \in (D\mathbb{Z})^n \cap C_N\}$-measurable set of discrete diameter $\leq r$ for the process $\{(\varphi_v^D, \rho) : v \in \mathbb{Z}^n \cap C_N\}$, then $B \cap R$ is $\rho_{C_N}$-measurable and has continuous diameter $\leq r + 2\varepsilon$, where we apply Lemma 2.6 for the process $\{(\varphi_v, \rho) : v \in C_N\}$.

Hence

$$\mathrm{sh}(\varphi, \rho, r+2\varepsilon) \leq |C_D|^{-1}\mathrm{sh}(\varphi^D, \rho, r).$$

Fixing $D$ and letting $r \to 0$ gives

$$\mathrm{sh}(\varphi, \rho, 2\varepsilon) \leq |C_D|^{-1}\mathrm{sh}(\varphi^D, \rho).$$

Let $\varepsilon \downarrow 0$ so that $D \downarrow 0$, as forced by $\varepsilon$. Then we have

$$\mathrm{sh}(\varphi, \rho) \leq \sup \lim_{D\downarrow 0}(|C_D|^{-1}\mathrm{sh}(\varphi^D, \rho)).$$

Now this estimate and inequality (2.9) allow us to complete the proof. □

We present some simple corollaries of Theorem 2.5.

COROLLARY 2.7. *Let $\varphi$, $\mathbb{R}^n$, $(X, \mathcal{B}, \mu)$, $\rho$ be as in the statement of Theorem 2.5, and let $p \in \mathbb{R}$, $p > 0$. Define the action $\varphi^p$ of $\mathbb{R}^n$ on $(X, \mathcal{B}, \mu)$ as follows:*

$$\varphi_v^p(x) = \varphi(pvx), \quad v \in \mathbb{R}^n, x \in X.$$

*Then $\mathrm{sh}(\varphi^p, \rho) = p^n\mathrm{sh}(\varphi, \rho)$ and $h(\varphi^p) = p^n h(\varphi)$.*

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*Proof.* Indeed,

$$\mathrm{sh}(\varphi^p, \rho) = \lim_{D\downarrow 0} |C_D|^{-1}\mathrm{sh}(\varphi^{pD}, \rho) = p^n \lim_{D\downarrow 0} |C_{pD}|^{-1}\mathrm{sh}(\varphi^{pD}, \rho) = p^n\mathrm{sh}(\varphi, \rho). \quad \square$$

Consider now an action $\varphi$ of the group $\mathbb{R}^n \times \mathbb{Z}^m$, $n, m \in \mathbb{N}$ on $(X, \mathcal{B}, \mu)$. Again let $\varphi^D$ be the $\mathbb{Z}^n \times \mathbb{Z}^m$-action obtained from $\varphi$ on $(D\mathbb{Z})^n \times \mathbb{Z}^m$. We need the following generalization of Theorem 2.5.

THEOREM 2.8. *Let $\varphi$ be an ergodic measure-preserving action of $\mathbb{R}^n \times \mathbb{Z}^m$ on a probability space $(X, \mathcal{B}, \mu)$, and $\rho$ a finite partition of $X$. If $\varphi^D$ is the restriction of $\varphi$ to $(D\mathbb{Z})^n \times \mathbb{Z}^m$, then*

$$\mathrm{sh}(\varphi, \rho) = \lim_{D\downarrow 0} |C_D|^{-1}\mathrm{sh}(\varphi^D, \rho).$$

To prove the theorem, one can apply the argument of the proof of Theorem 2.5.

COROLLARY 2.9. *Let $\varphi$ and $\rho$ be as in the statement of Theorem 2.8, and $\varphi^D$ be an ergodic action of $(D\mathbb{Z})^n \times \mathbb{Z}^m$. Then*

$$\mathrm{sh}(\varphi, \rho) = \lim_{i\to\infty} |C_{D_i}|^{-1} h_K(\varphi^{D_i}, \rho) \tag{2.10}$$

*where $D_i = 1/2^i D$, and $h_K(\varphi^D, \rho)$ is the Kolmogorov–Sinai entropy of the process $(\varphi^D, \rho)$.*

*Proof.* Since $\varphi^D$ is ergodic, $\varphi^{D_i}$ is also ergodic. Hence, $\mathrm{sh}(\varphi^{D_i}, \rho) = h_K(\varphi^{D_i}, \rho)$ by Proposition 2.4. Now the corollary follows from Theorem 2.8. $\square$

We now consider some generalizations of Theorems 2.5 and 2.8.

Let $\bar{D} = (D_1, \ldots, D_n)$, $D_i \in \mathbb{R}_+$, and $C_{\bar{D}}$ be the rectangle $C_{\bar{D}} = \{0 \le x_i \le D_i, 1 \le i \le n\}$ in $\mathbb{R}^n$. Consider the family of rectangles $C_{\bar{D}N}$, where $\bar{D}N = \{0 \le x_i \le ND_i, 1 \le i \le n\}$ and $N \in \mathbb{R}_+$. Notice that $|C_{\bar{D}N}| = N^n|C_{\bar{D}}|$. Let $\varphi$ be an ergodic action of $\mathbb{R}^n$ as above, and $\varphi^{\bar{D}N}$ be the restriction of $\varphi$ to the subgroup $\bigoplus_{i=1}^n D_i N\mathbb{Z}$ of $\mathbb{R}^n$, which is isomorphic to $\mathbb{Z}^n$.

PROPOSITION 2.10. *Let $\varphi$ be an ergodic action of $\mathbb{R}^n$, as in the statement of Theorem 2.5, and $\rho$ be a finite partition of $X$. Then*

$$\mathrm{sh}(\varphi, \rho) = \lim_{N\downarrow 0} |C_{\bar{D}N}|^{-1}\mathrm{sh}(\varphi^{\bar{D}N}, \rho),$$

*where $\varphi^{\bar{D}N}$ and $C_{\bar{D}N}$ are defined above.*

*The analogues of Corollary 2.7, Theorem 2.8 and Corollary 2.9 also hold in this setting.*

The proof of the proposition is as in Theorem 2.5.

Actually, we may give a stronger generalization of Theorem 2.5. Let $\{e_i\}$, $1 \le i \le n$ be a basis of the space $\mathbb{R}^n$, $\|e_i\| = 1$, not necessarily orthogonal. Then $\mathbb{R}^n = \mathbb{R}e_1 + \cdots + \mathbb{R}e_n$. Let $\bar{D} = (D_i)$, $1 \le i \le n$, where $D_i$ is a positive real number, and let $C_{\bar{D}}$ be the parallelepiped $\{x_i e_i, 0 \le x_i \le D_i, 1 \le i \le n\}$. Consider the family of parallelepipeds $C_{N\bar{D}} = \{x_i e_i : 0 \le x_i \le ND_i\}$, where $N$ ranges over $\mathbb{R}_+$. Again, $|C_{N\bar{D}}| = N^n|C_{\bar{D}}|$. Let $\varphi$ be the action of $\mathbb{R}^n$ as above, and $\varphi^{\bar{D}}$, $\bar{D} = (D_i)$ be the restriction $\varphi$ to the subgroup $D_1\mathbb{Z}e_1 + \cdots + D_n\mathbb{Z}e_n$ of $\mathbb{R}^n$, isomorphic to $\mathbb{Z}^n$. In fact, by [**33**, §II], any uniform lattice subgroup $\Gamma$ in $\mathbb{R}^n$ can be reduced to one of this form.

PROPOSITION 2.11. *Let $\mathbb{R}^n$, $\varphi$ and $\rho$ be as in the statement of Proposition 2.10; then*

$$\mathrm{sh}(\varphi, \rho) = \lim_{N\downarrow 0} |C_{N\bar{D}}|^{-1} \mathrm{sh}(\varphi^{N\bar{D}}, \rho).$$

*The analogues of Theorem 2.8 and Corollary 2.9 also hold.*

The next proposition describes a class of actions $\varphi$ of $\mathbb{R}^n$, for which we apply Theorem 2.5.

PROPOSITION 2.12. *Let $\varphi$ be an ergodic action of $\mathbb{R}^n \times \mathbb{Z}^m$ as above, $\Gamma$ a lattice subgroup of $\mathbb{R}^n \times \mathbb{Z}^m$, and $\varphi^\Gamma$ the restriction of $\varphi$ to $\Gamma$. If the spectrum of $\varphi$ does not contain a discrete component in $\mathcal{L}_0^2(X, \mu) = \{f \in \mathcal{L}^2 : \int f(x)\, d\mu(x) = 0\}$, then $\varphi^\Gamma$ is ergodic.*

*Proof.* Suppose that $\varphi$ is free and $\varphi^1$ is not ergodic. Then we can take a Borel decomposition of $\varphi^1$ into ergodic components (see for example [**24**, III, Theorem 18.5]). Then there is a Borel factor-space $(Y, \mathcal{B}_Y, \nu)$ of $X, \mathcal{B}, \mu)$ and the Borel partition $X_y y \in Y$ of $X$, where the Borel set $X_y$ is $\varphi^1$-invariant for each $y \in Y$, and the restriction $\varphi^1|_{X_y}$ of $\varphi^1$ to $X_y$ is ergodic. Furthermore, we have the integral decomposition $\mu = \int_Y \mu_y \, d\nu(y)$, where $\mu_y$ is the conditional measure on $X_y$, and $\nu$ is the restriction of $\mu$ to $Y$.

Observe that for all $t \in \mathbb{R}^n$, $\varphi^1_t X_y$ is again a $\varphi_1$-invariant subset of $X$. Thus, $(Y, \mathcal{B}_Y, \nu)$ is a $\varphi$-invariant factor-space of $(X, \mathcal{B}, \mu)$, and, furthermore, it is an ergodic factor-space because $\varphi$ is ergodic. Now it follows from the definition of $Y$ that the action $\varphi$ of $\mathbb{R}^n$ on $Y$ is transitive, and it reduces to the action of the group $\mathbb{T}^m = \mathbb{R}^n/\mathbb{R}^{n-m} \times \mathbb{Z}^m$, where $1 \le m \le n$. Hence, $(Y, \nu)$ coincides with $(\mathbb{T}^m, \nu')$, where $\nu'$ is the Haar measure for $\mathbb{T}^n$.

Thus, if $\varphi^1$ is not ergodic, then $\varphi$ has the factor-space $(\mathbb{T}^m, \nu')$. It is obvious that the restriction of $\varphi$ to this factor-space has a discrete spectrum. □

2.4. *Entropy of $\mathbb{R}^n$-actions and an analogue of the Abramov formula for the entropy of a flow.* In this section, we discuss a connection between the Ornstein–Weiss entropy $h(\varphi)$ of an $\mathbb{R}^n$-action $\varphi$ and the classical Kolmogorov entropy $h_K(\varphi^1)$ of an ergodic $\mathbb{Z}^n$-action $\varphi^1$ (Theorem 2.14). As a consequence of this theorem, we obtain a new proof of the Abramov formula [**1**] for entropies of ergodic actions of lattice subgroups of $\mathbb{R}$ and its generalization for $\mathbb{R}^n$ due to Conze [**5**].

PROPOSITION 2.13. *Let $G$ be a countably infinite amenable group, and $G_r$ a subgroup of $G$ of a finite index $r$, i.e. $[G : G_r] = r$. If $T$ is a measure-preserving action by automorphisms of $G$ on the probability space $(X, \mathcal{B}, \mu)$, then $h_K(G_r) = r h_K(G)$.*

This statement is well known for $\mathbb{Z}$ actions [**6, 15**] and for $\mathbb{Z}^n$-actions [**5**]. An alternative proof was given in [**7**].

*Proof.* Let $E_r$ be a finite subset of $G$ which meets each right $G_r$-coset exactly once. If $\alpha$ is a finite partition of $X$, we set $\alpha^r = \bigvee_{g\in E_r} g\alpha$. The following formula was proved in [**18**]:

$$h_K(G_r, \alpha^r) = r h_K(G, \alpha). \tag{2.11}$$

One can derive from (2.11) that $h_K(G) = \infty$ if and only if $h_K(G_r) = \infty$. Suppose that $h_K(G) < \infty$, and that $T$ acts freely on $(X, \mathcal{B}, \mu)$. Then there exists a finite

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generating partition $\rho$ of $X$ for the action of $G$, i.e. $\bigvee_{g\in G} g\rho$ generates the $\sigma$-algebra $\mathcal{B}$ (see [**8, 38**]). Furthermore, each such generating partition $\rho$ has the property that $h_K(G) = h_K(G, \rho)$ [**8, 38**]. It follows from the definition of $\rho^r$ that if $\rho$ is a generating partition for the $G$-action, then $\rho^r$ is a generating partition for the action of $G^r$, and $h_K(G_r) = h_K(G_r, \rho^r)$. Now one sees from (2.11) that $h_K(G_r) = rh_K(G)$.

Now suppose that the action $T$ of $G$ is not free. By [**46**], there exists a Bernoulli action $T^B$ of $G$ on $(Y, \mathcal{B}_Y, \mu_B)$ with entropy $0 < h_K(T^B) < \infty$, so that the action $T'_g = T_g \otimes T^B_g$, $(g \in G)$ is free and $h(T') = h_K(T) + h_K(T^B)$. It follows that $h_K(T'|_{G_r}) = rh_K(T')$, where $T'|_{G_r}$ is the restriction of $T'$ to $G_r$. Since [**46**], $h_K(T'|_{G_r}) = h_K(T|_{G_r}) + h_K(T^B|_{G_r})$, and $h(T^B|_{G_r}) = rh_K(T^B)$. We conclude that $h_K(T|_{G_r}) = rh_K(T)$. □

THEOREM 2.14. *Let $\varphi$ be an ergodic action of $\mathbb{R}^n$ on $(X, \mathcal{B}, \mu)$ as in the statement of Theorem 2.5, and suppose that the spectrum of $\varphi$ does not contain a discrete component (see Proposition 2.12). If $\varphi^1$ is the restriction of $\varphi$ to the lattice subgroup $\mathbb{Z}^n$ of $\mathbb{R}^n$, then*

$$h(\varphi) = h_K(\varphi^1),$$

*where $h(\varphi)$ is the Ornstein–Weiss entropy of $\varphi$, and $h_K(\varphi^1)$ is the classical entropy of $\varphi^1$.*

A more general statement of Theorem 2.14 was conjectured in [**30**, Appendix B]. We give the proof of it for our special case.

*Proof.* Recall that $\varphi^1$ is an ergodic action, and assume first that $h_K(\varphi^1) < \infty$: we will consider the other case below. It follows from (2.9) that $\mathrm{sh}(\varphi, \rho) \geq \mathrm{sh}(\varphi^1, \rho) = h_K(\varphi^1, \rho)$ for any finite partition $\rho$ of $X$. Hence, $h(\varphi) = \sup_\rho \mathrm{sh}(\varphi \cdot \rho) \geq \sup_\rho h_K(\varphi^1, \rho) = h_K(\varphi^1)$.

We claim that $h(\varphi) = h_K(\varphi^1)$. We will prove this by contradiction. Suppose the claim is not correct. Then we have $h(\varphi) > h_K(\varphi^1)$ and there exists $\epsilon > 0$ such that

$$h(\varphi) - \epsilon > h_K(\varphi^1). \tag{2.12}$$

It follows from the definition of $h(\varphi)$ (see §2.2 or [**30**]) that there is a finite partition $\alpha$ of $X$ such that $h(\varphi) \geq \mathrm{sh}(\varphi, \alpha) > h(\varphi) - \epsilon$. Since $\mathrm{sh}(\varphi, \alpha) = \lim_{i\to\infty} 2^{in} h_K(\varphi^{1/2^i}, \alpha)$, by Corollary 2.9, for $j$ sufficiently large, $2^{jn} h_K(\varphi^{1/2^j}, \alpha) > h(\varphi) - \epsilon$. Furthermore,

$$2^{jn} h_K(\varphi^{1/2^j}) > h(\varphi) - \epsilon.$$

Now observe that $\mathbb{Z}^n$ is a subgroup of $(1/2^j\mathbb{Z})^n$ of index $2^{jn}$. Hence, we have $2^{jn} h_K(\varphi^{1/2^j}) = h_K(\varphi^1)$ by Proposition 2.13. But then the following estimate follows: $h_K(\varphi^1) > h(\varphi) - \epsilon$. This contradicts (2.12), which shows that $h(\varphi) = h_K(\varphi^1)$ for ergodic $\varphi^1$ with $h_K(\varphi^1) < \infty$.

Now suppose that $h_K(\varphi^1) = \infty$. Then, for any sufficiently large real number $N > 0$, there exists a finite partition $\alpha$ such that $h_K(\varphi^1, \alpha) > N$. But $\mathrm{sh}(\varphi^1, \alpha) = h_K(\varphi^1, \alpha)$ by Proposition 2.4. Thus, we have

$$\mathrm{sh}(\varphi, \alpha) \geq \mathrm{sh}(\varphi^1, \alpha) > N$$

by (2.9). Since $N$ is arbitrary, we have $h(\varphi) = \infty$, and $h(\varphi) = h_K(\varphi^1) = \infty$. Thus, the statement is proved. □

The relation $h(\varphi) = h_K(\varphi^1)$ holds in more general situations. Let $G = \mathbb{R}^n$ and let $\Gamma$ be a uniform lattice subgroup of $G$: these subgroups were described in Propositions 2.10 and 2.11. Now let $\Gamma$ act on $(G, m_G)$ by shifts, where $m_G$ is the Haar measure on $G$.

Thus, there is a compact subset $G(\Gamma)$ of $G$ such that each $\Gamma$-orbit intersects with $G(\Gamma)$ in a unique point. Notice that $G(\Gamma)$ can be realized as a parallelepiped in $\mathbb{R}^n$, as we have seen above. Let $|G(\Gamma)|$ be the Haar measure of the set $G(\Gamma)$ in $G$.

COROLLARY 2.15. *Let $\varphi$ be as in the statement of Theorem 2.14, $\Gamma$ a lattice subgroup of $\mathbb{R}^n$, and $\varphi^\Gamma$ the restriction of $\varphi$ to $\Gamma$. Then*

$$h(\varphi) = |G(\Gamma)|^{-1} h_K(\varphi^\Gamma).$$

The proof follows exactly as in Theorem 2.14, using Propositions 2.10 and 2.11.

COROLLARY 2.16. (Abramov–Conze formula) *Let $\varphi$ and $\Gamma$ be as in the statement of Corollary 2.15. Then*

$$h_K(\varphi^1) = |G(\Gamma)|^{-1} h_K(\varphi^\Gamma).$$

Recall that this formula for the entropy of a flow $\{S_t, t \in \mathbb{R}\}$ was proved by Abramov [**1**]: $h_K(S_t) = |t| h_K(S_1)$. But he did not require $S_t$, $t \neq 0$ to be ergodic. Conze [**5**] extended this result for $n > 1$ as follows. Let $\{e_i\}$, $1 \leq i \leq n$, where $e_i = (0, \ldots, 0, 1_i, 0, \ldots, 0)$ is a basis in the space $\mathbb{R}^n$, and $\{\gamma_i\}$, $1 \leq i \leq n$ is the image of $\{e_i\}$, $1 \leq i \leq n$ by a real $n \times n$-matrix $M$. Then $h_K(\varphi^\Gamma) = |\det(M)| h_K(\varphi^1)$, where $\Gamma$ is a lattice subgroup of $\mathbb{R}^n$, generated by $\{\gamma_i\}$, $1 \leq i \leq n$.

Our proof of the formula of Corollary 2.16 given above uses a different approach from [**1, 5**]. We use our approach to give a similar formula for non-commutative nilpotent Lie groups from $\mathcal{ULG}$ in §4.

## 3. $\mathbb{R}^n$-*actions with positive entropy*

In §3.1, CPE and uniformly mixing actions of $\mathbb{R}^n \times \mathbb{Z}^m$ are considered. Spectral properties of these actions are studied in §3.2. Actions of these groups with a positive entropy, their Pinsker algebras and spectral properties are investigated in §3.3.

### 3.1. *CPE actions of* $\mathbb{R}^n$.

*Definition 3.1.* We will say that a free action $\varphi$ of $G = \mathbb{R}^n \times \mathbb{Z}^m$, $n, m \in \mathbb{N}$ on a probability space $(X, \mathcal{B}, \mu)$ has *completely positive entropy* (CPE) if the spatial entropy $\mathrm{sh}(\varphi, \rho)$ of the process $(\varphi, \rho)$ is positive for any finite partition $\rho$ of $X$.

If $G$ is a discrete group, then an action $\varphi$ of $G$ has CPE if $h_K(\varphi, \rho) > 0$ for any finite partition $\rho$ of $X$.

THEOREM 3.2. *Let $G$ and $\varphi$ be as above. Then $\varphi$ is a CPE action of $G$ if and only if for any uniform lattice subgroup $\Gamma$ of $G$ (see the remark after the statement of Theorem 2.10), the action $\varphi^\Gamma$ of $\Gamma$ is also CPE.*

*Proof.* Assume first that $\varphi$ is a CPE action of $\mathbb{R}^n$. Then it is obvious that $\varphi$ is ergodic. Let us show that $\varphi^{\bar{D}}$ is also ergodic for any $\bar{D} = (D_i)$, $D_i > 0$. Suppose $\varphi^{\bar{D}}$ is not ergodic for some $\bar{D}$. To simplify the notation, consider the situation of Theorem 2.5 and assume that $\varphi^1$ is a non-ergodic action of $\mathbb{Z}^n$. It follows from the proof of Proposition 2.12 that there exists a $\varphi$-invariant factor-space $(Y, \mathcal{B}_Y, \nu)$, where $Y = \mathbb{T}^n$ and $\nu$ is the Haar measure on $Y$. Recall that $\varphi$ is free. Furthermore, $(Y, \nu)$ also contains the factor-space $(Y', \nu')$,

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where $Y' = \mathbb{T}$. Now, if $\psi$ is the restriction of $\varphi$ to $Y'$ and $n \geq 2$, then it is obvious that $h(\psi) = 0$. If $n = 1$, then again $h(\psi) = 0$. This follows from the second part of the proof of Theorem 2.5 (see [**30**, Appendices B]).

Hence, if $\varphi^1$ is not ergodic on $(X, \mathcal{B}, \mu)$, then $\varphi$ is a non-CPE action. But this contradicts our assumption on $\varphi$, and hence $\varphi^1$ is ergodic. Thus, if $\varphi$ is a CPE action of $G$, and $\Gamma$ is a uniform lattice subgroup of $G$, then $\varphi^\Gamma$ is ergodic.

Let us show that $\varphi^\Gamma$ is also a CPE action of $\Gamma$. To illustrate the idea of the proof, consider the case $G = \mathbb{R}$. Then for any finite partition $\rho$ of $X$,

$$\mathrm{sh}(\varphi, \rho) = \lim_{n\to\infty} n h_K(\varphi^{1/n}, \rho),$$

by Corollary 2.9. As $\mathrm{sh}(\varphi, \rho) > 0$, our assumption implies that $h_K(\varphi^{1/n}, \rho) > 0$ for sufficiently large $n$. But $h_K(\varphi^{1/n}, \rho)$ is the classical entropy of the $(1/n)\mathbb{Z}$-action, hence we see that $h_K(\varphi^{1/n}, \rho) = H(\rho \mid \bigvee_{i=1}^{\infty} \varphi_{-i/n}\rho)$ (see [**6, 15**] for $\mathbb{Z}$-actions, and [**5**] for $\mathbb{Z}^n$-actions). This relation allows to derive the following estimate:

$$0 < H\left(\rho \,\middle|\, \bigvee_{i=1}^{\infty} \varphi_{-i/n}\rho\right) \leq H\left(\rho \,\middle|\, \bigvee_{i=1}^{\infty} \varphi^{-i}\rho\right) = h_K(\varphi^1, \rho).$$

Hence, $\mathrm{sh}(\varphi^1, \rho) = h_K(\varphi^1, \rho) > 0$, and $\varphi^1$ is a CPE action of $\mathbb{Z}$.

The general case can be treated similarly, using Propositions 2.10 and 2.11, and properties of the entropy of $\mathbb{Z}^n$-actions from [**5, 23**].

For the opposite direction, we suppose that $\varphi^D$ is CPE. Then $\varphi^D$ is ergodic, and $h_K(\varphi^D, \rho) = \mathrm{sh}(\varphi^D, \rho) > 0$ for any finite partition $\rho$ of $X$ by Proposition 2.4. Now it follows from (2.9) that $\mathrm{sh}(\varphi, \rho) \geq |C_D|^{-1}\mathrm{sh}(\varphi^D, \rho) > 0$. This shows that $\varphi$ is a CPE action of $\mathbb{R}^n$. □

*Definition 3.3.* An action of a locally compact group $G$ on a probability space $(X, \mathcal{B}, \mu)$ is called *uniformly mixing* if for any finite partition $\rho$ and for any $\varepsilon > 0$, there exists a compact subset $K \subset G$ such that for any finite set $F \subset G$ which is $K$-separated (i.e. $gh^{-1} \notin K$ for any two distinct $g, h \in F$) one has:

$$H(\rho) - \frac{1}{|F|} H\left(\bigvee_{g\in F} g\rho\right) < \varepsilon.$$

THEOREM 3.4. *Let $\varphi$ be a free, ergodic, measure-preserving action of the group $G = \mathbb{R}^n \times \mathbb{Z}^m$ on a probability space $(X, \mathcal{B}, \mu)$. Then $\varphi$ is a CPE action if and only if $\varphi$ is uniformly mixing.*

*Proof.* We will give the proof for $G = \mathbb{R}$: a similar proof holds in the general case. We suppose first that $\varphi$ is uniformly mixing and show that $\varphi$ is CPE. To show this, it suffices to check that $\varphi^1$ is a CPE action of $\mathbb{Z}$ by Theorem 3.2.

Let $\rho$ be a finite partition of $X$. Then for any $\varepsilon > 0$, there exists, by Definition 3.3, a compact set $K \subset \mathbb{R}$ such that for any finite set $F \subset \mathbb{R}$ which is $K$-separated (i.e. $g - h \notin K$ for $g, h \in F$), we have

$$H(\rho) - \frac{1}{|F|} H\left(\bigvee_{r\in F} (\varphi_r \rho)\right) < \varepsilon. \tag{3.1}$$

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Suppose that $K$ is a closed interval $K = [0, m]$, where $m \in \mathbb{N}$, and $F$ is a $K$-separated set of integers. Then it follows from (3.1) that $\varphi^1$ is a CPE action of $\mathbb{Z}$, by [**37**] (see also [**12, 47**]), and hence $\varphi$ is a CPE action of $\mathbb{R}$ by Theorem 3.2. Thus, we have proved the theorem in one direction.

Suppose that $\varphi$ is a CPE action of $\mathbb{R}$ on $(X, \mathcal{B}, \mu)$. We will show that $\varphi$ is uniformly mixing. Again, this proof can be easily extended to actions of $\mathbb{R}^n$, $n > 1$. Recall that if $\xi$ and $\eta$ are finite partitions of $X$, then we can set $d(\xi, \eta) = H(\xi/\eta) + H(\eta/\xi)$, where $d(\xi, \eta)$ is the Rokhlin metric on the set of all measurable partitions $\xi$ of $X$ with $H(\xi) < \infty$ (see [**36**, §6] or [**15**, Ch. 15, §3]).

Since $\varphi$ is a strongly continuous action of $\mathbb{R}$ in the operator topology in the Hilbert space, then for any finite partition $\rho$ of $X$ and any $\varepsilon > 0$, there exists an integer $m$ such that

$$d(\varphi_t \rho, \rho) < \varepsilon/2, \quad |t| < 1/m, \tag{3.2}$$

by [**15**, Lemma 15.9]. As $\varphi$ is a CPE action, $\varphi^{1/m}$ is also a CPE action of $(1/m)\mathbb{Z}$, by Theorem 3.2. Hence $\varphi^{1/m}$ is uniformly mixing by the main theorem of [**41**], and there exists for $\varepsilon > 0$ a finite subset $K = \{0, \pm 1, \ldots, \pm m_1\}$, $m_1 \in \mathbb{N}$ such that if a finite subset $F$ of $\mathbb{Z}$ is $K$-separated, then

$$H(\rho) - \frac{1}{|F|} H\biggl(\bigvee_{i \in F} \varphi_{i/m} \rho\biggr) < \varepsilon/2. \tag{3.3}$$

Let $p = (m_1 + 2)/m$, and $K_1 = [-p, p] \subset \mathbb{R}$. We claim that if $F_1$ is a finite subset of $\mathbb{R}$ and $F_1$ is $K_1$-separated, then

$$H(\rho) - \frac{1}{|F_1|} H\biggl(\bigvee_{r_i \in F_1} \varphi_{r_i} \rho\biggr) < \varepsilon. \tag{3.4}$$

To prove this, notice that each $r_i$ from $F_1$ can be written in the form $r_i = a_i/m + t_i$, where $a_i \in \mathbb{Z}$ and $|t_i| < 1/m$. Since $|r_i - r_j| > p$, then $|a_i/m - a_j/m| > m_1/m$, and hence it follows from (3.3) that

$$H(\rho) - \frac{1}{|F_1|} H\biggl(\bigvee_{r_i \in F_1} \varphi_{a_i/m} \rho\biggr) < \varepsilon/2. \tag{3.5}$$

We will use the following properties of the Rokhlin metric.

(i) $d(\varphi\rho, \varphi\rho') = d(\rho, \rho')$ for any finite partitions $\rho, \rho'$ of $X$.

(ii) If $\{\rho_i\}$ and $\{\rho'_i\}$ are two families of finite partitions of $X$, and $1 \le i \le n$, $n \in \mathbb{N}$, then

$$\biggl| H\biggl(\bigvee_i \rho_i\biggr) - H\biggl(\bigvee_i \rho'_i\biggr)\biggr| \le \sum_{i=1}^{n} d(\rho_i, \rho'_i).$$

Using these properties of the Rokhlin metric and (3.2), one can derive from (3.3) the following estimate:

$$H(\rho) - \frac{1}{|F_1|} H\biggl(\bigvee_{r_i \in F_1} \varphi_{r_i} \rho\biggr)$$
$$\le H(\rho) - \frac{1}{|F_1|} H\biggl(\bigvee_{r_i \in F_1} \varphi_{a_i/m} \rho\biggr) + \varepsilon/2 \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, (3.4) holds, and $\varphi$ is uniformly mixing. □

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COROLLARY 3.5. *Let $\varphi$ be a free CPE action of $G = \mathbb{R}^n \times \mathbb{Z}^m$ on $(X, \mathcal{B}, \mu)$, and $H$ a closed subgroup of $G$. Then the restriction $\varphi$ to $H$ is a CPE action of $H$ on $(X, \mathcal{B}, \mu)$.*

*Proof.* Let $\varphi^H$ be the restriction of $\varphi$ to $H$. Since $\varphi$ is uniformly mixing by Theorem 3.4, then $\varphi^H$ is also uniformly mixing. Notice that $H$ is isomorphic to the group $G_1 = \mathbb{R}^{n_1} \times \mathbb{Z}^{m_1}$ (see [**33**, Ch. II]), where $0 \le n_1 \le n$, $0 \le m_1 \le n + m - n_1$. Hence, it follows again from Theorem 3.4 that $\varphi^H$ is a CPE action of $H$. □

COROLLARY 3.6. *Let $G$, $\varphi$, $(X, \mathcal{B}, \mu)$ be as in Theorem 3.4, and $H$ a cocompact closed subgroup of $G$. Then $\varphi$ is a CPE action if and only if $\varphi^H$ is a CPE action.*

*Proof.* Notice that if $H$ is a cocompact closed subgroup of $\mathbb{R}^n$, then the structure of $H$ is worked out in [**33**]. More exactly, there is a decomposition of the space $\mathbb{R}^n$ as a direct sum of subspaces $V$ and $W$ and a basis $\{e_1, \ldots, e_p\}$ in $W$ such that $H = V \bigoplus (\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_p)$, $\dim V + p = n$. Hence, any uniform lattice subgroup of $H$ is a uniform lattice subgroup of $\mathbb{R}^n$. Now the corollary is obvious. □

To complete this subsection, we make some remarks about *Bernoulli actions of* $\mathbb{R}^n \times \mathbb{Z}^m$, which are important examples of CPE actions of these groups. If $G$ is an infinite discrete amenable group, then it is easy to construct a Bernoulli action of $G$ using the *von Neumann construction*. Let $(X, \mathcal{B}, \mu)$ be a probability space, and take a copy $(X_g, \mathcal{B}_g, \mu_g)$ of $(X, \mathcal{B}, \mu)$ for each $g \in G$. We define a space $(Y, \mathcal{B}_Y, \nu)$ as follows:

$$(Y, \mathcal{B}_Y, \nu) = \bigotimes_{g \in G} (X_g, \mathcal{B}_g, \mu_g),$$

where $y = (x_g)$, $x_g \in X_g$, is a point of $Y$ and $\nu = \bigotimes_g \mu_g$. The Bernoulli action $T$ of $G$ is defined as follows:

$$(T_g y)_h = y_{hg},$$

where $g, h \in G$. It is easily seen from this definition that $T$ is a uniformly mixing action of $G$, and hence $T$ is a CPE action of $G$ (see for example [**12**]). It follows from [**30**] that any Bernoulli action $T'$ of $G$ is isomorphic either to some $T$ as above or to a factor action of this $T$. Hence, any Bernoulli action of a discrete amenable group $G$ is CPE.

If $G$ is a locally compact unimodular amenable group, then a Bernoulli action $T$ of $G$ can be realized either as a Poisson process on $(G, m)$, where $m$ is the Haar measure of $G$, or as a factor of this process [**30**, III, §§4, 6]. Let $T$ be a Bernoulli action of $\mathbb{R}^n$. Since $\mathbb{Z}^n$ is a closed cocompact subgroup of $\mathbb{R}^n$, then the restriction $T$ to $\mathbb{Z}^n$ is a Bernoulli action of $\mathbb{Z}^n$ by [**30**], and hence this restriction is a CPE action of $\mathbb{Z}^n$ by the remark above. Now we can conclude that the *Bernoulli action $T$ is a CPE action* of $\mathbb{R}^n$ by Theorem 3.2. We will use these remarks in the next subsection.

## 3.2. *Spectral properties of CPE $\mathbb{R}^n$-actions.*

*Definition 3.7.* Let $G$ be a locally compact group, and $T$ be a Borel free measure-preserving action of $G$ on a standard Borel probability space $(X, \mathcal{B}, \mu)$. The action $T$ defines the unitary representation $g \to U_g$, $g \in G$, of the group $G$ on the Hilbert space $H_0 = L_0^2(X, \mu) = \{f : f \in L_2(X, \mu); \int_X f(x)\, d\mu(x) = 0\}$ by

$$(U_g f)(x) = f(T_g^{-1} x), \quad f \in H_0, g \in G.$$

We say that $T$ has infinite Lebesgue spectrum if the representation $g \mapsto U_g$ can be decomposed as a countably infinite direct sum of copies of the regular representations of $G$.

THEOREM 3.8. *Let $\varphi$ be a free CPE action of $G = \mathbb{R}^n \times \mathbb{Z}^m$, $n \geq 1$ on a probability space $(X, \mathcal{B}, \mu)$. Then $\varphi$ has infinite Lebesgue spectrum on $H_0$.*

To prove this theorem for $G = \mathbb{R}^n \times \mathbb{Z}^m$, we develop the approach which we used in [**11**] to study spectral properties of discrete amenable groups. Let $G = \mathbb{R}^n \times \mathbb{Z}^m$, and let $\hat{G}$ be the Pontryagin dual of $G$ with Haar measure $m_{\hat{G}}$. Suppose we have a Borel free action $T$ of $G$ on $(X, \mathcal{B}, \mu)$, as in Definition 3.7. As above, let $g \to U_g$ be the unitary representation of $G$ on $H_0$ induced by $T$. It follows from a generalized version of Stone's theorem [**35**] that $U_g = \int_{\hat{G}} \langle g, x\rangle dE_x$, where $\langle g, x\rangle$, $g \in G$ is the character of $G$ corresponding to $x \in \hat{G}$, and $E_x$ is the spectral measure of $g \to U_g$ on $H_0$ commuting with $U_g$, $g \in G$. Let $f \in H_0$, $\|f\| = 1$. Then we can define a Borel measure $m_f((\Delta)) = ((E_\Delta) f, f)$ on $\hat{G}$, where $\Delta$ is a Borel subset of $\hat{G}$. As is well known, there is a vector $f_0$, $\|f_0\| = 1$ in $H_0$ such that the measure $m_0 = m_{f_0}$ is the maximal measure in the following sense: any $m_k$ for $k \in H_0$ with $\|k\| = 1$ is subordinated to $m_0$, or, in symbols, $m_k \preceq m_0$. This means that if $A_0$ is the support of $m_0$ and $A_k$ is the support of $m_k$, then $A_k \subseteq A_0$. Recall that the support $A_f$ of $m_f$ is the smallest Borel subset of $\hat{G}$ such that $m_f(A_f) = 1$, $m_f(\hat{G} \backslash A_f) = 0$. Now we can realize $H_0$ as a direct integral of Hilbert spaces $H_x$, $x \in \hat{G}$, with respect to the measure $m_0$:

$$H_0 = \int_{\hat{G}} \bigoplus H_x \, dm_0(x)$$

(see [**9**] or [**10**, Appendice A]). If $f_i$, $i = 1, 2$ are vectors from $H_0$, then there are functions $f_i(x) \in H_x$ corresponding to $f_i$, or, in symbols, $x \sim \{f_i(x)\}$, $x \in \hat{G}$, such that functions $x \to (f_1(x), f_2(x))$, $x \to (f_i(x), f_i(x))$ are measurable, where $(f_1(x), f_2(x))$ is the inner product in the space $H_x$ for almost all $x \in \hat{G}$ by the measure $m_0$. Furthermore, $(f_1, f_2) = \int_{\hat{G}} (f_1(x), f_2(x))\, dm_0(x)$. Let $\nu(x) = \dim H_x$ be the dimension of $H_x$. The function $\nu(x)$ is measurable with values in $\mathbb{N} \cup \infty$.

PROPOSITION 3.9. *Let $G$, $\varphi$, $(X, \mathcal{B}, \mu)$ be as in the statement of Theorem 3.8. Then $\varphi$ has infinite Lebesgue spectrum if and only if the measure $m_0$ on $\hat{G}$ is equivalent to the Haar measure $m_{\hat{G}}$ of the group $\hat{G}$ and the dimension function is $\nu(x) = \infty$.*

The proposition is a simple reformulation of Theorem 3.8 in the terminology of direct integrals of Hilbert spaces (see [**9, 10**, Appendice A]). □

The next step in proving Theorem 3.8 is the following lemma.

LEMMA 3.10. *Let $\varphi$ be as in the statement of Theorem 3.8. Then the measure $m_0$ on $\hat{G}$, as above, is subordinated to the Haar measure $m_{\hat{G}}$ of the group $\hat{G}$, but is not necessarily equivalent to the Haar measure.*

*Proof.* Again, to simplify the notation, we treat the case $\mathbb{R}$; the general case can be treated similarly. In this case, $\hat{G}$ is again $\mathbb{R}$ and $m_{\hat{G}}$ is Lebesgue measure $l$ on $\mathbb{R}$. Now, if we have an action $S_t$, $t \in \mathbb{R}$ on $(X, \mathcal{B}, \mu)$, then we can define the unitary representation of $\mathbb{R}$ on $H_0 = L_0^2(X, \mu)$ by

$$U_t f(x) = f(S_{-t}x), \quad t \in \mathbb{R},\ f \in H_0.$$

Thus, we have a one-parameter strongly continuous group of unitary operators $\{U_t\}$, $t \in \mathbb{R}$ on the Hilbert space $H_0$. It follows that $\{U_t\}$, $t \in \mathbb{R}$ has the spectral representation

$$U_t = \int_{-\infty}^{\infty} e^{i\lambda t}\, dE_\lambda,$$

by Stone's theorem [**34**], where $E_\lambda$, $\lambda \in \mathbb{R}$ is the spectral family of projections on $H_0$ commuting with $U_t$ for all $t \in \mathbb{R}$.

Let $\psi$ be a finite function on $\mathbb{R}$. Then we can consider the operator $\psi(U)$ on $H_0$ given by

$$\psi(U) = \int \psi(t) U_t\, dt = \int \psi(t) e^{i\lambda t}\, dE_\lambda\, dt = \int \hat{\psi}(\lambda)\, dE_\lambda,$$

where $\hat{\psi}(\lambda) = \int_{\mathbb{R}} \psi(t) e^{i\lambda t}\, dt$ is the Fourier transformation of $\psi$. In particular, if $f \in H_0$, and $\|f\| = 1$, then

$$(\psi_1(U) f, \psi_2(U) f) = \int \hat{\psi}_1(\lambda) \bar{\hat{\psi}}_2(\lambda)\, d(E_\lambda f, f) \tag{3.6}$$

and

$$(U_s \psi(U) f, \psi_2(U) f) = \int e^{i\lambda s} \hat{\psi}_1(\lambda) \bar{\hat{\psi}}_2(\lambda)\, d(E_\lambda f, f), \tag{3.7}$$

where $s \in \mathbb{R}$.

Consider the subspace $H_f$ of $H_0$ generated by $\{U_s f\}$, $s \in \mathbb{R}$, where $f \in H_0$, $\|f\| = 1$. $H_f$ is an invariant subspace with respect to $U_s$, $s \in \mathbb{R}$. It follows from (3.6) and (3.7) that we can consider the inner product on $H_f$ in the following form: if $\psi_1, \psi_2 \in H_f$, then there exist measurable functions $\psi_1(\lambda)$, $\psi_2(\lambda)$, $\lambda \in \mathbb{R}$ such that

$$(\psi_1, \psi_2)' = \int_{\mathbb{R}} \psi_1(\lambda), \bar{\psi}_2(\lambda)\, d(E_\lambda f, f),$$

where $(\psi_i, \psi_i)' = \int_{\mathbb{R}} |\psi_i(\lambda)|^2\, d(E_\lambda f, f) < \infty$, $i = 1, 2$. Furthermore, we have $U_t(\psi(\lambda)) = e^{it\lambda} \psi(\lambda)$, $t \in \mathbb{R}$ for $\psi$ from $H_f$.

Recall that $m_f(\lambda_1 - \lambda_2) = (E_{(\lambda_1 - \lambda_2)} f, f)$, $\lambda_i$, $i = 1, 2$ is a Borel measure on $\mathbb{R}$. We would like to show that $m_f$ is subordinated to Lebesgue measure $l$ on $\mathbb{R}$. Suppose that $m_f$ is not subordinated to $l$. Then, as is well known, we have $m_f = m_{fl} + m_{fs}$, where $m_{fl}$ is subordinated to $l$, and $m_{fs}$ is singular, i.e. does not subordinate to $l$. Let the Borel subsets $A_l$ and $A_s$ of $\mathbb{R}$ be the supports of $m_{fl}$ and $m_{fs}$, respectively. Then we have $A_l \cup A_s \subseteq \mathbb{R}$, $A_l \cap A_s = \varnothing$, and $m_{fl}(R) = m(R \cap A_l)$, $m_{fs}(R) = m(R \cap A_s)$, where $R$ is a Borel subset of $\mathbb{R}$. Furthermore, $H_f$ is decomposed as $H_f = H_l \oplus H_s$, where $H_l$ and $H_s$ are orthogonal $S_t$-invariant subspaces of $H_f$, where $t \in \mathbb{R}$, corresponding to $A_l$ and $A_s$, respectively.

Consider the subgroup $S_n$, $n \in \mathbb{Z}$. Let $U_n$ be the corresponding unitary representation of this subgroup on $H_0$, and $U_n = \int_0^{2\pi} e^{in\alpha}\, dF_\alpha$ the spectral representation of this subgroup, where $\{F_\alpha\}$ is the family of spectral projections for $U_n$, $n \in \mathbb{Z}$, commuting with $U_n$, $n \in \mathbb{Z}$.

Since $S_t$, $t \in \mathbb{R}$ has a CPE action on $(X, \mathcal{B}, \mu)$, then $S_n$, $n \in \mathbb{Z}$ also has a CPE action on $(X, \mathcal{B}, \mu)$ by Theorem 3.2 above. Hence, $U_n$, $n \in \mathbb{Z}$ has Lebesgue spectrum on $H_0$ by [**11, 37**]. In particular, it has a spectrum on $H_s$, subordinated to the Lebesgue spectrum, because $H_s$ is an $(S_t, t \in \mathbb{R})$-invariant subspace of $H_0$. We use this observation below.

If now $\psi_1, \psi_2 \in H_s$, then we can rewrite the inner product of $\psi_1$ and $\psi_2$ as follows:

$$(\psi_1, \psi_2)'_s = \int \psi_1(\lambda)\bar{\psi}_2(\lambda)\, dm_s(\lambda).$$

It is obvious that the function $g(\lambda) = \chi_{[0,2\pi)\cap A_s}(\lambda)$, where $\chi_I$ is the indicator of the set $I \in \mathbb{R}$, belongs to $H_s$. Suppose that $(g(\lambda), g(\lambda))'_s > 0$, and consider the following relations

$$((F_\alpha g)(\lambda), g(\lambda)'_s) = \int_0^\alpha |g(\lambda)|^2\, dm_s(\lambda).$$

The left part of this equality defines the measure on $[0, 2\pi)$ which is subordinated to the Lebesgue measure in view of the spectral properties of $S_n$, $n \in \mathbb{Z}$. But the right part defines a singular measure on $[0, 2\pi)$. This contradiction shows that $(g(\lambda), g(\lambda))'_s = 0$, and $A_s \cap [0, 2\pi) = \varnothing$.

Hence, for any $k \in \mathbb{Z}$, we also have $A_s \cap [2k\pi, 2(k+1)\pi) = \varnothing$. Since $\{S_{i/m}\}$, $i \in \mathbb{Z}$, $m \in \mathbb{N}$ also has a CPE action on $(X, \mathcal{B}, \mu)$ by Theorem 3.2, then again $A_s \cap [2km\pi, 2(k+1)m\pi) = \varnothing$. Thus, $m_{fs} = 0$, and $m_f = m_{fl}$, and since $f$ is an arbitrary vector from $H_0$, we conclude that the maximal measure $m_0$ is subordinated to $l$. □

We now consider spectral properties of Bernoulli actions of the groups $G = \mathbb{R}^n \times \mathbb{Z}^m$, $n, m \in \mathbb{N}$, which are an important class of the CPE actions of these groups. First, we will prove a Lemma on the structure of these actions.

LEMMA 3.11. *Let $G$ be a countable discrete infinite abelian group so that $G = \mathbb{R}^n \times \mathbb{Z}^m$, and let $T$ be a measure-preserving Bernoulli action of $G$ on the probability space $(Y, \mathcal{B}, \nu)$. Let $(Z, \mathcal{B}_Z, \mu_Z) = \bigotimes_{i=1}^n (Y_i, \mathcal{B}_i, \nu_i)$, where $(Y_i, \mathcal{B}_i, \nu_i)$ is a copy of $(Y, \mathcal{B}, \nu)$, and $n \in \mathbb{N} \cup \infty$. If $\phi$ is the action of $G$ on $(Z, \mathcal{B}_Z, \mu_Z)$ defined by*

$$(\phi^h z)_i = T_h y_i, \quad h \in G,$$

*where $z = (y_i)$, $1 \le i \le n$, $y_i \in Y$, then $\phi$ is a Bernoulli action of $G$, and $h(\phi) = nh(T)$.*

*Proof.* First assume that $G$ is discrete. Since all Bernoulli actions of $G$ with the same entropy are isomorphic [**30**], the action $T$ of $G$ on $(Y, \mathbb{B}, \nu)$ is a von Neumann action, as discussed at the end of §3.1. Thus, we have

$$(Z, \mathcal{B}_Z, \mu_Z) = \bigotimes_{i=1}^n \bigotimes_{g\in G} (X_g^i, \mathcal{B}_g^i, \mu_g^i).$$

Consider a subspace $(Z_e, \mathcal{B}_e, \mu_e)$ of $(Z, \mathcal{B}_Z, \mu_Z)$, which we define as follows:

$$(Z_e, \mathcal{B}_e, \mu_e) = \bigotimes_{i=1}^n (X_e^i, \mathcal{B}_e^i, \mu_e^i).$$

For $g \in G$, we have

$$\phi^g(Z_e, \mathcal{B}_e, \mu_e) = \bigotimes_{i=1}^n (X_g^i, \mathcal{B}_g^i, \mu_g^i).$$

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Furthermore, it follows from the construction that subspaces $(Z_e, \mathcal{B}_e, \mu_e)$ and $\phi^g(Z_e, \mathcal{B}_e, \mu_e)$ are independent for $g \neq e$, and $\bigcup_{g\in G} \phi^g(Z_e, \mathcal{B}_e, \mu_e) = (Z, \mathcal{B}_Z, \mu_Z)$. Hence,

$$(Z, \mathcal{B}_Z, \mu_Z) = \bigotimes_{g\in G} \phi^g(Z_e, \mathcal{B}_e, \mu_e),$$

and one can see that $\phi$ acts according to the von Neumann construction. This means that $\phi$ is a Bernoulli action of $G$. The equality $h(\phi) = nh(T)$ is obvious.

Now suppose that $G = \mathbb{R}^n \times \mathbb{Z}^m$. Then $\Gamma = \mathbb{Z}^n \times \mathbb{Z}^m$ is a closed cocompact subgroup of $G$. It follows from [**30**] that any free action $S$ of $G$ is Bernoulli if and only if the restriction of $S$ to $\Gamma$ is Bernoulli. Since the restriction $\phi$ to $\Gamma$ is Bernoulli by the argument above, it follows that $\phi$ is also a Bernoulli action of $G$. The equality $h(\phi) = nh(T)$ now follows from the equality for actions of $\Gamma$, above, and Theorem 2.14. □

LEMMA 3.12. *Let $G$, $(X, \mathcal{B}, \mu)$ be as in the statement of Theorem 3.8, and let $\phi$ be a Bernoulli action of $G$ on $(X, \mathcal{B}, \mu)$. If $h(\phi) = \infty$, then $\phi$ has a Lebesgue spectrum.*

*Proof.* Consider first the case $G = \mathbb{R}$. The general case can be proved similarly. Let $S = \{S_t, t \in \mathbb{R}\}$ be a Bernoulli action on $(X, \mathcal{B}, \mu)$ with $h(S) = \infty$, and $t \to U_t$ the corresponding unitary representation of $S$ on $H_0$.

It follows from Stone's theorem [**34**] that $U_t = e^{itA}$, where $A$ is a self-adjoint operator on $H_0$. Recall that a complex number $\lambda$ is said to be in the *resolvent set* $\rho(A)$ of $A$ if the operator $R_\lambda = (\lambda I - A)^{-1}$ is bounded on $H_0$. If the number $\lambda$ does not belong to $\rho(A)$, then this number is said to be in the *spectrum* $\sigma(A)$ of $A$. Notice that $\sigma(A)$ is a closed subset of $\mathbb{R}$.

Now let $S^{-1} = \{S_{-t}, t \in \mathbb{R}\}$. Since $h_K(S_t) = h_K(S_{-t})$ for each $t$, then $h(S) = h(S^{-1})$ by Theorem 2.14, and, furthermore, $S^{-1}$ is Bernoulli, as $S_t$, $t \neq 0$ is Bernoulli. Hence, $S$ and $S^{-1}$ are isomorphic by [**30**], and therefore the unitary representations $t \to U_t = e^{itA}$ and $t \to U_{-t} = e^{-itA}$ are unitarily equivalent. This allows us to conclude that $\sigma(A) = \sigma(-A) = -\sigma(A)$, and, furthermore, if $\lambda \in \sigma(A)$, then $-\lambda \in \sigma(A)$.

Consider the action $S \otimes S = (S_t \otimes S_t)$, $t \in \mathbb{R}$ on $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu)$. This is a Bernoulli action of $\mathbb{R}$ with $h(S \otimes S) = \infty$, by Lemma 3.11. Then $S \otimes S$ is isomorphic to $S$ by [**30**], and hence the unitary representation $t \to U_t$ of $\mathbb{R}$ on $\mathcal{L}^2(X, \mu)$ is unitarily equivalent to the unitary representation $t \to U_t \otimes U_t$, $t \in \mathbb{R}$ on $\mathcal{L}^2(X \otimes X, \mu \otimes \mu)$. Since $U_t \otimes U_t = e^{itA} \otimes e^{itA} = e^{it(A\otimes I + I\otimes A)}$, $\sigma(A) = \sigma(A \otimes I + I \otimes A)$. Hence, if $\lambda, \xi \in \sigma(A)$, then $\lambda + \xi \in \sigma(A)$ too. Thus, $0 \in \sigma(A)$, and hence $\sigma(A)$ is a closed subgroup of $\mathbb{R}$ because $\sigma(A)$ is a closed subset of $\mathbb{R}$. It follows from Lemma 3.10 that $\sigma(A) = A_l$, and $l(\sigma(A)) = l(A_l) > 0$, where $l$ is Lebesgue measure on $\mathbb{R}$. Thus, $A$ is a closed subgroup of positive measure of $\mathbb{R}$. It is well known (and easily proved) that this implies $A = \mathbb{R}$; hence the spectrum of $S$ is Lebesgue.

We now give the details of the case $G = \mathbb{R}^2$: the proof for general $n$ is sufficiently similar to be left to the reader.

Let $S = (S^1, S^2)$ be a Bernoulli action of $G$ on $(X, \mathcal{B}, \mu)$ with $h(S^1, S^2) = \infty$, where $(t, 0) \to S_t^1$ and $(0, t) \to S_t^2$, $t \in \mathbb{R}$ are free actions of $\mathbb{R}$. Notice that if $t \to U_t^j$ is the unitary representation, corresponding to $S^j$, $j = 1, 2$, then $U_t^j$ has a form $U_t^j = e^{itA_j}$, where $A_j$ is the infinitesimal generator of $U^j$, and there is a common domain of essential

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self-adjointness to each operator $A_j$, $j = 1, 2$ (see [**34**, Theorems VIII. 12–13]). We claim that $t \to S_t^i$ is Bernoulli. Indeed, as $S$ is a Bernoulli action of $\mathbb{R}^2$, then $(S_n^1, S_m^2)$, $n, m \in \mathbb{Z}^2$ is a Bernoulli action of $\mathbb{Z}^2$, by [**30**, §III, Theorem 3.10], because $\mathbb{Z}^2$ is a closed cocompact subgroup of $\mathbb{R}^2$. But then each of the actions $S_n^1$ and $S_n^2$, $n \in \mathbb{Z}$ is also a Bernoulli action of $\mathbb{Z}$, by [**12**, Proposition 3.2]. Hence, $S_t^i$, $t \in \mathbb{R}$ is a Bernoulli action of $\mathbb{R}$ for each $i$, again by [**30**], and, furthermore, $h(S^i) = \infty$ for $i = 1, 2$.

Now let

$$H_0 = \int_{\mathbb{R}^2} \bigoplus H_{\lambda_1,\lambda_2}\, dm_0(\lambda_1, \lambda_2)$$

be the decomposition of $H_0$ corresponding to the representation $(t_1, t_2) \to (U_{t_1}^1, U_{t_2}^2)$ of $\mathbb{R}^2$ on $H_0$. It follows from Lemma 3.10 that $dm_0(\lambda_1, \lambda_2) = g(\lambda_1, \lambda_2)\, d\lambda_1\, d\lambda_2$, where $g(\lambda_1, \lambda_2)$ is a non-negative function from $\mathcal{L}_1(\mathbb{R}^2, l \times l)$ and $\int g(\lambda_1, \lambda_2)\, d\lambda_1\, d\lambda_2 = 1$.

Consider the direct integral of Hilbert spaces

$$H_{\lambda_1} = \int_{\mathbb{R}} \bigoplus H_{\lambda_1,\lambda_2}\, dm_{\lambda_1}(\lambda_2), \tag{3.8}$$

where $dm_{\lambda_1}(\lambda_2) = g(\lambda_1, \lambda_2) d\lambda_2$. Let $E_0 = \{\lambda_1 : \int_{\mathbb{R}} g(\lambda_1, \lambda_2)\, d\lambda_2 = 0\}$; then

$$H_0 = \int_{\mathbb{R}\setminus E_0} \bigoplus H_{\lambda_1}\, d\lambda_1$$

is the decomposition of $H_0$ corresponding to the representation $t \to U_t^1$. Since $t \to S_t^1$ is a CPE action of $\mathbb{R}$ by Corollary 3.5, the spectrum of $t \to S_t^1$ is Lebesgue by the argument above. Hence, $l(E_0) = 0$, and we obtain

$$H_0 = \int_{\mathbb{R}} H_{\lambda_1}\, d\lambda_1.$$

Since $t \to S_t^2$ has also a CPE action of $\mathbb{R}$ on $X$ by Corollary 3.5, the restriction of the representation $t \to U_t^2$ to $H_{\lambda_1}$, defined by (3.8), has Lebesgue spectrum for any $\lambda_1 \in E_1$, where $E_1 = \mathbb{R}\setminus E_0$. Thus, if $\lambda_1 \in E_1$, then $g(\lambda_1, \lambda_2) > 0$ for a.a. $\lambda_2 \in \mathbb{R}$. If now $N_0 = \{(\lambda_1, \lambda_2) : g(\lambda_1, \lambda_2) = 0\}$, then we have $l \times l(N_0) = 0$ by Fubini's theorem. This means that $(S^1, S^2)$ has Lebesgue spectrum, as claimed. □

LEMMA 3.13. *Let $G$, $(X, \mathcal{B}, \mu)$ be as in the statement of Theorem 3.8, and $\phi$ a Bernoulli action of $G$ on $(X, \mathcal{B}, \mu)$ with $0 < h(\phi) < \infty$. Then $\phi$ has Lebesgue spectrum.*

*Proof.* As usual, we first consider the case $G = \mathbb{R}$. Let $S = \{S_t, t \in \mathbb{R}, \}$ be the Bernoulli flow with $0 < h(\phi) < \infty$, and $t \to U_t$ the unitary representation on $H_0 = L_0^2(X, \mu)$ corresponding to $t \to S_t, t \in \mathbb{R}$. Then $U_t = e^{itA}$ by Stone's theorem, where $A$ is an unbounded self-adjoint operator on $H_0$. Notice that $A$ is unbounded because the spectrum of the lattice subgroup $1/n\mathbb{Z}$ is the set $[-n\pi, n\pi]$ for any $n \in \mathbb{N}$.

If $\sigma(A) \in \mathbb{R}$ is the spectrum of $A$, then $\sigma(A) = -\sigma(A)$ (see the proof of Lemma 3.12), and $l(\sigma(A)) > 0$ by Lemma 3.10. Consider also the flow $S^p = \{S_t^p = S_{pt}, t \in \mathbb{R}\}$, where $p \in \mathbb{R}$, $p > 0$. It is clear that $S^p$ is Bernoulli, and $h(S^p) = ph(S)$. The last formula follows from Corollary 2.7 above. It is obvious that the unitary representation $t \to U_t^p$ of $\mathbb{R}$, corresponding to the flow $S^p$, has a form $t \to U_t^p = e^{itpA}$. This shows that the spectrum of $S^p$ is the subset $\sigma(pA) = p\sigma(A)$ of $\mathbb{R}$.

Since $h(S^p) < h(S)$ for $0 < p < 1$, $S$ has factor-space $(Z_p, \mathcal{B}_p, \mu_p)$ such that the restriction $T_p$ of $S$ to $(Z_p, \mathcal{B}_p, \mu_p)$ is Bernoulli with entropy $h(T_p) = h(S^p) = ph(S)$ by [**30**]. As $T_p$ is Bernoulli with the entropy $ph(S)$, then $T_p$ is isomorphic to $S^p$, by [**30**], and hence the spectrum of $T_p$ coincides with the set $p\sigma(A)$, which is the spectrum of $S^p$. Since $T^p$, $0 < p < 1$ is the restriction of $S$ to the factor-space $(Z_p, \mathcal{B}_p, \mu_p)$, $p\sigma(A) \subseteq \sigma(A)$ for any $0 < p < 1$. Now, if $a \in \sigma(A)$, $0 < a < \infty$, and $p$ ranges over $0 < p < 1$, then the closed interval $[-a, a] \subseteq \sigma(A)$, because $\sigma(A)$ is a closed subset of $\mathbb{R}$. Since $\sigma(A)$ is unbounded in $\mathbb{R}$, we see that $\sigma(A) = \mathbb{R}$. Thus, the support of the measure $m_0$ in the statement of Lemma 3.10 is equal to $\mathbb{R}$, and $m_0$ is equivalent to Lebesgue measure on $\mathbb{R}$. This means that the flow $S$ with $0 < h(S) < \infty$ has Lebesgue spectrum.

To complete the proof of the lemma, we make some remarks about the case $G = \mathbb{R}^2$. In fact, we can apply the same argument as in the preceding proof. Furthermore, we see that if $S = (S^1, S^2)$ is a Bernoulli action of $G$ with $0 < h(S) < \infty$, then each $S^i$, $i = 1, 2$ will be a Bernoulli action of $\mathbb{R}$ with $h(S_i) = \infty$. Thus, one can repeat the argument of Lemma 3.12 for $G = \mathbb{R}^2$. The case $\mathbb{R} \times \mathbb{Z}$ can be treated similarly. □

For the next lemma, we recall some properties of direct integrals of Hilbert spaces (see [**9, 10**], Appendice A). Let $M$ be the commutative von Neumann algebra on $H_0$ generated by the unitary operators $U_g$, $g \in G$, where the group $G$ is as in the statement of Theorem 3.8, and let $g \to U_g$ be the unitary representation of $G$ described in Definition 3.7. Denote by $M'$ the commutant of $M$ on the space $H_0$: a bounded operator $b$ on $H_0$ belongs to $M'$ if $bm = mb$ for any $m \in M$. If $b \in M'$, then there is a correspondence $b \sim \{b(x)\}$, $x \in \hat{G}$, where $b(x)$ is a bounded operator on $H_x$ for a.e. $x \in \hat{G}$ (a.e. denotes almost every), such that for any vector $f \sim \{f(x)\}$, $x \in \hat{G}$, from $H_0$, we have $bf \sim \{b(x)f(x)\}$. Furthermore, the function $x \to \|b(x)\|$ is measurable, and $\|b\| = \sup_x \|b(x)\|$.

Thus, we have realized our space $H_0$ as the direct integrals of Hilbert spaces $H_0 = \int_{\hat{G}} \bigoplus H_x \, dm_0(x)$. Now we have $(U_g f) \sim < g, x > f(x)$, where $g \to U_g$ is the representation of the group $G$ on $H_0$, $< g, x >$ is the character of $\hat{G}$, corresponding to $g \in G$, and $f \sim f(x)$ is a vector from $H_0$. If $b \in M'$ and $f \in H_0$, then $bf \sim b(x)f(x)$. This means that in our case the von Neumann algebra $M'$ is realized as direct integrals of von Neumann algebras $M'_x$, or $M' = \int_{\hat{G}} \bigoplus M'_x \, dm_0(x)$. It is important to note that in our case $M'_x = B(H_x)$, where $B(H_x)$ is the algebra of all bounded operators on the space $H_x$.

LEMMA 3.14. *Let $\phi$ be a Bernoulli action of $G = \mathbb{R}^n \times \mathbb{Z}^m$ on $(X, \mathcal{B}, \mu)$. Then $\phi$ has Lebesgue spectrum with infinite multiplicity.*

*Proof.* It suffices, as above, to consider the case $G = \mathbb{R}$. This simplifies the notation, and the proof of the general case follows by the same arguments. Thus, let $S = \{S_t, t \in \mathbb{R}\}$ be a Bernoulli flow, and first consider the case $h(S) = \infty$. Consider the unitary representation $t \to U_t$, on the Hilbert space $H_0 = \mathcal{L}^2_0(X, \mu)$, corresponding to the action $t \to S_t$ on $(X, \mathcal{B}, \mu)$. Then we have $H_0 = \int_{\mathbb{R}} \bigoplus H_\lambda \, d\lambda$, by Lemma 3.12.

Let $K$ be a cyclic finite group with $k$ a generator: $K = \{k^i, 1 \le i \le n-1, k^n = e\}$. Consider the space

$$(Y, \mathcal{B}_Y, \nu) = \bigotimes_{s \in K} (X_s, \mathcal{B}_s, \mu_s), \quad s = e, k, \ldots, k^{n-1},$$

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where each $(X_s, \mathbb{B}_s, \mu_s)$ is a copy of $(X, \mathcal{B}, \mu)$. Define the action $S' = \{S'_t, t \in \mathbb{R}\}$ on $(Y, \mathcal{B}_Y, \nu)$ by

$$(S'_t y)_s = S_t x_s,$$

where $y = (x_s)$, $s \in K$ is a point in $Y$. It follows from Lemma 3.11 that $S'$ is Bernoulli with $h(S') = \infty$.

Now $K$ also has an action on $(Y, \mathcal{B}_Y, \nu)$ defined by

$$(L_k y)_s = x_{ks}, \quad s \in K.$$

It is easy to check that $L_r S'_t = S'_t L_r$ for $t \in \mathbb{R}$, $r \in K$.

Let $t \to U'_t$, $t \in \mathbb{R}$ be the unitary representation of $\mathbb{R}$ on the space $H'_0 = L^2_0(Y, \nu)$, corresponding to $t \to S'_t$, and let $s \to V_s$ be the unitary representation of $K$ on $H'_0$, corresponding to $s \to L_s$. Notice that $s \to V_s$ is isomorphic to the left regular representation of $K$.

We may write $U'_t$, $t \in \mathbb{R}$ in diagonal form $U'_t = \int_{\mathbb{R}} e^{it\lambda}\, dE'_\lambda$ and realize $H'_0$ as the direct integral of Hilbert spaces $H'_0 = \int_{\mathbb{R}} \bigoplus H'_\lambda\, d\lambda$. To show that $\dim H'_\lambda \geq n$ for a.e. $\lambda \in \mathbb{R}$, consider the subspace $H^K_0 = (\bigotimes_{s \in K} H^s_0)$ of $H'_0$, where $H^s_0$ is the copy of $H_0$, and notice that $H^K_0$ is invariant with respect to $U'$ and $V$.

Consider the restriction $U'_t$ and $V_k$ to $H^K_0$ and retain the same notations for them. Observe that $U'_t$, $t \in \mathbb{R}$ can be considered on the space $H^K_0$ as the diagonal $n \times n$-matrix with operator $U_t$ on the diagonal and the remaining coefficients are zero, and $V_k$ as the $n \times n$-matrix with coefficients 0 and $I_{H_0}$, where $I_{H_0}$ is the unit operator in $H_0$, such that $V_k U'_t V_k^{-1} = U'_t$, $t \in \mathbb{R}$.

Consider also the $S'_t$-invariant subspace $Y_s = (X_s, \mathcal{B}_s, \mu_s)$ of $(X, \mathcal{B}, \nu)$ and let $E_s$ be the conditional expectation $E_s$ from $Y$ onto $Y_s$. It is obvious that $S'_t E_s = E_s U_t = E_s S'_t$, and $E_s$ defines the orthogonal projection $P_s$ from $H^K_0$ onto $H^s_0$, and, furthermore, $U'_t P_s = P_s U'_t$, $t \in \mathbb{R}$. The projection $P_s$ can be also considered as the $n \times n$-matrix $\{a_{ij}, i, j = 1, 2, \ldots, n\}$, where $a_{ss} = P_s$, and the rest $a_{ij} = 0$. It is obvious that $V_k P_s V_k^{-1} = P_{ks}$.

Recall that $H_K$ is $\{U_t, t \in \mathbb{R}\}$,-invariant. Hence, we can realize $H^K_0$ in the following form:

$$H^K_0 = \int_{\mathbb{R}} \bigoplus H^K_\lambda\, d\lambda.$$

Since $V_k$ and $P_s$ commute with $U'_t$, $t \in \mathbb{R}$, then there exist measurable operator functions $\{V_k(\lambda)\}$ and $\{P_s(\lambda)\}$, corresponding to $V_k$ and $P_s$, respectively, where $V(\lambda)$ is a unitary operator and $P(\lambda)$ is a projection for a.a. $\lambda$. Moreover, $V_k(\lambda) P_s(\lambda) V_k^{-1}(\lambda) = P_{ks}(\lambda)$ for a.a $\lambda$. Recall that $S_t$, $t \in \mathbb{R}$ is a Bernoulli flow, and it has the Lebesgue spectrum by Lemmas 3.12 and 3.13. Hence, $P_s(\lambda) \neq 0$ for a.e. $\lambda$. Then we have that the relation $V_k(\lambda) P_s(\lambda) V_k^{-1}(\lambda) = P_{ks}(\lambda)$ is not zero for a.e. $\lambda$. But the operators $\{V_k(\lambda), P_s(\lambda)\}$ act on the space $H^K_\lambda$, hence one can conclude that $\dim H^K_\lambda \geq n$ for a.e. $\lambda$. Finally, since $n$ is arbitrary, we have that $\dim H^K_\lambda = \infty$ for a.e. $\lambda$.

But $H^K_0$ is a subspace of $H'_0 = \int_{\mathbb{R}} \bigoplus H'_\lambda\, d\lambda$, so $H^K_\lambda$ is a subspace of $H'_\lambda$ for a.e. $\lambda$. Thus, $\dim H'_\lambda \geq \dim H^K_\lambda \geq \infty$ for a.e. $\lambda$.

Returning now to $S = \{S_t, t \in \mathbb{R}\}$ and $S' = \{S_t, t \in \mathbb{R}\}$, we see that as $S$ and $S'$ are Bernoulli actions of $\mathbb{R}$ with the same entropy $h(S) = h(S') = \infty$, they are isomorphic by [**30**] and their unitary representations $t \to U_t$ and $t \to U'_t$ on the spaces $H_0$ and $H'_0$, respectively, are unitarily equivalent. But $H_0 = \int_{\mathbb{R}} \bigoplus H_\lambda \, d\lambda$, and we can deduce from the estimate above that $\dim H_\lambda = \infty$ for a.e. $\lambda$. This means that the Bernoulli action $S$ of $\mathbb{R}$ with $h(S) = \infty$ has infinite Lebesgue spectrum.

Now let $S$ be a Bernoulli action of $\mathbb{R}$ with $0 < h(S) < \infty$. Then there exists a Bernoulli action $S_m$ of $\mathbb{R}$ with entropy $h(S_m) = 1/mh(S)$ for any $m \in \mathbb{N}$ (see [**30**, III]). Let $S' = (\otimes S_m)^m$. It follows from Lemma 3.11 that $S'$ is also Bernoulli with $h(S') = h(S)$. Hence, $S$ and $S'$ are isomorphic, and one can apply a similar argument to show that $S$ has infinite Lebesgue spectrum. □

Now we can complete the proof of Theorem 3.8.

*Proof of Theorem 3.8.* Let $S$ be a CPE action of $\mathbb{R}$ on $(X, \mathcal{B}, \mu)$. There exists a factor-space $(Y, \mathcal{B}_Y, \nu)$ of $(X, \mathcal{B}, \mu)$ such that the restriction $S'$ of $S$ to $(Y, \mathcal{B}_Y, \nu)$ is Bernoulli with $0 < h(S') \leq h(S)$ (see [**30**, III, §3]). Moreover, there exists a conditional expectation $E$ from $(X, \mathcal{B}, \mu)$ onto $(Y, \mathcal{B}_Y, \nu)$ such that $ES_t = S_t E = S'_t$ for any $t \in \mathbb{R}$. The last equality shows that $E$ and $S_t$ commute for any $t$.

It is obvious that $E$ can be extended to an orthogonal projection $P_E$ from $H_0^X = \mathcal{L}_0^2(X, \mu)$ onto $H_0^Y = L_0^2(Y, \nu)$. Furthermore, if $t \to U_t$ is a unitary representation of $t \to S_t$ on $H_0^X$ and $t \to U'_t$ is a unitary representation on $H_0^Y$ of $t \to U'_t$, then $P_E U_t = U_t P_E = U'_t$ for any $t$. Consider the decomposition $H_0^X = \int_{\mathbb{R}} H_\lambda \, dm_0(\lambda)$ of $H_0^X$ corresponding to the diagonal presentation of $U_t = \int e^{it\lambda} \, dE_\lambda$, where $m_0$ is the Borel measure on $\mathbb{R}$, defined in the statement of Lemma 3.10 and subordinated to the Lebesgue measure $l$ on $\mathbb{R}$. Let $M$ be a commutative von Neumann algebra on $H_0^X$ generated by $U_t$, $t \in \mathbb{R}$. Then $P_E$ belongs to the commutant $M'$ of $M$, and there is a measurable field $\{P_\lambda\}$, where $P_\lambda$ is an orthogonal projection from $B(H_\lambda)$ for a.a. $\lambda \in \mathbb{R}$, such that $P_E \sim \{P_\lambda\}$ (see [**9**]).

Since $P_E H_0^X = H_0^Y$, we have $H_0^Y = \int_{\mathbb{R}} P_\lambda H_\lambda \, dm_0(\lambda)$. But this is a decomposition of $H_0^Y$ with respect to $t \to U'_t$, and since $t \to S'_t$ is Bernoulli, $m_0$ is equivalent to the Lebesgue measure $l$ by Lemmas 3.12 and 3.13. Furthermore, $\dim P_\lambda H_\lambda = \infty$ for a.e. $\lambda \in \mathbb{R}$, by Lemma 3.14. Hence, $\dim H_\lambda \geq \dim P_\lambda H_\lambda = \infty$ for a.e. $\lambda$, which means that $S$ has infinite Lebesgue spectrum. □

3.3. *The Pinsker algebras of $\mathbb{R}^n$-actions.* In this section, we consider ergodic actions $\varphi$ of $\mathbb{R}^n$ with positive entropy, and we also suppose that action $\varphi^\Gamma$ is ergodic on $(X, \mathcal{B}, \mu)$ for any lattice subgroup $\Gamma$ of $\mathbb{R}^n$. We will show that the Pinsker algebra $\Pi(\varphi)$ of the action $\varphi$ exists and coincides with the Pinsker algebra of any action $\varphi^\Gamma$ of a lattice subgroup $\Gamma$ of $\mathbb{R}^n$ (see Theorem 3.17). Then we will describe the spectral properties of such actions (see Theorem 3.18).

*Definition 3.15.* Let $\varphi$ be a free action of an amenable group $G$ on $(X, \mathcal{B}, \mu)$, and let $\Pi(\varphi)$ be the minimal $\varphi$-invariant $\sigma$-subalgebra of $\mathcal{B}$ which contains each finite partition $\rho$ of $X$ such that $h(\varphi, \rho) = 0$, where we assume that $h(\varphi, \rho)$ is the Kolmogorov–Sinai entropy if $G$ is a discrete group, and the spatial entropy in other cases.

PROPOSITION 3.16. *Let $\varphi$ be a free action of $\mathbb{R}^n$ on $(X, \mathcal{B}, \mu)$, and let $\varphi^1$ and $\varphi^\Gamma$ be, as above, the restriction of $\varphi$ to $\mathbb{Z}^n$ and $\Gamma$, respectively, where $\Gamma$ is a lattice subgroup of $\mathbb{R}^n$,* *and $\Gamma \neq \mathbb{Z}^n$. If $\Pi(\varphi^1)$ and $\Pi(\varphi^\Gamma)$ are the Pinsker algebras of $\varphi^1$ and $\varphi^\Gamma$, respectively, then $\Pi(\varphi^1) = \Pi(\varphi^\Gamma)$. In particular, $\varphi^\Gamma$ is a CPE action if and only if $\varphi^1$ is a CPE action.*

*Proof.* It follows from the properties of the entropy $h_K(\varphi^1, \rho)$ (see [**15, 36**]) that $\Pi(\varphi^\Gamma)$ is $\varphi$-invariant. Thus, it is also $\varphi^1$-invariant. As the restriction of $\varphi^\Gamma$ to $\Pi(\varphi^\Gamma)$ has zero entropy, the restriction of $\varphi^1$ to $\Pi(\varphi^\Gamma)$ also has entropy zero, by the Abramov–Conze formula (Corollary 2.16). It follows that $\Pi(\varphi^1) \subseteq \Pi(\varphi^\Gamma)$, and the opposite inclusion follows from symmetry. □

Notice that results analogous to Proposition 3.16 were discussed in [**3, 19**] for the case $n = 1$; we believe that the case $n > 1$ has not been considered before. In the next theorem, we show that $\Pi(\varphi) = \Pi(\varphi^\Gamma)$ if any $\varphi^\Gamma$ is ergodic.

THEOREM 3.17. *Let $\varphi$ be a free ergodic action of $\mathbb{R}^n$, $n < \infty$, on $(X, \mathcal{B}, \mu)$ with positive entropy $h(\varphi)$, and let $\varphi^\Gamma$ be ergodic for any lattice subgroup $\Gamma$ of $\mathbb{R}^n$. Then $\Pi(\varphi) = \Pi(\varphi^\Gamma)$.*

*Proof.* In view of Proposition 3.16, it is enough to show that $\Pi(\varphi) = \Pi(\varphi^1)$. Since $\Pi(\varphi^1)$ is $\varphi$-invariant (see proof of Proposition 3.16), we can consider the restriction $\psi$ of $\varphi$ to $\Pi(\varphi^1)$. It follows from the definition of $\psi$ that $h_K(\psi^1) = 0$, but $h(\psi) = h_K(\psi^1)$ by Theorem 2.14. Hence, $h(\psi) = 0$. This shows that $\Pi(\varphi^1) \subseteq \Pi(\varphi)$.

For the opposite direction, consider a finite partition $\rho$ of $X$ such that $\mathrm{sh}(\varphi, \rho) = 0$. It follows from the estimate of equation (2.9) that $\mathrm{sh}(\varphi, \rho) \geq \mathrm{sh}(\varphi^1, \rho)$. Since $\varphi^1$ is ergodic by our assumptions, then $h_K(\varphi^D, \rho) = \mathrm{sh}(\varphi^D, \rho)$ by Proposition 2.4, and we have $h_K(\varphi^D, \rho) \leq \mathrm{sh}(\varphi, \rho) = 0$. Hence, $\Pi(\varphi) \subseteq \Pi(\varphi^1)$ and we have $\Pi(\varphi) = \Pi(\varphi^1)$.

The case of an arbitrary uniform subgroup $\Gamma$ uses the same argument, if one takes into account Propositions 2.10 and 2.11, which generalize Theorem 2.5. □

We next analyse the spectral properties of $\mathbb{R}^n$-actions with positive entropy.

THEOREM 3.18. *Let $\varphi$ be a free ergodic action of $\mathbb{R}^n$, $n < \infty$ on $(X, \mathcal{B}, \mu)$ with entropy $h(\varphi) > 0$ and with Pinsker algebra $\Pi(\varphi)$, and let $\varphi^\Gamma$ be ergodic for any lattice subgroup of $\mathbb{R}^n$. Then $\varphi$ has infinite Lebesgue spectrum on the space $\mathcal{L}_0^2(X, \mu) \ominus \mathcal{L}_0^2(\Pi(\varphi))$, where $\mathcal{L}_0^2(\Pi(\varphi))$ is the subspace of $\mathcal{L}_0^2(X, \mu)$ consisting of all $\Pi(\varphi)$-measurable functions.*

*Proof.* Since $\varphi^1$ is an ergodic action of $(\mathbb{Z})^n$, $\varphi^{1/2^k}$ is an ergodic action of $(1/2^k\mathbb{Z})^n$, and, furthermore, $\Pi(\varphi) = \Pi(\varphi^{1/2^k})$ by Theorem 3.17. But $\varphi^{1/2^k}$ has countable Lebesgue spectrum on $H_0^X = \mathcal{L}_0^2(X, \mu) \ominus \mathcal{L}_0^2(\Pi(\varphi^{1/2^k}))$, by [**11**, Theorem 5.4]. This observation allows us to apply the argument of Lemma 3.10 to show that $H_0^X = \int_{\mathbb{R}^n} H_\lambda \, dm_0(\lambda)$, where $m_0$ is a Borel measure on $\mathbb{R}^n$, subordinated to the Haar measure on $\mathbb{R}^n$. Now we need to show that $m_0$ is equivalent to the Lebesgue measure on $\mathbb{R}^n$ and $\dim H_\lambda = \infty$ for a.a. $\lambda$.

To do this, we observe that $\varphi$ has a Bernoulli subfactor in view of our assumptions on it. This means that there exists a $\varphi$-invariant subspace $(Y, \mathcal{B}_Y)$, such that the restriction $\varphi|_Y$ of $\varphi$ to $(Y, \mathcal{B}_Y, \mu_Y)$ is Bernoulli with $0 < h(\varphi|_Y) \leq h(\varphi)$, and even $h(\varphi|_Y) < \infty$, according to [**30**, III, §3, Theorem].

Then $\varphi' = (\varphi|_Y)^1$ also has a Bernoulli action on $(Y, \mathcal{B}_Y, \mu_Y)$, by [**30**, III, §6, Theorem 10], and there exists a finite partition $\rho$ of $(Y, \mathcal{B}_Y, \mu_Y)$ such that the family $\{\varphi'_\gamma \rho, \gamma \in \mathbb{Z}^n\}$ is independent for $\gamma \neq \gamma'$ and generates $\mathcal{B}_Y$. If $\beta$ is any finite partition

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of $X$ from $\Pi(\varphi) = \Pi(\varphi^1)$, then $h_K(\varphi^1, \beta) = 0$, and it follows from [**5**, Theorem 4.2] that $\rho$ and $\beta$ are independent. Since $\mathcal{B}_Y$ is generated by $\{\varphi'_\gamma \rho, \gamma \in \mathbb{Z}^n\}$, any $A \in \mathcal{B}_Y$ and $B \in \Pi(\varphi)$ is independent. One concludes from this observation that if $f \in \mathcal{L}^2_0(Y, \mu_Y)$, then also $f \in H^X_0 = \mathcal{L}^2_0(X, \mu) \ominus \mathcal{L}^2_0(\Pi(\varphi))$, and hence $\mathcal{L}^2_0(Y, \mu_Y)$ is a $\varphi$-invariant subspace of $H^X_0$. But since $\varphi|_Y$ is Bernoulli, then $\varphi$ has infinite Lebesgue spectrum on $\mathcal{L}^2_0(Y, \mu_Y)$, by Lemma 3.14. Thus, we are in the situation of Theorem 3.8, and the argument given there shows that $\varphi$ has infinite Lebesgue spectrum on $H^X_0$. This completes the proof of the theorem. □

## 4. *Entropy of nilpotent Lie group actions*

We described in the Introduction the class of unicommutator Lie groups ($\mathcal{ULG}$). Recall that any connected, simply connected nilpotent Lie group can be realized as a closed subgroup of the group $\mathcal{U}T_n(\mathbb{R})$ of upper triangular unipotent $n \times n$-matrices over $\mathbb{R}$ for some $n \in \mathbb{N}$ [**33**]. The simplest example of a non-commutative group from $\mathcal{ULG}$ is the Heisenberg group $\mathcal{H} = \mathcal{UT}_3(\mathbb{R})$. In this section, we consider free ergodic positive entropy actions of elements of $\mathcal{ULG}$. We will see that all the results on CPE actions and actions with positive entropy which were proved in §3 extend to this class of nilpotent groups. Detailed proofs are mostly given for $\mathcal{H}$, but these generalize easily to all of $\mathcal{ULG}$.

### 4.1. *Classical entropy for actions of $\mathcal{UT}_n((Z))$ and its subgroups.*

In this section, we briefly consider the entropy of actions of nilpotent countable torsion-free groups with a finite number of generators [**17**]. Recall that each such group can be faithfully represented in $\mathcal{UT}_n(\mathbb{Z})$ for some $n \in \mathbb{Z}$, by a well-known theorem of Malcev.

Let $G$ be a countable two-step nilpotent matrix group,

$$\mathbf{G} = \begin{pmatrix} 1 & n_3 & n_1 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix},$$

where $n_i \in \mathbb{Z}$, $i = 1, 2, 3$. We fix generators of $G$: let $T_i$, $(i = 1, 2, 3)$ be the matrix such that $n_i = 1$ and $n_j = 0$ if $j \neq i$. Here $T_1$ generates the centre $Z$ of $G$. We define the linear order $T_3 > T_2 > T_1$, together with the associated lexicographical linear relation on $G : T_3^{j_3} T_2^{j_2} T_1^{j_1} < T_3^{k_3} T_2^{k_2} T_1^{k_1}$, if $(j_3, j_2, j_1)$ is lexicographically less than $(k_3, k_2, k_1)$. This order relation is invariant with respect to the left translations of $G$, so we have the notion of the 'past' in $G$ defined as a subset of all elements of $G$ which are less than the identity in $G$.

Consider the sequence of rectangles $F_n = \{T_3^{i_3} T_2^{i_2} T_1^{i_1}, m_s(n) \leq i_s \leq M_s(n), s = 1, 2, 3\}$, where $(M_s(n) - m_s(n)) \to \infty$, as $n \to \infty$, and

$$\frac{M_1(n) - m_1(n)}{M_s(n) - m_s(n)} \to \infty, \quad s = 2, 3,$$

as $n \to \infty$. Now, $(F_n)$, $n \in \mathbb{N}$ is a Følner sequence of sets in $G$ [**17**]. In particular, we can let $M_2(n) = M_3(n) = n$, $M_1(n) = n^2$, $m_s(n) = 0$, $s = 1, 2, 3$.

Now suppose that we have a free measure-preserving action $S$ of $G$ on the probability space $(X, \mathcal{B}, \mu)$, and a partition $\rho$ of $X$ with $H(\rho) < \infty$. Then we can consider the

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Kolmogorov entropy

$$h_K(S,\rho)=\lim_{n\to\infty}\frac{1}{|F_n|}H(\rho^{F_n})$$

of the process $(S,\rho)$, where $\{F_n\}$, $n\in\mathbb{N}$ is a sequence of sufficiently invariant subsets of $G$, as above. Note that $h_K(S,\rho)$ does not depend on the choice of the sequence $\{F_n\}$.

Since $G$ has a past, we can calculate $h_K(S,\rho)$ in a different form, using conditional entropy, which will be useful for us below.

To do this, we introduce some new notation. Let

$$\rho^-_{T_i}=\bigvee_{k=1}^{\infty}T_i^{-k}\rho,\quad \rho_{T_i}=\bigvee_{-\infty}^{\infty}T_i^{k}\rho,$$

where $T_i$ and $\rho$ are as above. We also set $\rho^-_G=\bigvee T_3^{k_3}T_2^{k_2}T_1^{k_1}$, where the join is taken over all triples $(k_1,k_2,k_3)\in\mathbb{Z}^3$, which are lexicographically less than $(0,0,0)$. More precisely,

$$\rho^-_G=\rho^-_{T_1}(\rho_{T_1})^-_{T_2}(\rho_{(T_1,T_2)})^-_{T_3}. \tag{4.1}$$

Now we have $h_K(S,\rho)=H(\rho|\rho^-_G)$, by [**32, 42**]. A similar formula for the entropy of $\mathbb{Z}^n$-actions was introduced in [**5**] (see §4). Now we can extend the methods of that section to non-commutative groups from $\mathcal{ULG}$.

Let $\rho$ be a partition of $X$ and $H(\rho)<\infty$. We set

$$\hat{\rho}=\bigwedge_n(T_1^{-n}\rho^-_{T_1}\vee T_2^{-n}(\rho_{T_1})^-_{T_2}\vee T_3^{-n}(\rho_{(T_1,T_2)})^-_{T_3}).$$

The next proposition will be useful in the following.

PROPOSITION 4.1. *Let $S$ be a free action of $G$ on $(X,\mathcal{B},\mu)$ with positive entropy, and let $\rho^1$ and $\rho^2$ be partitions of $X$ with $H(\rho^i)<\infty$ for $i=1,2$. Assume that $\hat{\rho}^1=\nu$, where $\nu=\{X,\varnothing\}$, and $h_K(S,\rho^2)=0$. Then the partitions $\rho^1$ and $\rho^2$ are independent.*

This proposition is an analogue of [**5**, Theorem 4.2], where it was proved for $\mathbb{Z}^n$-actions

*Proof.* Let $G_{p_k}$ be the subgroup of $G$ generated by $T_2^{p_k}$, $T_3^{p_k}$ and $T_1^{p_k^2}$, where $p_k$ divides $p_{k+1}$. Then $G_{p_{k+1}}$ is a subgroup of $G_{p_K}$, and the sequence of partitions $\rho^1_k=(\rho^1)^-_{G_{p_k}}$ is decreasing. Since $\hat{\rho}^1=\nu$, $\bigwedge_k\rho^1_k=\nu$, and hence $\lim_{k\to\infty}H(\rho^1|\rho^1_k)=H(\rho^1)$. Now, since $h_K(S,\rho^2)=0$, we have the equality $H(\rho^1|\rho^1_k)=H(\rho^1|\rho^1_k\vee\rho^2)$, the proof of which is contained in the proof of [**17**, Theorem 2.6]. Hence,

$$H(\rho^1|\rho^2)\geq H(\rho^1|\rho^1_k\vee\rho^2)=H(\rho^1|\rho^1_k),$$

and thus $H(\rho^1|\rho^2)\geq H(\rho^1)$. But since $H(\rho^1|\rho^2)\leq H(\rho^1)$, we can deduce $H(\rho^1)=H(\rho^1|\rho^2)$. It is well known that this equality implies that $\rho^1$ and $\rho^2$ are independent. □

COROLLARY 4.2. *Let $S$ be a free ergodic action of $G$ on $(X,\mathcal{B},\mu)$ with positive entropy $h_K(S)$, $(Y,\mathcal{B}_Y,\mu_Y)$ a Bernoulli factor of $(X,\mathcal{B},\mu)$ for $S$, and $h_K(S|_Y)<\infty$. If $\rho$ is a finite partition of $X$ from $\Pi(S)$, i.e. $h_K(S,\rho)=0$, and $\alpha$ is a finite $\mathcal{B}_Y$-measurable partition of $Y$, then $\rho$ and $\alpha$ are independent.*

Notice that this assertion for $\mathbb{Z}$-actions was first proved in [**31**], then in a more general form for $\mathbb{Z}^n$-actions in [**44**], and finally for countable amenable groups in [**16**]. We present a simple proof of this statement for our case.

*Proof.* Since the restriction of $S$ to $(Y, \mathcal{B}_Y, \nu)$ is Bernoulli and $h_K(S|_Y) < \infty$, there exists a finite partition $\gamma$ of $Y$ such that all partitions $g\gamma$, $g \in G$ are independent and generate the $\sigma$-algebra $\mathcal{B}_Y$. It follows from these properties of $\gamma$ that $\hat{\gamma} = \nu$. Hence, $\gamma$ and $\rho$ are independent by Proposition 4.1. The same holds for $g\gamma$ and $\rho$ for any $g \in G$. But $g\gamma$, $g \in G$ generate $\mathcal{B}_Y$, and as $\alpha$ is $\mathcal{B}_Y$-measurable, it follows that $\alpha$ and $\rho$ are independent. □

4.2. *Spatial entropy for actions of $\mathcal{ULG}$-groups.* Let us describe the structure of a nilpotent Lie group from the class $\mathcal{ULG}$, defined in the Introduction (Definition 1.2). If a group $G$ belongs to this class, then its Lie algebra $\mathfrak{g}$ has a basis $\{e_i\}_1^N$ whose commutators satisfy the following conditions:

(i) $[e_i, e_j] = 0$ or $[e_i, e_j] = e_{k(i,j)}$; and

(ii) $[e_i, e_{k(i,j)}] = [e_j, e_{k(i,j)}] = 0$.

These properties imply some strong conditions on the group $G$ and its Lie algebra $\mathfrak{g}$. Let $M(n, \mathbb{R})$ be the space of all $n \times n$-matrices over $\mathbb{R}$, and introduce in $M(n, \mathbb{R})$ the usual Lie bracket: $[x, y] = xy - yx$ for $x, y \in M(n, \mathbb{R})$. Consider $M(n, \mathbb{R})$ as the Lie algebra of the group $\mathrm{GL}(n, \mathbb{R})$ and it follows from the Ado–Iwasawa theorem [**33**] that there is a faithful representation $\rho$ of $\mathfrak{g}$ on the subalgebra of all upper triangular nilpotent matrices of $M(n, \mathbb{R})$ for some $n \in \mathbb{N}$. We will identify $\rho(\mathfrak{g})$ with $\mathfrak{g}$; we can then consider $e_i$, $1 \le i \le N$ as upper triangular nilpotent matrices from $M(n, \mathbb{R})$.

The exponential mapping exp from $M(n, \mathbb{R})$ to $\mathrm{GL}(n, \mathbb{R})$ is given by

$$\exp X = I + X + 1/2X^2 + \cdots$$

for $X \in M(n, \mathbb{R})$. If $X \in \mathfrak{g} \subset M(n, \mathbb{R})$, then $\exp X$ contains only a finite number of terms, since $X$ is a nilpotent matrix and hence $\exp \mathfrak{g}$ is a connected simply connected nilpotent Lie subgroup of $\mathcal{UT}_n(\mathbb{R})$, isomorphic to $G$ [**33**]. We identify $\exp \mathfrak{g}$ with $G$.

Now let $\mathfrak{n}$ be the $\mathbb{Z}$-linear span of $\{e_i\}_1^N$, so that $\mathfrak{n}$ is a lattice in $\mathfrak{g}$ [**33**], and, furthermore, $\mathfrak{n}$ is a Lie subalgebra of $M(n, \mathbb{Z})$ in view of the above commutation relations for $\{e_i\}_1^N$. To see this, one can use the argument of the proof of [**33**, Theorem 2.12]. Now, as in [**33**, Theorem 2.12], we can conclude that $\exp \mathfrak{n}$ is a lattice subgroup of $G$.

We now introduce the new *assumption (A)* on our Lie algebra $\mathfrak{g}$. We assume that each matrix $e_i$ from $\mathfrak{g}$ satisfies $e_i^2 = 0$, $1 \le i \le N$.

First notice that condition (A) does not follow from the above commutations relations. It is easy to construct examples of this. If condition (A) does hold, then we have $\exp te_i = I + te_i \in M(n, \mathbb{R})$ for $t \in \mathbb{R}$ and, in particular, $\exp e_i = I + e_i \in \mathcal{UT}_n(\mathbb{Z})$. Recall that $\mathcal{UT}_n(\mathbb{Z})$ is the group of all unipotent $n \times n$-matrices over $\mathbb{Z}$. Hence, the lattice subgroup $\exp \mathfrak{n}$ of $G$ is also a subgroup of $\mathcal{UT}_n(\mathbb{Z})$. This will be important for us in Theorem 4.7 below.

Clearly, the simplest situation is if $e_i$ is a matrix unit $u_{ks}$, $1 \le k < s \le n$ of $M(n, \mathbb{Z})$ for some $i$ or, of course, for all $i = 1, 2, \ldots, N$. Observe that if $\mathfrak{g}$ is the subalgebra of all upper triangular nilpotent matrices from $M(n, \mathbb{R})$, then the basis of $\mathfrak{g}$ can be chosen as follows: $u_{ks}$, $1 \leq k < s \leq n$, where $u_{ks}$ is a matrix unit from $M(n, \mathbb{Z})$. In this case, $\exp \mathfrak{g} = \mathcal{UT}_n(\mathbb{R})$ and $\exp \mathfrak{n} = \mathcal{UT}_n(\mathbb{Z})$.

We have introduced $\mathcal{ULG}$ as a simple class of connected nilpotent Lie groups where it is easy to study the entropy of actions of the group and its lattice subgroups. We will try to describe all phenomena which occur in this situation. We feel sure that many of our theorems can be extended to a wider class of groups.

We consider in detail the Heisenberg group $\mathcal{H} = \mathcal{UT}_3(\mathbb{R})$, the simplest example of a non-commutative group from $\mathcal{ULG}$, and the entropy of its actions. The properties we develop actually hold for any group from $\mathcal{ULG}$, in view of the above structure theorems, and this allows one to apply similar methods:

$$\mathcal{H} = \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix},$$

where $x_{ij} \in \mathbb{R}$, $i, j = 1, 2$. Let $\varphi$ be an ergodic action of $\mathcal{H}$ by automorphisms of $(X, \mathcal{B}, \mu)$, preserving a probability measure $\mu$, and $\rho$ a finite measurable partition of $X$. We introduce the metric $d_C^\rho(x, y) = d_C(x, y)$, where $x, y \in X$, and $C$ is a measurable subset of $\mathcal{H}$, as follows: $d_C(x, y) = (1/|C|)|\{v \in \mathcal{H} : \rho(\varphi_v x) \neq \rho(\varphi_v y)\}|$, where $v \in \mathcal{H}$, $|C|$ is the Haar measure of $C$ in $\mathcal{H}$, and $\rho(\varphi_v x)$ is defined in §2.1. Now, for a positive real number $N$, we take the rectangle $C_N$ in $\mathcal{H}$ of the form $C_N = \{O \leq x_{12} \leq N, 0 \leq x_{23} \leq N, 0 \leq x_{13} \leq N^2\}$, and consider the analogue of the $(\rho, N, r)$-family of disjoint subsets of $X$ for the action of $\mathbb{R}^n$ in §2.3. We can now define the $r$-spatial entropy $\mathrm{sh}(\varphi, \rho, r)$ and the spatial entropy $sp(\varphi, \rho)$ of the process $(\varphi, \rho)$ as in §2.3.

Let $D$ be a positive real number. Denote by $\mathcal{H}^D$ the subgroup of $\mathcal{H}$ of the form $x_{12} = Dn_{12}$, $x_{13} = D^2 n_{13}$, $x_{23} = Dn_{23}$, where $n_{ij} \in \mathbb{Z}$. One easily sees that $\mathcal{H}^D$ is a uniform subgroup of $\mathcal{H}$, isomorphic to $UT_3(\mathbb{Z})$. Thus, if $\varphi$ is a free ergodic action of $\mathcal{H}$ on $(X, \mathcal{B}, \mu)$, then $\varphi$ induces a free action $\varphi^D$ of $\mathcal{H}^D$ on $(X, \mathcal{B}, \mu)$, and one can define the classical Kolmogorov entropy $h_K(\varphi^D, \rho)$ and the spatial entropy $\mathrm{sh}(\varphi^D, \rho)$ of the process $(\varphi^D, \rho)$. The next theorem describes the connections between entropy $\mathrm{sh}(\varphi, \rho)$ and entropies $\mathrm{sh}(\varphi^D, \rho)$ as $D \downarrow 0$. This is an analogue of Theorem 2.5 for spatial entropies of $\mathbb{R}^n$-actions.

THEOREM 4.3. *Let $\varphi$ be a measure-preserving ergodic action of $\mathcal{H} = \mathcal{UT}_3(\mathbb{R})$ on a $(X, \mathbb{B}, \mu)$, and let $\varphi^D$ be the restriction of $\varphi$ to the lattice subgroup $\mathcal{H}^D$ of $\mathcal{H}$. Then*

$$\mathrm{sh}(\varphi, \rho) = \lim_{D \downarrow 0} |C_D|^{-1} \mathrm{sh}(\varphi^D, \rho),$$

*where $\rho$ is a finite partition of $X$.*

*Proof.* Let $u \in C_N$ and $D > 0$ be such that $N/D$ is an integer. Then the matrix $u$ can be decomposed as a product $u = vw$, where $w \in C_D$, or $0 < w_{12} \leq D$, $0 < w_{23} \leq D$, $0 < w_{13} \leq D^2$, and $v$ has the following coefficients: $v_{12} = i_{12} D$, $v_{23} = i_{23} D$, $v_{13} = i_{13} D^2$. Here, the integers $i_{ij}$ satisfy the following inequalities:

$$0 < i_{12} \leq N/D, \quad 0 < i_{23} \leq N/D, \quad -N/D - 1 < i_{13} \leq N^2/D^2. \tag{4.2}$$

Observe that the inequalities (4.2) define the rectangle $F_N^D$ in the lattice subgroup $\mathcal{H}^D$, and, furthermore, $\{F_N^D\}$ is a Følner sequence of subsets in $\mathcal{H}^D$ for fixed $D$, $N/D \in \mathbb{Z}$

and $N/D \to \infty$. Now we apply the same approach as in the proof of Theorem 2.5 to compute the *continuous* $d^\rho$-distance on $C_N$ between $x$ and $y$ from $X$ by taking the *discrete* $d^\rho$-distance between $\varphi x$ and $\varphi y$ over the $C_D$-lattice points in $C_N$, and taking the normalized integral of this as $w$ ranges over $C_D$. More exactly,

$$d^\rho_{C_N}(x, y) = \int_{C_D} \frac{dw}{|C_D|}\bigg(\frac{1}{|F_N^D|} \sum_{v \in F_N^D} |\{w : \rho(\varphi_{vw} x) \neq \rho(_{vw} y)\}|\bigg).$$

The rest of the proof is sufficiently similar to the proof of Theorem 2.5 above to be left to the reader. □

Again we have an analogue of Proposition 2.12.

PROPOSITION 4.4. *Let $\varphi$ be an ergodic action of $\mathcal{H} = \mathcal{UT}_3(\mathbb{R})$ on $(X, \mathcal{B}, \mu)$, and $\varphi^D$ the restriction of $\varphi$ to the lattice subgroup $\mathcal{H}^D$. If the spectrum of $\varphi$ does not contain a discrete component in $\mathcal{L}_0^2(X, \mu) = \{f \in \mathcal{L}_0^2(X, \mu) : \int_X f(x)\, d\mu(x) = 0\}$, then $\varphi^D$ is ergodic for any $D > 0$.*

To prove this proposition, we use the same argument as in Proposition 2.12.

The next assertion is a consequence of Theorem 4.3.

COROLLARY 4.5. *Let $\varphi$ and $\rho$ be as in the statement of Theorem 4.3, and suppose that $\varphi^D$ is ergodic. Then*

$$\mathrm{sh}(\varphi, \rho) = \lim_{i\to\infty} |C_{D_i}|^{-1} h_K(\varphi^{D_i}, \rho),$$

*where $D_i = 1/2^i D$, and $h_K(\varphi^D, \rho)$ is the Kolmogorov–Sinai entropy of the process $(\varphi^D, \rho)$.*

This corollary is an analogue of Corollary 2.9, and can be proved similarly.

There is also an analogue of Proposition 2.10 for actions of $\mathcal{H} = \mathcal{UT}_3(\mathbb{R})$. Let $\bar{D} = (D_1, D_2)$, where $D_i$ is a positive real number. Consider the discrete subgroups of $\mathcal{H}$, $\mathcal{H}^{\bar{D}}$, where $x_{12} = i_{12} D_1$, $x_{13} = i_{13} D_1 D_2$, $x_{23} = i_{23} D_2$, and $i_{kl} \in \mathbb{Z}$ . It is easy to see that $\mathcal{H}^{\bar{D}}$ is a uniform subgroup of $\mathcal{H}$ in each case.

Let $N\bar{D} = (ND_1, ND_2)$, where $N \in \mathbb{R}_+$, and let $C_{\bar{D}}$ be the rectangle in $H$ defined by

$$C_{\bar{D}} = \{0 \le x_{12} \le D_1, 0 \le x_{23} \le D_2, 0 \le x_{13} \le D_1 D_2\}.$$

It is obvious that $|C_{\bar{D}}| = (D_1 D_2)^2$.

Notice that if $\Gamma$ is a lattice subgroup of $\mathcal{H}$, then one can show that $\Gamma$ is conjugated to $\mathcal{H}^{\bar{D}}$ for some $\bar{D} = (D_1, D_2)$, $D_i > 0$, $i = 1, 2$. This follows from [**33**, Proposition 2.17]. A similar statement holds for any group in $\mathcal{ULG}$.

PROPOSITION 4.6. *Let $\varphi$ and $\rho$ be as in the statement of Theorem 4.3. Then*

$$\mathrm{sh}(\varphi, \rho) = \lim_{N\downarrow 0} |C_{N\bar{D}}|^{-1} \mathrm{sh}(\varphi^{N\bar{D}}, \rho).$$

*The analogues of Proposition 4.4 and Corollary 4.5 also hold in this situation.*

Again, all the above results of this subsection hold for $\mathcal{ULG}$.

Now we are ready to prove an analogue of Theorem 2.14.

THEOREM 4.7. *Let $\varphi$ be an ergodic measure-preserving action of $\mathcal{H} = \mathcal{UT}_3(\mathbb{R})$ on the probability space $(X, \mathcal{B}, \mu)$, and let the spectrum of $\varphi$ not contain a discrete component (see Proposition 4.4 above). If $\varphi^1$ is the restriction of $\varphi$ to the subgroup $\mathcal{H}^1 = \mathcal{UT}_3(\mathbb{Z})$ of $\mathcal{H}$, then*

$$h(\varphi) = h_K(\varphi^1),$$

*where $h(\varphi)$ is the Ornstein–Weiss entropy of the action $\varphi$ (see §2.2), and $h_K(\varphi^1)$ is the classical Kolmogorov entropy of the action $\varphi^1$.*

It is important to note that the statement of Theorem 4.7 makes sense for any nilpotent Lie group $G$ from $\mathcal{ULG}$ because $G \cap \mathcal{UT}_n(\mathbb{Z})$ is a lattice subgroup of $G$, as explained in the beginning of this subsection.

*Proof.* The proof is based upon Theorem 2.14. To demonstrate this, we recall that $\varphi^1$ is ergodic. We present several steps of the proof, sufficient to convey the ideas. Notice that the analogue of (2.9) for the present case is $\mathrm{sh}(\varphi, \rho) \geq \mathrm{sh}(\varphi^1, \rho)$, where $\rho$ is a finite partition of $X$. One can deduce from this, as in the proof of Theorem 2.14, that $h(\varphi) \geq h_K(\varphi^1)$.

To prove the opposite inequality, we apply the relation $\mathrm{sh}(\varphi, \rho) = \lim_{i\to\infty} 16^i h_K(\varphi^{1/2^i}, \rho)$ from Corollary 4.5, where $\rho$ is a finite partition of $X$. Furthermore, notice that $\mathcal{H}^1$ is a subgroup of $\mathcal{H}^{1/2}$ of index 16; in symbols, $[\mathcal{H}^{1/2} : \mathcal{H}^1] = 16$. This observation shows that $[\mathcal{H}^{1/2^i} : \mathcal{H}^1] = 16^i$, and it follows from Proposition 2.13 that $h_K(\varphi^1) = 16^i h_K(\varphi^{1/2^i})$. We can derive from these, as in the proof of Theorem 2.14, that $h_K(\varphi^1) \geq h(\varphi)$, and hence $h(\varphi) = h_K(\varphi^1)$ holds. □

As in §2.4, let us consider a general version of the relation $h(\varphi) = h_K(\varphi^1)$. Let $\Gamma$ be a lattice subgroup of $\mathcal{H}$. Then there is, as in §2.4, a compact subset $H(\Gamma)$ of $\mathcal{H}$ such that each $\Gamma$-orbit in $G$ meets $H(\Gamma)$ in a unique point. If $|H(\Gamma)|$ is the Haar measure of $H(\Gamma)$ and if $\Gamma = H^{\bar{D}}$, then $|H(\Gamma)| = (D_1 D_2)^2$, where $\bar{D} = (D_1, D_2)$.

COROLLARY 4.8. *Let $\varphi$ be as in the statement of Theorem 4.7, $\Gamma$ a lattice subgroup of $\mathcal{H}$, and let $\varphi^\Gamma$ be the restriction of $\varphi$ to $\Gamma$. Then*

$$h(\varphi) = |H(\Gamma)| h_K(\varphi^\Gamma).$$

The proof is analogous to the proof of Theorem 4.7, taking into consideration Proposition 4.6.

Now we are ready to present our analogues of the Abramov formula for the entropies of lattice subgroups of actions of a nilpotent Lie group from $\mathcal{ULG}$.

COROLLARY 4.9. *Let $\varphi$, $\Gamma$ be as in Corollary 4.8, then*

$$h_K(\varphi^1) = |H(\Gamma)| h_K(\varphi^\Gamma).$$

4.3. *CPE actions of $\mathcal{ULG}$-groups and uniform mixing.* Recall that the definition of CPE actions for a group $G$ from $\mathcal{ULG}$ is given in Definition 3.1 above.

THEOREM 4.10. *Let $\varphi$ be a free measure-preserving action of a group $G \in \mathcal{ULG}$ on a space $(X, \mathcal{B}, \mu)$, and let $\Gamma$ be a lattice subgroup of $G$. Then the action $\varphi$ is CPE if and only if the restriction $\varphi^\Gamma$ of $\varphi$ to $\Gamma$ is CPE.*

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*Proof.* The proof is similar to the proof of Theorem 3.2. However, we make some comments on it for the case of the group $\mathcal{H} = \mathcal{UT}_3(\mathbb{R})$. It follows from Theorem 4.7 that the action $\varphi^\Gamma$ is ergodic on $X$ for any lattice $\Gamma$. This allows one to apply Corollary 4.5 to prove that $\varphi^\Gamma$ is a CPE action if $\varphi$ is CPE. In proving this, we use the formula $h_K(\varphi^1, \rho) = H(\rho \mid \bigvee_1^\infty \varphi_{-i}\rho)$ (see proof of Theorem 3.2), where $\rho$ is a finite partition of $X$. Now we use a similar formula $h_K(\varphi^1, \rho) = H(\rho|\rho^-_{\mathcal{H}^1})$, where $\rho^-_{\mathcal{H}^1}$ is defined in (4.1).

Now we make several remarks about the proof that a CPE action of $\varphi^\Gamma$ implies that the action of $\varphi$ is CPE. Recall that $\Gamma$ has a special structure, namely $\Gamma = \mathcal{H}^{\bar{D}}$ (see the remarks following Corollary 4.5). This allows us to apply the estimate of (2.9) to complete this proof along the same lines as the proof of Theorem 3.2. □

As in §3.1, we now consider uniformly mixing actions of groups from $\mathcal{ULG}$. The definition of these actions is given in Definition 3.3. Theorem 4.10 allows to prove a well-known conjecture about the connection between CPE and uniformly mixing actions of groups from $\mathcal{ULG}$.

THEOREM 4.11. *Let $\varphi$ be a free measure-preserving ergodic action of a group $G$ from $\mathcal{ULG}$-class on a probability space $(X, \mathcal{B}, \mu)$. Then $\varphi$ is CPE if and only if $\varphi$ is uniformly mixing.*

*Proof.* The proof is similar to that of Theorem 3.4. For the case $G = \mathcal{H}$, there are some slight differences. If $\varphi$ is uniformly mixing, $\varphi^1$ is also uniformly mixing for the action of $\mathcal{H}^1 = \mathcal{UT}_3(\mathbb{Z})$. But then $\varphi^1$ is a CPE action of $\mathcal{H}^1$ by [**12, 18, 47**]. Thus, $\varphi$ is also a CPE action of $\mathcal{H}$ by Theorem 4.10. On the other hand, if $\varphi$ is a CPE action of $\mathcal{H}$, then $\varphi^1$ is a CPE action by Theorem 4.10, and hence $\varphi^1$ is uniformly mixing by [**41**]. Since $\varphi$ is a strongly continuous action of $\mathcal{H}$ and $\varphi^1$ is uniformly mixing, it follows that $\varphi$ is also uniformly mixing using the properties of the Rokhlin metric given in the proof of Theorem 3.4. □

COROLLARY 4.12. *Let $\varphi$ be a CPE action of a group $G \in \mathcal{ULG}$ on a space $(X, \mathcal{B}, \mu)$, and let $K$ be a discrete closed subgroup or closed $\mathcal{ULG}$-subgroup of $G$. Then the restriction $\varphi^K$ of $\varphi$ to $K$ is a CPE action on $(X, \mathcal{B}, \mu)$.*

*Proof.* It follows from Theorem 4.11 that $\varphi$ and $\varphi^K$ are uniformly mixing. If $K$ is discrete, then $\varphi^K$ is a CPE action, by [**12**]. If $K$ is a $\mathcal{ULG}$-group, then $\varphi^K$ has CPE by Theorem 4.11. □

PROPOSITION 4.13. *Let $G \in \mathcal{ULG}$, $\mathfrak{g}$ be its Lie algebra, and $\{e_i\}_1^n$ the canonical basis of $\mathfrak{g}$, described at the beginning of §4.2. Let $K$ be a closed cocompact subgroup of $G$ such that the connected component of unity $K_0$ is a normal subgroup of $G$, and its Lie algebra $\mathfrak{k}_0$ has a basis $\{f_i\}_1^m$, $m < n$ such that $\{f_i\}_1^m$ is a subset of $\{e_i\}_1^n$. If $\varphi$ is a free measure-preserving ergodic action of $G$ on $(X, \mathcal{B}, \mu)$, then $\varphi$ is a CPE action of $G$ if and only if $\varphi^K$ is a CPE action of $K$.*

*Proof.* It follows from the assumptions that $K$ contains a uniform lattice subgroup $\Gamma$ which is a lattice subgroup of $G$. Now it is possible to develop the theory of §4.2 for actions of the groups $K$ and their uniform lattice subgroups $\Gamma$, and prove, in particular, analogues of Theorems 4.10 and 4.11. The subsequent argument is obvious. □

4.4. *Spectral properties of CPE actions of $\mathcal{ULG}$-groups.* A locally compact group $G$ is of type I if any of its factor-representations by unitary operators in a Hilbert space is of type I in the von Neumann classification [**10**]. For such groups, Mackey [**28**] defines the dual object $\hat{G}$, elements of which are equivalence classes of irreducible unitary representations of $G$. Mackey introduced a Borel structure in $\hat{G}$, and he showed that if $g \to U_g$ is a unitary representation of $G$ in a Hilbert space $H$, then there exists a decomposition

$$H = \int_{\hat{G}}^{\oplus} H_x \, d\mu(x) \quad \text{and} \quad U_g = \int_{\hat{G}}^{\oplus} U_g(x) \, d\mu(x),$$

where $x \to H_x$ is a Borel field of Hilbert spaces, $x \mapsto U_g(x)$ is a Borel field of factor-representations of type I of $G$, and $\mu$ is a Borel measure on $\hat{G}$.

Furthermore, $H_x = H_x^1 \otimes H_x^2$ and $U_g(x) = U_g^1(x) \otimes I_{n(x)}$, where $x \to H_x^i$, $i = 1, 2$, are Borel fields of Hilbert spaces, $x \to U_g^1(x)$ is a Borel field of irreducible representations of $G$, and $I_{n(x)}$ is the identity operator in $H_x^2$ such that $\dim H_x^2 = n(x)$ for a.e. $x \in \hat{G}$.

Thus, it is possible to define for every unitary representation $g \to U_g$ of the group $G$, a Borel measure $\mu$ on $\hat{G}$ and a Borel multiplicity function $x \to n(x)$ on $\hat{G}$. Mackey proved that two unitary representations $g \mapsto U_g^1$, $i = 1, 2$ of $G$ are unitarily equivalent if and only if $\mu_1 \sim \mu_2$ and $n_1(x) = n_2(x)$ for a.e. $x \in \hat{G}$.

Let us recall some examples of groups of type I which are considered in this paper. If $G$ is an abelian locally compact group, then its dual $\hat{G}$ coincides with the Pontryagin dual group $\hat{G}$. The other important, for us, class of groups is nilpotent connected Lie groups which are also of type I. If $G$ is such a group, and $Z_G$ is its centre, then $\hat{G} = \hat{Z_G}$.

More detailed information and references on type I groups can be found in Dixmier's monograph [**10**, 13.11.12], where a large class of Lie groups of type I is described, including connected nilpotent Lie groups. Notice also that Dixmier (§18) introduced the notion of the Plancherel measure $m_G$ on $\hat{G}$ for the decomposition of the regular (left and right) representation of $G$. This measure allows us to obtain a version of the Plancherel formula for non-commutative groups of type I. If $G$ is abelian, then $m_G$ coincides with the Haar measure of $\hat{G}$, and if $G$ is a connected nilpotent Lie group, then $m_G$ coincides with the Haar measure of $\hat{Z_G}$.

Now we are ready to present our results about the spectrum of CPE action of $\mathcal{ULG}$-groups.

THEOREM 4.14. *Let $\varphi$ be a free CPE action of $G \in \mathcal{ULG}$ on $(X, \mathcal{B}, \mu)$. Then the action of $\varphi$ on $\mathcal{L}_0^2(X, \mu)$ has Lebesgue spectrum with infinite multiplicity.*

*Proof.* Let $Z_G$ be the centre of $G$. Then $Z_G$ is a closed subgroup of $G$, and it follows from Corollary 4.12 that $\varphi^{Z_G}$, the restriction $\varphi$ to $Z_G$, also has a CPE action. But $Z_G$ is isomorphic to $\mathbb{R}^n$ for some integer $n$, in view of our assumptions on $\mathcal{ULG}$-groups. Hence, $\varphi_G^Z$ has infinite Lebesgue spectrum by Theorem 3.8, and the measure corresponding to $\varphi$, is the Plancherel measure $m_G$ of $\hat{Z_G}$, as above. Thus, the spectrum of $\varphi$ is Lebesgue.

To prove infinite multiplicity of the spectrum of $\varphi$, assume first that $\varphi$ is a Bernoulli action of $G$, as in the case of Theorem 3.8.

LEMMA 4.15. *If $\varphi$ is a Bernoulli action of $G$ as above, then $\varphi$ is a CPE action and it has Lebesgue spectrum with infinite multiplicity.*

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*Proof.* Notice first that $\varphi$ is a CPE action. Indeed, if $\varphi$ is Bernoulli, then $\varphi^1$ is a Bernoulli action of the lattice subgroup $\Gamma_1$ of $G$, by [**30**, III, §6]. But since any two Bernoulli actions of $\Gamma_1$ with the same entropy are isomorphic [**30**, III], then we can realize it using a von Neumann construction (see the end of §3.1). It is easy to see from this construction that $\varphi^1$ is CPE. Then it follows from Theorem 4.10 that $\varphi$ is also the CPE action of $G$, and hence $\varphi$ has the Lebesgue spectrum by the result above.

Now let $\varphi$ act on the space $(X, \mathcal{B}, \mu)$, and consider the unitary representation $g \to U_{\varphi_g}$, $g \in G$ on the Hilbert space $H_0 = (f \in \mathcal{L}_0^2(X, \mu) : \int_X f(x)\, d\mu(x) = 0)$. If we reduce the representation $g \to U_{\varphi_g}$, $g \in Z_G$ to the diagonal form, then $H_0$ is realized as

$$H_0 = \int_{\hat{Z}_G}^{\oplus} H_z \, dm_G(z),$$

and we have the following decomposition of the representation $g \to U_{\varphi_g}$, $g \in G$ of $G$,

$$U_{\varphi_g} = \int_{\hat{Z}_G}^{\oplus} U_g(z)\, dm_G(z),$$

where $g \to U_g(z)$ is the irreducible $I_\infty$-representation of $G$ on the space $H_z$ for a.e. $z$.

Let $M$ be the commutant of $U_{\varphi_g}$, $g \in G$, i.e. $M$ contains all bounded operators $m$ on $H_0$ such that $mU_{\varphi_g} = U_{\varphi_g} m$. There is a decomposition of $M$, $M = \int_{\hat{Z}_G}^{\oplus} M_z \, dm_G(z)$ (see [**9**, Appendice A]), where $M_z$ is a factor of type $I_{n(z)}$, where $n(z)$ is a measurable function from $\hat{Z}_G$ to $\mathbb{N}$.

Since $\varphi$ is a Bernoulli action of $G$, then we can apply the same approach as in the proof of Lemma 3.14 and prove that $M_z$ contains for a.a. $z$ a subfactor of type $I_n$ for any natural number $n$. This means that $M_z$ is also $I_\infty$-factor, and our representation $g \to U_{\varphi_g}$ has Lebesgue spectrum with infinite multiplicity. □

Let us return to the proof of Theorem 4.14. Since $\varphi$ is a free CPE action of $G$, then it contains a Bernoulli subfactor, by [**30**, III, §3]. Thus, $\varphi$ has the Lebesgue spectrum and has a Bernoulli subfactor which has Lebesgue spectrum with infinite multiplicity. Hence, $\varphi$ also has Lebesgue spectrum with infinite multiplicity (see proof of Theorem 3.8 for details). □

4.5. *Actions of $\mathcal{ULG}$-groups with positive entropy and their Pinsker algebras.* In this section, we consider an action $\varphi$ of a $\mathcal{ULG}$-group $G$ with positive entropy, assuming that $\varphi$ does not contain a discrete component, and hence the action $\varphi^\Gamma$ is ergodic for any lattice subgroup $\Gamma$ of $G$. We will show that the Pinsker algebra $\Pi(\varphi)$ exists for such an action, and that $\Pi(\varphi) = \Pi(\varphi^\Gamma)$ (see Theorem 4.17). Then we will describe the spectral properties for the action $\varphi$ (see Theorem 4.18 below).

Recall that we assume that $G$ is a subgroup of $\mathcal{UT}_n(\mathbb{R})$, and $\Gamma_1 = G \cap \mathcal{UT}_n(\mathbb{Z})$ is a lattice subgroup of $G$, and below we will write $\varphi^1$ instead of $\varphi^{\Gamma_1}$. Notice also that the definition of the Pinsker algebra for action $\varphi$ is given in Definition 3.15 above.

PROPOSITION 4.16. *Let $\varphi$ be a free action of $G$ on $(X, \mathcal{B}, \mu)$, with positive entropy $h(\varphi) > 0$, and let the spectrum $\varphi$ not contain a discrete component. Then the Pinsker algebras $\Pi(\varphi^1)$ and $\Pi(\varphi^\Gamma)$ of the actions $\varphi^1$ and $\varphi^\Gamma$, respectively, coincide. In particular, $\varphi^\Gamma$ is a CPE action of $\Gamma$ if and only if $\Gamma_1$ is a CPE action of $\Gamma_1$.*

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*Proof.* The proof is based on Corollary 4.9, and can be proved by the same argument as the proof of Proposition 3.16. □

THEOREM 4.17. *Let $\varphi$ and $\Gamma_1$ be as in the statement of Proposition 4.16. Then $\Pi(\varphi) = \Pi(\varphi^1)$.*

*Proof.* The proof is the same as the proof of Theorem 3.17, but now we apply Proposition 4.16 instead of Proposition 3.16. Let us check only that $\Pi(\varphi^1)$ is $\varphi$-invariant. Without loss of generality, it is enough to consider the case $G = \mathcal{H}$. In this case, $\Gamma_1 = \mathcal{H}^1 = \mathcal{UT}_3(\mathbb{Z})$.

Let $N$ be a normal subgroup of $\mathcal{H}$ such that $x_{12}, x_{23} \in \mathbb{Z}$, and $x_{13} \in \mathbb{R}$. It is easy to see that $\mathcal{H}^1$ is a cocompact lattice subgroup of $N$. The same argument as in the proof of Proposition 3.16 shows that $\Pi(\varphi^1)$ is $\varphi^N$-invariant, and $\Pi(\varphi^1) = \Pi(\varphi^N)$. If now $g \in \mathcal{H}$, then $\Gamma = g\mathcal{H}^1 g^{-1}$ is also a cocompact lattice subgroup of $N$ and $\Pi(\varphi^\Gamma) = \Pi(\varphi^N)$. It is clear that $\Pi(\varphi^\Gamma) = g\Pi(\varphi_1)g^{-1}$. Hence,

$$g\Pi(\varphi^1)g^{-1} = \Pi(\varphi^\Gamma) = \Pi(\varphi^N) = \Pi(\varphi^1).$$

Since $g$ is an arbitrary element from $\mathcal{H}$, $\Pi(\varphi^1)$ is $\varphi$-invariant. □

Consider now the spectral properties of an action $\varphi$ of $G$, as above, with positive entropy. Again, we aim to reduce the problem to the restriction of $\varphi$ to the centre $Z_G$ of $G$ and apply the results of Mackey [**28**] and Dixmier [**10**] outlined in §4.4.

THEOREM 4.18. *Let $\varphi$ be a free ergodic action of a group $G$, as above, on a space $(X, \mathcal{B}, \mu)$ with positive entropy $h(\varphi) > 0$ and Pinsker algebra $\Pi(\varphi)$. Then $\varphi$ has infinite Lebesgue spectrum on the Hilbert space $H_0^X = \mathcal{L}_0^2(X, \mu) \ominus \mathcal{L}_0^2(\Pi(\varphi))$, where $\mathcal{L}_0^2(\Pi(\varphi))$ is a subspace of $\mathcal{L}_0^2(X, \mu)$ consisting of all $\Pi(\varphi)$-measurable functions.*

*Proof.* First, we can show that $H_0^X = \int_{\hat{Z_G}} H_\lambda \, dm_0(\lambda)$, where $m_0(\lambda)$ is a Borel measure on $\hat{Z_G}$ subordinated to the Haar measure on $\hat{Z_G}$, by applying the argument of the beginning of the proof of Theorem 3.18.

Then we again use the same approach as in the proof of Theorem 3.18, applying Corollary 4.2 instead of [**5**, Theorem 5]. This corollary states the existence of the Bernoulli factor of $\varphi$ such that it is independent, in some sense, from $\Pi(\varphi)$. Since this factor has infinite Lebesgue spectrum by Lemma 4.15, the measure $m_0(\lambda)$ on $\hat{Z_G}$ is subordinated to the Haar measure, and then one can conclude, using the same argument as in the proof of Theorem 3.18, that $\varphi$ has also infinite Lebesgue spectrum on the space $H_0$. □

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