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Susan J. Sierra and Alexey V. Petukhov

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ABSTRACT

Let $W = \mathbb{C}[t, t^{-1}]\partial_t$ be the *Witt algebra* of algebraic vector fields on \mathbb{C}^\times and let Vir be the *Virasoro algebra*, the unique nontrivial central extension of W . In this paper, we study the Poisson ideal structure of the symmetric algebras of Vir and W , as well as several related Lie algebras. We classify prime Poisson ideals and Poisson primitive ideals of $S(Vir)$ and $S(W)$. In particular, we show that the only functions in W^* which vanish on a nontrivial Poisson ideal (that is, the only maximal ideals of $S(W)$ with a nontrivial Poisson core) are given by linear combinations of derivatives at a finite set of points; we call such functions *local*. Given a local function $\chi \in W^*$, we construct the associated Poisson primitive ideal through computing the algebraic symplectic leaf of χ , which gives a notion of coadjoint orbit in our setting. As an application, we prove a structure theorem for subalgebras of Vir of finite codimension and show, in particular, that any such subalgebra of Vir contains the central element z , substantially generalising a result of Ondrus and Wiesner on subalgebras of codimension one. As a consequence, we deduce that $S(Vir)/(z - \zeta)$ is Poisson simple if and only if $\zeta \neq 0$.

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1. Introduction

Let G be a connected algebraic group over \mathbb{C} with Lie algebra \mathfrak{g} , and consider the coadjoint action of G on \mathfrak{g}^* . This is a beautiful classical topic, with profound connections to areas from geometric representation theory to combinatorics to physics. Algebraic geometry tells us that coadjoint orbits in \mathfrak{g}^* correspond to G -invariant radical ideals in the symmetric algebra $S(\mathfrak{g})$.

These can also be defined using the Kostant–Kirillov *Poisson bracket* on $S(\mathfrak{g})$:

$$\{f, g\} = \sum_{i,j} \frac{\partial f}{\partial e_i} \frac{\partial g}{\partial e_j} [e_i, e_j],$$

where $\{e_i\}$ is a basis of \mathfrak{g} . Recall that an ideal I of $S(\mathfrak{g})$ is a *Poisson ideal* if I is also a Lie ideal for the Poisson bracket. A basic fact is that I is G -invariant if and only if I is Poisson.

Thus, to compute the closure of the coadjoint orbit of $\chi \in \mathfrak{g}^*$, let \mathfrak{m}_χ be the kernel of the evaluation morphism

$$\text{ev}_\chi : S(\mathfrak{g}) \rightarrow \mathbb{k},$$

and let $P(\chi)$ be the *Poisson core* of \mathfrak{m}_χ : the maximal Poisson ideal contained in \mathfrak{m}_χ . By definition, an ideal of the form $P(\chi)$ is called a *Poisson primitive*; by a slight abuse of notation, we refer to $P(\chi)$ as the *Poisson core of χ* . The closure of the coadjoint orbit of χ is defined by $P(\chi)$:

$$\overline{G \cdot \chi} = V(P(\chi)) := \{\nu \in \mathfrak{g}^* \mid \text{ev}_\nu(P(\chi)) = 0\}, \quad (1.0.1)$$

and so $\chi, \nu \in \mathfrak{g}^*$ are in the same G -orbit if and only if $P(\chi) = P(\nu)$. In the case of algebraic Lie algebras over \mathbb{C} or \mathbb{R} , coadjoint orbits are symplectic leaves for the respective Poisson structure.

In this paper, we investigate how this theory extends to the *Witt algebra* $W = \mathbb{C}[t, t^{-1}]\partial_t$ of algebraic vector fields on \mathbb{C}^\times , and to its central extension the *Virasoro algebra* $Vir = \mathbb{C}[t, t^{-1}]\partial_t \oplus \mathbb{C}z$, with Lie bracket given by

$$[f\partial_t, g\partial_t] = (fg' - f'g)\partial_t + \text{Res}_0(f'g'' - g'f'')z, \quad z \text{ is central.}$$

(We also consider some important Lie subalgebras of W .) These infinite-dimensional Lie algebras, of fundamental importance in representation theory and in physics, have no adjoint group [Lem97], but one can still study the Poisson cores of maximal ideals and, more generally, the Poisson ideal structure of $S(W)$ and $S(Vir)$. Motivated by (1.0.1), we say that functions $\chi, \nu \in Vir^*$ or in W^* are in the same *pseudo-orbit* if $P(\chi) = P(\nu)$. These (coadjoint) pseudo-orbits can be considered as algebraic symplectic leaves in Vir^* or W^* .

Taking the previous discussion as our guide, we focus on prime Poisson ideals and Poisson primitive ideals of $S(\text{Vir})$ and $S(W)$. Important questions here, which for brevity we ask in the introduction only for Vir , include the following.

- (i) Given $\chi \in \text{Vir}^*$, can we compute the Poisson core $P(\chi)$ and the pseudo-orbit of χ ? When is $P(\chi)$ nontrivial?
- (ii) How can we understand prime Poisson ideals of $S(\text{Vir})$? Can we parameterise them in a reasonable fashion, ideally in a way which gives us further information about the ideal? How does one distinguish Poisson primitive ideals from other prime Poisson ideals?
- (iii) It is known, see [LS08, Corollary 5.1], that $S(\text{Vir})$ satisfies the ascending chain condition on prime Poisson ideals. The augmentation ideal of $S(\text{Vir})$, that is, the ideal generated by $\text{Vir} \subset S(\text{Vir})$, is clearly a maximal Poisson ideal. What are the others? Conversely, does any nontrivial prime Poisson ideal have finite height?
- (iv) Do prime Poisson ideals induce any reasonable algebraic geometry on the uncountable-dimensional vector space Vir^* ?

We answer all of these questions, almost completely working out the structure of the Poisson spectra of $S(\text{Vir})$ and $S(W)$.

Let us begin by discussing the idea of algebraic geometry on Vir^* . *A priori*, this seems completely intractable as Vir^* is an uncountable-dimensional affine space; little interesting can be said about $S(\mathfrak{a})$ where \mathfrak{a} is a countable-dimensional *abelian* Lie algebra. However, Vir and W are extremely noncommutative and so Poisson ideals in their symmetric algebras are very large: in particular, by a result of Iyudu and the second author [IS20, Theorem 1.3], if I is a nontrivial Poisson ideal of $S(W)$ (respectively, a noncentrally generated Poisson ideal of $S(\text{Vir})$), then $S(W)/I$ (respectively, $S(\text{Vir})/I$) has polynomial growth. This suggests that we might hope that a Poisson primitive ideal, and, more generally, a prime Poisson ideal, would correspond to a finite-dimensional algebraic subvariety of Vir^* , which we might be able to investigate using tools from affine algebraic geometry. We show that this is indeed the case.

From the discussion, it is important to characterise which functions $\chi \in \text{Vir}^*$ have nontrivial Poisson cores. One striking result, proved in this paper, is that such χ must vanish on the central element z . Further, the induced function $\bar{\chi} \in W^*$ is given by evaluating *local* behaviour on a proper (that is, finite) subscheme of \mathbb{C}^\times . We have the following result.

THEOREM 1.1 (Theorem 3.3.1). *Let $\chi \in \text{Vir}^*$. The following are equivalent.*

- (1) *The Poisson core of χ is nontrivial: that is, $P(\chi) \supsetneq (z - \chi(z))$.*
- (2) *We have $\chi(z) = 0$ and the induced function $\bar{\chi} \in W^*$ is a linear combination of functions of the form*

$$f \partial_t \mapsto \alpha_0 f(x) + \dots + \alpha_n f^{(n)}(x),$$

where $x \in \mathbb{C}^\times$ and $\alpha_0, \dots, \alpha_n \in \mathbb{C}$.

- (3) *The isotropy subalgebra Vir^χ of χ has finite codimension in Vir .*

We call functions $\chi \in \text{Vir}^*$ satisfying the equivalent conditions of Theorem 1.1 *local functions* as by condition (2) they are defined by local data.

Motivated by condition (3) of Theorem 1.1, we investigate subalgebras of Vir of finite codimension. We prove the following.

THEOREM 1.2 (Proposition 3.3.3). *Let $\mathfrak{k} \subseteq \text{Vir}$ be a subalgebra of finite codimension. Then there is $f \in \mathbb{C}[t, t^{-1}] \setminus \{0\}$ so that $\mathfrak{k} \supseteq \mathbb{C}z + f\mathbb{C}[t, t^{-1}]\partial_t$. In particular, any finite codimension subalgebra of Vir contains z .*

As an immediate corollary of Theorem 1.2, we show the following result.

COROLLARY 1.3 (Corollary 3.3.5). *If $0 \neq \zeta \in \mathbb{C}$, then $S(\text{Vir})/(z - \zeta)$ is Poisson simple: it has no nontrivial Poisson ideals.*

We then study the pseudo-orbits of local functions on Vir , W and related Lie algebras; we describe our results for Vir in the introduction. If $\chi \in \text{Vir}^*$ is local, then by combining Theorem 1.1 and [IS20, Theorem 1.3], $S(\text{Vir})/P(\chi)$ has polynomial growth and we thus expect the pseudo-orbit of χ to be finite-dimensional. We show that pseudo-orbits of local functions in Vir^* are, in fact, orbits of a finite-dimensional solvable algebraic (Lie) group acting on an affine variety which maps injectively to Vir^* , and we describe these orbits explicitly (Section 4.2). This allows us to completely determine the pseudo-orbit of an arbitrary local function in Vir^* (Theorem 4.3.1) and thus also determine the Poisson primitive ideals of $S(\text{Vir})$ (Remark 4.3.9). We also classify maximal Poisson ideals in $S(\text{Vir})$ (Corollary 4.3.18): they are the augmentation ideal, the ideals $(z - \zeta)$ for $\zeta \in \mathbb{C}^\times$, and the defining ideals of all but one of the two-dimensional pseudo-orbits.

Through this analysis, we obtain a nice combinatorial description of pseudo-orbits in W^* : pseudo-orbits of local functions on W and, thus, Poisson primitive ideals of $S(W)$, correspond to a choice of a partition λ and a point in an open subvariety of a finite-dimensional affine space \mathbb{A}^k , where k can be calculated from λ . (See Remark 4.3.3.) In Theorem 5.2.8 and Remark 5.2.13, we expand this correspondence to obtain a parameterisation of all prime Poisson ideals of $S(W)$ and $S(\text{Vir})$. We also study the related Lie algebra $W_{\geq -1} = \mathbb{C}[t]\partial_t$, and prove (Corollary 5.2.14) that Poisson primitive and prime Poisson ideals of $S(W_{\geq -1})$ are induced by restriction from $S(W)$.

Our understanding of prime Poisson ideals allows us to determine exactly which prime Poisson ideals of $S(\text{Vir})$ obey the *Poisson Dixmier–Moeglin equivalence* (PDME), which generalises the characterisation of primitive ideals in enveloping algebras of finite-dimensional Lie algebras due to Dixmier and Moeglin. The central question is when a Poisson primitive ideal of $S(\text{Vir})$ is *Poisson locally closed*: that is, locally closed in the Zariski topology on Poisson primitive ideals. (If $\dim \mathfrak{g} < \infty$, then a prime Poisson ideal of $S(\mathfrak{g})$ is Poisson primitive if and only if it is Poisson locally closed [LL19, Theorem 2].) We show (Theorem 5.3.1) that (z) is the only Poisson primitive ideal of $S(\text{Vir})$ which is not Poisson locally closed. We further prove (Corollary 5.4.2) that $S(W)$ has no nonzero prime Poisson ideals of finite height.

One part of the proof of Theorem 1.2 is to show that, given a subalgebra \mathfrak{k} of finite codimension in W , there are a finite collection of points S (the ‘support’ of \mathfrak{k}) and $n \in \mathbb{Z}_{\geq 0}$ such that all vector fields in \mathfrak{k} vanish at all points of S and so that \mathfrak{k} contains all vector fields vanishing to order $\geq n$ at every point of S . Based on this result, we classify subalgebras of Vir of codimension ≤ 3 in §6.

Our original motivation for studying Poisson ideals of $S(\text{Vir})$ was to study two-sided ideals in the universal enveloping algebra $U(\text{Vir})$, and we turn to enveloping algebras in §7. In the finite-dimensional setting, Kirillov’s orbit method gives a correspondence between primitive ideals of $U(\mathfrak{g})$ and coadjoint orbits in \mathfrak{g}^* . We conjecture that a similar correspondence exists for Vir and related Lie algebras, and in §7.1 we show that pseudo-orbits of dimension two in Vir^* quantise to give a family of primitive ideals, kernels of well-known maps from $U(\text{Vir})$ through $U(W)$ to the localised Weyl algebra. We end the paper with some conjectures about (two-sided) ideals of the universal enveloping algebra $U(\text{Vir})$ which are motivated by our work on $S(\text{Vir})$; these are the subject of further research.

To end the introduction, let us briefly discuss the classical (continuous) version of the theory of coadjoint orbits of the Virasoro algebra. If one considers the real Lie algebra of continuous

vector fields on the circle and its central extension the real Virasoro algebra $Vir_{\mathbb{R}}$, then the group of diffeomorphisms of the circle acts on $Vir_{\mathbb{R}}$ and its continuous dual. There are, of course, notions of coadjoint orbits in this context, see [Kir81, Wit88]. However, it happens that the corresponding orbits do not define an interesting (Poisson) ideal in the symmetric algebra $S(Vir_{\mathbb{R}})$; in fact, this can be viewed as a somewhat informal result of our paper. Note that the local functions which appear in Theorem 1.1 and which we study in this paper can be thought of as a product of a point-based distribution with a vector field and, hence, are very far from being continuous.

2. Poisson ideals and pseudo-orbits

In this section, we recall the general notions of Poisson algebra, Poisson ideal and Poisson primitive ideal. We then consider how these concepts behave for the symmetric algebras of the infinite-dimensional Lie algebras in which we are interested.

Throughout, let \mathbb{k} be an uncountable algebraically closed field of characteristic zero. Let us define the Lie algebras of interest in this paper. The *Witt algebra* $W = \mathbb{k}[t, t^{-1}]\partial$ is the Lie algebra of vector fields on $\mathbb{k}^{\times} := \mathbb{k} \setminus \{0\}$; here $\partial = \partial_t = d/dt$. It is graded by setting $\deg t^n \partial = n - 1$. Then $W_{\geq -1}$ is the subalgebra $\mathbb{k}[t]\partial$ of W and $W_{\geq 1}$, sometimes called the *positive Witt algebra*, is the subalgebra $t^2\mathbb{k}[t]\partial$; $W_{\geq 0}$ stands for $t\mathbb{k}[t]\partial$.

The *Virasoro algebra* Vir is isomorphic as a vector space to $\mathbb{k}[t, t^{-1}]\partial \oplus \mathbb{k}z$. It is endowed with a Lie algebra structure by the formula

$$[f\partial + c_1z, g\partial + c_2z] = (fg' - f'g)\partial + \text{Res}_0(f'g'' - f''g')z.$$

(Here $\text{Res}_0(f)$ denotes the algebraic residue of f at 0, i.e. the coefficient of t^{-1} in the Laurent expansion of f at 0.) It is well-known that Vir is the unique nontrivial one-dimensional central extension of W . There is a canonical Lie algebra homomorphism $Vir \rightarrow W$ given by factoring out z .

Let V be a \mathbb{k} -vector space. We use $S(V)$ to denote the symmetric algebra of V ; that is, polynomial functions on V^* . For $\chi \in V^*$ we denote by ev_{χ} the induced homomorphism $S(V) \rightarrow \mathbb{k}$ defined by $\text{ev}_{\chi}(f) = f(\chi)$.

Let \mathfrak{g} be one of the Lie algebras $Vir, W, W_{\geq -1}$ or $W_{\geq 1}$. Our assumption on the cardinality of the field \mathbb{k} means that the following extended Nullstellensatz applies to $S(\mathfrak{g})$; see [MR87, Corollary 9.1.8, Lemma 9.1.2].

THEOREM 2.0.1 (Extended Nullstellensatz). *Let A be a commutative \mathbb{k} -algebra such that $\dim_{\mathbb{k}} A < |\mathbb{k}|$. Then:*

- *A is a Jacobson ring: every radical ideal is an intersection of a family of maximal ideals;*
- *if $\mathfrak{m} \subseteq A$ is a maximal ideal, then the canonical map $\mathbb{k} \rightarrow A/\mathfrak{m}$ is an isomorphism;*
- *if $A = S(V)$ for a vector space V with $\dim V < |\mathbb{k}|$, then the maximal ideals of A are all of the form $\mathfrak{m}_{\chi} = \ker \text{ev}_{\chi}$ for some $\chi \in V^*$.*

Let A be a commutative \mathbb{k} -algebra. (Note that we make no noetherianity or finiteness assumption on A .) Denote by $\text{MSpec } A$ the set of maximal ideals of A , which we consider as a (potentially infinite-dimensional) variety. We denote by \mathfrak{m}_x the maximal ideal corresponding to a point $x \in \text{MSpec } A$. As usual, $\text{MSpec } A$ is a topological space under the Zariski topology. Given an ideal N of A , we denote the corresponding closed subset of $\text{MSpec } A$ by

$$V(N) := \{\mathfrak{m} \in \text{MSpec } A \mid N \subseteq \mathfrak{m}\}.$$

Given $X \subseteq \text{MSpec } A$, we denote the corresponding radical ideal of A by

$$I(X) := \bigcap_{x \in X} \mathfrak{m}_x.$$

As we have enlarged the class of varieties somewhat beyond the usual, we make the convention that for us an *algebraic variety* is a classical variety: a (nonempty) integral separated scheme of finite type over \mathbb{k} , as in [Har77, p. 105]. If A, B are commutative \mathbb{k} -algebras which are domains, an algebraic map or *morphism of varieties* $\phi : \text{MSpec } A \rightarrow \text{MSpec } B$ has the usual meaning: a function so that the pullback ϕ^* defines an algebra homomorphism $B \rightarrow A$.

Let $\{\cdot, \cdot\} : A \times A \rightarrow A$ be a skew-symmetric \mathbb{k} -bilinear map. We say that $(A, \{\cdot, \cdot\})$ is a *Poisson algebra* if $\{\cdot, \cdot\}$ satisfies the Leibniz rule on each input and the Jacobi identity. An ideal I of A is *Poisson* if $\{I, A\} \subseteq I$. It is clear that the sum of all Poisson subideals of any ideal I of A is the maximal Poisson ideal inside I ; we denote this ideal by $\text{Core}(I)$ and refer to it as the *Poisson core* of I . Note that if I is radical, respectively prime, then $\text{Core}(I)$ inherits this property; see [PS20, Lemmata 2.6 and 2.8].

A Poisson ideal I is called *Poisson primitive* if $I = \text{Core}(\mathfrak{m})$ for a maximal ideal \mathfrak{m} of A ; Poisson primitive ideals are prime. We denote the set of Poisson primitive ideals of A by $\text{PSpec}_{\text{prim}} A$, and the set of prime Poisson ideals by $\text{PSpec } A$. Both are given the Zariski topology, where the closed subsets are defined by Poisson ideals of A .

Consider a Lie algebra \mathfrak{g} with $\dim \mathfrak{g} < |\mathbb{k}|$. It is well-known that $S(\mathfrak{g})$ possesses a canonical Poisson algebra structure, induced by defining $\{u, v\} = [u, v]$ for any $u, v \in \mathfrak{g}$. As $\dim \mathfrak{g} < |\mathbb{k}|$, then by the Nullstellensatz $\text{MSpec}(S(\mathfrak{g}))$ can be canonically identified with \mathfrak{g}^* :

$$\chi \in \mathfrak{g}^* \leftrightarrow \mathfrak{m}_\chi := \ker \text{ev}_\chi \in \text{MSpec}(S(\mathfrak{g})).$$

Thus, any Poisson primitive ideal of $S(\mathfrak{g})$ is equal to $\text{Core}(\mathfrak{m}_\chi)$ for some $\chi \in \mathfrak{g}^*$. Set $P(\chi) := \text{Core}(\mathfrak{m}_\chi)$. For any Lie algebra \mathfrak{g} , the ideal $\mathfrak{m}_0 = \text{Core}(\mathfrak{m}_0) = P(0)$ (the augmentation ideal) is the Poisson core of $0 \in \mathfrak{g}^*$ and so is Poisson primitive.

Even in the absence of an adjoint group to \mathfrak{g} , the Poisson primitives $P(\chi)$ give analogues of coadjoint orbits.

DEFINITION 2.0.2. Let \mathfrak{g} be any Lie algebra. The *pseudo-orbit* of $\chi \in \mathfrak{g}^*$ is

$$\mathbb{O}(\chi) := \{\nu \in \mathfrak{g}^* \mid P(\nu) = P(\chi)\}.$$

The *dimension* of $\mathbb{O}(\chi)$ is defined to be $\text{GKdim } S(\mathfrak{g})/P(\chi)$. (Here if R is a \mathbb{k} -algebra, then $\text{GKdim } R$ denotes the *Gelfand–Kirillov dimension* of R ; see [KL00].)

Remark 2.0.3. If $\mathfrak{g} = \text{Lie } G$ is the Lie algebra of a connected algebraic group and $\chi \in \mathfrak{g}^*$, then $\mathbb{O}(\chi)$ is the coadjoint orbit of χ . In our setting, \mathfrak{g} is not the Lie algebra of any algebraic group. However, we show that we can still define algebraic group actions on pieces of \mathfrak{g}^* that allow us to recover pseudo-orbits as actual orbits.

Part (b) of the next result is an analogue of (1.0.1) for pseudo-orbits.

LEMMA 2.0.4. Let \mathfrak{g} be a Lie algebra with $\dim \mathfrak{g} < |\mathbb{k}|$.

- (a) Any radical Poisson ideal in $S(\mathfrak{g})$ is equal to an intersection of Poisson primitive ideals. Explicitly, given a radical Poisson ideal I we have

$$I = \bigcap \{P(\chi) \mid \chi \in \mathfrak{g}^*, \text{ev}_\chi(I) = 0\}.$$

(b) Assume now that $\dim \mathfrak{g}$ is countable and let $\chi \in \mathfrak{g}^*$ be nonzero. Then

$$P(\chi) = \bigcap_{\nu \in \mathbb{O}(\chi)} \mathfrak{m}_\nu.$$

Proof. Part (a) is well-known, but we give a proof for completeness. As I is radical and Poisson,

$$I \subseteq \bigcap \{P(\chi) \mid \chi \in \mathfrak{g}^*, \text{ev}_\chi(I) = 0\},$$

and this is contained, by definition, in $\bigcap_{\chi \in \mathfrak{g}^*, \text{ev}_\chi(I)=0} \mathfrak{m}_\chi$. By the Nullstellensatz, this last part is equal to I .

For part (b), set $R := S(\mathfrak{g})/P(\chi)$. Thanks to the proof of [LS08, Theorem 6.3], which uses only that \mathfrak{g} is countably generated (see also [BLLM17, Theorem 3.2]), there is a sequence of nonzero Poisson ideals L_1, L_2, \dots of R such that if $\mu \in \mathfrak{g}^*$ with $P(\mu)$ strictly containing $P(\chi)$, then $P(\mu)/P(\chi)$ contains L_i for at least one i .

This is equivalent to the following statement: there is a sequence $f_1, f_2, \dots \in R \setminus \{0\}$ such that if $P(\mu)$ strictly contains $P(\chi)$ then $f_i(\mu) = 0$ for at least one i . Therefore, $R' = (S(\mathfrak{g})/P(\chi))[f_1^{-1}, f_2^{-1}, \dots]$ is Poisson simple. On the other hand, R' is clearly at most countable-dimensional and, hence, $(0) \subset R'$ is an intersection of a family of maximal ideals of R' by the Nullstellensatz. The definition of R' and f_i guarantees that $P(\mu) = P(\chi)$ for each $\mu \in \mathfrak{g}^*$ defining such an ideal. We thus obtain a family of maximal ideals of $S(\mathfrak{g})$ contained in the pseudo-orbit of χ whose intersection is $P(\chi)$. This implies the desired result. \square

2.1 Some results on pseudo-orbits

We now give several results which are well-known for finite-dimensional algebraic Lie algebras, but require proof in our setting. Throughout this subsection, let \mathfrak{g} be a Lie algebra with $\dim \mathfrak{g} < |\mathbb{k}|$.

For every $\chi \in \mathfrak{g}^*$ define the skew-symmetric bilinear form $B_\chi(x, y) := \chi([x, y])$. The kernel of B_χ is a Lie subalgebra of \mathfrak{g} , which we denote by \mathfrak{g}^χ . Observe that \mathfrak{g}^χ is precisely $\{v \in \mathfrak{g} \mid v \cdot \chi = 0\}$; that is, \mathfrak{g}^χ is the isotropy subalgebra of χ under the (coadjoint) action of \mathfrak{g} on \mathfrak{g}^* . As B_χ induces a nondegenerate skew-symmetric form on $\mathfrak{g}/\mathfrak{g}^\chi$, we have $\text{rk } B_\chi = \dim \mathfrak{g}/\mathfrak{g}^\chi$.

LEMMA 2.1.1. *The dimension of $\mathbb{O}(\chi)$ is at least $\text{rk } B_\chi$, i.e.*

$$\text{GKdim}(S(\mathfrak{g})/P(\chi)) \geq \text{rk } B_\chi.$$

Before proving Lemma 2.1.1, we establish some notation, which we need for several results. For $u_1, \dots, u_n, v_1, \dots, v_n \in \mathfrak{g}$ set $D(u_1, \dots, u_n; v_1, \dots, v_n) \in S(\mathfrak{g})$ to be the determinant

$$\begin{vmatrix} [u_1, v_1] & \cdots & [u_1, v_n] \\ \cdots & \cdots & \cdots \\ [u_n, v_1] & \cdots & [u_n, v_n] \end{vmatrix}. \tag{2.1.2}$$

Proof of Lemma 2.1.1. Pick $r \in \mathbb{Z}_{\geq 0}$ such that $r \leq \text{rk } B_\chi$. Then there is an r -dimensional subspace V of \mathfrak{g} so that $\text{rk}(B_\chi|_V) = r$; that is, if u_1, \dots, u_r is a basis of V , then $\text{ev}_\chi(D(u_1, \dots, u_r, u_1, \dots, u_r)) \neq 0$. We show that u_1, \dots, u_r are algebraically independent in $S(\mathfrak{g})/P(\chi)$; that is, that $\text{GKdim}(S(\mathfrak{g})/P(\chi)) \geq r$.

Assume to the contrary that the u_i are not algebraically independent modulo $P(\chi)$. This means that there is some nonzero $P \in \mathbb{k}[u_1, \dots, u_r] \cap P(\chi)$, and we may assume P is of minimal degree among such elements. As $P(\chi)$ is Poisson,

$$\{u_i, P\} = \sum_j \{u_i, u_j\} \partial_j P \in P(\chi), \tag{2.1.3}$$

where $\partial_j P = \partial P / \partial u_j$.

Let Q be the field of fractions of $S(\mathfrak{g})/P(\chi)$ (recall that $P(\chi)$ is prime). We may rewrite (2.1.3) as the matrix equation

$$(\{u_i, u_j\})_{ij} \begin{pmatrix} \partial_1 P \\ \vdots \\ \partial_r P \end{pmatrix} = \vec{0}$$

over Q . By minimality of $\deg P$, the vector $\begin{pmatrix} \partial_1 P \\ \vdots \\ \partial_r P \end{pmatrix} \in Q^r$ is nonzero. Thus, $(\{u_i, u_j\})_{ij} \in M_{r \times r}(Q)$ is singular, so $D(u_1, \dots, u_r; u_1, \dots, u_r) \in P(\chi) \subseteq \mathfrak{m}_\chi$. This contradicts the first paragraph of the proof. \square

It is not necessarily easy to calculate $P(\chi)$, but we can sometimes use GK dimension to show that a Poisson ideal contained in \mathfrak{m}_χ is, in fact, equal to $P(\chi)$. This is given by the following lemma.

LEMMA 2.1.4. *Let \mathfrak{g} be a Lie algebra and A a commutative \mathbb{k} -algebra that is a domain. Let $\phi : S(\mathfrak{g}) \rightarrow A$ be an algebra homomorphism so that $\ker \phi$ is Poisson. Let $\chi \in \mathfrak{g}^*$ be such that*

$$\ker \phi \subseteq \mathfrak{m}_\chi \quad \text{and} \quad \text{rk } B_\chi \geq \text{GKdim } A. \tag{2.1.5}$$

Then $\ker \phi = P(\chi)$ and, thus, is Poisson primitive; further, $\text{rk } B_\chi = \text{GKdim } A = \dim \mathcal{O}(\chi)$.

Proof. Certainly $\ker \phi \subseteq P(\chi)$. If the containment is strict, then as A is a domain,

$$\text{GKdim}(S(\mathfrak{g})/P(\chi)) < \text{GKdim } A \leq \text{rk } B_\chi,$$

contradicting Lemma 2.1.1. The final equality is direct from Lemma 2.1.1. \square

COROLLARY 2.1.6. *If both χ_1, χ_2 satisfy (2.1.5) for the same ϕ , then $P(\chi_1) = P(\chi_2)$.*

We use Lemma 2.1.4 and Corollary 2.1.6 to describe primitive ideals in $S(\mathfrak{g})$ and the corresponding pseudo-orbits in \mathfrak{g}^* explicitly. In the situation of Lemma 2.1.4, A may not necessarily be a Poisson algebra. We abuse notation slightly, however, and say that if $\ker \phi$ is a Poisson ideal, then ϕ is a *Poisson morphism*. Further, given a morphism of varieties

$$\psi : \text{MSpec}(A) \rightarrow \text{MSpec}(S(\mathfrak{g})) = \mathfrak{g}^*,$$

we say that ψ is *Poisson* if $\psi^* : S(\mathfrak{g}) \rightarrow A$ is Poisson.

It is important to know under which conditions maps are Poisson and the answer is given in Proposition 2.1.7. To explain the setup we need to use some concepts from (affine) algebraic geometry. As mentioned, we, somewhat loosely, refer to infinite-dimensional vector spaces such as \mathfrak{g}^* as varieties, enlarging the class from standard usage. For any variety X and any point $x \in X$ let $T_x X = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ denote the tangent space to X at x ; this definition makes sense for $X = \mathfrak{g}^*$ as well. Without further comment, we identify $T_x X$ with elements of \mathfrak{m}_x^* which vanish on \mathfrak{m}_x^2 . If there is a map $\psi : X \rightarrow Y$ between algebraic varieties X and Y , then, for all $x \in X$, there is an induced map $\psi_x : T_x X \rightarrow T_{\psi(x)} Y$.

We canonically identify the tangent space to \mathfrak{g}^* at $\chi \in \mathfrak{g}^*$ with \mathfrak{g}^* . For any derivation D of $S(\mathfrak{g})$ and any $\chi \in \mathfrak{g}^*$ we denote by D_χ the tangent vector defined by D at χ ; that is, $D_\chi(f) = \chi(Df)$ for any $f \in \mathfrak{m}_\chi$. Now, \mathfrak{g} acts by derivations on $S(\mathfrak{g})$. For $u \in \mathfrak{g}$ and $\chi \in \mathfrak{g}^*$ let u_χ be the tangent vector defined by u at χ . If $v \in \mathfrak{g}$, then $u_\chi(v) = \chi([u, v])$ and, thus, $u_\chi \in T_\chi(\mathfrak{g}^*) = \mathfrak{g}^*$ is identified with $u \cdot \chi$. If $\mathfrak{u} \subseteq \mathfrak{g}$ let $\mathfrak{u}_\chi = \{u_\chi \mid u \in \mathfrak{u}\}$. Thus, $\mathfrak{g}_\chi \cong \mathfrak{g} \cdot \chi \cong \mathfrak{g}/\mathfrak{g}^\chi$ and $\text{rk } B_\chi = \dim \mathfrak{g}_\chi$.

PROPOSITION 2.1.7. *Let \mathfrak{g} be a Lie algebra with $\dim \mathfrak{g} < |\mathbb{k}|$. Let X be an affine variety and let $\phi : X \rightarrow \mathfrak{g}^*$ be a morphism of varieties. Then ϕ is Poisson if and only if $\mathfrak{g}_{\phi(x)} \subseteq \phi_x(\mathbb{T}_x X)$ for all $x \in X$.*

Proposition 2.1.7 is a direct consequence of the following lemma.

LEMMA 2.1.8. *Let \mathfrak{g}, X, ϕ be as in the statement of Proposition 2.1.7 and let D be a derivation of $S(\mathfrak{g})$. Then $\ker \phi^*$ is D -stable if and only if $D_{\phi(x)} \in \phi_x(\mathbb{T}_x X)$ for all $x \in X$.*

Proof. Suppose that $D_{\phi(x)} \in \phi_x(\mathbb{T}_x X)$ for all $x \in X$. Let $f \in \ker \phi^*$. We show that $\phi^*(Df) = 0$ or, equivalently, that $Df \in \mathfrak{m}_{\phi(x)}$ for all $x \in X$. Fix x and let $\ell \in \mathbb{T}_x X$ be such that $D_{\phi(x)} = \phi_x(\ell)$. Then

$$(Df)(\phi(x)) = \ell(\phi^* f) = \ell(0) = 0,$$

as needed.

Conversely, let $x \in X$ and let $\mathfrak{n}_x := \mathfrak{m}_x \cap \phi^*(S(\mathfrak{g}))$. If $\ker \phi^*$ is D -stable, then D induces a derivation \bar{D} on $\phi^*(S(\mathfrak{g}))$, defined by

$$\bar{D}(\phi^* f) = \phi^*(Df),$$

and, thus, defines an element $\bar{D}_x \in (\mathfrak{n}_x/\mathfrak{n}_x^2)^*$. Let $\ell \in \mathbb{T}_x X$ be any extension of \bar{D}_x to $\mathfrak{m}_x/\mathfrak{m}_x^2$. Then for $f \in \mathfrak{m}_{\phi(x)}$ we have

$$D_{\phi(x)}(f) = \text{ev}_{\phi(x)}(Df) = \phi^*(Df)(x) = \bar{D}_x \phi^* f = \ell \phi^* f,$$

showing that $D_{\phi(x)} \in \phi_x(\mathbb{T}_x X)$. □

2.2 Some pseudo-orbits in W^* and $W_{\geq -1}^*$

This subsection is effectively an extended example, where we use the methods of the previous subsection, particularly Lemma 2.1.4, to compute the Poisson cores of some particular functions in W^* and $W_{\geq -1}^*$. We show later (Proposition 4.3.14) that these give all of the prime Poisson ideals of $S(W)$ and $S(W_{\geq -1})$ of co-GK-dimension two.

Throughout the subsection fix $x, \alpha, \gamma \in \mathbb{k}$ with $x \neq 0$ and α, γ not both zero. Let $\chi := \chi_{x;\alpha,\gamma} \in W^*$ be defined by

$$f\partial \mapsto \alpha f(x) + \gamma f'(x).$$

Further, given $g \in \mathbb{k}[t, t^{-1}]$ let $W(g) := \mathbb{k}[t, t^{-1}]g\partial \subseteq W$. Both sets of notation are generalised and used more extensively in § 3.

We first compute the isotropy subalgebra of χ .

LEMMA 2.2.1. *We have*

$$W^\chi = \begin{cases} W((t-x)^2) & \gamma = 0, \\ \{g\partial \mid g(x) = \alpha g'(x) + \gamma g''(x) = 0\} & \gamma \neq 0. \end{cases}$$

Proof. Recall that χ defines a bilinear form B_χ on W by $B_\chi(v, w) = \chi([v, w])$, and that $W^\chi = \ker B_\chi$. For all χ , that is, for all choices of x, α, γ , therefore, $W^\chi \neq W$.

First assume that $\gamma = 0$. If $g\partial \in W((t-x)^2)$, then

$$\chi([g\partial, f\partial]) = \alpha(g(x)f'(x) - f(x)g'(x)) = 0,$$

so $W((t-x)^2) \subseteq W^\chi$. However, B_χ defines a nondegenerate bilinear form on W/W^χ , so $\dim W/W^\chi \geq 2$ and, thus, $W((t-x)^2) = W^\chi$.

Now suppose $\gamma \neq 0$. Let

$$V := \{g \in \mathbb{k}[t, t^{-1}] \mid g(x) = \chi(g'\partial) = 0\},$$

which is a codimension-two subspace of $\mathbb{k}[t, t^{-1}]$ and let $g \in V, f \in \mathbb{k}[t, t^{-1}]$. Then

$$\chi([f\partial, g\partial]) = \alpha(f(x)g'(x) - f'(x)g(x)) + \gamma(f(x)g''(x) - f''(x)g(x)) = f(x)\chi(g'\partial) - g(x)\chi(f'\partial),$$

which is zero by assumption on g . Thus, $W^\chi \supseteq V\partial$ and, as before, the two must be equal. \square

Note that in all cases $W^\chi \supseteq W((t-x)^3)$ and that if $\lambda \neq 0$, then $W^\chi = W^{\chi_{x;\lambda\alpha,\lambda\gamma}}$.

We now compute $P(\chi)$. Let $B = \mathbb{k}[t, t^{-1}, y]$, and define a Poisson bracket on B induced from defining $\{y, t\} = 1$.

Define $p_\gamma : S(W) \rightarrow B$ as the algebra homomorphism induced by defining

$$p_\gamma(f\partial) = fy + \gamma f'. \tag{2.2.2}$$

We verify

$$\{p_\gamma(f\partial), p_\gamma(g\partial)\} = \{fy + \gamma f', gy + \gamma g'\} = y(fg' - f'g) + \gamma(fg'' - f''g) = p_\gamma((fg' - f'g)\partial).$$

Thus, p_γ respects Poisson brackets, so $\ker p_\gamma$ is a Poisson ideal of $S(W)$.

LEMMA 2.2.3. *The Poisson core of χ is equal to $\ker p_\gamma$ and, in particular, depends only on γ as long as $(\alpha, \gamma) \neq (0, 0)$.*

Proof. First, $\chi(f\partial) = p_\gamma(f\partial)|_{t=x, y=\alpha}$, and it follows immediately that if we extend χ to a homomorphism $\text{ev}_\chi : S(W) \rightarrow \mathbb{k}$, it factors through p_γ . As a result, $\text{ev}_\chi(\ker p_\gamma) = 0$. As, in all cases, W^χ has codimension $2 = \text{GKdim}(B)$ by Lemma 2.2.1, the result is a direct consequence of Lemma 2.1.4. \square

Remark 2.2.4. Let $x, \alpha, \gamma \in \mathbb{k}$ with α, γ not both zero, and define $\nu \in W_{\geq -1}^*$ analogously to χ : that is, $\nu(f\partial) = \alpha f(x) + \gamma f'(x)$. One may similarly prove that $P(\nu) = S(W_{\geq -1}) \cap \ker p_\gamma$ and, in particular, that

$$P(\chi|_{W_{\geq -1}}) = P(\chi) \cap S(W_{\geq -1}).$$

We show in Proposition 4.3.6 that this is true for all elements of W^* . Likewise, $P(\chi|_{W_{\geq 1}}) = P(\chi) \cap S(W_{\geq 1})$.

2.3 Pseudo-orbits versus orbits

We wish to relate the pseudo-orbits from § 2.1 to orbits of an algebraic group acting on an appropriate algebraic variety X . The next result gives us a general technique to do this.

PROPOSITION 2.3.1. *Let \mathfrak{g} be a Lie algebra with $\dim \mathfrak{g} < |\mathbb{k}|$. Let X be an irreducible affine algebraic variety acted on by a connected algebraic group H with Lie algebra \mathfrak{h} and let $U \subseteq X$ be an open affine subset. Fix a morphism of varieties $\phi : U \rightarrow \mathfrak{g}^*$. Assume that for every $x \in U$ we have $\mathfrak{g}_{\phi(x)} \subseteq \phi_x(\mathfrak{h}_x)$.*

- (a) *For all $x \in U$ the pseudo-orbit of $\phi(x)$ is contained in $\overline{\phi(Hx \cap U)}$, recalling that the topology on \mathfrak{g}^* is the Zariski topology.*
- (b) *For $x \in U$, let $I^U(Hx)$ be the defining ideal in $\mathbb{k}[U]$ of $\overline{Hx} \cap U$. Let $x \in U$ be such that $\dim \mathfrak{g}_{\phi(x)} = \dim \mathfrak{h}_x$. Then $P(\phi(x))$ is equal to the kernel of the induced homomorphism*

$$S(\mathfrak{g}) \rightarrow \mathbb{k}[U]/I^U(Hx) = \mathbb{k}[\overline{Hx} \cap U], \tag{2.3.2}$$

and

$$\dim \mathbb{O}(\phi(x)) = \text{GKdim } S(\mathfrak{g})/P(\phi(x)) = \dim \mathfrak{g}_{\phi(x)}. \tag{2.3.3}$$

In particular, if $y \in U$ is such that $Hx = Hy$ and $\dim \mathfrak{g}_{\phi(y)} = \dim \mathfrak{h}_y$, then $P(\phi(x)) = P(\phi(y))$.

- (c) Let $x, y \in U$ be such that $\dim \mathfrak{g}_{\phi(x)} = \dim \mathfrak{h}_x$ and $\dim \mathfrak{g}_{\phi(y)} = \dim \mathfrak{h}_y$. Then $P(\phi(x)) = P(\phi(y))$ if and only if there are open subsets $U_x \subseteq Hx \cap U$ and $U_y \subseteq Hy \cap U$ such that $\phi(U_x) = \phi(U_y)$.

Proof. (a) The kernel of (2.3.2) is Poisson by Proposition 2.1.7 and is contained in $\mathfrak{m}_{\phi(x)}$ by definition. Thus, it is contained in $P(\phi(x))$, which is what we need. Note also that this statement is completely trivial if \mathfrak{g} is the Lie algebra of a finite-dimensional algebraic group.

(b) Let K be the kernel of (2.3.2). Then K is Poisson by part (a). We have

$$\text{GKdim } S(\mathfrak{g})/K \leq \dim Hx = \dim \mathfrak{h}_x = \text{rk } B_{\phi(x)},$$

so $K = P(\phi(x))$ by Lemma 2.1.4. Certainly

$$\text{GKdim } S(\mathfrak{g})/P(\phi(x)) \leq \text{GKdim } \mathbb{k}[U]/I^U(Hx) = \dim \mathfrak{h}_x = \dim \mathfrak{g}_{\phi(x)},$$

and the two are equal by Lemma 2.1.4. The final statement follows from Corollary 2.1.6.

(c) For any dense subset $U_x \subseteq Hx \cap U$, by part (b) $P(\phi(x))$ is equal to the kernel of the induced map $S(\mathfrak{g}) \rightarrow \mathbb{k}[U_x]$. This is determined by $\phi(U_x)$, so if $\phi(U_x) = \phi(U_y)$, then $P(\phi(x)) = P(\phi(y))$.

Suppose now that $P(\phi(x)) = P(\phi(y))$. Consider the induced maps $Hx \rightarrow \mathfrak{g}^*, Hy \rightarrow \mathfrak{g}^*$ and the respective fibre product $(Hx) \times_{\mathfrak{g}^*} (Hy)$. Note that $(Hx) \times_{\mathfrak{g}^*} (Hy)$ is a closed subset of $(Hx) \times (Hy)$ and, hence, the ideal defining $(Hx) \times_{\mathfrak{g}^*} (Hy)$ is generated by a finite collection of elements. This implies that there is a finite-dimensional subspace $V \subset \mathfrak{g}$ such that

$$(Hx) \times_{\mathfrak{g}^*} (Hy) = (Hx) \times_{V^*} (Hy).$$

As $P(\phi(x)) = P(\phi(y))$, the images of Hx and Hy in V^* have the same closure in V^* ; call it Z . The image of Hx in V^* may not be open in Z , but it is constructible and, thus, contains an open subset $U(x)$ of Z . Likewise the image of Hy in V^* contains an open subset $U(y)$ of Z . We pick U_x to be the preimage of $U(x) \cap U(y)$ in Hx and let U_y be the preimage of $U(x) \cap U(y)$ in Hy ; then they satisfy the desired property. \square

3. Primitive ideals and local functions

We now specialise to let \mathfrak{g} be one of $W, W_{\geq 1}, W_{\geq -1}$ or Vir . The main goal of this section is to determine which functions $\chi \in \mathfrak{g}^*$ have nontrivial Poisson core $P(\chi)$. We show that these χ are precisely those χ which measure the local behaviour of $f\partial \in \mathfrak{g}$ at a finite collection of points; we call these functions *local*. (For example, the functions $\chi \in W^*$ of § 2.2 are local.) We provide several equivalent characterisations of local functions and apply these to classify subalgebras of \mathfrak{g} of finite codimension. As a consequence, we show that $S(Vir)/(z - \zeta)$ is Poisson simple for $\zeta \neq 0$.

More formally, consider $\mathfrak{g} = W_{\geq -1}$. Let $x, \alpha_0, \dots, \alpha_n \in \mathbb{k}$ with $(\alpha_0, \dots, \alpha_n) \neq \vec{0}$. Define a linear function $\chi_{x;\alpha_0, \dots, \alpha_n} \in W_{\geq -1}^*$ by

$$\chi_{x;\alpha_0, \dots, \alpha_n} : f\partial \mapsto \alpha_0 f(x) + \alpha_1 f'(x) + \dots + \alpha_n f^{(n)}(x). \tag{3.0.1}$$

The same formula defines elements of $W_{\geq 1}^*$ and W^* , although in the last case we need to require that $x \neq 0$.

We now formally define local functions.

DEFINITION 3.0.2.

- (a) A local function on $W_{\geq -1}$ or $W_{\geq 1}$ is a sum of finitely many functions of the form (3.0.1) with (possibly) distinct x .
- (b) A local function on W is a sum of finitely many functions of the form (3.0.1) with (possibly) distinct $x \neq 0$.
- (c) A local function on Vir is the pullback of a local function on W via the canonical map $Vir \rightarrow W$.

A local function of the form (3.0.1) is called a *one-point local function*. Let $\chi = \chi_{x;\alpha_0,\dots,\alpha_n}$ be a one-point local function. We say that $\{x\}$ is the *support* of χ and that x is the *base point* of χ . If $\alpha_n \neq 0$, we say that n is the *order* of χ .

Let χ be an arbitrary local function. The *support* of χ is the union of the supports of the component one-point local functions. Further, the orders of the component one-point local functions give rise to a partition $\lambda(\chi)$. More explicitly, write $\chi = \chi_1 + \dots + \chi_r$, where the χ_i are one-point local functions based at distinct points. Let m_i be the order of χ_i . By reordering the χ_i if necessary, we may assume that $m_1 \geq m_2 \geq \dots \geq m_r$. The partition

$$\lambda(\chi) := (m_1 + 1, \dots, m_r + 1)$$

is called the *order partition* of χ . (We add 1 here so that the partition (0) corresponds to the zero function.) We call m_1 the *order* of χ .

It follows from the Chinese remainder theorem that a local function is zero if and only if it is zero pointwise and, thus, any local function χ on W , $W_{\geq -1}$ or Vir has a unique presentation as a sum of nonzero one-point local functions with distinct base points. This also shows that the partition $\lambda(\chi)$ and the order of χ are well-defined. For $W_{\geq 1}$ it is easy to see that $\chi_{0;1} = \chi_{0;0,1} = 0$ and the presentation is unique under the assumption that the coefficients of $f(0)$ and $f'(0)$ are zero.

Remark 3.0.3. Let \mathfrak{g} be Vir , W , $W_{\geq -1}$ or $W_{\geq 1}$. Then \mathfrak{g}^* and the subspace of local functions are both uncountable-dimensional vector spaces; for local functions observe that any set of one-point local functions with distinct base points is linearly independent. On the other hand, clearly ‘most’ elements of \mathfrak{g}^* are not local. In fact, we show in Remark 5.2.3 that local functions are parameterised by a countable union of algebraic varieties.

For a specific example of a nonlocal function, let $\alpha_0, \alpha_1, \dots \in \mathbb{k}$ be algebraically independent over \mathbb{Q} , and define $\varkappa \in W_{\geq -1}^*$ by $\varkappa(t^i \partial) = \alpha_i$.

As local functions are defined similarly for W , $W_{\geq 1}$, $W_{\geq -1}$ and Vir we sometimes discuss all of them simultaneously. When we do so, we assume without comment whenever we talk about a one-point local function on W or Vir that $x \neq 0$.

Pick $\chi \in W^*$ or $W_{\geq 1}^*$ or $W_{\geq -1}^*$. In this section, we show that $P(\chi)$ is nonzero if and only if χ is local and prove a similar statement for Vir^* . The starting point here is the following result, due in its strongest form to Iyudu and the second author.

THEOREM 3.0.4. *Let \mathfrak{g} be W , $W_{\geq 1}$ or $W_{\geq -1}$ and let I be a nonzero Poisson ideal of $S(\mathfrak{g})$. Then*

$$\text{GKdim}(S(\mathfrak{g})/I) < \infty.$$

In particular, if $\chi \in \mathfrak{g}^$ is such that $(0) \neq P(\chi)$, then $\text{GKdim}(S(\mathfrak{g})/P(\chi)) < \infty$.*

Further, if $\chi \in Vir^$ is such that $P(\chi) \neq (z - \chi(z))$, then*

$$\text{GKdim}(S(Vir)/P(\chi)) < \infty.$$

Proof. See [PS20, Theorem 1.4] for $W_{\geq 1}$ and [IS20, Theorem 1.3] for all other \mathfrak{g} . □

Theorem 3.0.4 has the following extremely useful consequence.

COROLLARY 3.0.5. *Let \mathfrak{g} be as before and let $\chi \in \mathfrak{g}^*$. If $\mathfrak{g} = W, W_{\geq -1}$ or $W_{\geq 1}$, assume that $P(\chi) \neq (0)$, and if $\mathfrak{g} = Vir$, assume that $P(\chi) \neq (z - \chi(z))$. Then $\dim \mathfrak{g} \cdot \chi < \infty$.*

Proof. Combine Theorem 3.0.4 and Lemma 2.1.1. □

We also recall the following result.

PROPOSITION 3.0.6. *[LS08, Corollary 5.1] Let \mathfrak{g} be $Vir, W, W_{\geq -1}$ or $W_{\geq 1}$. Then $S(\mathfrak{g})$ satisfies the ascending chain condition on radical Poisson ideals and every Poisson ideal has finitely many minimal primes above it, each of which is Poisson.*

It is not known for any of these Lie algebras whether $S(\mathfrak{g})$ satisfies the ascending chain condition on arbitrary Poisson ideals.

Although we use similar notation for $W_{\geq -1}, W_{\geq 1}, W$ and Vir , the details are slightly different, so we analyse local functions in each of these cases separately.

3.1 Local functions on $W_{\geq -1}$ and $W_{\geq 1}$

In this subsection we set $\mathfrak{g} = W_{\geq -1} = \mathbb{k}[t]\partial$. It is useful to consider Lie subalgebras of $W_{\geq -1}$ of a particular form. For any $f \in \mathbb{k}[x] \setminus \{0\}$ denote by $W_{\geq -1}(f)$ the space of vector fields of the form

$$\{gf\partial\}_{g \in \mathbb{k}[x]}.$$

In other words, $W_{\geq -1}(f) = fW_{\geq -1}$ under the obvious notation. It is clear that $W_{\geq -1}(f)$ is a Lie subalgebra of $W_{\geq -1} = W_{\geq -1}(1)$.

We give five equivalent conditions for local functions. Similar conditions will hold for the other Lie algebras we consider, see Theorems 3.2.1 and 3.3.1.

THEOREM 3.1.1. *Let $f \in \mathbb{k}[t] \setminus \{0\}$ and $\chi \in W_{\geq -1}(f)^*$. Then the following conditions are equivalent:*

- (1) χ is the restriction of a local function on $W_{\geq -1}$;
- (2) $P(\chi) \neq (0)$;
- (3) $\dim W_{\geq -1}(f)/W_{\geq -1}(f)^\chi = \dim W_{\geq -1}(f) \cdot \chi < \infty$;
- (4) there exists $h \in \mathbb{k}[t]$ such that $\chi|_{W_{\geq -1}(fh)} = 0$;
- (5) $W_{\geq -1}(f)^\chi \neq (0)$.

Remark 3.1.2. Fix a basis $\{ft^i\partial \mid i \geq -1\}$ of $W_{\geq -1}(f)$ and consider a local function $\chi \in W_{\geq -1}(f)^*$. Then χ can be identified with a sequence

$$\chi_0 = \chi(f\partial), \chi_1 = \chi(ft\partial), \chi_2 = \chi(ft^2\partial), \chi_3 = \chi(ft^3\partial), \dots \in \mathbb{k}. \tag{3.1.3}$$

Condition (3) can be restated as follows:

$$a_n\chi_{m+n} + \dots + a_0\chi_m = 0$$

for all $m \geq 0$, where $h = a_nt^n + \dots + a_0$. Therefore, local functions on $W_{\geq -1}(f)$ can be identified with sequences (3.1.3) obeying a linear recurrence relation. This shows, in particular, that the function \varkappa defined in Remark 3.0.3 is not local.

Part of the proof of Theorem 3.1.1 is a general technique that can allow us to show that Poisson cores of elements of \mathfrak{g}^* are nontrivial for any Lie algebra \mathfrak{g} .

LEMMA 3.1.4. Let \mathfrak{g} be an arbitrary Lie algebra and let $n \in \mathbb{Z}_{\geq 0}$. There is a Poisson ideal $I(n)$ with the property that

$$I(n) \subseteq \mathfrak{m}_\chi \iff \dim \mathfrak{g} \cdot \chi \leq n \tag{3.1.5}$$

for any $\chi \in \mathfrak{g}^*$.

Proof. Recall the determinant $D(u_1, \dots, u_n; v_1, \dots, v_n)$ from (2.1.2), and note that $\dim \mathfrak{g} \cdot \chi \leq n - 1$ if and only if for all $u_1, \dots, u_n, v_1, \dots, v_n \in \mathfrak{g}$, (2.1.2) evaluated at χ is degenerate, i.e.

$$\text{ev}_\chi(D(u_1, u_2, \dots, u_n; v_1, v_2, \dots, v_n)) = 0.$$

Let $I(n)$ be the ideal generated by the $D(u_1, u_2, \dots, u_{n+1}; v_1, v_2, \dots, v_{n+1})$ for all possible tuples

$$u_1, \dots, u_{n+1}, v_1, \dots, v_{n+1} \in \mathfrak{g}.$$

By the previous paragraph,

$$I(n) \subseteq \mathfrak{m}_\chi \iff \dim \mathfrak{g} \cdot \chi \leq n.$$

Let $w \in \mathfrak{g}$. It is easy to check that

$$\begin{aligned} & \{D(u_1, u_2, \dots; v_1, v_2, \dots), w\} \\ &= D([u_1, w], u_2, u_3, \dots; v_1, v_2, \dots) + D(u_1, [u_2, w], u_3, \dots; v_1, v_2, \dots) + \dots \\ &+ D(u_1, u_2, \dots; [v_1, w], v_2, v_3, \dots) + D(u_1, u_2, \dots; v_1, [v_2, w], v_3, \dots) + \dots \in I(n). \end{aligned}$$

It follows that $I(n)$ is Poisson. □

Proof of Theorem 3.1.1. That (0) \iff (3) is a straightforward application of the Chinese remainder theorem. It is clear that (2) \implies (4); Corollary 3.0.5 gives that (1) \implies (2). We show that (4) \implies (3) \implies (2) \implies (1). This will complete the proof.

We first show that condition (4) implies condition (3). Let $hf\partial \in W_{\geq -1}(f)^\times \setminus \{0\}$ with $h \in \mathbb{k}[t]$. Then

$$0 = \chi([hf\partial, hfr\partial]) = \chi(h^2 f^2 r' \partial) \tag{3.1.6}$$

for all $r \in \mathbb{k}[t]$. This is equivalent to $\chi|_{W_{\geq -1}(h^2 f^2)} = 0$ as needed.

Next, we show that condition (3) implies condition (2). Let h satisfy condition (3). As

$$[W_{\geq -1}(f^2 h^2), W_{\geq -1}] \subseteq W_{\geq -1}(fh),$$

we have $W_{\geq -1}(h^2 f^2) \subseteq W_{\geq -1}(f)^\times$ and, thus, $\dim W(f)/W(f)^\times < \infty$ as needed.

Finally we show that condition (2) implies condition (1). Suppose that $\chi \in W_{\geq -1}(f)$ satisfies condition (2). Let $n = \dim W_{\geq -1}(f)/W_{\geq -1}(f)^\times = \dim W_{\geq -1}(f) \cdot \chi$ and let $I(n)$ be the ideal defined in Lemma 3.1.4. By that lemma, $I(n)$ is Poisson and $I(n) \subseteq \mathfrak{m}_\chi$, so $I(n) \subseteq P(\chi)$.

Therefore, if $I(n) \neq 0$, then $P(\chi) \neq 0$. To show that $I(n) \neq 0$ it suffices to find $\omega \in W_{\geq -1}(f)^*$ with $\dim W_{\geq -1}(f) \cdot \omega > n$. In fact, we find $\omega \in W_{\geq -1}^*$ with

$$\dim W_{\geq -1}(f) \cdot \omega = \infty.$$

Indeed, as (2) \implies (3), if $\dim W_{\geq -1}(f) \cdot \omega < \infty$, then ω can be represented by a linearly recurrent sequence by Remark 3.1.2. On the other hand, the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots$ is clearly not linearly recurrent. □

Remark 3.1.7. Similarly to Remarks 3.0.3 and 5.2.3, we should expect that ‘most’ sequences are not linearly recurrent and that the linearly recurrent sequences are parameterised by a countable union of affine varieties, although we do not formalise these notions here.

Note that $W_{\geq -1}(t^2) = W_{\geq 1}$ and that $W_{\geq -1}(t)$ is equal to the *nonnegative Witt algebra* $W_{\geq 0}$. Therefore, Theorem 3.1.1 gives a complete characterisation of local functions on $W_{\geq 1}$ and $W_{\geq 0}$.

3.2 Local functions on W and applications

In this subsection we set $\mathfrak{g} = \mathbb{k}[t, t^{-1}]\partial$ and define $W(f)$ similarly to $W_{\geq 1}(f)$. A partial analogue of Theorem 3.1.1 holds for W .

THEOREM 3.2.1. *For any $f \in \mathbb{k}[t] \setminus \{0\}$ and $\chi \in W(f)^*$ the following conditions are equivalent:*

- (1) χ is the restriction of a local function on W ;
- (2) $P(\chi) \neq (0)$;
- (3) $\dim W(f)/W(f)^\chi < \infty$;
- (4) there exists $h \in \mathbb{k}[x]$ such that $\chi|_{W(fh)} = 0$.

Remark 3.2.2. The reason that Theorem 3.2.1 differs slightly from Theorem 3.1.1 is that the function $\text{Res}_0(\cdot) \in W^*$ satisfies condition (4) of Theorem 3.1.1 but does not satisfy the other conditions (0), (1), (2), and (3).

Before proving Theorem 3.2.1 we give two lemmata on functions defined by residues. Denote by $\mathbb{k}((t))$ the field of formal Laurent power series in t . Fix $f \in \mathbb{k}((t))$ and consider the map

$$(a, b) \mapsto (a, b)_f := \text{Res}_0(f(ab' - a'b)), \tag{3.2.3}$$

which defines a skew-symmetric bilinear form on $\mathbb{k}((t))$.

LEMMA 3.2.4. *The kernel of $(a, b)_f$ is one-dimensional if f is a perfect square in $\mathbb{k}((t))$ and is trivial otherwise. In the first case, the kernel is generated by $1/\sqrt{f}$.*

Proof. Let a be in the kernel of $(\cdot, \cdot)_f$. Then

$$\text{Res}_0(fa^2r') = \text{Res}_0(f(a(ar)') - a'(ar)) = (a, ar)_f = 0 \tag{3.2.5}$$

for all $r \in \mathbb{k}((t))$. This implies that fa^2 is constant. Thus, f is a perfect square in $\mathbb{k}((t))$ and a is proportional to $1/\sqrt{f}$. Equation (3.2.5) also gives that if f is a perfect square, then $1/\sqrt{f}$ belongs to the kernel of $(\cdot, \cdot)_f$. \square

The second lemma proves part of (a more general version of) Remark 3.2.2.

LEMMA 3.2.6. *Let $g \in \mathbb{k}[t, t^{-1}] \setminus \{0\}$ and define $\omega \in W(g^2)^*$ by*

$$\omega = \text{Res}_0\left(\frac{\cdot}{g^2}\right).$$

Then $\dim W(g^2) \cdot \omega = \infty$.

Proof. For $a, b \in \mathbb{k}[t, t^{-1}]$ we have

$$B_\omega(g^2a\partial, g^2b\partial) = \text{Res}_0(g^2(ab' - a'b)).$$

Suppose that $a \notin \mathbb{k} \cdot 1/g$. Then by Lemma 3.2.4 there is a formal Laurent series \hat{b} so that

$$(a, \hat{b})_{g^2} = \text{Res}_0(g^2(a\hat{b}' - a'\hat{b})) \neq 0.$$

However, the computation of $\text{Res}_0(g^2(a\hat{b}' - a'\hat{b}))$ needs only finitely many terms in the Laurent expansion of \hat{b} and so we may replace \hat{b} by $b \in \mathbb{k}[t, t^{-1}]$ so that $\text{Res}_0(g^2(ab' - a'b)) \neq 0$. Thus $ag^2\partial \notin \ker B_\omega = W(g^2)^\omega$. This means that $W(g^2)^\omega \subseteq \mathbb{k} \cdot g\partial$ and $\dim W(g^2) \cdot \omega = \infty$. \square

Proof of Theorem 3.2.1. The proofs of (0) \iff (3), (1) \implies (2) and (3) \implies (2) \implies (1) are very similar to the corresponding steps of the proof of Theorem 3.1.1. The only part which is significantly different is (2) \implies (3).

Pick χ satisfying condition (2) and $h \in W(f)^{\times} \setminus \{0\}$. Then (3.1.6) holds for all $r \in \mathbb{k}[t, t^{-1}]$. Unfortunately, this is not enough to show that χ vanishes on $W(h^2 f^2)$, as the map $r \mapsto r'$ is not surjective on $\mathbb{k}[t, t^{-1}]$.

Consider $\chi|_{W(f^2 h^2)}$: we have

$$\chi(f^2 h^2 p \partial) = \chi\left(\frac{f^2 h^2}{t} \partial\right) \text{Res}_0(p)$$

for all $p \in \mathbb{k}[t, t^{-1}]$. Suppose that $\chi((f^2 h^2/t)\partial) \neq 0$. Then $\dim W(f^2 h^2) \cdot \chi|_{W(f^2 h^2)} = \infty$ by Lemma 3.2.6. Thus, $\dim W(h^2 f^2) \cdot \chi = \infty$, as $\dim W(f)/W(f^2 h^2) < \infty$. This contradicts our assumption that χ satisfies condition (2) and so $\chi((f^2 h^2/t)\partial) = 0$, i.e. $\chi|_{W(f^2 h^2)} = 0$. \square

To end the subsection, we apply Theorem 3.2.1 to obtain a structure result on finite codimension subalgebras of W . For a polynomial $f \in \mathbb{k}[t, t^{-1}]$ set

$$\text{rad}(f) := \prod \{(t - x) \mid x \in \mathbb{k}^{\times}, f(x) = 0\}.$$

PROPOSITION 3.2.7. *Let \mathfrak{k} be a subalgebra of W of finite codimension. Then:*

- (a) *there exists $f \in \mathbb{k}[t, t^{-1}]$ so that $W(\text{rad}(f)) \supseteq \mathfrak{k} \supseteq W(f)$;*
- (b) *we can choose f satisfying (a) so that $f \in \mathbb{k}[t]$, f is monic and $f(0) \neq 0$;*
- (c) *if we assume that f is of minimal degree then such a choice of f is unique.*

Proof. The inclusion $\mathfrak{k} \subseteq W$ induces the dual map $W^* \rightarrow \mathfrak{k}^*$. We identify $(W/\mathfrak{k})^*$ with the kernel of this map; that is, with elements of W^* which vanish on \mathfrak{k} , so \mathfrak{k} is the set of common zeros of $(W/\mathfrak{k})^* \subseteq W^*$. Let χ_1, \dots, χ_s be a basis of $(W/\mathfrak{k})^*$. Fix $i \in \{1, \dots, s\}$; by definition, we have $B_{\chi_i}(\mathfrak{k}, \mathfrak{k}) = 0$, so \mathfrak{k} is an isotropic subspace of W with respect to B_{χ_i} . Hence, the rank of B_{χ_i} is at most $2 \dim(W/\mathfrak{k})$ and, thus, is finite. By Theorem 3.2.1, χ_i is local.

Theorem 3.2.1 implies that for all i there is $h_i \in \mathbb{k}[t, t^{-1}] \setminus \{0\}$ with $\chi_i(W(h_i)) = 0$. Therefore $W(h_1 \cdots h_s)$ is annihilated by all χ_i and therefore $W(h_1 \cdots h_s) \subseteq \mathfrak{k}$ as desired.

Let $f \in \mathbb{k}[t, t^{-1}] \setminus \{0\}$ with $\mathfrak{k} \supseteq W(f)$; we may assume without loss of generality that $f \in \mathbb{k}[t]$, f is monic and $f(0) \neq 0$ as $W(f)$ corresponds to an ideal of $\mathbb{k}[t, t^{-1}]$. Suppose, in addition, that f has minimal degree among all such polynomials with $\mathfrak{k} \supseteq W(f)$. Thus, if $\mathfrak{k} \supseteq W(h)$, then by the Euclidean algorithm $f|h$. This justifies uniqueness of f .

Write $f = \prod_{i=1}^n (t - x_i)^{a_i}$ with the $x_i \neq 0$ distinct and $a_i > 0$; set $h := \text{rad}(f) = (t - x_1) \cdots (t - x_n)$. It is clear that $h \mid f \mid h^{\max(a_i)}$. Let $k\partial \in \mathfrak{k}$. We wish to show that $h \mid k$. Indeed, for all $r \in \mathbb{k}[t, t^{-1}]$ the element $[k\partial, fr\partial] = (k(fr' + f'r) - frk')\partial$ is in \mathfrak{k} . As $\mathfrak{k} \supseteq W(f)$, thus $kf'r \in \mathfrak{k}$ for all $r \in \mathbb{k}[t, t^{-1}]$. Thus, $\mathfrak{k} \supseteq W(kf')$ and so $f \mid f'k$. This forces k to vanish at all roots of f , which is equivalent to $h \mid k$. \square

3.3 Local functions on *Vir*

In this subsection we set $\mathfrak{g} = \text{Vir}$.

The natural map $\text{Vir} \rightarrow W$ extends to the morphism $S(\text{Vir}) \rightarrow S(W)$; the kernel is the Poisson ideal (z) of $S(\text{Vir})$. The main goal of the subsection is to prove the following analogue of Theorem 3.2.1 for *Vir*.

THEOREM 3.3.1. For $\chi \in \text{Vir}^*$ we have

$$P(\chi) \neq (z - \chi(z)) \iff \chi \text{ is local} \iff \dim \text{Vir}/\text{Vir}^\chi < \infty.$$

In particular, if $P(\chi) \neq (z - \chi(z))$, then $\chi(z) = 0$.

Remark 3.3.2. It follows from Theorems 3.2.1 and 3.3.1 that $\chi \in \text{Vir}^*$ is local if and only if there is some $f \in \mathbb{k}[t, t^{-1}]$ such that χ vanishes on $W(f) + \mathbb{k} \cdot z \subseteq \text{Vir}$.

Before proving Theorem 3.3.1, we consider arbitrary subalgebras of Vir of finite codimension and show they are strongly constrained.

PROPOSITION 3.3.3. If \mathfrak{k} is a subalgebra of Vir of finite codimension, then $z \in [\mathfrak{k}, \mathfrak{k}]$. Thus, $\dim \text{Vir}/\mathfrak{k} < \infty$ if and only if \mathfrak{k} contains some $W(f) + \mathbb{k} \cdot z$, where $f \in \mathbb{k}[t, t^{-1}] \setminus \{0\}$.

This result generalises [OW18, Proposition 2.3], which considered subalgebras of Vir of codimension one. We also note that [OW18] refers to subalgebras of Vir of the form $W(f) + \mathbb{k} \cdot z$ as polynomial subalgebras.

Proof of Proposition 3.3.3. Let $\bar{\mathfrak{k}}$ be the image of \mathfrak{k} in W . By Proposition 3.2.7 there is some $f \in \mathbb{k}[t, t^{-1}] \setminus \{0\}$ so that $\bar{\mathfrak{k}} \supseteq W(f)$. Thus, for all $p \in \mathbb{Z}$, there is $\zeta_p \in \mathbb{k}$ so that the element

$$v_p := ft^p\partial + \zeta_p z$$

is in \mathfrak{k} . Therefore, $[\mathfrak{k}, \mathfrak{k}]$ contains the elements

$$\frac{1}{q-p}[v_p, v_q] = f^2 t^{p+q-1} \partial + \text{Res}_0(t^{p+q-3}(2t^2(f')^2 + ff't(p+q-1) - t^2 ff'' + f^2 pq))z \quad (3.3.4)$$

for all $p, q \in \mathbb{Z}$. Fix $d = p + q$ and consider $p = d - q$ as a function of q . The only part of (3.3.4) that varies with q is $q(d - q) \text{Res}_0(t^{d-3} f^2)$.

If $\text{Res}_0(t^{d-3} f^2)$ is not zero, then

$$\frac{1}{d - 2q_1}[v_{d-q_1}, v_{q_1}] - \frac{1}{d - 2q_2}[v_{d-q_2}, v_{q_2}] = (q_1 - q_2)(d - q_1 - q_2) \text{Res}_0(t^{d-3} f^2)z$$

is a nonzero scalar multiple of z for almost all q_1, q_2 . If $z \notin [\mathfrak{k}, \mathfrak{k}]$, we therefore have $\text{Res}_0(t^d f^2) = 0$ for all $d \in \mathbb{Z}$. This implies that $f^2 = 0$, contradicting our assumption on f .

The final sentence is an immediate consequence of Proposition 3.2.7. □

Proof of Theorem 3.3.1. Let $\chi \in \text{Vir}^*$. If χ is local, then by definition χ descends to a local function $\bar{\chi}$ on $\text{Vir}/(z) \cong W$. By Theorem 3.2.1, $P(\chi) \not\supseteq (z)$ and $\dim \text{Vir}/\text{Vir}^\chi = \dim W/W^{\bar{\chi}} < \infty$.

If $\dim \text{Vir}/\text{Vir}^\chi < \infty$, then by Proposition 3.3.3, $z \in [\text{Vir}^\chi, \text{Vir}^\chi]$ and so $\chi(z) = 0$, as χ vanishes, by definition, on $[\text{Vir}^\chi, \text{Vir}^\chi]$. We may, thus, factor out z and apply Theorem 3.2.1 again to conclude that χ is local.

Finally, suppose that $P(\chi) \neq (z - \chi(z))$. Then by Corollary 3.0.5 $\dim \text{Vir} \cdot \chi = \dim \text{Vir}/\text{Vir}^\chi < \infty$. □

As an immediate corollary of Theorem 3.3.1, we obtain a powerful result on Poisson ideals of $S(\text{Vir})$.

COROLLARY 3.3.5. If $\zeta \in \mathbb{k}^\times$, then $S(\text{Vir})/(z - \zeta)$ is Poisson simple, i.e. contains no nontrivial Poisson ideals.

Proof. Let $\zeta \in \mathbb{k}$. If $S(\text{Vir})/(z - \zeta)$ is not Poisson simple, then $(z - \zeta)$ is strictly contained in some proper Poisson ideal J of $S(\text{Vir})$. By the Nullstellensatz there is some $\chi \in \text{Vir}^*$ with $J \subseteq \mathfrak{m}_\chi$; thus, $\chi(z) = \zeta$ as $z - \zeta \in \mathfrak{m}_\chi$. Further $P(\chi) \supseteq J \not\supseteq (z - \zeta)$ and so by Theorem 3.3.1 we have $\zeta = 0$. □

We thus show that almost all prime Poisson ideals of $S(\text{Vir})$ contain z .

COROLLARY 3.3.6. *Let Q be a prime Poisson ideal of $S(\text{Vir})$. Then either:*

- (i) $Q = (0)$;
- (ii) $Q = (z - \zeta)$ for some $\zeta \in \mathbb{k}^\times$;
- (iii) $Q \supseteq (z)$.

Proof. By Corollary 3.3.5 and primeness of Q , it suffices to prove that if $Q \neq (0)$, then Q contains a nonzero element of $\mathbb{k}[z]$. Let $h \in Q \setminus \{0\}$; using primeness of Q we may assume that h is not a multiple of any element of $\mathbb{k}[z] \setminus \mathbb{k}$. Let $\chi \in \text{Vir}^*$ so that $Q \subseteq \mathfrak{m}_\chi$. As $h \in P(\chi)$, we see that $P(\chi) \neq (z - \chi(z))$. By Theorem 3.3.1 $P(\chi) \supseteq (z)$. Thus, applying Lemma 2.0.4(a),

$$Q = \bigcap \{P(\chi) \mid \text{ev}_\chi(Q) = 0\} \supseteq (z). \quad \square$$

Given Corollary 3.3.5 it is natural to conjecture as follows.

CONJECTURE 3.3.7. If $\zeta \neq 0$, then $U(\text{Vir})/(z - \zeta)$ is simple.

However, we as yet have no proof of Conjecture 3.3.7. Note that the obvious strategy of proof by taking the associated graded of an ideal $(z - \zeta) \subsetneq J \triangleleft U(\text{Vir})$ does not work, because in this case $\text{gr } J \ni z$ and Corollary 3.3.5 is not directly relevant.

4. Pseudo-orbits and Poisson primitive ideals for the algebras of interest

Let \mathfrak{g} be one of Vir , W , $W_{\geq -1}$ or $W_{\geq 1}$. In this section, we describe the pseudo-orbits for \mathfrak{g} , using the results on local functions from the previous section and the general strategy of Proposition 2.3.1, and derive some consequences for the Poisson primitive spectrum of $S(\mathfrak{g})$.

We begin by describing the pseudo-orbits of non-local functions, where the results of §3 quickly give the answer.

PROPOSITION 4.0.1. *For $\mathfrak{g} = W, W_{\geq -1}$ or $W_{\geq 1}$, the nonlocal functions in \mathfrak{g}^* form a pseudo-orbit. If $\mathfrak{g} = \text{Vir}$, then for any $\zeta \in \mathbb{k}$ the nonlocal functions χ with $\chi(z) = \zeta$ form a pseudo-orbit.*

Proof. This is immediate from Theorems 3.1.1 ($\mathfrak{g} = W_{\geq -1}$ or $W_{\geq 1}$), 3.2.1 ($\mathfrak{g} = W$) or 3.3.1 ($\mathfrak{g} = \text{Vir}$). By those results, if $\mathfrak{g} = W, W_{\geq -1}$ or $W_{\geq 1}$, then $\chi \in \mathfrak{g}^*$ is not local if and only if $P(\chi) = 0$; and if $\mathfrak{g} = \text{Vir}$, then χ is not local if and only if $P(\chi) = (z - \chi(z))$. \square

We may thus restrict to considering pseudo-orbits of local functions. By Proposition 4.0.1 if $\chi \in \mathfrak{g}^*$ is local and $\omega \in \mathcal{O}(\chi)$, then ω is also local. As, by definition, local functions on Vir vanish on z , the pseudo-orbits for W directly determine those for Vir .

Thus, for the rest of the section we let \mathfrak{g} be $W, W_{\geq -1}$ or $W_{\geq 1}$. In §4.1 we introduce a finite-dimensional action which determines the pseudo-orbits of one-point local functions on \mathfrak{g} , in §4.2 we describe the orbits of this action explicitly and in §4.3 we use this action to describe pseudo-orbits of arbitrary local functions.

4.1 An algebraic group acting on local functions

Set \mathfrak{g} to be $W, W_{\geq -1}$, or $W_{\geq 1}$. We fix notation for the subsection.

DEFINITION 4.1.1. For $x \in \mathbb{k}$, $n \in \mathbb{Z}_{\geq 0}$, let $\text{Loc}_x^{\leq n}$ denote the subspace of \mathfrak{g}^* consisting of one-point local functions based at x and of order $\leq n$. Let $\text{Loc}_x := \bigcup_{n \geq 0} \text{Loc}_x^{\leq n}$.

Define $Loc_{\mathfrak{g}}^{\leq n} = \bigcup_{x \in \mathbb{k}^\times} Loc_x^{\leq n}$ if $\mathfrak{g} = W$, and $Loc_{\mathfrak{g}}^{\leq n} = \bigcup_{x \in \mathbb{k}} Loc_x^{\leq n}$ if $\mathfrak{g} = W_{\geq -1}$ or $W_{\geq 1}$. Let

$$\widetilde{Loc}_{\mathfrak{g}}^{\leq n} = \begin{cases} \mathbb{k}^\times \times \mathbb{k}^{n+1} & \mathfrak{g} = W \\ \mathbb{k} \times \mathbb{k}^{n+1} & \mathfrak{g} = W_{\geq -1} \\ \mathbb{k} \times \mathbb{k}^{n+1} & \mathfrak{g} = W_{\geq 1}. \end{cases}$$

For all \mathfrak{g} there is a canonical map $\pi_{\mathfrak{g}}^{\leq n} : \widetilde{Loc}_{\mathfrak{g}}^{\leq n} \rightarrow Loc_{\mathfrak{g}}^{\leq n}$ (this is, in fact, birational, cf. Proposition 5.1.1). If $\mathfrak{g} = W, W_{\geq -1}$, then $\pi_{\mathfrak{g}}^{\leq n}$ is an isomorphism away from $\chi = 0$. If $\mathfrak{g} = W_{\geq 1}$ it is an isomorphism away from $\chi = 0$ and $x = 0$, due to the fact that $\chi_{0;\beta_0,\beta_1} = 0$ for all $\beta_0, \beta_1 \in \mathbb{k}$ in this case. Formally, we let $\phi_{\mathfrak{g}}^{\leq n} : \widetilde{Loc}_{\mathfrak{g}}^{\leq n} \rightarrow \mathfrak{g}^*$ be the composition of $\pi_{\mathfrak{g}}^{\leq n}$ with the inclusion $Loc_{\mathfrak{g}}^{\leq n} \subseteq \mathfrak{g}^*$. There is an induced pullback morphism $(\phi_{\mathfrak{g}}^{\leq n})^* : S(\mathfrak{g}) \rightarrow \mathbb{k}[\widetilde{Loc}_{\mathfrak{g}}^{\leq n}]$.

If χ is a local function on \mathfrak{g} and $v \in \mathfrak{g}$, then $v \cdot \chi$ is local and, in fact, the coadjoint action of \mathfrak{g} preserves Loc_x for all x . We now study the action of \mathfrak{g} on Loc_x , and relate it to a finite-dimensional action using Proposition 2.3.1.

We begin by defining a group action. Fix $x \in \mathbb{k}$ (if $\mathfrak{g} = W$ we assume that $x \neq 0$). Let $\tilde{t} := t - x$. Clearly, (3.0.1) makes sense for every formal power series $f\partial \in \mathbb{k}[[\tilde{t}]]\partial$, and so $Loc_x^{\leq n}$ also gives elements of $(\mathbb{k}[[\tilde{t}]]\partial)^*$. For every $s \in \mathbb{k}[[\tilde{t}]]$ with $s(x) = x$ we introduce a local change of coordinates endomorphism $\text{End}_{t \rightarrow s}(\cdot)$ of $\mathbb{k}[[\tilde{t}]]$ through the formula $t \rightarrow s$. Note that $\text{End}_{t \rightarrow s}$ is invertible if $s'(x) \neq 0$. Let $DLoc_x$, the group of formal local diffeomorphisms at x , denote the group of all $\text{End}_{t \rightarrow s}$ with $s(x) = x$ and $s'(x) \neq 0$. The group $DLoc_x$ has a subgroup of transformations of the form $t \rightarrow \zeta t + (1 - \zeta)x$, for $\zeta \in \mathbb{k}^\times$; we also write this transformation as $\tilde{t} \rightarrow \zeta \tilde{t}$ and let Dil_x denote the group of such transformations, which we term dilations at x .

Pick $s \in \mathbb{k}[[\tilde{t}]]$ with $s(x) = x$ and $s'(x) \neq 0$. We extend $\text{End}_{t \rightarrow s}$ to an automorphism of $\mathbb{k}[[\tilde{t}]]\partial$ via the formulae

$$t \mapsto s, \quad \partial \mapsto \frac{1}{s'}\partial (= \partial_s). \tag{4.1.2}$$

This gives actions of $DLoc_x$ on $\mathbb{k}[[\tilde{t}]]\partial$ and on $(\mathbb{k}[[\tilde{t}]]\partial)^*$. We may consider $Loc_x^{\leq n}$ as a subset of $(\mathbb{k}[[\tilde{t}]]\partial)^*$, and this subset is preserved by the $DLoc_x$ -action.

Denote by $DLoc_x^{\leq n}$ the image of $DLoc_x$ in the group $\text{Aut}(Loc_x^{\leq n})$ of linear automorphisms of $Loc_x^{\leq n}$. Although $DLoc_x$ is infinite-dimensional, its image in $\text{Aut}(Loc_x^{\leq n})$ is a finite-dimensional solvable algebraic group. Let us consider the action of the corresponding (finite-dimensional) Lie algebra, which we denote by $\text{lie}_x^{\leq n}$.

LEMMA 4.1.3. Identify $\text{lie}_x^{\leq n}$ with the tangent space to $DLoc_x^{\leq n}$ at the identity. Let $s \in \mathbb{k}[[t]]$ with $s(x) = 0$ and denote by ξ_s the tangent direction at the identity defined by the line $h \rightarrow \text{End}_{t \rightarrow t+hs}(\cdot)$ for $h \in \mathbb{k}$. The action of $DLoc_x^{\leq n}$ derives to an action of ξ_s on $Loc_x^{\leq n}$. Then:

- (a) $\text{lie}_x^{\leq n}$ consists of vectors of the form ξ_s with $s(x) = 0$;
- (b) $\xi_s \cdot \chi = s\partial \cdot \chi$ for all $\chi \in Loc_x$.

Proof. Part (a) is clear from the definition of $DLoc_x^{\leq n}$. For part (b), as the action of ξ_s on Loc_x is induced by the action (4.1.2) on $\mathbb{k}[[\tilde{t}]]\partial$, it is enough to check that $\xi_s \cdot f\partial = [s\partial, f\partial]$ for

all $f\partial \in \mathfrak{g}$. However,

$$\begin{aligned} \xi_s \cdot f\partial &= (\xi_s \cdot f)\partial + f(\xi_s \cdot \partial) \\ &= (\partial_h f(t + hs))|_{h=0}\partial + f(\partial_h(\text{End}_{t \rightarrow t+hs} \partial))|_{h=0} \\ &= (f'(t + hs)s)|_{h=0}\partial + f\left(\partial_h\left(\frac{1}{1 + hs'}\partial\right)\right)\Big|_{h=0} \\ &= f's\partial - fs'\partial = [s\partial, f\partial], \end{aligned}$$

as needed. □

We now let $\mathfrak{g} = W_{\geq -1}$. For all x, n , the coadjoint action of $W_{\geq -1}(t - x)$ preserves $Loc_x^{\leq n}$, and Lemma 4.1.3 shows that

$$W_{\geq -1}(t - x) \cdot \chi = \text{lie}_x^{\leq n} \cdot \chi \tag{4.1.4}$$

for $\chi \in Loc_x^{\leq n}$. For $z \in \mathbb{k}$ define

$$Shift_z(\chi_{x;\alpha_0, \alpha_1, \dots}) := \chi_{x+z; \alpha_0, \alpha_1, \dots}$$

The set $\{Shift_z \mid z \in \mathbb{k}\}$ forms a one-dimensional algebraic group, which we denote by $Shifts$; clearly $Shifts \cong \mathbb{k}^+$. Deriving the action of $Shifts$ on $Loc_{W_{\geq -1}}$ and on $Loc_{\widehat{W}_{\geq -1}}^{\leq n}$, we have

$$\begin{aligned} \partial_h Shift_h(\chi_{x;\alpha_0, \dots, \alpha_n}(f\partial))|_{h=0} &= \partial_h(\alpha_0 f(x + h) + \alpha_1 f'(x + h) + \dots + \alpha_n f^{(n)}(x + h))|_{h=0} \\ &= \alpha_0 f'(x) + \dots + \alpha_n f^{(n+1)}(x) = (\partial \cdot \chi_{x;\alpha_0, \dots, \alpha_n})(f\partial) \end{aligned} \tag{4.1.5}$$

and, thus, we may without ambiguity identify the Lie algebra of $Shifts$ with $\mathbb{k} \cdot \partial$.

For all $x, z \in \mathbb{k}$, $Shift_z$ gives a continuous homomorphism $\mathbb{k}[[t - x]] \rightarrow \mathbb{k}[[t - x - z]]$ and, hence, induces an isomorphism

$$Shift_z : DLoc_x^{\leq n} \rightarrow DLoc_{x+z}^{\leq n}.$$

In particular, we can take $x = 1$, and then for $\mathfrak{g} = W_{\geq -1}$ and for all n the action map $Shifts \times Loc_1^{\leq n} \rightarrow Loc_{\widehat{W}_{\geq -1}}^{\leq n}$ is clearly bijective. This allows us to introduce the action of $\widehat{DLoc}^{\leq n} := Shifts \times DLoc_1^{\leq n}$ and $\widehat{DLoc} := Shifts \times DLoc_1$ on

$$Loc_{\widehat{W}_{\geq -1}}^{\leq n} \cong \mathbb{k} \times Loc_1^{\leq n}$$

componentwise. Note that \widehat{DLoc} and $\widehat{DLoc}^{\leq n}$ also act on $\widetilde{Loc}_{\widehat{W}_{\geq -1}}^{\leq n}$ in such a way that $\pi_{\widehat{W}_{\geq -1}}^{\leq n}$ is equivariant.

Let $\widehat{\text{lie}}^{\leq n}$ denote the Lie algebra of $\widehat{DLoc}^{\leq n}$. From Lemma 4.1.3 we have the following result.

LEMMA 4.1.6. *Let $\mathfrak{g} = W, W_{\geq -1}$ or $W_{\geq 1}$ and fix x, n and $\chi \in Loc_x^{\leq n}$ (recall that $x \neq 0$ if $\mathfrak{g} = W$).*

(a) *If $\mathfrak{g} = W_{\geq -1}$, then*

$$\mathfrak{g} \cdot \chi = \text{lie}_x^{\leq n} \cdot \chi + \mathbb{k}\partial_x \cdot \chi.$$

(b) *If $\mathfrak{g} = W_{\geq -1}$, then $\mathfrak{g} \cdot \chi = \widehat{\text{lie}}^{\leq n} \cdot \chi$ for all $\chi \in Loc_{\mathfrak{g}}^{\leq n}$.*

(c) *If $x \neq 0$, then we can identify $Loc_x^{\leq n}$ for $W, W_{\geq -1}$ and $W_{\geq 1}$. If $\mathfrak{g} = W$ or $\mathfrak{g} = W_{\geq 1}$, under this identification we have $\mathfrak{g} \cdot \chi = W_{\geq -1} \cdot \chi$.*

(d) *If $\mathfrak{g} = W_{\geq 1}$ and $x = 0$, then $\mathfrak{g} \cdot \chi = \text{lie}_x^{\leq n} \cdot \chi$.*

Proof. Parts (a), (b) and (d) are straightforward from the previous discussion. For part (c) note that the image of $f\partial$ in $Loc_x^{\leq n}$ depends only on the Taylor series expansion of f around x up to degree $n + 1$, which is not affected by the behaviour of f at 0. \square

It is useful to have a more detailed description of $DLoc_x$ and $DLoc_x^{\leq n}$. The main point here is that $DLoc_x^{\leq n}$ is a connected solvable algebraic group which has a filtration by normal subgroups with one-dimensional quotients.

Pick $k \geq 2$. Let $x \in \mathbb{k}$ (or $x \in \mathbb{k}^\times$ if $\mathfrak{g} = W$), and recall that \tilde{t} denotes $t - x$. Elements of the form

$$\text{End}_{\tilde{t} \rightarrow \tilde{t} + \tilde{t}^k h}, \quad h \in \mathbb{k}[[\tilde{t}]],$$

constitute a subgroup of $DLoc_x$ and we denote this subgroup by $DLoc_x^{k+}$. We use the notation $DLoc_x^{\leq n, k+}$ for the image of $DLoc_x^{k+}$ in $DLoc_x^{\leq n}$. The following lemma is straightforward.

LEMMA 4.1.7. *Let $x \in \mathbb{k}$ and let $\mathfrak{g} = W, W_{\geq 1}$ or $W_{\geq -1}$ with $x \neq 0$ for $\mathfrak{g} = W$.*

- (a) *The group $DLoc_x^{2+, \leq n}$ is the unipotent radical of $DLoc_x^{\leq n}$ and Dil_x is a maximal reductive subgroup of $DLoc_x^{\leq n}$. In particular, the natural map*

$$Dil_x \rightarrow DLoc_x^{\leq n} / DLoc_x^{2+, \leq n}$$

is an isomorphism.

- (b) *The group $DLoc_x^{k+, \leq n} / DLoc_x^{k+1+, \leq n}$ is either isomorphic to \mathbb{k} or $\{0\}$ if $2 \leq k \leq n$. Moreover, the natural map*

$$\mathbb{k} \rightarrow DLoc_x^{k+, \leq n} / DLoc_x^{k+1+, \leq n}$$

induced by sending $h \mapsto \text{End}_{\tilde{t} \rightarrow \tilde{t} + h\tilde{t}^n}(\cdot)$ is surjective.

- (c) *The group $DLoc_x^{k+, \leq n}$ is trivial for $k \geq n$.*

4.2 Explicit description of $DLoc_x^{\leq n}$ -orbits

We now compute the $DLoc_x$ -orbits of one-point local functions; in the next subsection we see that this allows us to compute the pseudo-orbit of an arbitrary local function. If $\mathfrak{g} = W_{\geq -1}$, then the action of $DLoc_x$ is clearly homogeneous in x for all $x \in \mathbb{k}$, and similarly for W for $x \neq 0$. If $x = 0$ and $\mathfrak{g} = W_{\geq 1}$, the story is a bit more delicate.

Let \mathfrak{g} be $W_{\geq -1}, W$ or $W_{\geq 1}$ and let $x \in \mathbb{k}$. If $\mathfrak{g} = W$ or $W_{\geq -1}$, we additionally assume that $x \neq 0$. For $i \in \mathbb{Z}_{\geq 0}$ we set $e_i(x) = (t - x)^{i+1}\partial$ and define $e_i(x)^*$ by the formula

$$f\partial \mapsto \frac{f^{i+1}(x)}{(i + 1)!},$$

so that $e_i(x)^*(e_j(x)) = \delta_{ij}$. (We view the $e_i(x)^*$ as elements either of $(\mathbb{k}[[t - x]]\partial)^*$ or \mathfrak{g}^* , depending on context.) The main goal of this subsection is to prove the following theorem, and to consider its consequences.

THEOREM 4.2.1. *Assume $\mathfrak{g} = W_{\geq -1}, W$ or $W_{\geq 1}$ and let $x \in \mathbb{k}$. If $\mathfrak{g} = W$ or $W_{\geq 1}$ we additionally assume that $x \neq 0$. Fix $n, \beta_0, \dots, \beta_n$ with $\beta_n \neq 0$ and set $\chi = \chi_{x; \beta_0, \dots, \beta_n}$. Let $e_i := e_i(x)$ and $e_i^* := e_i(x)^*$ for all $i \in \mathbb{Z}_{\geq 0}$.*

- (a) *If n is even, then $DLoc_x^{\leq n}\chi = DLoc_x^{\leq n}e_{n-1}^*$ and $\dim DLoc_x^{\leq n}\chi = n + 1$.*
- (b) *If n is odd and $n > 1$, then there is β such that $DLoc_x^{\leq n}\chi = DLoc_x^{\leq n}(e_{n-1}^* + \beta e_k^*)$ where $k = (n - 1)/2$. We have $\dim DLoc_x^{\leq n}\chi = n$.*
- (b') *If $n = 1$, then $DLoc_0^{\leq n}\chi = DLoc_x^{\leq n}(\beta_1 e_0^*)$ and $\dim DLoc_x^{\leq n}\chi = 1$.*

- (c) Pick $\beta_1, \beta_2 \in \mathbb{k}$ and $k \in \mathbb{Z}_{\geq 1}$. Then $DLoc_x^{\leq n}(e_{2k}^* + \beta_1 e_k^*) = DLoc_x^{\leq n}(e_{2k}^* + \beta_2 e_k^*)$ if and only if $\beta_1 = \pm \beta_2$.
- (c') Pick $\beta_1, \beta_2 \in \mathbb{k}$. Then $DLoc_x^{\leq 1}(\beta_1 e_0^*) = DLoc_x^{\leq 1}(\beta_2 e_0^*)$ if and only if $\beta_1 = \beta_2$.

Proof. The proof is the same for any base point; to reduce notation we give the proof for $\mathfrak{g} = W_{\geq -1}$ and $x = 0$.

First we compute the dimensions of the corresponding orbits. Note that

$$\dim DLoc_0^{\leq n} \chi = \dim \text{lie}_0^{\leq n} \cdot \chi = \dim W_{\geq 0} \cdot \chi,$$

see (4.1.4). Moreover, (4.1.5) implies that $W_{\geq -1} \cdot \chi \not\subseteq Loc_0^{\leq n}$. Thus, $\dim W_{\geq -1} \cdot \chi = \dim W_{\geq 0} \cdot \chi + 1$. Further $\dim W_{\geq -1} \cdot \chi = \text{rk } B_\chi$. The rank of this form can be evaluated explicitly. Indeed,

$$\chi_{0; \beta_0, \dots, \beta_n}([e_i, e_j]) = \begin{cases} 0 & \text{if } i + j > n - 1, \\ (j - i)(i + j + 1)! \beta_{i+j+1} & \text{if } i + j \leq n - 1. \end{cases}$$

From this formula it is clear that $B_\chi(e_i, e_j) = 0$ if $i \geq n + 1$ or $j \geq n + 1$. Thus, the rank of this form can be evaluated on the first $(n + 2) \times (n + 2)$ entries corresponding to $-1 \leq i, j \leq n$. This block is skew-upper-triangular. If n is even, then all values on the skew-diagonal line are nonzero (because $\beta_n \neq 0$); hence, the rank is $n + 2$ in this case. If n is odd, then this skew-diagonal line contains zero only in the position corresponding to $i = j = (n - 1)/2$; hence the rank is $n + 1$ in this case. This provides the desired dimensions.

The idea of the rest of the proof is to use the subnormal series in Lemma 4.1.7 to reduce the number of coefficients which we have to consider step by step.

Let $i \in \mathbb{Z}_{\geq 0}$. For $j \geq 1$ we have

$$\begin{aligned} \text{End}_{t \rightarrow t + \alpha t^{j+1}}(e_i) &= \frac{(t + \alpha t^{j+1})^{i+1}}{1 + \alpha(j + 1)t^j} \partial = (t^{i+1} + \alpha(i - j)t^{i+j+1} + \text{higher}) \partial \\ &= e_i + \alpha(i - j)e_{i+j} + \text{higher}. \end{aligned}$$

Thus,

$$\text{End}_{t \rightarrow t + \alpha t^{j+1}}(e_i^*) = e_i^* + \alpha(i - 2j)e_{i-j}^* + \text{a linear combination of } e_{<i-j}^*. \tag{4.2.2}$$

Likewise

$$\text{End}_{t \rightarrow \zeta t}(e_i^*) = \zeta^i e_i^*. \tag{4.2.3}$$

We apply these transformations to

$$\chi_{0; \beta_0, \dots, \beta_n} = \beta_0 e_{-1}^* + \dots + \beta_n (n!) e_{n-1}^*, \tag{4.2.4}$$

noting that the coefficient of e_{n-1}^* in (4.2.4) is nonzero.

By applying $\text{End}_{t \rightarrow t + \alpha_1 t^2}$ with appropriate α_1 we may cancel the coefficient of e_{n-2}^* , using (4.2.2). This does not affect the coefficient of e_{n-1}^* . Then by applying the appropriate $\text{End}_{t \rightarrow t + \alpha_2 t^3}$ we can cancel the coefficient of e_{n-3}^* without changing the coefficient of e_{n-2}^* or e_{n-1}^* . Repeating, we may cancel the coefficients of all e_{n-1-k}^* with $1 \leq k \leq n$, unless $k = (n - 1)/2$.

Thus, in cases (a) or (b') we obtain that $\omega = \beta_n e_{n-1}^* \in DLoc_0^{\leq n}(\chi)$. In case (b) we obtain some

$$\omega = \alpha e_{(n-1)/2}^* + \beta_n e_{n-1}^* \in DLoc_0^{\leq n}(\chi).$$

Now applying (4.2.3) to ω we may rescale the coefficient of e_{n-1}^* by any ζ^{n-1} , so if $n \neq 1$ we may set the coefficient of e_{n-1}^* to be 1. This proves the rest of the cases (a), (b) and (b').

We now prove case (c'). It follows from (4.2.2) that applying any nontrivial element of the unipotent radical of $DLoc_0^{\leq 1}$ to e_0^* will give a nonzero e_{-1}^* term. Thus, we must simply consider acting by Dil_0 , and rescaling as in (4.2.3) does not affect the coefficient of e_0^* .

Finally, we prove case (c). Formula (4.2.3) implies that if $\beta_1 = \pm\beta_2$, then

$$DLoc_0^{\leq n}(e_{2k}^* + \beta_1 e_k^*) = DLoc_0^{\leq n}(e_{2k}^* + \beta_2 e_k^*).$$

Pick $\beta_1, \beta_2 \in \mathbb{k}$ and set $\chi_1 = e_{2k}^* + \beta_1 e_k^*, \chi_2 = e_{2k}^* + \beta_2 e_k^*$. Assume that $DLoc_0^{\leq n}(\chi_1) = DLoc_0^{\leq n}(\chi_2)$, so there exists $s \in \mathbb{k}[[t]]$ with $\text{End}_{t \rightarrow s}(\chi_1) = \chi_2, s(0) = 0$ and $s'(0) \neq 0$.

If

$$s = s'(0)t, \tag{4.2.5}$$

then the statement of case (c) is straightforward. Assume to the contrary that $s \neq s'(0)t$. Then

$$s(t) \neq s'(0)t \pmod{t^d}$$

for some d , and we choose d to be minimal.

If $d \geq 2k + 2$, then linear operators $\text{End}_{t \rightarrow s}$ and $\text{End}_{t \rightarrow s'(0)t}$ coincide after the restriction to $Loc_0^{\leq n}$, i.e. we can replace s by $s'(0)t$, which is case (4.2.5).

If $d \leq 2k + 1$, then

$$s = s'(0)t + \frac{s^{(d)}(0)}{d!}t^d \pmod{t^{d+1}}$$

with $s^{(d)}(0) \neq 0$; set $\gamma = s'(0), \tau = s^{(d)}(0)/d!$. Further (4.2.2) and (4.2.3) imply

$$\text{End}_{t \rightarrow s}(\chi_1) = \gamma^{-2k}e_{2k} + \gamma^{i-2j-1}\tau(2k - 2d)e_{2k-d} \pmod{Loc_0^{\leq 2k-d}}.$$

This cannot give χ_2 unless $d = k$.

Thus, suppose $d = k$ and note that

$$\text{End}_{t \rightarrow t + \alpha t^{k+1} + (\alpha^2(k+1) - \alpha(\beta/2))t^{2k+1}}(e_{2k}^* + \beta e_k^*) = (e_{2k}^* + \beta e_k^*)$$

for every $\alpha \in \mathbb{k}$. Thus, we can replace s by

$$s + \alpha s^{k+1} + \left(\alpha^2(k+1) - \alpha \frac{\beta}{2}\right)s^{2k+1} \tag{4.2.6}$$

for an arbitrary α . Pick $\alpha = -s^{(d+1)}(0)/(d+1)!(s'(0))^{d+1}$. Then (4.2.6) is equal to $s'(0)t \pmod{t^{d+1}}$ and we have reduced this case to the previous one. \square

Remark 4.2.7. If $\chi \in W^*$ is a one-point local function, the beginning of the proof of Theorem 4.2.1 provides an explicit description of W^χ . Indeed, if $\chi = \chi_{x;\beta_0,\beta_1,\dots,\beta_n}$ with $\beta_n \neq 0$, then $W^\chi = W((t-x)^{n+1})$ if n is even, and W^χ is the sum of $W((t-x)^{n+1})$ with a one-dimensional subspace if n is odd.

A modification of Theorem 4.2.1 holds for $\mathfrak{g} = W_{\geq 1}$ and $x = 0$ with a very similar proof, which we leave to the reader.

THEOREM 4.2.8. *Assume $\mathfrak{g} = W_{\geq 1}$. Fix $n \geq 2$ and β_2, \dots, β_n with $\beta_n \neq 0$ and set $\chi = \chi_{0;0,0,\beta_2,\dots,\beta_n}$.*

- (a) *If n is even, then $DLoc_0^{\leq n}\chi = DLoc_0^{\leq n}e_{n-1}^*$. We have $\dim DLoc_0^{\leq n}\chi = n - 1$.*
- (b) *If n is odd and $n > 1$, then there is β such that $DLoc_0^{\leq n}\chi = DLoc_0^{\leq n}(e_{n-1}^* + \beta e_k^*)$ where $k = (n - 1)/2$. We have $\dim DLoc_0^{\leq n}\chi = n - 2$.*
- (c) *Pick $\beta_1, \beta_2 \in \mathbb{k}$ and $k \in \mathbb{Z}_{\geq 1}$. Then $DLoc_0^{\leq n}(e_{2k}^* + \beta_1 e_k^*) = DLoc_0^{\leq n}(e_{2k}^* + \beta_2 e_k^*)$ if and only if $\beta_1 = \pm\beta_2$.*

We next describe the ‘Bruhat order’ on $DLoc_x^{\leq n}$ orbits (that is, the inclusions of orbit closures): it turns out that it can almost be computed just from the dimension of the orbit.

COROLLARY 4.2.9. *Let \mathfrak{g}, x be as in the statement of Theorem 4.2.1. Let χ^-, χ^+ be local functions on \mathfrak{g} based at x and of order $\leq n$.*

Then the following conditions are equivalent:

- (1) $DLoc_x^{\leq n} \cdot \chi^-$ is contained in the closure of $DLoc_x^{\leq n} \cdot \chi^+$ and $DLoc_x^{\leq n} \cdot \chi^- \neq DLoc_x^{\leq n} \cdot \chi^+$;
- (2) $P(\chi^+) \not\subseteq P(\chi^-)$;
- (3) $\dim DLoc_x^{\leq n} \cdot \chi^- < \dim DLoc_x^{\leq n} \cdot \chi^+$ and χ^+ is not of order one.

Proof. We again give the proof for $\mathfrak{g} = W_{\geq -1}$ and $x = 0$. We show that (2) \Rightarrow (3) \Leftrightarrow (1) \Rightarrow (2). Next, without loss of generality we assume that n is the order of χ^+ .

We first show that (3) implies (1). As in Theorem 4.2.1, pick the related presentation $\chi_{0; \beta_0^+, \beta_1^+, \dots, \beta_n^+}$ for χ^+ , with $\beta_n^+ \neq 0$. If n is even, then the closure of $DLoc_0^{\leq n} \cdot \chi^+$ equals $Loc_0^{\leq n}$. On the other hand, all orbits of strictly smaller dimension belong to $Loc_0^{\leq n}$, so this completes the proof in this case.

If $n = 2k + 1$ is odd, then $k \geq 1$ by the assumptions of condition (3). The closure of $DLoc_0^{\leq n} \cdot \chi^+$ is an irreducible subvariety of $Loc_0^{\leq n}$ of codimension one, i.e. a hypersurface defined by some function F_+ . We claim that this function is semi-invariant with respect to $DLoc_0^{\leq n}$.

To prove that F_+ is semi-invariant, we describe it in more detail. Let $R = \mathbb{k}[\beta_0, \beta_1, \dots, \beta_{n-1}, \beta_n^{\pm 1}]$, where $\beta_0, \beta_1, \dots, \beta_n$ are free variables. Denote by $(\cdot)_{\mathbb{k} \rightarrow R}$ the base change from \mathbb{k} to R . The arguments of Theorem 4.2.1(b) imply that there exists a group element $g \in (DLoc_0^{\leq n})_{\mathbb{k} \rightarrow R}$ and $h \in R$ such that

$$g \cdot \underbrace{\chi_{0; 0, \dots, 0, h, 0, \dots, 0, \beta_n}}_{k \text{ times}} = \chi_{0; \beta_0, \beta_1, \dots, \beta_n}.$$

Recall that $k = (n - 1)/2 \geq 1$. By replacing each β_i by $e_{i-1}/i!$ (cf. (4.2.4)) we may identify R with $\mathbb{k}[Loc_0^{\leq n}][e_{2k}^{-1}]$. (Here we regard the e_i as functions on $Loc_0^{\leq n}$ in the obvious way.) Then $h = f/e_{2k}^\ell$ for some $f \in \mathbb{k}[Loc_0^{\leq n}]$ and $\ell \in \mathbb{Z}_{\geq 0}$.

Let $\chi, \chi' \in Loc_0^{\leq n}$ with $e_{2k}(\chi), e_{2k}(\chi') \neq 0$. As in the proof of Theorem 4.2.1(c), $DLoc_0^{\leq n} \cdot \chi = DLoc_0^{\leq n} \cdot \chi'$ if and only if

$$e_{2k}(\chi)^{-1} \frac{f^2(\chi)}{e_{2k}(\chi)^{2\ell}} = e_{2k}(\chi')^{-1} \frac{f^2(\chi')}{e_{2k}(\chi')^{2\ell}}.$$

The rational function

$$F := f^2/e_{2k}^{2\ell+1} \tag{4.2.10}$$

is, thus, $DLoc_0^{\leq n}$ -invariant and separates orbits. Therefore, $DLoc_0^{\leq n} \cdot \chi^+$ is the hypersurface in $Loc_0^{\leq n} \setminus Loc_0^{\leq n-1}$ defined by

$$F_+ := e_{2k}(\chi^+)^{2\ell+1} f^2 - f^2(\chi^+) e_{2k}^{2\ell+1}.$$

Note that F_+ is semi-invariant, as claimed.

All orbits of dimension more than n belong to $Loc_0^{\leq n-2}$. We check that the closure of $DLoc_0^{\leq n} \cdot \chi^+$ contains $Loc_0^{\leq n-2}$. The orbit closure $DLoc_0^{\leq n} \chi^+$ is defined in $Loc_0^{\leq n}$ by F_+ , so we must show that $F_+|_{Loc_0^{\leq n-2}} = 0$. The restriction of F_+ to $Loc_0^{\leq n-1}$ is also a semi-invariant function; in particular, this restriction is invariant with respect to $DLoc_0^{1+, \leq n}$, see Lemma 4.1.7. The arguments of Theorem 4.2.1 imply that all n -dimensional $DLoc_0^{1+, \leq n}$ -invariant hypersurfaces

of $Loc_0^{\leq n-1}$ are defined by the equality $e_{n-1} = \beta_{n-1}$ for some $\beta_{n-1} \in \mathbb{k}$. This implies that the restriction of F_+ is a polynomial in e_{n-1} . The fact that F_+ is $DLoc_0^{\leq n}$ -semi-invariant implies that $F_+|_{Loc_0^{\leq n-1}} = c(e_{n-1})^m$ for some $m \geq 0$ and $c \in \mathbb{k}$. This implies the desired condition that $F_+|_{Loc_0^{\leq n-2}} = 0$.

We now show that condition (1) implies condition (3), so suppose that condition (1) holds. The dimension inequality in condition (3) is clear. Thus, we are left to deal with the case when χ^+ has order one; we want to show this cannot happen.

Suppose now that $\chi^+ = \chi_{x;\alpha,\beta}$ has order one, so $\beta \neq 0$. By the dimension inequality, $\chi^- = 0$. Now, Theorem 4.2.1(b') shows that $DLoc_0^{\leq 1} \cdot \chi^+ = DLoc_0^{\leq 1} \cdot (\beta e_0^*)$ consists of all $\chi_{0;\alpha',\beta}$ for $\alpha' \in \mathbb{k}$, and so is defined in $Loc_0^{\leq 1} \setminus Loc_0^{\leq 0} = \text{Spec } \mathbb{k}[\beta_0, \beta_1, \beta_1^{-1}]$ by $\beta_1 = \beta$. Thus, $\overline{DLoc_0^{\leq 1} \cdot \chi^+}$ is defined in $Loc_0^{\leq 1} \cong \text{Spec } \mathbb{k}[\beta_0, \beta_1]$ by $\beta_1 = \beta$ and does not contain $0 = \chi_{0;0,0}$, a contradiction.

We now show that (1) \Rightarrow (2). This is an application of Proposition 2.3.1. Let $X = \widehat{Loc}_{\mathfrak{g}}^{\leq n}$ (recall that $\mathfrak{g} = W_{\geq -1}$) and let $H = \widehat{DLoc}^{\leq n}$. Let $\mathfrak{h} = \widehat{\text{lie}}^{\leq n} = \text{Lie}(H)$. Let $\phi = \phi_{\mathfrak{g}}^{\leq n} : X \rightarrow \mathfrak{g}^*$. Choose $x^+ \in \phi^{-1}(\chi^+)$ and $x^- \in \phi^{-1}(\chi^-)$; note here that ϕ is bijective onto $Loc_{\mathfrak{g}}^{\leq n} \setminus \{0\}$ so x^{\pm} may be uniquely determined. By Lemma 4.1.6, $\dim \mathfrak{h}_{x^{\pm}} = \dim \mathfrak{g}_{\chi^{\pm}}$. Thus, by Proposition 2.3.1, $P(\chi^+)$ is the kernel of the map $\phi^+ : S(\mathfrak{g}) \rightarrow \mathbb{k}[\overline{H \cdot x^+}]$ induced from $\phi^* : S(\mathfrak{g}) \rightarrow \mathbb{k}[X]$ and $P(\chi^-)$ is the kernel of $\phi^- : S(\mathfrak{g}) \rightarrow \mathbb{k}[\overline{H \cdot x^-}]$. However, by assumption, $H \cdot x^- \subseteq \overline{H \cdot x^+} \subseteq X$, giving the following commutative diagram.

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\phi^+} & \mathbb{k}[\overline{H \cdot x^+}] \\ & \searrow \phi^- & \downarrow \\ & & \mathbb{k}[\overline{H \cdot x^-}] \end{array}$$

Thus, $P(\chi^-) = \ker \phi^- \supseteq \ker \phi^+ = P(\chi^+)$. The two are clearly distinct.

We still must show condition (2) implies condition (3). If condition (2) holds, then $\dim \mathcal{O}(\chi^-) < \dim \mathcal{O}(\chi^+)$. By the previous discussion and (2.3.3),

$$\dim \mathcal{O}(\chi^{\pm}) = \dim \mathfrak{g}_{\chi^{\pm}} = \dim DLoc_x^{\leq n} \cdot \chi^{\pm} + 1.$$

Finally, note that if $\chi^+ = \chi_{x;\alpha,\gamma}$ has order zero or one, by the dimension inequality we must have $\chi^- = 0$. We have seen in Lemma 2.2.3 that $P(\chi^+)$ is the kernel of

$$p_{\gamma} : S(\mathfrak{g}) \rightarrow \mathbb{k}[t, t^{-1}, y], \quad f\partial \mapsto fy + \gamma f'.$$

We compute

$$p_{\gamma}([(t-x)\partial]^2 - \partial[(t-x)^2\partial]) = ((t-x)y + \gamma)^2 - y((t-x)^2y + 2\gamma(t-x)) = \gamma^2$$

and so $[(t-x)\partial]^2 - \partial[(t-x)^2\partial] - \gamma^2 \in \ker p_{\gamma}$. This element is contained in $\mathfrak{m}_0 = P(0)$ if and only if $\gamma = 0$, so χ^+ cannot have order one. \square

For future reference, we record that Corollary 4.2.9 also gives information about $\widehat{DLoc}^{\leq n}$ orbits. Let

$$Loc_{W_{\geq -1}}^n := Loc_{W_{\geq -1}}^{\leq n} \setminus Loc_{W_{\geq -1}}^{\leq n-1}$$

and similarly define Loc_W^n . We identify $\mathbb{k}[Loc_{W_{\geq -1}}^n]$ with $\mathbb{k}[x, \beta_0, \dots, \beta_n, \beta_n^{-1}]$. Note that each $Loc_{W_{\geq -1}}^n$ is $\widehat{DLoc}^{\leq n}$ -stable.

The action of *Shifts* and, thus, the action of \widehat{DLoc} on Loc_W is only partially defined, as we cannot shift the base point to 0. We refer to *orbits* of this partial action with the obvious meaning: the intersection of an orbit in $Loc_{W_{\geq -1}}$ with W^* .

COROLLARY 4.2.11. Let $\mathfrak{g} = W$ or $W_{\geq -1}$.

- (a) If n is even, then $Loc_{\mathfrak{g}}^n$ is a single $\widehat{DLoc}^{\leq n}$ -orbit.
- (b) If $n \geq 3$ is odd, there is $F_n \in \mathbb{k}[x, x^{-1}, \beta_0, \dots, \beta_n, \beta_n^{-1}]$ (if $\mathfrak{g} = W$) or in $\mathbb{k}[x, \beta_0, \dots, \beta_n, \beta_n^{-1}]$ (if $\mathfrak{g} = W_{\geq -1}$) so that the $\widehat{DLoc}^{\leq n}$ -orbits are precisely the fibres of the morphism $F_n : Loc_{\mathfrak{g}}^n \rightarrow \mathbb{A}^1$.
- (c) Define $F_1 : Loc_{\mathfrak{g}}^1 \rightarrow \mathbb{A}^1 \setminus \{0\}$ by $\chi_{x;\alpha,\gamma} \mapsto \gamma$. Then the fibres of F_1 are exactly the orbits of $\widehat{DLoc}^{\leq 1}$ on $Loc_{\mathfrak{g}}^1$.

Proof. This result is a consequence of Theorem 4.2.1 and the proof of Corollary 4.2.9. If $n \geq 3$ is odd, note that, by definition, applying an element of *Shifts* changes x but not any of the β_i . Thus, we may take F_n to be the polynomial F defined in (4.2.10), regarded as an element of $\mathbb{k}[x, \beta_0, \dots, \beta_n, \beta_n^{-1}]$. The proof for $n = 1$ is similar. \square

It is instructive to compute that for $n = 3$, the polynomial F_3 given by Corollary 4.2.11 is a scalar multiple of β_2^2/β_3 . In particular, x does not occur.

Remark 4.2.12. The dimension of a $\widehat{DLoc}^{\leq n}$ -orbit in Corollary 4.2.11 is

$$2 \left\lfloor \frac{n+2}{2} \right\rfloor = \begin{cases} n+2 & n \text{ even,} \\ n+1 & n \text{ odd.} \end{cases}$$

4.3 Implicit description of pseudo-orbits

Let $\mathfrak{g} = W, W_{\geq -1}$ or $W_{\geq 1}$. We now consider arbitrary local functions on \mathfrak{g} . We provide every pair of local functions $\chi, \nu \in \mathfrak{g}^*$ with an affine variety X acted on by an algebraic group H , an open subset $U \subseteq X$, a Poisson map $\phi : U \rightarrow \mathfrak{g}^*$ and a pair of points $x, y \in X$ which will satisfy the conditions of Proposition 2.3.1. Thus, we reduce the mysterious infinite-dimensional case to the more easily comprehensible action of an algebraic group on an affine variety. We use this to determine the pseudo-orbits in \mathfrak{g}^* .

THEOREM 4.3.1. Let $\mathfrak{g} = Vir, \mathfrak{g} = W$ or $W_{\geq -1}$. Let χ^I and χ^{II} be local functions in \mathfrak{g}^* represented by sums

$$\chi^I = \sum_{i=1}^{\ell} \chi_i^I, \quad \chi^{II} = \sum_{i=1}^{\ell} \chi_i^{II}, \quad \chi_i^I \in Loc_{x_i^I}, \quad \chi_i^{II} \in Loc_{x_i^{II}},$$

such that $x_i^I \neq x_j^I$ and $x_i^{II} \neq x_j^{II}$ for $i \neq j$. Then χ^I and χ^{II} are in the same pseudo-orbit if and only if it is possible to reorder x_i^I and x_i^{II} in such a way that

$$DLoc_1(Shift_{1-x_i^I} \chi_i^I) = DLoc_1(Shift_{1-x_i^{II}} \chi_i^{II}) \quad \text{for all } i; \tag{4.3.2}$$

that is,

$$\widehat{DLoc}(\chi_i^I) = \widehat{DLoc}(\chi_i^{II}) \quad \text{for all } i.$$

In particular, if $\mathbb{O}(\chi^I) = \mathbb{O}(\chi^{II})$, then $\lambda(\chi^I) = \lambda(\chi^{II})$ and, hence, the supports of χ^I and χ^{II} have the same cardinality.

Remark 4.3.3. Let $\mathfrak{g} = W$ or $W_{\geq -1}$. Fix a partition $\lambda = (m_1, \dots, m_r)$ and consider the set $V_{\mathfrak{g}}^\lambda$ of all local functions $\chi \in \mathfrak{g}^*$ with $\lambda(\chi) = \lambda$. Let $\chi = \chi_1 + \dots + \chi_r, \nu = \nu_1 + \dots + \nu_r \in V_{\mathfrak{g}}^\lambda$, where each χ_i (respectively, ν_i) is a one-point local function of order $m_i - 1$. By Theorem 4.3.1, $\mathbb{O}(\chi) = \mathbb{O}(\nu) \iff \mathbb{O}(\chi_i) = \mathbb{O}(\nu_i)$ for $1 \leq i \leq r$ (up to permutation of indices). Combining Corollary 4.2.11 and Theorem 4.3.1 applied to one-point local functions we have

- (a) if m_i is odd, then $Loc_{\mathfrak{g}}^{m_i-1}$ is a single pseudo-orbit;
- (b) if m_i is even and $m_i \neq 2$, then the pseudo-orbits in $Loc_{\mathfrak{g}}^{m_i-1}$ are parameterised by \mathbb{A}^1 ;
- (c) the pseudo-orbits in $Loc_{\mathfrak{g}}^1$ are parameterised by $\mathbb{A}^1 \setminus \{0\}$.

Thus, pseudo-orbits in $V_{\mathfrak{g}}^\lambda$ (and the corresponding Poisson primitive ideals of $S(\mathfrak{g})$) are parameterised by a symmetric product $(\mathbb{A}^1)^{\times(k-k_2)} \times (\mathbb{A}^1 \setminus \{0\})^{\times k_2}$, where k is the total number of even parts of λ and $k_2 := |\{i \mid m_i = 2\}|$.

More specifically, let

$$\mathbb{A}(\lambda) := \begin{cases} \mathbb{A}^k & \text{if } k_2 = 0, \\ \mathbb{A}^{k-1} \times (\mathbb{A}^1 \setminus \{0\}) & \text{if } k_2 \neq 0. \end{cases}$$

Let $\mathfrak{S}_\lambda = \{\sigma \in \mathfrak{S}_k \mid m_i = m_{\sigma(i)} \text{ for all } i\}$. The pseudo-orbits in $V_{\mathfrak{g}}^\lambda$ are parameterised by

$$[(\mathbb{A}^1)^{\times(k-k_2)} \times (\mathbb{A}^1 \setminus \{0\})^{\times k_2}] / \mathfrak{S}_\lambda \cong \mathbb{A}(\lambda).$$

In § 5.2 we refine this parameterisation to apply to arbitrary prime Poisson ideals of $S(\mathfrak{g})$.

To prove Theorem 4.3.1 we need two preparatory results.

LEMMA 4.3.4. *Let $\chi \in W^*$ be a local function with $\chi = \chi_1 + \dots + \chi_s$ and $\chi_i \in Loc_{x_i}$ such that $x_i \neq x_j$ if $i \neq j$. Then $W \cdot \chi = \bigoplus_i W \cdot \chi_i$. Likewise, if $\chi \in W_{\geq -1}^*$, then $W_{\geq -1} \cdot \chi = \bigoplus_i W_{\geq -1} \cdot \chi_i$.*

Proof. This statement is basically implied by the Chinese remainder theorem, but we give the details for W . It is easy to verify that $W \cdot \chi = \{B_\chi(u, \cdot) \mid u \in W\}$. As χ is local, there are $d_1, \dots, d_s \in \mathbb{Z}_{\geq 1}$ so that $\chi|_{W(\prod_i (t-x_i)^{d_i})} = 0$. Let $f := \prod_i (t-x_i)^{d_i+1}$. Then $B_\chi|_{W(f)} = 0$. By the Chinese remainder theorem,

$$\mathbb{k}[t, t^{-1}] / \left(\prod_i (t-x_i)^{d_i+1} \right) \cong \bigoplus_i \mathbb{k}[t] / (t-x_i)^{d_i+1}. \tag{4.3.5}$$

This leads to the decomposition

$$W/W(f) \cong \bigoplus_i [W(f_i)/W(f)], \quad \text{where } f_i := \frac{f}{(t-x_i)^{d_i+1}}.$$

It is straightforward to verify that the partial restriction $B_{\chi_i}(W(f_j), \cdot)$ is nonzero only if $i = j$. Hence,

$$W \cdot \chi = \{B_\chi(u, \cdot) \mid u \in W\} = \bigoplus_i \{B_{\chi_i}(u_i, \cdot) \mid u_i \in W(f_i)\}.$$

This implies the desired result. □

We next apply the general results from § 2 and the group action from § 4.1 to the Poisson cores of local functions and deduce some consequences.

PROPOSITION 4.3.6.

- (a) *Let $\chi \in W^*$. Then*

$$P(\chi) \cap S(W_{\geq -1}) = P(\chi|_{W_{\geq -1}}).$$

(b) Let \mathfrak{g} be W , $W_{\geq -1}$ or $W_{\geq 1}$ and pick $\chi \in \mathfrak{g}^*$. Then

$$\dim \mathbb{O}(\chi) = \text{GKdim } S(\mathfrak{g}) / P(\chi) = \dim \mathfrak{g} \cdot \chi.$$

We alert the reader that the proof of Proposition 4.3.6 establishes notation that is also used in the proof of Theorem 4.3.1. (This notation generalises that used in the proof of the last part of Corollary 4.2.9.)

Proof. (a) If χ is not local, then by Theorems 3.2.1 and 3.1.1 $P(\chi)$ and $P(\chi|_{W_{\geq -1}})$ are the zero ideals, respectively, of $S(W)$ and $S(W_{\geq -1})$.

Thus, we may assume that χ is local. Write $\chi = \sum_{i=1}^{\ell} \chi_i$ with $\chi_i \in \text{Loc}_{x_i}$ and $x_i \neq x_j$ if $i \neq j$; further, assume that no $\chi_i = 0$. Let n be the order of χ .

Let $X := (\widehat{\text{Loc}}_{W_{\geq -1}}^{\leq n})^{\times \ell}$, let $Y := (\widehat{\text{Loc}}_W^{\leq n})^{\times \ell}$ and let $U \subseteq Y$ be the complement of all the diagonals Δ_{ij} in Y , where for $i \neq j$ we define

$$\Delta_{ij} = \{(x^1, \underline{\alpha}^1, \dots, x^{\ell}, \underline{\alpha}^{\ell}) \in Y \mid x^i = x^j\},$$

where $\underline{\alpha}^i$ stands for the sequence $\alpha_0^i, \alpha_1^i, \dots$. That is, U gives representations of local functions as sums of one-point local functions based at distinct points. Note that U, Y are open subsets of X , and $H := (\widehat{DLoc}^{\leq n})^{\times \ell}$ acts on X . Let $\Sigma_W : (W^*)^{\times \ell} \rightarrow W^*$ be the summation map, and likewise define $\Sigma_{W_{\geq -1}}$. Let $\phi_W : U \rightarrow W^*$ be $\Sigma_W \circ (\phi_W^{\leq n})^{\times \ell}$ and let $\phi_{W_{\geq -1}} : X \rightarrow W_{\geq -1}^*$ be $\Sigma_{W_{\geq -1}} \circ (\phi_{W_{\geq -1}}^{\leq n})^{\times \ell}$. There is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi_{W_{\geq -1}}} & W_{\geq -1}^* \\ \subseteq \uparrow & & \uparrow \\ U & \xrightarrow{\phi_W} & W^* \end{array} \tag{4.3.7}$$

where the right vertical arrow is restriction. This induces maps

$$\begin{array}{ccc} \mathbb{k}[X] & \xleftarrow{\phi_{W_{\geq -1}}^*} & S(W_{\geq -1}) \\ \subseteq \downarrow & & \downarrow \subseteq \\ \mathbb{k}[U] & \xleftarrow{\phi_W^*} & S(W). \end{array} \tag{4.3.8}$$

By Lemma 4.3.4, $\dim W \cdot \chi = \oplus_i \dim W \cdot \chi_i$, which is $\dim((\widehat{\text{lie}}^{\leq n})^{\times \ell}) \cdot \chi$ by Lemma 4.1.6. Let $y \in \phi_W^{-1}(\chi)$. By Proposition 2.3.1, $P(\chi)$ is the full preimage under ϕ_W^* of $I^U(Hy)$ and, likewise, $P(\chi|_{W_{\geq -1}})$ is the full preimage under $\phi_{W_{\geq -1}}^*$ of $I^U(Hy)$. By commutativity of (4.3.8), this is $P(\chi) \cap S(W_{\geq -1})$.

(b) If $\dim \mathfrak{g} \cdot \chi = \infty$, then the result follows from Lemma 2.1.1. Thus, we can assume $\mathfrak{g} \cdot \chi < \infty$. For $\mathfrak{g} = W$ or $W_{\geq -1}$ this follows from the previous discussion and Proposition 2.3.1; see, in particular, (2.3.3). The proof for $\mathfrak{g} = W_{\geq 1}$ is similar. \square

We now give the proof of Theorem 4.3.1.

Proof of Theorem 4.3.1. We give the proof for the case $\mathfrak{g} = W$.

By definition, $\mathbb{O}(\chi^I) = \mathbb{O}(\chi^{II})$ if and only if $P(\chi^I) = P(\chi^{II})$. Thus, we compute the Poisson cores of χ^I and χ^{II} .

First, by changing ℓ and switching χ^I and χ^{II} if necessary, we may assume that none of the χ_i^I are zero. Let n be the maximum of the orders of χ^I and of χ^{II} ; thus, both are in the image of $(Loc_W^{\leq n})^{\times \ell}$.

Define U, V, X, H as in the proof of Proposition 4.3.6, and set up maps as in (4.3.7) and (4.3.8). Let $\phi = \phi_W$. Then $\chi^I = \phi(y^I)$, $\chi^{II} = \phi(y^{II})$ for some $y^I, y^{II} \in U$. As in the proof of Proposition 4.3.6, $P(\chi^I)$ is the full preimage under ϕ^* of the defining ideal of $I^U(Hy^I)$. Likewise, $P(\chi^{II})$ is the full preimage under ϕ^* of the defining ideal of $I^U(Hy^{II})$.

Note that ϕ factors through the dominant map $U \rightarrow U/\mathfrak{S}_\ell$, where \mathfrak{S}_ℓ is the symmetric group. It is clear that if we can reorder the χ_i^I and χ_i^{II} as described, then $P(\chi^I) = P(\chi^{II})$.

Suppose now that $P(\chi^I) = P(\chi^{II})$. Then by Proposition 2.3.1(c), there are open sets $U^I \subseteq Hy^I \cap U$ and $U^{II} \subseteq Hy^{II} \cap U$ such that their images in W^* are the same. In particular, there are $h_1^I, h_1^{II}, \dots, h_\ell^I, h_\ell^{II} \in \widehat{DLoc}^{\leq n}$, with $(h_1^I \chi_1^I, \dots, h_\ell^I \chi_\ell^I) \in U$ and so that $\sum h_i^I \chi_i^I - \sum h_i^{II} \chi_i^{II} = 0$. Now, a set of one-point local functions with distinct base points is linearly independent. As, by the definition of U , the base points of the one-point local functions $h_i^I \chi_i^I$ are distinct, the only option is that the supports of $\sum h_i^I \chi_i^I$ and $\sum h_i^{II} \chi_i^{II}$ are equal and we can reorder $\chi_1^I, \dots, \chi_\ell^I$ so that $h_i^I \chi_i^I = h_i^{II} \chi_i^{II}$ for all i . This is clearly equivalent to (4.3.2). \square

Remark 4.3.9. Let $\mathfrak{g} = Vir, W$ or $W_{\geq -1}$ and let $\chi \in \mathfrak{g}^*$. If χ is not local, then $P(\chi) = (0)$ (if $\mathfrak{g} = W$ or $W_{\geq -1}$) or $P(\chi) = (z - \chi(z))$ (if $\mathfrak{g} = Vir$). If χ is local, then by Lemma 2.0.4(a) we may construct $P(\chi) = \bigcap \{m_\nu \mid \nu \in \mathbb{O}(\chi)\}$ from Theorem 4.3.1.

Further combining Corollary 4.2.11 with Theorem 4.3.1 one may, in principle, compute all the Poisson primitive ideals of $S(W)$ or $S(W_{\geq -1})$; the function F_{2k+1} whose fibres by Corollary 4.2.11 give the pseudo-orbits of a one-point local function of order $2k + 1$ may be worked out using Lemma 4.1.7 for any k . There is an example of this in § 2.2, although we have not given a fully general formula.

Note also that we have not studied generators (in any sense) for the Poisson prime ideals $P(\chi)$, and this might be an interesting subject of research.

A version of Theorem 4.3.1 holds for $W_{\geq 1}$ and $W_{\geq 0}$ with almost the same proof. We state it as follows.

THEOREM 4.3.10. *Let $\mathfrak{g} = W_{\geq 0}$ or $W_{\geq 1}$. Let $\chi^I, \chi^{II} \in \mathfrak{g}^*$ be local functions. Then there are unique local functions $\chi_W^I, \chi_W^{II} \in W^*$ and $\chi_{[0]}^I, \chi_{[0]}^{II} \in Loc_0$ such that*

$$\chi^I = \chi_W^I|_{\mathfrak{g}} + \chi_{[0]}^I, \quad \chi^{II} = \chi_W^{II}|_{\mathfrak{g}} + \chi_{[0]}^{II}.$$

Moreover, $P(\chi^I) = P(\chi^{II})$ if and only if $P(\chi_W^I) = P(\chi_W^{II})$ and $DLoc_0(\chi_{[0]}^I) = DLoc_0(\chi_{[0]}^{II})$.

The description of pseudo-orbits in Theorem 4.3.1 has several immediate applications. We start by comparing pseudo-orbits in W^* and $W_{\geq -1}^*$.

COROLLARY 4.3.11. *Let $\chi^I, \chi^{II} \in W^*$ be local functions. Then*

$$\mathbb{O}(\chi^I) = \mathbb{O}(\chi^{II}) \iff \mathbb{O}(\chi^I|_{W_{\geq -1}}) = \mathbb{O}(\chi^{II}|_{W_{\geq -1}}).$$

Further, restriction induces a bijection

$$PSpec_{\text{prim}} S(W) \rightarrow PSpec_{\text{prim}} S(W_{\geq -1}). \tag{4.3.12}$$

In Corollary 5.2.14 we show that restriction also provides a bijection between Poisson prime ideals of $S(W)$ and $S(W_{\geq -1})$.

Proof. The first statement is immediate from Theorem 4.3.1. By that result, if $\chi \in W_{\geq -1}^*$ is a local function, then there is $\nu \in \mathcal{O}(\chi)$ so that 0 is not in the support of ν . Thus, restriction of local functions induces a bijection on pseudo-orbits.

By Lemma 2.0.4 there are bijections

$$\begin{aligned} \text{nonzero Poisson primitive ideals of } S(W) &\leftrightarrow \text{pseudo-orbits in } W^* \\ &\leftrightarrow \text{pseudo-orbits in } W_{\geq -1}^* \leftrightarrow \text{nonzero Poisson primitive ideals of } S(W_{\geq -1}), \end{aligned}$$

where the middle bijection comes from restriction of local functions as in the first paragraph of the proof. By Proposition 4.3.6(a) $P(\chi) \cap S(W_{\geq -1}) = P(\chi|_{W_{\geq -1}})$. Thus, the bijection from left to right above agrees with restriction of ideals from $S(W)$ to $S(W_{\geq -1})$. \square

Remark 4.3.13. Let $P = P(\chi)$ be a Poisson primitive ideal of $S(W_{\geq -1})$, where $\chi \in (W_{\geq -1})^*$ is local. As we may apply elements of *Shifts* to χ without affecting $P(\chi)$, we may assume that χ does not involve any element of Loc_0 . Then χ extends uniquely to a local function $\tilde{\chi}$ on W , and we have seen that $P(\chi) = P(\tilde{\chi}) \cap S(W_{\geq -1})$.

Let $\gamma \in \mathbb{k}$ and recall the definition of the map p_γ of (2.2.2) and the fact, proved in Lemma 2.2.3 that $\ker p_\gamma = P(\chi_{x;\alpha,\gamma})$.

PROPOSITION 4.3.14. *The ideals $\ker p_\gamma$ are all of the Poisson prime ideals of $S(W)$ of co-GK 2.*

Proof. Let $J(\gamma) = \ker p_\gamma$ for all $\gamma \in \mathbb{k}$. Let J be a prime Poisson ideal of $S(W)$ with $\text{GKdim } S(W)/J = 2$. By Lemma 2.0.4

$$J = \bigcap \{P(\chi) \mid \chi \in W^*, \text{ev}_\chi(J) = 0\}.$$

Fix such χ . By Lemma 2.1.1 $\text{rk } B_\chi$, which must be even as B_χ is an alternating form, is zero or two. The only $\chi \in W^*$ with $\text{rk } B_\chi = 0$ is $\chi = 0$; as $J \neq P(0) = \mathfrak{m}_0$ there is some $\nu \in W^*$ with $\text{ev}_\nu(J) = 0$ and $\text{rk } B_\nu = 2$.

If $J \subsetneq P(\nu)$, then $2 = \text{GKdim } S(W)/J > \text{GKdim } S(W)/P(\nu)$, but this last is $\dim W \cdot \nu$ by Proposition 4.3.6, which must be an even positive integer. Thus, $J = P(\nu)$ and $\dim W \cdot \nu = 2$.

Lemma 4.3.4 implies that ν is a one-point local function, and by Theorem 4.2.1 we must have $\nu = \chi_{x;\alpha,\gamma}$ for some $x \in \mathbb{k}^\times, \alpha, \gamma \in \mathbb{k}$. Thus, $J = P(\nu) = J(\gamma)$. \square

Remark 4.3.15. The codomain of the maps p_γ of (2.2.2) is the localised Poisson–Weyl algebra $B = \mathbb{k}[t, t^{-1}, y]$ with $\{y, t\} = 1$. Define a Poisson bracket on $B[s]$ by setting s to be Poisson central, and define

$$\Phi : S(W) \rightarrow B[s], \quad f\partial \mapsto fy + sf'.$$

As in §2.2, it is easy to check that Φ is a Poisson map. One can show using the methods of the proof of Proposition 4.3.14 that $\ker \Phi$ is the unique prime Poisson ideal of $S(W)$ of co-GK 3. This is proved in Remark 5.2.11 with a different argument, so we do not give a proof here.

Let \mathfrak{g} be a Lie algebra and let $\chi \in \mathfrak{g}^*$. The closure of $\mathcal{O}(\chi)$ is, by definition, $V(P(\chi))$. We call $V(P(\chi)) = \overline{\mathcal{O}(\chi)}$ the *orbit closure* of χ . (Technically, this term should probably be ‘pseudo-orbit closure’; we have used wording which we find more pleasant at the cost of a slight abuse of terminology.) Note that μ is in the orbit closure of χ if and only if $P(\mu) \supseteq P(\chi)$.

The orbit closure relations for one-point local functions are essentially given by Corollary 4.2.9, but for arbitrary local functions they are quite complex. Consider the following example, which for simplicity we give for $W_{\geq -1}$ only.

Example 4.3.16. Let $\mathbb{k} = \mathbb{C}$, let $x \neq y \in \mathbb{C}$ and let $a, b, c, d \in \mathbb{C}$ with (a, b) not both zero and likewise for (c, d) . Let $\chi = \chi_{x;a,b} + \chi_{y;c,d}$. We claim that $Loc_{W_{\geq -1}}^{\leq 1} \subseteq \mathbb{O}(\chi)$ and, in particular, $\overline{\mathbb{O}(\chi)}$ contains the functions $0, \chi_{x;a,b}$ and $\chi_{y;c,d}$.

To see this, let $\gamma, s \in \mathbb{C}^\times$. Consider the local function

$$\chi(\gamma, s) := \chi_{s;\gamma/s,b} + \chi_{0;-\gamma/s,d} \in W_{\geq -1}^*$$

By Theorems 4.3.1 and 4.2.1, the $\chi(\gamma, s)$ are all in $\mathbb{O}(\chi)$. For $f \in \mathbb{k}[t]$, we have

$$\lim_{s \rightarrow 0} \chi(\gamma, s)(f\partial) = \lim_{s \rightarrow 0} \gamma \frac{f(s) - f(0)}{s} + bf'(s) + df'(0) = (\gamma + b + d)f'(0).$$

As Zariski closed sets are closed in the complex analytic topology,

$$\lim_{s \rightarrow 0} \chi(\gamma, s) = \chi_{0;0,\gamma+b+d} \in \overline{\mathbb{O}(\chi)}$$

for any γ . Applying Corollary 4.2.9, $\overline{\mathbb{O}(\chi)}$ contains all of $Loc_{W_{\geq -1}}^{\leq 1}$.

The same statement holds for arbitrary \mathbb{k} by the Lefschetz principle.

Although arbitrary orbit closures are complicated, we are able to give a partial description of orbit closures of general local functions in the next corollary.

COROLLARY 4.3.17. *Let $\mathfrak{g} = Vir, W, W_{\geq -1}$ or $W_{\geq 1}$.*

- (a) *Let $\mu, \chi \in \mathfrak{g}^*$ be local functions with disjoint support and let ν be in the orbit closure of μ . Then $\nu + \chi$ is in the orbit closure of $\mu + \chi$.*
- (b) *Let χ be a local function. Then 0 is not in the orbit closure of χ if and only if $\lambda(\chi) = (2)$, i.e. $\chi = \chi_{x;\alpha,\beta}$ for some $\beta \in \mathbb{k}^\times$.*

Proof. For part (a), by Theorem 4.3.1, the pseudo-orbit of $\mu + \chi$ consists of all local functions of the form $\mu' + \chi'$ where $\mu' \in \mathbb{O}(\mu)$, $\chi' \in \mathbb{O}(\chi)$ and the supports of μ' and χ' are disjoint. A modification of Corollary 4.2.9 for the multipoint case implies that the orbit closure $\overline{\mathbb{O}(\mu + \chi)}$ contains $\nu + \chi$.

For part (b), Corollary 4.2.9 gives that if $\lambda(\chi) = (2)$, then $0 \notin \overline{\mathbb{O}(\chi)}$.

Assume now that $0 \notin \overline{\mathbb{O}(\chi)}$. Write

$$\chi = \mu_1 + \cdots + \mu_\ell + \nu_1 + \cdots + \nu_r,$$

where the μ_i, ν_j are 1-point local functions supported at distinct points, the μ_i have order zero or one and the ν_j have order $n_j > 1$. By part (a), 0 must not be in the orbit closure of at least one of the component 1-point local functions of χ , so $\ell \geq 1$ by Corollary 4.2.9.

Write $\mu_i = \chi_{x_i;\beta_i,\gamma_i}$ and $\nu_j = \chi_{y_j;\alpha_0^j, \dots, \alpha_{n_j}^j}$, where $\alpha_{n_j}^j \neq 0$. If $r \geq 1$, then let $\mu'_i = \chi_{y_1;\beta_i,\gamma_i}$. As we may move points within an orbit,

$$\mu'_1 + \cdots + \mu'_\ell + \nu_1,$$

which is a 1-point local function of order $n_1 > 1$, is in the orbit closure of $\mu_1 + \cdots + \mu_\ell + \nu_1$. By part (a) and the previous paragraph, then 0 is in the orbit closure of χ , a contradiction. Thus, $r = 0$.

If $\ell \geq 2$, then, applying Example 4.3.16 and part (a) repeatedly, we have $0 \in \overline{\mathbb{O}(\chi)}$. Thus, $\ell = 1$, and by Corollary 4.2.9 $\chi = \mu_1$ must have order 1. □

We now apply our results to classify maximal Poisson ideals of $S(Vir)$.

COROLLARY 4.3.18. *The maximal Poisson ideals of $S(Vir)$ are the ideals $(z - \zeta)$ for $\zeta \in \mathbb{k}^\times$, the ideals $P(\chi_{1;0,\gamma})$ for $\gamma \in \mathbb{k}^\times$, and the augmentation ideal $P(0)$.*

Proof. That $(z - \zeta)$ is a maximal Poisson ideal if $\zeta \in \mathbb{k}^\times$ is Corollary 3.3.5. If Q is any other maximal Poisson ideal, then we must have $z \in Q$; it follows from the Nullstellensatz and Theorem 3.2.1 that there is a local function χ with $Q = P(\chi)$. If $\lambda(\chi) \neq (2)$, then by Corollary 4.3.17 we have $P(\chi) \subseteq P(0)$, so we must have $\chi = 0$ and Q must be the augmentation ideal $P(0)$.

If $\lambda(\chi) = (2)$, then $P(\chi)$ is maximal among Poisson primitive ideals by Corollaries 4.2.9 and 4.3.17, and thus maximal among Poisson ideals. \square

5. Prime Poisson ideals

In this section we apply our results on Poisson primitive ideals to the structure of arbitrary Poisson prime ideals of $S(\mathfrak{g})$, where \mathfrak{g} is one of our Lie algebras of interest. After a preliminary subsection on birational maps in our context, we show that the partition data associated to local functions may be used to parameterise Poisson primes of $S(W)$ and $S(W_{\geq 1})$. This will allow us to show that the bijection $\text{PSpec}_{\text{prim}} S(W) \rightarrow \text{PSpec}_{\text{prim}} S(W_{\geq -1})$ of Corollary 4.3.11 extends to prime Poisson ideals. We then investigate when the Poisson analogue of the Dixmier–Moeglin equivalence holds for $S(\mathfrak{g})$ and, in particular, when Poisson primitive ideals are locally closed in $\text{PSpec} S(\mathfrak{g})$; we show that this is almost always, but not always, the case. Finally, we answer a question of León Sánchez and the second author [LS08] on heights of Poisson prime ideals, and in the process show that $\text{PSpec} S(W)$ has the somewhat counterintuitive property that every proper radical Poisson ideal contains a proper Poisson primitive ideal.

5.1 Birational maps

Chevalley’s fundamental result that images of algebraic maps are constructible holds for morphisms of finite presentation [SP22, Theorem 10.29.10, Tag 00FE] and, thus, applies to any homomorphism from $S(\mathfrak{g})$ to an affine algebra. The main result of this subsection is related to Chevalley’s theorem, and allows us to conclude that two domains are birational, even in our non-noetherian context.

PROPOSITION 5.1.1. *Let A, B be commutative \mathbb{k} -algebras which are domains, and let $\phi : A \rightarrow B$ be an injective homomorphism. Assume also that B is affine, (we make no additional assumption on A) and that the map $\text{MSpec } B \rightarrow \text{MSpec } A$ is also injective. Then there is $f \in A \setminus \{0\}$ so that the natural map $A[f^{-1}] \rightarrow B[f^{-1}]$ is an isomorphism.*

Before proving Proposition 5.1.1 we provide a preliminary lemma.

LEMMA 5.1.2. *Let A, B be commutative \mathbb{k} -algebras which are domains, and let $\phi : A \rightarrow B$ be an injective homomorphism. Assume also that B is affine (we make no additional assumption on A). Let $b \in B \setminus \{0\}$. There is $a \in A \setminus \{0\}$ so that if $\chi : A \rightarrow \mathbb{k}$ is a homomorphism with $\chi(a) \neq 0$, then there is some $\chi' : B \rightarrow \mathbb{k}$ with $\chi'\phi = \chi$ and $\chi'(b) \neq 0$.*

Proof. Without loss of generality $A \subseteq B$ and ϕ is the inclusion. Denote by $B[b^{-1}]$ the localisation of B by b . As B is affine, $B[b^{-1}]$ is finitely generated over A . By Chevalley’s theorem the image of $\text{MSpec } B[b^{-1}]$ contains an open subset $\{\mathfrak{m} \in \text{MSpec } A \mid a \notin \mathfrak{m}\}$ of $\text{MSpec } A$ defined by some element $a \in A \setminus \{0\}$. This implies that $a \in A$ satisfies the desired property. \square

Proof of Proposition 5.1.1. As $\dim_{\mathbb{k}} A \leq \dim_{\mathbb{k}} B \leq \aleph_0 < |\mathbb{k}|$, A is a Jacobson ring and the Nullstellensatz applies. In particular, if $\mathfrak{m} \in \text{MSpec } A$, then the natural map $\mathbb{k} \rightarrow A/\mathfrak{m}$ is an isomorphism.

Without loss of generality $A \subseteq B$ and ϕ is the inclusion. Let $Q(A)$ and $Q(B)$ be the quotient fields of A and B , respectively. Assume that $Q(B)$ is not algebraic over $Q(A)$. Then there exists $x \in B$ so that x is not algebraic over $Q(A)$, i.e. the natural map $A[t] \rightarrow B, t \mapsto x$ is injective; we denote the image of this map by $A[x]$.

Choose $p \in A[x] \setminus \{0\}$ to be the element defined by Lemma 5.1.2 for $A = A[x]$, $B = B$ and $b = 1$. Let $p_0 \in A$ be a nonzero coefficient of p . Then, for every $\mathfrak{m} \in \text{MSpec } A$ with $p_0 \notin \mathfrak{m}$ there are at most finitely many $x_0 \in \mathbb{k}$ such that $p(x_0) \in \mathfrak{m}$. Every other x_0 provides a maximal ideal $(\mathfrak{m}, x - x_0)$ of $A[x]$ which extends to a maximal ideal of B thanks to Lemma 5.1.2. Thus, every such \mathfrak{m} has uncountably many preimages in $\text{MSpec } B$, contradicting our assumption.

Thus, we can assume that $Q(B)$ is algebraic over $Q(A)$. Let B be generated as a \mathbb{k} -algebra by b_1, \dots, b_k . If $Q(A) = Q(B)$, then every b_i can be written a_i^+ / a_i^- with $a_i^\pm \in A \setminus \{0\}$; further, we have that $A[f^{-1}] = B[f^{-1}]$ for $f = a_1^- \cdots a_k^-$. Thus, we can assume that $Q(A) \neq Q(B)$ and, hence, there exists $x \in B \setminus Q(A)$.

As x is algebraic over $Q(A)$ there are $a_0, a_1, \dots, a_n \in A$ with $a_0 \neq 0, a_n \neq 0$ such that

$$F(x) := a_0x^0 + a_1x^1 + a_2x^2 + \cdots + a_nx^n = 0$$

with $n \geq 2$. Let $p = p^+ / a_n^d$ with $p^+ \in A \setminus \{0\}$, $d \in \mathbb{Z}_{\geq 0}$ be the resultant of F . Consider $\mathfrak{m} \in \text{MSpec } A$ with $a_n, p^+ \notin \mathfrak{m}$. Then it is well-known that there are exactly n maximal ideals \mathfrak{m}' of $A[x]$ with $\mathfrak{m} = \mathfrak{m}' \cap A$. Pick $y \in A[x]$ to be the element defined by Lemma 5.1.2 for $A = A[x]$, $B = B$ and $b = 1$. This element y is algebraic over A and thus there are $a'_0, a'_1, \dots \in A$ with $a'_0 \neq 0$ such that

$$a'_0y^0 + a'_1y^1 + a'_2y^2 + \cdots = 0.$$

If \mathfrak{m}' is a maximal ideal of $A[x]$ such that $a'_0 \notin \mathfrak{m} = \mathfrak{m}' \cap A$, then we have $y \notin \mathfrak{m}'$.

Pick $\mathfrak{m} \in \text{MSpec } A$ such that $a_n p^+ a'_0 \notin \mathfrak{m}$. Then there are at least n maximal ideals \mathfrak{m}'' of B such that $\mathfrak{m}'' \cap A = \mathfrak{m}$. This contradicts the injectivity of $\text{MSpec } B \rightarrow \text{MSpec } A$. \square

5.2 Parameterising Poisson primes

Throughout this subsection, let $\mathfrak{g} = W$ or $W_{\geq -1}$. We refine Remark 4.3.3 to a parameterisation of Poisson primes of $S(\mathfrak{g})$; we use this to show in Corollary 5.2.14 that the bijection of Corollary 4.3.11 extends to prime Poisson ideals.

Given a local function χ , recall the definition of the partition $\lambda(\chi)$ from § 3. Write $\lambda(\chi) = (m_i)$ and define

$$D(\lambda(\chi)) := 2 \sum \left\lfloor \frac{m_i + 1}{2} \right\rfloor.$$

By Proposition 4.3.6 and Remark 4.2.12,

$$\dim \mathbb{O}(\chi) = D(\lambda(\chi)), \tag{5.2.1}$$

and, in particular, $\dim \mathbb{O}(\chi)$ depends only on $\lambda(\chi)$.

Fix a partition $\lambda = (m_1 \geq m_2 \geq \cdots \geq m_r)$, where $m_i \in \mathbb{Z}_{\geq 1}$, and let

$$V_{\mathfrak{g}}^\lambda := \{\chi \in \mathfrak{g}^* \mid \lambda(\chi) = \lambda\}.$$

Let

$$\text{Loc}_{\mathfrak{g}}^\lambda := \prod_{i=1}^r \text{Loc}_{\mathfrak{g}}^{m_i-1},$$

where recall that $\text{Loc}_{\mathfrak{g}}^m = \{\chi_{x;\alpha_0, \dots, \alpha_m} \in \mathfrak{g}^* \mid \alpha_m \neq 0\}$. There is a natural summation map $\Sigma_{\mathfrak{g}}^\lambda : \text{Loc}_{\mathfrak{g}}^\lambda \rightarrow \mathfrak{g}^*$ defined by $\Sigma_{\mathfrak{g}}^\lambda(\chi_1, \dots, \chi_r) = \sum \chi_i$. As in the proof of Proposition 4.3.6, let

$$U_{\mathfrak{g}}^\lambda := \{(\chi_{x_1; \underline{\alpha}^1}, \dots, \chi_{x_r; \underline{\alpha}^r}) \in \text{Loc}_{\mathfrak{g}}^\lambda \mid \text{the } x_i \text{ are all distinct}\}$$

(here $\underline{\alpha}^i$ stands for a sequence $\alpha_0^i, \alpha_1^i, \dots$). Note that $\Sigma_{\mathfrak{g}}^\lambda(U_{\mathfrak{g}}^\lambda)$ is precisely $V_{\mathfrak{g}}^\lambda$, whereas $\Sigma_{\mathfrak{g}}^\lambda(\text{Loc}_{\mathfrak{g}}^\lambda)$ usually includes other local functions.

Let the partition

$$\lambda^e := (m_i \mid m_i \text{ is even})$$

consist of the even parts occurring in λ (with the same multiplicity); set $k := \text{len}(\lambda^e)$, $k_2 := |\{i \mid m_i = 2\}|$. For any $m_i \in \lambda^e$ let

$$\pi_i : \text{Loc}_{\mathfrak{g}}^{m_i-1} \rightarrow \begin{cases} \mathbb{A}^1 & \text{if } m_i > 2 \\ \mathbb{A}^1 \setminus \{0\} & \text{if } m_i = 2 \end{cases}$$

be the morphism F_{m_i-1} defined in Corollary 4.2.11. The fibres of π_i are pseudo-orbits (that is, $\widehat{DLoc}^{\leq m_i-1}$ -orbits) in $\text{Loc}_{\mathfrak{g}}^{m_i-1}$.

Let

$$\pi_{\mathfrak{g}}^{\lambda} : \text{Loc}_{\mathfrak{g}}^{\lambda} \rightarrow \mathbb{A}^{k-k_2} \times (\mathbb{A}^1 \setminus \{0\})^{k_2}$$

be the composition

$$\text{Loc}_{\mathfrak{g}}^{\lambda} \xrightarrow{\text{pr}} \text{Loc}_{\mathfrak{g}}^{\lambda^e} \xrightarrow{\prod \pi_i} [\mathbb{A}^1 \times \dots] \times [(\mathbb{A}^1 \setminus \{0\}) \times \dots] = \mathbb{A}^{k-k_2} \times (\mathbb{A}^1 \setminus \{0\})^{\times k_2},$$

where pr denotes projection.

Now, $\widehat{DLoc}^{\leq m_i-1}$ -orbits in $\text{Loc}_{\mathfrak{g}}^{m_i-1}$ are nonsingular and all are of dimension $2\lfloor(m_i + 1)/2\rfloor$. Further, π_i is flat: any torsion-free module over $\mathbb{k}[x]$ or $\mathbb{k}[x, x^{-1}]$ is flat. Thus, by [Har77, Theorem III.10.2], π_i is smooth, and so $\pi_{\mathfrak{g}}^{\lambda}$ is also smooth.

Given a partition $\lambda = (m_i)$ as before, we also write

$$\lambda = (\underbrace{n_1, \dots, n_1}_{j_1}, \underbrace{n_2, \dots, n_2}_{j_2}, \dots, \underbrace{n_{\ell}, \dots, n_{\ell}}_{j_{\ell}}),$$

where $n_1 > n_2 > \dots > n_{\ell}$. Note that $m_1 = \dots = m_{j_1} = n_1$, $m_{j_1+1} = m_{j_1+2} = \dots = m_{j_1+j_2} = n_2$, etc., and $r = \text{len}(\lambda) = \sum_{i=1}^{\ell} j_i$. The multisymmetric group $\mathfrak{S}_{\lambda} := \mathfrak{S}_{j_1} \times \dots \times \mathfrak{S}_{j_{\ell}}$ acts on $\text{Loc}_{\mathfrak{g}}^{\lambda}$ and on $U_{\mathfrak{g}}^{\lambda}$ by permuting the factors. Further, \mathfrak{S}_{λ} also acts on $\mathbb{A}^{k-k_2} \times (\mathbb{A}^1 \setminus 0)^{\times k_2}$ by permutations (if n_i is odd, the factor \mathfrak{S}_{j_i} acts trivially). We name the action map

$$\mu_{\lambda} : \mathfrak{S}_{\lambda} \times \mathbb{A}^{k-k_2} \times (\mathbb{A}^1 \setminus 0)^{\times k_2} \rightarrow \mathbb{A}^{k-k_2} \times (\mathbb{A}^1 \setminus 0)^{\times k_2} \tag{5.2.2}$$

because we need to refer to it later. As in Remark 4.3.3, denote the quotient $[(\mathbb{A}^1)^{\times(k-k_2)} \times (\mathbb{A}^1 \setminus \{0\})^{\times k_2}] / \mathfrak{S}_{\lambda}$ by $\mathbb{A}(\lambda)$.

Note that $\pi_{\mathfrak{g}}^{\lambda}$ is \mathfrak{S}_{λ} -equivariant. There is thus an induced map

$$\overline{\pi}_{\mathfrak{g}}^{\lambda} : U_{\mathfrak{g}}^{\lambda} / \mathfrak{S}_{\lambda} \rightarrow \mathbb{A}(\lambda).$$

We remark that $U_{\mathfrak{g}}^{\lambda} / \mathfrak{S}_{\lambda}$ and $\mathbb{A}(\lambda)$ are affine. Further, $\mathbb{A}(\lambda)$ is nonsingular (in fact, isomorphic to \mathbb{A}^k or $\mathbb{A}^{k-1} \times (\mathbb{A}^1 \setminus \{0\})$) because \mathfrak{S}_{λ} is a reflection group, see also Remark 4.3.3. The fibres of $\overline{\pi}_{\mathfrak{g}}^{\lambda}$ are isomorphic to fibres of the smooth morphism $\pi_{\mathfrak{g}}^{\lambda}$ and are thus nonsingular. Therefore, $U_{\mathfrak{g}}^{\lambda} / \mathfrak{S}_{\lambda}$ is also nonsingular.

The summation map $\Sigma_{\mathfrak{g}}^{\lambda}$ factors through $\text{Loc}_{\mathfrak{g}}^{\lambda} / \mathfrak{S}_{\lambda}$. This induces a morphism of varieties

$$\psi_{\mathfrak{g}}^{\lambda} : U_{\mathfrak{g}}^{\lambda} / \mathfrak{S}_{\lambda} \rightarrow \mathfrak{g}^*,$$

which is injective, as any collection of one-point functions with distinct supports is linearly independent. The image of $\psi_{\mathfrak{g}}^{\lambda}$ is precisely the set $V_{\mathfrak{g}}^{\lambda}$ of local functions with order partition λ . Further note that $\psi_{\mathfrak{g}}^{\lambda}$ maps the tangent space to a \widehat{DLoc} -orbit of $\chi \in \mathfrak{g}^*$ at χ^* to $\mathfrak{g}_{\psi_{\mathfrak{g}}^{\lambda}(\chi)}$ (see Lemma 4.1.6 and 4.3.4), i.e. it satisfies the conditions of Propositions 2.1.7 and 2.3.1.

Remark 5.2.3. We may identify $V_{\mathfrak{g}}^\lambda$ with $U_{\mathfrak{g}}^\lambda/\mathfrak{S}_\lambda$ as a set. This construction allows us to write the set of local functions on \mathfrak{g} as a countable union of the affine varieties $U_{\mathfrak{g}}^\lambda/\mathfrak{S}_\lambda$, further justifying the statement in Remark 3.0.3 that most elements of \mathfrak{g}^* are not local.

We abuse notation slightly, and identify Loc_W^λ , via restriction, with the subset of $Loc_{W_{\geq -1}}^\lambda$ consisting of functions whose support does not contain 0. The diagram

$$\begin{array}{ccc}
 W^* & \xrightarrow{\text{res}} & W_{\geq -1}^* \\
 \psi_W^\lambda \uparrow & & \uparrow \psi_{W_{\geq -1}}^\lambda \\
 U_W^\lambda/\mathfrak{S}_\lambda & \xrightarrow{\subset} & U_{W_{\geq -1}}^\lambda/\mathfrak{S}_\lambda \\
 \searrow \overline{\pi_W^\lambda} & & \swarrow \overline{\pi_{W_{\geq -1}}^\lambda} \\
 & \mathbb{A}(\lambda) &
 \end{array} \tag{5.2.4}$$

commutes, and both maps to $\mathbb{A}(\lambda)$ are surjective.

Let $\widehat{DLoc}^\lambda := \widehat{DLoc}^{\leq m_1-1} \times \cdots \times \widehat{DLoc}^{\leq m_r-1}$. The group \widehat{DLoc}^λ acts on $Loc_{W_{\geq -1}}^\lambda$, and there are partial \widehat{DLoc}^λ -actions on Loc_W^λ , on U_W^λ , and on $U_{W_{\geq -1}}^\lambda$; as usual, we speak of \widehat{DLoc}^λ -orbits on these varieties to mean the intersections with full orbits in $Loc_{W_{\geq -1}}^\lambda$. This (partial) action does not commute with the \mathfrak{S}_λ -action. Nevertheless, it is compatible with the natural action of \mathfrak{S}_λ on \widehat{DLoc}^λ and together these provide an action of the semidirect product $\widehat{DLoc}^\lambda \rtimes \mathfrak{S}_\lambda$. Further, by Corollary 4.2.11 the fibres of $\overline{\pi_{\mathfrak{g}}^\lambda}$ are precisely the images in $U_{\mathfrak{g}}^\lambda/\mathfrak{S}_\lambda$ of $\widehat{DLoc}^\lambda \rtimes \mathfrak{S}_\lambda$ -orbits in $U_{\mathfrak{g}}^\lambda$. (In particular, they all have dimension $D(\lambda)$.) It follows from Theorem 4.3.1 that these fibres correspond under $\psi_{\mathfrak{g}}^\lambda$ precisely with pseudo-orbits of local functions with partition λ . We refer to the image in $U^\lambda/\mathfrak{S}_\lambda$ of a $\widehat{DLoc}^\lambda \rtimes \mathfrak{S}_\lambda$ -orbit in $U_{\mathfrak{g}}^\lambda$ as simply a \widehat{DLoc}^λ -orbit quotient. We abuse terminology and say a subset or subvariety of $U^\lambda/\mathfrak{S}_\lambda$ is \widehat{DLoc}^λ -invariant if it is a union of \widehat{DLoc}^λ -orbit quotients.

We can use the maps $\psi_{\mathfrak{g}}^\lambda$ defined previously to strengthen Proposition 2.1.7 in our setting.

PROPOSITION 5.2.5. *Let $\mathfrak{g} = W$ or $\mathfrak{g} = W_{\geq -1}$. Let Q be a proper radical ideal of $S(\mathfrak{g})$ and let $Z := V(Q)$. Suppose also that for some λ the set $V_{\mathfrak{g}}^\lambda \cap Z$ is (Zariski) dense in Z . Then Q is Poisson if and only if $(\psi_{\mathfrak{g}}^\lambda)^{-1}(Z \cap V_{\mathfrak{g}}^\lambda)$ is \widehat{DLoc}^λ -invariant.*

Proof. Let $\psi := \psi_{\mathfrak{g}}^\lambda$, let $U := U_{\mathfrak{g}}^\lambda/\mathfrak{S}_\lambda$, let $X := \overline{\psi^{-1}(Z)} \subseteq U$ and let \tilde{X} be the preimage of X in U_W^λ . As $Z \cap V^\lambda$ is dense in Z , thus Q is the kernel of the composition

$$S(\mathfrak{g}) \xrightarrow{\psi^*} \mathbb{k}[U] \longrightarrow \mathbb{k}[X] \hookrightarrow \mathbb{k}[\tilde{X}]$$

and by Proposition 2.1.7, Q is Poisson if and only if $\mathfrak{g}_{\psi(x)} \subseteq \psi_x(T_x X)$ for all $x \in X$.

Denote by A the image of $S(\mathfrak{g})$ in $\mathbb{k}[X]$. The map $U \rightarrow \mathfrak{g}^*$ is injective and therefore the map $X \rightarrow \text{MSpec } A$ is also injective. Let $f \in A$ be the element given by Proposition 5.1.1. Then the open subsets $\text{MSpec}(A[f^{-1}])$ and $\text{MSpec}(\mathbb{k}[X][f^{-1}])$ of $\text{MSpec}(A)$ and X , respectively, are isomorphic to each other. This implies that ψ_x identifies the tangent spaces $T_x X$ and $T_{\psi(x)} Z$ for every $x \in \text{MSpec}(\mathbb{k}[X][f^{-1}])$.

Moreover, $\widetilde{X} \rightarrow X$ is finite and so by generic smoothness there are dense open affine subsets $X' \subseteq X$, $\widetilde{X}' \subseteq \widetilde{X}$ so that $\widetilde{X}' \rightarrow X'$ is étale and so identifies tangent spaces.

Let $X'' = X' \cap \text{MSpec}(\mathbb{k}[X][f^{-1}])$ and let $\widetilde{X}'' \subseteq \widetilde{X}$ be the preimage of X'' , which is also affine. For every $x \in X''$ and every preimage $\tilde{x} \in \widetilde{X}''$ of x , the map $\widetilde{X} \rightarrow \mathfrak{g}^*$ gives an isomorphism of the tangent spaces $T_{\tilde{x}}\widetilde{X}$ and $T_{\psi(x)}Z$.

Let \mathfrak{h} be the Lie algebra of \widehat{DLoc}^λ . The proofs of Lemma 4.1.6 and Lemma 4.3.4 show that $\psi_x(\mathfrak{h}_x) = \mathfrak{g}_{\psi(x)}$ for any $x \in U$. By Proposition 2.1.7 applied to the morphism $X'' \rightarrow \mathfrak{g}^*$ and the previous discussion, the ideal Q is Poisson if and only if $\mathfrak{h}_x \subseteq T_x\widetilde{X}$ for all $x \in \widetilde{X}''$. To finish the proof we note that the following conditions are equivalent:

- (a) $\mathfrak{h}_x \subseteq T_x\widetilde{X}$ for all $x \in \widetilde{X}''$ (that is, tangent spaces to \widehat{DLoc}^λ -orbits of points of \widetilde{X}'' are tangent to \widetilde{X});
- (b) $\widetilde{X} = \widetilde{X}''$ is \widehat{DLoc}^λ -invariant; that is, a union of \widehat{DLoc}^λ -orbits;
- (c) the defining ideal of $X = \overline{X''}$ is \widehat{DLoc}^λ -invariant. □

Let $J_{\mathfrak{g}}^\lambda := I(V_{\mathfrak{g}}^\lambda)$. We immediately obtain the following.

COROLLARY 5.2.6. *The ideal $J_{\mathfrak{g}}^\lambda$ is a prime Poisson ideal.*

Proof. That $J_{\mathfrak{g}}^\lambda$ is Poisson is an immediate application of Proposition 5.2.5. To see that it is prime, note that $U_{\mathfrak{g}}^\lambda/\mathfrak{S}_\lambda$ is irreducible and that $J_{\mathfrak{g}}^\lambda$ is the kernel of $(\psi_{\mathfrak{g}}^\lambda)^* : \mathbb{S}(\mathfrak{g}) \rightarrow \mathbb{k}[U_{\mathfrak{g}}^\lambda/\mathfrak{S}_\lambda]$. □

Remark 5.2.7. Recall that $V_{\mathfrak{g}}^\lambda \subset \mathfrak{g}^*$ is the image of the affine variety $U_{\mathfrak{g}}^\lambda/\mathfrak{S}_\lambda$. Thus, by Chevalley’s theorem $V_{\mathfrak{g}}^\lambda$ contains a nonempty open subset of $\overline{V_{\mathfrak{g}}^\lambda}$. Let U be the union of all open subsets of $\overline{V_{\mathfrak{g}}^\lambda}$ contained in $V_{\mathfrak{g}}^\lambda$. It is surely true that U is Poisson, in the sense that the defining ideal of $\overline{V_{\mathfrak{g}}^\lambda} \setminus U$ is a Poisson ideal. However, we cannot prove this at this point.

We next show how to parameterise Poisson primes of $\mathbb{S}(\mathfrak{g})$.

THEOREM 5.2.8. *Let $\mathfrak{g} = W$ or $\mathfrak{g} = W_{\geq -1}$.*

- (a) *Let Q be a nonzero proper prime Poisson ideal of $\mathbb{S}(\mathfrak{g})$. There is a unique $\lambda := \lambda(Q)$ so that $V(Q) \cap V_{\mathfrak{g}}^\lambda$ contains a nonempty open subset of $V(Q)$. Further, $Y(Q) := \overline{\pi_{\mathfrak{g}}^\lambda}(\psi_{\mathfrak{g}}^\lambda)^{-1}(V(Q))$ is a (nonempty) irreducible closed subvariety of $\mathbb{A}(\lambda)$, and*

$$\text{GKdim } \mathbb{S}(\mathfrak{g})/Q = \dim Y(Q) + D(\lambda(Q)).$$

- (b) *The function $Q \mapsto (\lambda(Q), Y(Q))$ defines a bijection*

$$\Psi_{\mathfrak{g}} : \text{PSpec } \mathbb{S}(\mathfrak{g}) \rightarrow \{(\lambda, Y) \mid Y \text{ is an irreducible subvariety of } \text{Var}_{\lambda(Q)}\}.$$

- (c) *If Q, Q' are Poisson primes of $\mathbb{S}(\mathfrak{g})$ with $\lambda(Q) = \lambda(Q')$, then $Q \subseteq Q'$ if and only if $Y(Q) \supseteq Y(Q')$.*

We call the partition $\lambda(Q)$ defined in Theorem 5.2.8 the *generic order partition of Q* , in other words the partition of a generic orbit in the associated variety. It is easy to check that the construction of Theorem 5.2.8(a) guarantees that a Poisson prime Q is the kernel of the map

$$\mathbb{S}(\mathfrak{g}) \rightarrow \mathbb{k}[\overline{(\pi_{\mathfrak{g}}^\lambda)^{-1}(Y(Q))}], \tag{5.2.9}$$

where $\lambda = \lambda(Q)$. This provides the explicit inverse to the map $\Psi_{\mathfrak{g}}$ of Theorem 5.2.8(b).

Proof. Let Q be a nonzero proper prime Poisson ideal of $S(\mathfrak{g})$ and let $Z := V(Q)$. As Q is prime, there can be at most one λ so that $Z \cap V_{\mathfrak{g}}^{\lambda}$ contains a nonempty open subset of Z ; it remains to establish existence of some such λ .

By Theorem 3.0.4 $d := \text{GKdim}(S(\mathfrak{g})/Q) < \infty$. By (5.2.1) there are thus only finitely many partitions λ such that $\dim \mathbb{O}(\chi) \leq d$ for any χ with order partition λ , and so for some $\lambda^1, \dots, \lambda^n$ we have

$$Z \subseteq V_{\mathfrak{g}}^{\lambda^1} \cup \dots \cup V_{\mathfrak{g}}^{\lambda^n}.$$

Fix i and let $\lambda = \lambda^i$. Let Q' be a minimal prime of $Q + J_{\mathfrak{g}}^{\lambda}$, which exists by Proposition 3.0.6, and let $Z' = V(Q')$, which is an irreducible component of $Z \cap \overline{V_{\mathfrak{g}}^{\lambda}}$. Let $X' = (\psi_{\mathfrak{g}}^{\lambda})^{-1}(Z')$. By Chevalley's theorem $V_{\mathfrak{g}}^{\lambda} \cap Z$ contains a dense subset U_i which is locally closed in Z . As $\bigcup U_i = Z$, by primeness of Q some U_j has $\overline{U_j} = Z$: in other words, U_j is open in its closure Z . Thus, $\lambda(Q) = \lambda_j$.

Let $\lambda := \lambda(Q) = \lambda_j$, and let $X := (\psi_{\mathfrak{g}}^{\lambda})^{-1}(Z)$. This is an irreducible closed subvariety of $U_{\mathfrak{g}}^{\lambda}/\mathfrak{S}_{\lambda}$, and by Proposition 5.2.5 it is a union of \widehat{DLoc}^{λ} -orbits. It therefore follows that $Y(Q) := \overline{\pi_{\mathfrak{g}}^{\lambda}(X)}$ is closed in $\mathbb{A}(\lambda)$. This proves all but the last statement of part (a).

We now prove part (b). Fix a partition λ and let $k := \text{len}(\lambda^e)$. Let \mathfrak{S}_{λ} acts as in (5.2.2). Let Y be an irreducible closed subvariety of $\mathbb{A}(\lambda)$. Let $X := (\overline{\pi_{\mathfrak{g}}^{\lambda}})^{-1}(Y)$. As the fibres of $\overline{\pi_{\mathfrak{g}}^{\lambda}}$ are \widehat{DLoc}^{λ} -orbit quotients and are therefore irreducible (note that \widehat{DLoc}^{λ} is a connected algebraic group), X is an irreducible \widehat{DLoc}^{λ} -invariant closed subvariety of $U_{\mathfrak{g}}^{\lambda}/\mathfrak{S}_{\lambda}$ with $Y = \overline{\pi_{\mathfrak{g}}^{\lambda}(X)}$. Let Q be the kernel of $S(\mathfrak{g}) \rightarrow \mathbb{k}[X]$; clearly Q is prime, and by Proposition 5.2.5 Q is a Poisson ideal of $S(\mathfrak{g})$. Let $Z := V(Q)$. By applying Chevalley's theorem to the morphism $\psi_{\mathfrak{g}}^{\lambda} : X \rightarrow Z$ we see that $\psi_{\mathfrak{g}}^{\lambda}(X)$ contains a nonempty open subset of Z . This shows, in particular, that $\lambda = \lambda(Q)$, and it is clear that $Y = Y(Q)$.

Conversely, given Q apply the procedure in the previous paragraph to $\lambda(Q)$ and $Y(Q)$; one recovers Q , see also (5.2.9). This shows that $\Psi_{\mathfrak{g}}$ is bijective, completing the proof of part (b).

Given a nonzero proper Poisson prime Q of $S(\mathfrak{g})$ let $\lambda := \lambda(Q)$ and $Y := Y(Q)$. Let $X := (\overline{\pi_{\mathfrak{g}}^{\lambda}})^{-1}(Y)$. We have seen that the fibres of $\overline{\pi_{\mathfrak{g}}^{\lambda}}$ have dimension $D(\lambda)$ and so $\dim X = \dim Y + D(\lambda)$. It is shown in the proof of Proposition 5.2.5 that $S(\mathfrak{g})$ and $\mathbb{k}[X]$ are birational, so $\dim X = \text{GKdim } S(\mathfrak{g})/Q$. This completes part (a).

For part (c), let $\lambda = \lambda(Q) = \lambda(Q')$ and let $\psi = \psi_{\mathfrak{g}}^{\lambda}$ and $\pi = \overline{\pi_{\mathfrak{g}}^{\lambda}}$. Now let $Z = V(Q)$, $X = \psi^{-1}(Z) = \psi^{-1}(Z \cap V_{\mathfrak{g}}^{\lambda})$ and $Y = Y(Q) = \pi(X)$. Likewise define $Z' = V(Q')$, $X' = \psi^{-1}(Z')$ and $Y' = Y(Q')$.

By the Nullstellensatz,

$$Q \subseteq Q' \iff Z \supseteq Z'.$$

Thus, if $Q \subseteq Q'$ we have $Z \cap V_{\mathfrak{g}}^{\lambda} \supseteq Z' \cap V_{\mathfrak{g}}^{\lambda}$ and we see immediately that $Y \supseteq Y'$. Conversely, if $Y \supseteq Y'$, then $\psi(X) = Z \cap V_{\mathfrak{g}}^{\lambda} \supseteq \psi(X') = Z' \cap V_{\mathfrak{g}}^{\lambda}$. The proof of part (b) showed that $Z = \overline{Z \cap V_{\mathfrak{g}}^{\lambda}}$, which clearly contains $\overline{Z' \cap V_{\mathfrak{g}}^{\lambda}} = Z'$. □

It follows from Theorem 5.2.8 that $\text{PSpec}(W)$ and $\text{PSpec}(W_{\geq -1})$ are partitioned into countably many affine strata, corresponding to partitions. Given a partition λ , the corresponding stratum consists of prime Poisson ideals Q with $\lambda(Q) = \lambda$ and is homeomorphic to the affine variety $\mathbb{A}(\lambda)$, an open subset of a finite-dimensional affine space. However, we do not know how to

tell in terms of the parameterisation $Q \leftrightarrow (\lambda, Y)$ when $Q \subseteq Q'$ for arbitrary Q, Q' . In fact, we cannot answer this question completely even when Q, Q' are Poisson primitive; see Corollary 4.3.17 and Example 4.3.16 for some discussion of the complexities.

COROLLARY 5.2.10 (Cf. [PS20, Lemma 2.9]). *Let $\mathfrak{g} = W$ or $\mathfrak{g} = W_{\geq -1}$ and let Q be a prime Poisson ideal of $S(\mathfrak{g})$. Then there exists $f \in S(\mathfrak{g}) \setminus Q$ such that $(S(\mathfrak{g})/Q)[f^{-1}]$ is a finitely generated Poisson algebra.*

Proof. It is clear that $(S(\mathfrak{g})/Q)[f^{-1}]$ is a Poisson algebra for every such f and, thus, we need to choose f in such a way that $(S(\mathfrak{g})/Q)[f^{-1}]$ is finitely generated. Set $A = S(\mathfrak{g})/Q$, $\lambda = \lambda(Q)$ and $B = \mathbb{k}[(\pi_{\mathfrak{g}}^{\lambda})^{-1}(Y(Q))]$; note that B is finitely generated as it is easy to check that $(\pi_{\mathfrak{g}}^{\lambda})^{-1}(Y(Q))$ is affine. By (5.2.9) we have an injective map $A \rightarrow B$ and the induced map $\text{MSpec } B \rightarrow \text{MSpec } A$ is injective as well because it is a restriction of an injective map $U_{\mathfrak{g}}^{\lambda}/\mathfrak{S}_{\lambda} \rightarrow \mathfrak{g}^*$. This together with Proposition 5.1.1 guarantees the existence of f . \square

Remark 5.2.11. Let Q be a prime Poisson ideal of $S(W)$ of co-GK 3. By Theorem 5.2.8 we have

$$3 = \dim Y(Q) + D(\lambda(Q)),$$

and, recalling that $D(\lambda(Q))$ is even by definition, we must have $\dim Y(Q) = 1$ and $D(\lambda(Q)) = 2$. (The case $\dim Y(Q) = 3, D(\lambda(Q)) = 0$ cannot occur.) Thus, $\lambda(Q) = (2)$ and $\mathbb{A}(\lambda) = \mathbb{A}^1 \setminus \{0\} = Y(Q)$.

Now, if $\chi \in W^*$, then $\lambda(\chi) = (2)$ if and only if $\chi = \chi_{x;\alpha,\gamma}$ for some $x, \gamma \in \mathbb{k}^*, \alpha \in \mathbb{k}$, and so

$$Q = \bigcap_{\substack{x,\gamma \in \mathbb{k}^* \\ \alpha \in \mathbb{k}}} P(\chi_{x;\alpha,\gamma}) = \bigcap_{\gamma} \ker p_{\gamma},$$

using Lemma 2.2.3. However, from the definition of p_{γ} in (2.2.2) we see that $\bigcap \ker p_{\gamma}$ is precisely the kernel of the map Φ from Remark 4.3.15. Thus, $\ker \Phi$ is the only prime Poisson ideal of $S(W)$ of co-GK 3.

We thank the anonymous referee for drawing our attention to this fact.

Remark 5.2.12. Fix a partition λ . Taking $Y = \mathbb{A}(\lambda)$ in Theorem 5.2.8 we see that the pair (λ, Y) corresponds to the ideal $J_{\mathfrak{g}}^{\lambda}$. This is another way to see that $J_{\mathfrak{g}}^{\lambda}$ is prime.

Remark 5.2.13. By Corollary 3.3.6 any noncentrally generated prime Poisson ideal of $S(\text{Vir})$ strictly contains (z) and thus corresponds to a nontrivial prime Poisson ideal of $S(W)$ as given in Theorem 5.2.8. On the other hand, centrally generated prime Poisson ideals of course correspond to prime ideals of $\mathbb{k}[z]$. Thus, $\text{PSpec } S(\text{Vir})$ is also partitioned into countably many strata, each homeomorphic either to a finite-dimensional affine space or to some $\mathbb{A}^k \times (\mathbb{A}^1 \setminus \{0\})$.

We now prove that restriction from $S(W)$ to $S(W_{\geq -1})$ induces a bijection on arbitrary prime Poisson ideals, not just Poisson primitive ideals. We conjecture that this bijection is, in fact, a homeomorphism between $\text{PSpec}_{\text{prim}}(S(W))$ and $\text{PSpec}_{\text{prim}}(S(W_{\geq -1}))$, but at this point we cannot prove it.

COROLLARY 5.2.14. *Restriction induces a bijection between prime Poisson $S(W)$ and prime Poisson ideals of $S(W_{\geq -1})$, and a bijection between irreducible closed subsets of*

$$\text{PSpec}_{\text{prim}} S(W_{\geq -1}) \quad \text{and of} \quad \text{PSpec}_{\text{prim}} S(W).$$

Proof. We must show that restriction gives a bijection

$$\text{res} : \{\text{prime Poisson ideals of } S(W)\} \rightarrow \{\text{prime Poisson ideals of } S(W_{\geq -1})\}.$$

Certainly, if J is a prime Poisson ideal of $S(W)$, then J is closed under all $\{v, -\}$ for $v \in W_{\geq -1}$ and so $J \cap S(W_{\geq -1})$ is a prime Poisson ideal. Thus, res is well-defined.

Let $P \in \text{PSpec } S(W)$. Let $(\lambda, Y) := \Psi_W(P)$ and let $Q := \Psi_{W_{\geq -1}}^{-1}(\lambda, Y)$. We claim that $Q = \text{res}(P)$.

For each $y \in Y$ choose a representative $\chi_y \in W^*$ lying in the pseudo-orbit corresponding via $\overline{\pi_W^\lambda}$ to y . Then Theorem 5.2.8 gives that

$$P = \Psi_W^{-1}(\lambda, Y) = \bigcap \{\mathfrak{m}_\chi \mid \chi \in W^*, \lambda(\chi) = \lambda, \overline{\pi_W^\lambda}(\chi) \in Y\}.$$

By Lemma 2.0.4, this is $\bigcap_{y \in Y} P(\chi_y)$. Likewise, Q is equal to

$$\Psi_{W_{\geq -1}}^{-1}(\lambda, Y) = \bigcap \{\mathfrak{m}_\chi \mid \chi \in W_{\geq -1}^*, \lambda(\chi) = \lambda, \overline{\pi_{W_{\geq -1}}^\lambda}(\chi) \in Y\} = \bigcap_{y \in Y} P(\chi_y|_{W_{\geq -1}}).$$

By Proposition 4.3.6(a), this is

$$\bigcap_{y \in Y} P(\chi_y) \cap S(W_{\geq -1}) = P \cap S(W_{\geq -1}) = \text{res}(P).$$

As res acts on nonzero Poisson prime ideals as the composition of the bijections Ψ_W and $\Psi_{W_{\geq -1}}^{-1}$, it is a bijection. □

5.3 The PDME

We next consider which Poisson prime ideals satisfy the PDME. This is the Poisson version of the equivalent conditions for primitive ideals in enveloping algebras of finite-dimensional Lie algebras, which are due to Dixmier and Moeglin. We describe the conditions here.

Let Q be a Poisson prime ideal in a Poisson algebra A , which we assume to be a domain. Then Q is *Poisson locally closed* if it is locally closed in the Zariski topology on $\text{PSpec}(A)$. We say Q is *Poisson rational* if the Poisson centre of the field of fractions of A/Q is algebraic over \mathbb{k} . We say the *PDME holds for A* if for any prime Poisson ideal Q of A , the conditions

- Q is Poisson locally closed,
- Q is Poisson primitive,
- Q is Poisson rational,

are equivalent. If \mathfrak{g} is a finite-dimensional Lie algebra, then the PDME holds for $S(\mathfrak{g})$ (see [LL19, Theorem 2]).

On the other hand, the next result shows that the PDME fails for $S(\text{Vir})$, $S(W)$ and $S(W_{\geq -1})$. However, it *almost* holds: for all these algebras, there is only one Poisson prime for which the PDME fails.

THEOREM 5.3.1. *The PDME holds for all prime Poisson ideals of $S(\text{Vir})$ except for (z) : that is, if $Q \neq (z)$ is a prime Poisson ideal of $S(\text{Vir})$, then Q is locally closed in the Poisson spectrum if and only if Q is Poisson primitive, if and only if Q is Poisson rational. However, (z) is Poisson primitive and Poisson rational but not Poisson locally closed. Thus, the PDME for $S(W)$ fails but holds for all prime Poisson ideals except for (0) . Likewise, the PDME for $S(W_{\geq -1})$ fails but holds for all prime Poisson ideals except for (0) .*

Remark 5.3.2. By Corollary 3.3.5, the Poisson spectrum of $S(\text{Vir})[z^{-1}]$, which is in natural bijection with the set of prime Poisson ideals of $S(\text{Vir})$ that do not contain z , consists of (0) and

the ideals $(z - \lambda)$ for $\lambda \in \mathbb{k}^*$: in other words, $\text{PSpec}(S(\text{Vir})[z^{-1}]) \cong \mathbb{A}^1 \setminus \{0\}$. Therefore, the most interesting structure of $S(\text{Vir})$ is concentrated above the ideal (z) , which Theorem 5.3.1 shows to be pathological in some sense.

To prove Theorem 5.3.1, we need notation for the Zariski topology on $\text{PSpec } S(\mathfrak{g})$. If A is a Poisson algebra and N is a Poisson ideal of A , we denote the corresponding closed subset of $\text{PSpec } A$ by

$$V_P(N) := \{P \in \text{PSpec } A \mid N \subseteq P\}.$$

We caution the reader that although $V(N) \subseteq \text{MSpec } A$ consists of maximal ideals, $V_P(N) \subseteq \text{PSpec } A$ consists of prime (Poisson) ideals.

Before proving Theorem 5.3.1, we note that the standard equivalent condition for a Poisson prime to be locally closed also holds in this infinite-dimensional setting.

LEMMA 5.3.3. *Let A be a (possibly non-noetherian) Poisson algebra, and let Q be a prime Poisson ideal of A . Then Q is locally closed in $\text{PSpec } A$ if and only if there is some $f \in A \setminus Q$ so that $(A/Q)[f^{-1}]$ is Poisson simple.*

We leave the proof to the reader.

Proof of Theorem 5.3.1. Let \mathfrak{g} be any countable-dimensional Lie algebra. By [LS08, Theorem 6.3], for prime Poisson ideals of $S(\mathfrak{g})$, Poisson locally closed implies Poisson primitive and Poisson primitive is equivalent to Poisson rational. Thus, to prove the theorem for $S(\text{Vir})$ and $S(W)$, it suffices to prove the following.

- (a) Let Q be a prime Poisson ideal of $S(\text{Vir})$ with $Q \neq (z)$. If Q is Poisson primitive, then Q is Poisson locally closed.
- (b) The ideal (z) is Poisson primitive but not Poisson locally closed.

We first prove part (b). Let $\nu \in \text{Vir}^*$ be any nonlocal function with $\nu(z) = 0$ (for example, we can take ν to induce the nonlocal function \varkappa on W given in Remark 3.0.3). By Theorem 3.3.1, $P(\nu) = (z)$ and so (z) is Poisson primitive.

To prove that (z) is not Poisson locally closed, by Lemma 5.3.3 it is enough to prove that $S(W)[f^{-1}]$ is not Poisson simple for any $f \in S(W) \setminus \{0\}$. Suppose that $S(W)[f^{-1}]$ is Poisson simple. Then f is contained in all proper Poisson ideals of $S(W)$. Let $\chi \in W^*$ be a local function. The Poisson core $P(\chi)$ of χ is nontrivial, so $f \in P(\chi) \subseteq \mathfrak{m}_\chi$. In other words, for any local function $\chi \in W^*$, we have $\text{ev}_\chi(f) = 0$, which is ridiculous.

We now prove part (a). Let $\chi \in \text{Vir}^*$ with $P(\chi) \neq (z)$. We show that $P(\chi)$ is Poisson locally closed.

The ideals $(z - \zeta)$ with $\zeta \neq 0$ are maximal in $\text{PSpec } S(\text{Vir})$ by Corollary 3.3.5 and are, thus, closed points of $\text{PSpec } S(\text{Vir})$. Thus, we may assume that $P(\chi) \neq (z - \chi(z))$. By Theorem 3.3.1 χ is therefore local and $d := \dim \text{Vir}/\text{Vir}^\chi < \infty$.

Let $I(d - 1)$ be the Poisson ideal of $S(\text{Vir})$ defined in Lemma 3.1.4. We claim that

$$V_P(P(\chi)) \setminus V_P(I(d - 1)) = \{P(\chi)\}$$

so $P(\chi)$ is Poisson locally closed.

First, by (3.1.5) $I(d - 1) \not\subseteq \mathfrak{m}_\chi$ so $P(\chi) \notin V_P(I(d - 1))$. By Proposition 4.3.6 $d = \text{GKdim } S(\text{Vir})/P(\chi)$. Thus, if $\nu \in \mathbb{O}(\chi) \setminus \mathbb{O}(\chi)$, we have

$$d - 1 \geq \text{GKdim } S(\text{Vir})/P(\nu) = \dim \text{Vir} \cdot \nu,$$

where we have used Proposition 4.3.6 again for the last equality. Thus, $I(d - 1) \subseteq \mathfrak{m}_\nu$ by (3.1.5), so $I(d - 1) \subseteq P(\nu)$. It follows that if P is a Poisson prime ideal of $S(\text{Vir})$ with $P \not\subseteq P(\chi)$,

then writing

$$P = \bigcap \{P(\nu) \mid \mathfrak{m}_\nu \supseteq P\},$$

we have $P \supseteq I(d - 1)$, establishing the claim.

The proof for $S(W_{\geq -1})$ is almost identical. □

We remark that we have used relatively few pieces of structure theory of Vir , W and $W_{\geq -1}$ in the proof of Theorem 5.3.1 (the inputs are essentially that $\dim \mathbb{O}(\chi) = \dim \mathfrak{g} \cdot \chi$ and that $\dim \mathfrak{g} \cdot \chi < \infty$ for all χ in which we are interested). Thus, similar results may hold for a wider class of Lie algebras.

5.4 Radical Poisson ideals contain Poisson primitive ideals

Let $\mathfrak{g} = W$ or $W_{\geq -1}$. From the bijection in Theorem 5.2.8 it is tempting to think of Poisson primitive ideals of $S(\mathfrak{g})$ as analogous to closed points: by that result, a prime Poisson ideal Q is Poisson primitive if and only if $Y(Q)$ is a single point. Note, however, that Theorem 5.3.1 also tells how to distinguish Poisson primitive ideals in $\text{PSpec } S(W)$, and we saw there that, in general, they are only locally closed. The next result shows how far most of the Poisson primitive ideals are from being maximal in $\text{PSpec } S(\mathfrak{g})$; see also Corollary 4.3.18.

PROPOSITION 5.4.1. *Let $\mathfrak{g} = W$, $W_{\geq -1}$ or $W_{\geq 1}$ and let I be a nonzero radical Poisson ideal of $S(\mathfrak{g})$. Then there is some local function $\nu \in \mathfrak{g}^*$ with $P(\nu) \subseteq I$.*

If I is prime and $\mathfrak{g} = W$ or $W_{\geq -1}$ this may be deduced from the bijection Ψ of Subsection 5.2. We provide a direct proof, however.

Proof. Thanks to Theorem 3.0.4 we have that $\text{GKdim}(S(\mathfrak{g})/I) < \infty$; pick $d \in \mathbb{Z}_{\geq 1}$ with $2d > \text{GKdim}(S(\mathfrak{g})/I)$. By Lemma 2.0.4 I is an intersection of a family of primitive ideals $P(\mu)$, and

$$\text{GKdim}(S(\mathfrak{g})/P(\mu)) \leq \text{GKdim}(S(\mathfrak{g})/I)$$

for each μ with $\mathfrak{m}_\mu \supseteq I$. We show that there exists $\nu = \nu_d \in \mathfrak{g}^*$ such that if $\mu \in \mathfrak{g}^*$ with

$$\text{GKdim}(S(\mathfrak{g})/P(\mu)) < 2d,$$

then $P(\nu) \subseteq P(\mu)$.

Pick $\mu \in \mathfrak{g}^*$ with $\dim \mathfrak{g} \cdot \mu = \text{GKdim}(S(\mathfrak{g})/P(\mu)) < 2d$, see Proposition 4.3.6(b). Then μ is local and, thus, is a sum of several (say ℓ) nonzero one-point local functions $\mu_i \in \mathfrak{g}^*$ with distinct supports x_i . Recall that $\dim \mathfrak{g} \cdot \mu = \sum_{i=1}^{\ell} \dim \mathfrak{g} \cdot \mu_i$ by Lemma 4.3.4 and therefore $\dim \mathfrak{g} \cdot \mu_i < 2d$ together with $\ell < 2d$. Pick $n \in \mathbb{Z}_{\geq 0}$ such that $n \geq 2d$ and $\mu_i \in \text{Loc}_{x_i}^{\leq n}$ for all i . Thanks to Lemmata 4.1.3 and 4.1.6 we have $\dim D\text{Loc}_{x_i}^{\leq n} \mu_i < 2d$ for all i .

Let $\tilde{\chi}_i := \chi_{x_i; \underbrace{0, 0, \dots, 0}_{2d \text{ times}}, 1}$. Thanks to Theorem 4.2.1 we have $\dim D\text{Loc}_{x_i}^{\leq n} \tilde{\chi}_i = 2d + 1 >$

$\dim D\text{Loc}_{x_i}^{\leq n} \mu_i$. This together with Corollary 4.2.9 implies that $P(\tilde{\chi}_i) \subseteq P(\mu)$.

By Corollary 4.3.17, $P(\sum_{i=1}^{\ell} \tilde{\chi}_i) \subseteq P(\mu)$.

Set $\tilde{\chi}_i' := \chi_{x_i; \underbrace{0, 0, \dots, 0}_{2d \text{ times}}, 1}$. By Theorem 4.3.1, $P(\sum_{i=1}^{\ell} \tilde{\chi}_i) = P(\sum_{i=1}^{\ell} \tilde{\chi}_i')$ and hence, applying

Corollary 4.3.17, $\nu = \sum_{i=1}^{2d} \tilde{\chi}_i'$ satisfies the desired properties.

By Lemma 2.0.4, $P(\nu) \subseteq I$. □

The statement of the above corollary can be rephrased as follows: every proper algebraic collection of pseudo-orbits is (strictly) contained in the closure of a single finite-dimensional pseudo-orbit. If $\dim \mathfrak{g} < \infty$, this means that \mathfrak{g}^* contains a dense coadjoint orbit; such Lie algebras

are called Frobenius Lie algebras and there are very few of them. For infinite-dimensional Lie algebras this statement seems quite counterintuitive. We expect that only a few Lie algebras satisfy it.

The next corollary answers Question 6.8 of [LS08].

COROLLARY 5.4.2. *Let $\mathfrak{g} = W$, $W_{\geq -1}$ or $W_{\geq 1}$. Then $S(\mathfrak{g})$ has no nonzero prime Poisson ideals of finite height.*

Proof. For $d \in \mathbb{Z}_{\geq 0}$ let $\nu_d \in \mathfrak{g}^*$ be the local function defined in the proof of Proposition 5.4.1. The crucial property of ν_d , established in that proof, is that

$$\text{if } \text{GKdim}(S(\mathfrak{g})/P(\mu)) < 2d, \text{ then } P(\nu_d) \subseteq P(\mu). \tag{5.4.3}$$

Let Q be a nonzero Poisson prime ideal of $S(\mathfrak{g})$ and let $d_1 := \text{GKdim} S(\mathfrak{g})/Q$, which is finite by Theorem 3.0.4. By the proof of Proposition 5.4.1, $P(\nu_{d_1}) \subseteq Q$. Then define d_i by induction: let $d_{i+1} := \text{GKdim}(S(\mathfrak{g})/P(\nu_{d_i})) + 2$. By (5.4.3), each $P(\nu_{d_{i+1}}) \subsetneq P(\nu_{d_i})$. \square

By Corollary 3.3.5, if $\zeta \in \mathbb{k}^\times$, then (0) is the only prime Poisson ideal of $S(\text{Vir})$ which is contained in $(z - \zeta)$; thus, the maximal Poisson ideal $(z - \zeta)$ has height 1 as a prime Poisson ideal.

6. Subalgebras of finite codimension

In this section we sharpen earlier results to classify subalgebras of Vir of small codimension. By Proposition 3.3.3 we know any such subalgebra contains z , so we may reduce to considering the corresponding subalgebra of W ; by Proposition 3.2.7 this contains some $W(f)$ with $f \neq 0$. We refine these results and provide more precise statements on subalgebras of Vir and of W of codimensions one, two and three.

Throughout this section we assume that \mathfrak{k} is a subalgebra of W of finite codimension. Let $f_{\mathfrak{k}} \in \mathbb{k}[t]$ be the lowest-degree monic polynomial with $\mathfrak{k} \supseteq W(f)$, which exists by Proposition 3.2.7. Proposition 3.2.7, in fact, gives us that

$$W(f_{\mathfrak{k}}) \subseteq \mathfrak{k} \subseteq W(\text{rad}(f_{\mathfrak{k}})), \tag{6.0.1}$$

where recall that $\text{rad}(f) = \prod\{(t - x) \mid f(x) = 0\}$. Thus,

$$\text{codim}_W \mathfrak{k} \geq \text{deg rad}(f_{\mathfrak{k}}) = |\{x \in \mathbb{k}^\times \mid f_{\mathfrak{k}}(x) = 0\}|. \tag{6.0.2}$$

By the Euclidean algorithm, $W(h) \subseteq \mathfrak{k}$ if and only if $f_{\mathfrak{k}} \mid h$.

We immediately obtain a classification of subalgebras of codimension one, a more conceptual proof of a result originally due to Ondrus and Wiesner [OW18, Proposition 2.3].

COROLLARY 6.0.3. *Let \mathfrak{h} be a subalgebra of Vir of codimension one. Then there is $x \in \mathbb{k}^\times$ so that*

$$\mathfrak{h} = (t - x)\mathbb{k}[t, t^{-1}]\partial \oplus \mathbb{k}z.$$

Proof. As remarked previously, by Proposition 3.3.3 $z \in \mathfrak{h}$ so it suffices to prove that $\mathfrak{k} := \mathfrak{h}/(z)$, which is a subalgebra of W of codimension one, is equal to some $W(t - x)$. By (6.0.2) $f_{\mathfrak{k}}$ must be equal to some $(t - x)^a$ and by (6.0.1) $\mathfrak{k} \subseteq W(t - x)$. However, $W(t - x)$ already has codimension one. \square

The problem of classifying general cofinite-dimensional subalgebras of Vir is equivalent, by Proposition 3.3.3 and (6.0.1), to the problem of classifying subalgebras of an arbitrary Lie algebra of the form $W(\text{rad}(f))/W(f)$; note that $W(f)$ is a Lie ideal of $W(\text{rad}(f))$ and that

$W(\text{rad}(f))/W(f)$ is finite-dimensional and nilpotent. In §§ 6.2 and 6.3 we give a complete classification of Lie subalgebras $\mathfrak{k} \subseteq W$ of codimension two and three, illustrating the complexity of the problem.

6.1 Notation and concepts

We begin by establishing some needed notation. We call the full list of subalgebras of W of codimension one the *spectrum* of W . Corollary 6.0.3 implies that every such subalgebra is of the form $W(t - x)$ for an appropriate $x \in \mathbb{k}^\times$ and, thus, we can identify the spectrum of W with \mathbb{k}^\times . Pick a subalgebra \mathfrak{k} of W of finite codimension. Denote by $\text{supp}(\mathfrak{k})$ the list of $x \in \mathbb{k}^\times$ so that $\mathfrak{k} \subseteq W(t - x)$. By (6.0.1) $\text{supp}(\mathfrak{k})$ is nonempty and finite. The picture is parallel to that of ideals in a commutative \mathbb{k} -algebra, where subalgebras of codimension one (maximal subalgebras) are the natural analogues of ideals of codimension one (maximal ideals).

It will be useful below to carry out a more detailed analysis of subalgebras with $|\text{supp}(\mathfrak{k})| = 1$. We establish notation for various invariants of such \mathfrak{k} .

Notation 6.1.1. Let \mathfrak{k} be a finite-codimension subalgebra of W with $|\text{supp}(\mathfrak{k})| = 1$. We define the following invariants of \mathfrak{k} .

Let $d := d(\mathfrak{k}) := \text{codim}_W \mathfrak{k}$. As $\text{supp}(\mathfrak{k}) = \{x\}$ for some $x \in \mathbb{k}^\times$ we have $f_{\mathfrak{k}} = (t - x)^a$ for some $a \geq d$. Let $a(\mathfrak{k}) := a$. If $a = d$, then $\mathfrak{k} = W((t - x)^d)$. By the definition of $f_{\mathfrak{k}}$, if $a \neq d$, then

$$(t - x)^{a-1} \partial \notin \mathfrak{k}. \tag{6.1.2}$$

Assume that $a \neq d$. Set $\tilde{t} := t - x$. Let f_1, f_2, \dots, f_{a-d} be elements of $\mathbb{k}[t, t^{-1}]$ so that the images of the $f_i \partial$ give a basis for $\mathfrak{k}/W(f)$. We may write each f_i as $\tilde{t}^{n_i} k_i$ where $k_i(x) = 1$. By cancelling leading terms in the Taylor expansion of f_i around $\tilde{t} = 0$ we may assume that

$$1 \leq n_1 < n_2 < \dots < n_{a-d} < a - 1,$$

where we used (6.1.2) for the last inequality. Write $\{1, \dots, a - 1\} \setminus \{n_1, \dots, n_{a-d}\} = \{g_1, \dots, g_{d-1}\}$ where $g_1 < g_2 < \dots$. We say that

$$\text{ldeg}(\mathfrak{k}) := \{n_1 - 1, \dots, n_{a-d} - 1\}$$

are the *leading degrees* of \mathfrak{k} . We say that $\text{gaps}(\mathfrak{k}) := \{g_1 - 1, \dots, g_{d-1}\}$ are *gaps* of \mathfrak{k} .

Note that we do not allow $\text{gaps}(\mathfrak{k}) = \{0, \dots, a - 2\}$ as this would mean that $\text{ldeg}(\mathfrak{k}) = \emptyset$, contradicting the assumption that $a(\mathfrak{k}) \neq d(\mathfrak{k})$.

LEMMA 6.1.3.

- (a) Let \mathfrak{k} satisfy $|\text{supp}(\mathfrak{k})| = 1$ with $a := a(\mathfrak{k}) > d := d(\mathfrak{k})$. Pick $1 \leq i \neq j \leq a - d$. Then either $(n_i - 1) + (n_j - 1) \geq (a - 1)$ or $(n_i - 1) + (n_j - 1) \in \text{ldeg}(\mathfrak{k})$.
- (b) Let $S = \{n_1, \dots, n_\ell\}$ be a subset of $\{1, \dots, a - 2\}$ satisfying the conclusion of part (a). Then there exists a subalgebra $\tilde{\mathfrak{k}}$ of W with $a(\tilde{\mathfrak{k}}) = a$ and $\text{ldeg}(\tilde{\mathfrak{k}}) = S$.

Proof. For part (a), consider

$$\tilde{f}_{i,j} \partial := [f_i \partial, f_j \partial] = [\tilde{t}^{n_i} k_i \partial, \tilde{t}^{n_j} k_j \partial] = (j - i) \tilde{t}^{n_i+n_j-1} k_i k_j + \tilde{t}^{n_i+n_j} (k_i k'_j - k'_i k_j) \in \mathfrak{k}.$$

It is easy to verify that $k_{i,j} = \tilde{f}_{i,j} / \tilde{t}^{n_i+n_j-1} \in \mathbb{k}[t, t^{-1}]$ and $k_{i,j}(x) \neq 0$. This implies part (a).

For part (b), it is clear that the space $\tilde{\mathfrak{k}}$ defined as the span of $W(\tilde{t}^a)$ and $\tilde{t}^i \partial$ with $i \in S$ is a Lie subalgebra of W . □

The reason for subtracting 1 in the definition of the leading degrees of \mathfrak{k} is that Lemma 6.1.3 shows that $\text{ldeg}(\mathfrak{k})$ has the structure of a partial semigroup under addition.

We deduce from Lemma 6.1.3 the following fact.

TABLE 1. Subalgebras of W of codimension two.

Code	$\text{gaps}(\mathfrak{k})$	$f_{\mathfrak{k}}$	Additional generators
$W((t-x)(t-y))$	–	$(t-x)(t-y)$	–
$W_{x;\alpha}^{2;1}$	1	$(t-x)^3$	$(t-x)\partial + \alpha(t-x)^2\partial$
$W_{x;\alpha}^{2;2}$	2	$(t-x)^4$	$(t-x)\partial + \alpha(t-x)^3\partial,$ $(t-x)^2\partial$

LEMMA 6.1.4. *Let \mathfrak{k} satisfy $|\text{supp}(\mathfrak{k})| = 1$ with $a := a(\mathfrak{k}) > d := d(\mathfrak{k})$. If $g_1 = 1$, then $g_i \leq 2i - 1$ for all $i, 1 \leq i \leq d - 1$. If $g_1 \neq 1$, then $g_i \leq 2i + 1$ for all $i, 1 \leq i \leq d$.*

Proof. First assume that $g_1 = 1$. Next, assume to the contrary that $g_i = 2i - 1 + \delta$ with $\delta > 0$. Consider the list of pairs

$$(2, 2i - 2 + \delta), (3, 2i - 3 + \delta), \dots, (i, i + \delta). \tag{6.1.5}$$

For each pair (p, q) in (6.1.5) we have $p + q - 1 = g_i$; further, Lemma 6.1.3 implies that either $p - 1$ or $q - 1$ belongs to $\text{gaps}(\mathfrak{k})$ for each pair (p, q) from (6.1.5), it must be in $\{g_2, \dots, g_{i-1}\}$ as $g_1 = 1 < p, q$. There are $i - 1$ pairs in (6.1.5) but only $i - 2$ gaps from g_2 to g_{i-1} . This contradiction completes the case $g_1 = 1$. In the case $g_1 > 1$ we have to do the same thing with a minor modification: we have to add g_1 to the list $\{g_2, \dots, g_{i-1}\}$. \square

Remark 6.1.6. Note that $g_d = a - 1 \leq 2d + 1$ and, hence, one can enumerate all the pairs (S, a) satisfying the conclusion of condition (a) of Lemma 6.1.3 for a given codimension d . We believe that such pairs (S, a) are in bijection with irreducible components of the moduli space of subalgebras \mathfrak{k} of codimension d with $|\text{supp}(\mathfrak{k})| = 1$.

6.2 Subalgebras of codimension two

The goal of this subsection is to show that all subalgebras of W of codimension two are listed in Table 1.

Here x, y, α are parameters taking values in \mathbb{k} with $x, y \neq 0$.

The proof consists of the following two statements.

PROPOSITION 6.2.1.

- (a) *If $\text{codim}_W \mathfrak{k} = 2$, then $|\text{supp}(\mathfrak{k})| \in \{1, 2\}$.*
- (b) *If $|\text{supp}(\mathfrak{k})| = 2$, then $\mathfrak{k} = W((t-x)(t-y))$ and $\text{supp}(\mathfrak{k}) = \{x, y\}$ for some $x \neq y \in \mathbb{k}^\times$.*

PROPOSITION 6.2.2.

- (a) *If $\text{codim}_W \mathfrak{k} = 2$ and $|\text{supp}(\mathfrak{k})| = 1$ (so $\text{supp}(\mathfrak{k}) = \{x\}$ for some $x \in \mathbb{k}^\times$), then either $\mathfrak{k} = W((t-x)^2)$ or $\text{gaps}(\mathfrak{k})$ is $\{1\}$ or $\{2\}$.*
- (b) *If $\text{gaps}(\mathfrak{k}) = \{1\}$, then $\mathfrak{k} = W_{x;\alpha}^{2;1}$ for a unique $\alpha \in \mathbb{k}$.*
- (c) *If $\text{gaps}(\mathfrak{k}) = \{2\}$, then $\mathfrak{k} = W_{x;\alpha}^{2;2}$ for a unique $\alpha \in \mathbb{k}$.*

Proof of Proposition 6.2.1. Theorem 3.2.7 implies $|\text{supp}(\mathfrak{k})| \geq 1$, and (6.0.2) implies that $|\text{supp}(\mathfrak{k})| \leq 2$. This proves part (a).

Assume $\text{supp}(\mathfrak{k}) = \{x, y\}$ for some distinct $x, y \in \mathbb{k}^\times$. We have $\mathfrak{k} \subseteq W((t-x)(t-y))$ and

$$\text{codim}_W W((t-x)(t-y)) = \text{codim}_W \mathfrak{k} = 2.$$

Hence, $\mathfrak{k} = W((t-x)(t-y))$ and part (b) is complete. \square

Proof of Proposition 6.2.2. We may assume that $\mathfrak{k} \neq W((t-x)^2)$. Adopt the terminology of Notation 6.1.1. We have $|\text{gaps}(\mathfrak{k})| = \text{codim}(\mathfrak{k}) - 1$ and, hence, $|\text{gaps}(\mathfrak{k})| = \{g_1 - 1\}$ for a certain positive integer g_1 . Note that $g_1 = a(\mathfrak{k}) - 1$ by (6.1.2). Thanks to Lemma 6.1.3 we have $g_1 \leq 3$. This proves part (a).

Note that $g_1 \neq 1$ as by convention $\text{gaps}(\mathfrak{k})$ cannot be equal to $\{0, \dots, a - 2\}$. Assume $g_1 = 2$, so $a(\mathfrak{k}) = 3$. Recall that $\tilde{t} = t - x$. Let $f_1 = \tilde{t}k_1$ be as in Notation 6.1.1; as $a(\mathfrak{k}) - d(\mathfrak{k}) = 1$, the image of $f_1\partial$ in $\mathfrak{k}/W(\tilde{t}^3)$ provides a generator of this one-dimensional vector space. We can assume that f_1 is monic in \tilde{t} and contains no terms in \tilde{t} of degree three or more. Thus, $f_1 = \tilde{t} + \alpha\tilde{t}^2$ for some $\alpha \in \mathbb{k}$ and $\mathfrak{k} = W_{x;\alpha}^{2;1}$. Further, α is unique as $\dim \mathfrak{k}/W(\tilde{t}^3) = 1$.

Assume $g_1 = 3$, so $a(\mathfrak{k}) = 4$. Let $f_1 = \tilde{t}k_1, f_2 = \tilde{t}^2k_2 \in \mathbb{k}[t] = \mathbb{k}[\tilde{t}]$ be as in Notation 6.1.1, so the images of $f_1\partial, f_2\partial$ in $\mathfrak{k}/W((t-x)^3)$ give a basis. We can assume that f_1, f_2 contain no terms in \tilde{t} of degree four or more and f_1 contains no terms in \tilde{t} of degree two. Thus, $f_1 = \tilde{t} + \alpha\tilde{t}^3, f_2 = \tilde{t}^2 + \beta\tilde{t}^3$ for some $\alpha, \beta \in \mathbb{k}$. We have

$$\mathfrak{k} \ni [f_1\partial, f_2\partial] = \tilde{t}^2\partial + 2\beta\tilde{t}^3\partial \pmod{W(\tilde{t}^4)}.$$

Hence, $[f_1\partial, f_2\partial] = f_2\partial$ modulo $W(\tilde{t}^4)$. This implies $\beta = 0$ and, therefore, $\mathfrak{k} = W_{x;\alpha}^{2;2}$. Again α is unique. □

Remark 6.2.3. Similar proofs give a classification of subalgebras of $W_{\geq -1}$ of codimension two: these are either of the form $W_{\geq -1}((t-x)(t-y))$ or may be written $(W_{\geq -1})_{x;\alpha}^{2;1}$ or $(W_{\geq -1})_{x;\alpha}^{2;2}$, where these last two are defined similarly to the analogous subalgebras of W . (Here, of course, we allow x, y to be zero.) These subalgebras are all deformations of $W_{\geq -1}(t^2) = W_{\geq 1}$.

Deformations of $W_{\geq 1}$ are classified in [Fia83] (see also [FF97]). These papers show that up to isomorphism there are three such deformations, denoted in [Fia83] by $L_1^{(1)}, L_1^{(2)}, L_1^{(3)}$. It can be shown that

$$L_1^{(1)} \cong W(t(t-y)), \quad L_1^{(2)} \cong (W_{\geq -1})_{0;\alpha}^{2;1}, \quad L_1^{(3)} \cong (W_{\geq -1})_{0;\alpha}^{2;2}$$

for appropriate y, α . We thank Lucas Buzaglo for explaining this to us.

6.3 Subalgebras of codimension three

It can also be shown that all subalgebras of W of codimension three are listed in Table 2. Because the methods are similar to those in § 6.2 we omit the proof.

The notation here is as follows:

- \mathfrak{k} is given in the final column as either an intersection of two explicitly given subalgebras or is spanned by $W(f_{\mathfrak{k}})$ and a few more explicit generators;
- α, β are parameters taking arbitrary values in \mathbb{k} for all cases but $W_{x,y;\alpha,\beta}^{3A}$ (for this case the required restrictions on α, β are given in the table);
- x, y are parameters taking arbitrary values in \mathbb{k}^\times except when restrictions are given in the table.

It is easy to verify that a subalgebra \mathfrak{k} of W of codimension three belongs to only one type and, moreover, the parameters defining \mathfrak{k} are unique if $\mathfrak{k} \neq W_{x,y;\alpha,\beta}^{3A}$ and unique up to scaling $(\alpha, \beta) \rightarrow (\lambda\alpha, \lambda\beta)$ in the case $\mathfrak{k} = W_{x,y;\alpha,\beta}^{3A}$.

Remark 6.3.1. (1) Recall the analogy between subalgebras of W of codimension one and maximal ideals. It is natural to ask whether or not an analogue of the Lasker–Noether primary decomposition theorem holds in this setting. However, this statement fails as we can easily see that the Lie algebra $W_{x,y;\alpha,\beta}^{3A}$ is not an intersection of subalgebras \mathfrak{k}_x and \mathfrak{k}_y with $\text{supp}(\mathfrak{k}_x) = \{x\}$ and $\text{supp}(\mathfrak{k}_y) = \{y\}$.

TABLE 2. Subalgebras of W of codimension three.

Code	gaps(\mathfrak{k})	$f_{\mathfrak{k}}$	Additional generators or description
$W(f_{\mathfrak{k}})$	–	$(t-x)(t-y)(t-z)$	–
$W_{x,y;\alpha,\beta}^{3A}$	–	$(t-x)^2(t-y)^2$	$(t-x)(t-y)(\alpha t + \beta)\partial,$ $\alpha x + \beta, \alpha y + \beta \neq 0, x \neq y$
$W_{x,y;\alpha}^{3B1}$	–	$(t-x)^3(t-y)$	$W_{x;\alpha}^{2;1} \cap W(t-y),$ $x \neq y$
$W_{x,y;\alpha}^{3B2}$	–	$(t-x)^4(t-y)$	$W_{x;\alpha}^{2;2} \cap W(t-y),$ $x \neq y$
$W_{x;\alpha}^{3C1}$	0, 2	$(t-x)^4$	$(t-x)^2\partial + \alpha(t-x)^3\partial$
$W_{x;\alpha,\beta}^{3C2}$	1, 2	$(t-x)^4$	$(t-x)\partial + \alpha(t-x)^2\partial + \beta(t-x)^3\partial$
$W_{x;\alpha,\beta}^{3C3}$	1, 3	$(t-x)^5$	$(t-x)\partial + \alpha(t-x)^2\partial + \beta(t-x)^4\partial,$ $(t-x)^3\partial - \alpha(t-x)^4\partial$
$W_{x;\alpha,\beta}^{3C4}$	1, 4	$(t-x)^6$	$(t-x)\partial + \alpha(t-x)^2\partial + \beta(t-x)^5\partial,$ $(t-x)^3\partial - \alpha^2(t-x)^5\partial,$ $(t-x)^4\partial - 2\alpha(t-x)^5\partial$
$W_{x;\alpha,\beta}^{3C5}$	2, 3	$(t-x)^5$	$(t-x)\partial + \alpha(t-x)^3\partial + \beta(t-x)^4\partial,$ $(t-x)^2\partial + \frac{\alpha}{2}(t-x)^4\partial$

(2) Recall that every subalgebra \mathfrak{k} of finite codimension in W lies between $W(f_{\mathfrak{k}})$ and $W(\text{rad}(f_{\mathfrak{k}}))$. One can construct a sequence of polynomials

$$h_0 = \text{rad} f_{\mathfrak{k}}, h_1, h_2, \dots, h_s = f_{\mathfrak{k}}$$

with $\deg h_{i+1} = \deg h_i + 1$ and $h_i \mid h_{i+1}$ and consider the filtration of $W(\text{rad} f_{\mathfrak{k}})$ by the Lie ideals $W(h_i)$. The associated graded algebra

$$\bigoplus_{i \geq 0} ([W(h_i) \cap \mathfrak{k}] / [W(h_{i+1}) \cap \mathfrak{k}])$$

is isomorphic to a graded algebra \mathfrak{k} with

$$W(h_0) \supseteq \mathfrak{k} \supseteq W(f_{\mathfrak{k}}), \text{codim}_W \mathfrak{k} = \text{codim}_W \mathfrak{k} \text{ and } f_{\mathfrak{k}} = f_{\mathfrak{k}}.$$

The subalgebras \mathfrak{k} can be described in purely combinatorial terms and, thus, they give a collection of discrete invariants for \mathfrak{k} . This generalises the notation of gaps and leading degrees for subalgebras with one-point support.

6.4 Some general comments on finite codimension subalgebras of W

The lattice of subalgebras \mathfrak{k} with $\text{supp}(\mathfrak{k}) = \{x\}$ and $a(\mathfrak{k}) \leq a$ can be naturally identified with the subalgebras of

$$W(t-x)/W((t-x)^a) \cong W(t)/W(t^a);$$

in particular, the isomorphism class of this lattice is independent of x .

A similar result holds true in a greater generality. Pick $s \geq 0, a \geq 1$ and distinct $x_1, \dots, x_s \in \mathbb{k}^\times$; set $h := (t-x_1) \cdots (t-x_s)$. Consider subalgebras \mathfrak{k} satisfying $W(h) \supset \mathfrak{k} \supset W(h^a)$. It is clear that $W(h^a)$ is a Lie ideal of $W(h)$ and the quotient $W(h)/W(h^a)$ is a finite-dimensional solvable

Lie algebra. A version of the Chinese remainder theorem implies that

$$W(h)/W(h^a) \cong \oplus_i W(t - x_i)/W((t - x_i)^a) \cong [W(t)/W(t^a)]^{\oplus s}. \tag{6.4.1}$$

In particular, the isomorphism class of $W(h)/W(h^a)$ depends only on a and s but not on the particular choice of the x_i . This immediately gives the following corollaries.

COROLLARY 6.4.2. *Let \mathfrak{k} be a subalgebra of finite codimension of $\mathfrak{g} = W, W_{\geq -1}, W_{\geq 1}$ or Vir . Then there are subalgebras $\mathfrak{k}^+, \mathfrak{k}^-$ with*

$$\mathfrak{k}^- \subseteq \mathfrak{k} \subseteq \mathfrak{k}^+ \quad \text{and} \quad \text{codim}_{\mathfrak{g}}(\mathfrak{k}^-) + 1 = \text{codim}_{\mathfrak{g}}(\mathfrak{k}) = \text{codim}_{\mathfrak{g}}(\mathfrak{k}^+) - 1.$$

Proof. A similar statement is well-known for subalgebras of solvable Lie algebras so (6.4.1) implies the desired result if $\mathfrak{k} \neq W(f_{\mathfrak{k}})$. If $\mathfrak{k} = W(f_{\mathfrak{k}})$ the result follows from a similar fact on ideals in $\mathbb{k}[t, t^{-1}]$. □

COROLLARY 6.4.3. *The lattices of subalgebras of finite codimension of Vir , of W and of $W_{\geq -1}$ are all isomorphic.*

Proof. Lemma 3.3.3 implies that the lattices of subalgebras of finite codimension are isomorphic for W and Vir ; hence, we left to show that the lattices are isomorphic for W and $W_{\geq -1}$.

Theorem 3.2.7 implies that the lattices of subalgebras of finite codimension for both W and $W_{\geq -1}$ are direct limits of the sublattices of subalgebras \mathfrak{k} containing $W((t - x_1)^a \cdots (t - x_s)^a)$ for all tuples $(a; x_1, \dots, x_s)$; the only difference between W and $W_{\geq -1}$ here is that in the first case $x_i \neq 0$. These lattices are isomorphic for W and $W_{\geq -1}$ and the embeddings between them are the same for W and $W_{\geq -1}$ (they essentially depend only on the integer-valued parameters s and a and on the cardinality of \mathbb{k}). □

7. Implications of our results for $U(\mathfrak{g})$

In this final section, we shift, for the first time in this paper, to considering the universal enveloping algebra $U(\mathfrak{g})$ of one of our Lie algebras of interest. We apply a version of the orbit method to relate the Poisson primitive ideals $\ker p_{\gamma} = P(\chi_{x;\alpha;\gamma})$ of $S(\mathfrak{g})$ to primitive ideals of $U(W)$ obtained as kernels of maps to the (localised) Weyl algebra. We end with some conjectures about ideals in $U(Vir)$, $U(W)$, and $U(W_{\geq -1})$.

7.1 Constructing primitive ideals through the orbit method

Kirillov’s orbit method for nilpotent and solvable Lie algebras attaches to $\chi \in \mathfrak{g}^*$ the annihilator in $U(\mathfrak{g})$ of the module induced from a polarisation of χ and the induced character. We apply the same construction to $\mathfrak{g} = W$ and $\chi = \chi_{x;\alpha;\gamma}$. We denote the corresponding induced W -modules by $M_{x;\gamma}$.

A description of the annihilators of $M_{x;\gamma}$ is given in Proposition 7.1.5. The main result here is that

$$\text{Ann}_{U(W)} M_{x;\gamma}$$

depends only on γ ; thanks to Theorem 4.2.1 (or Lemma 2.2.3) the same holds for $P(\chi_{x;\alpha;\gamma})$. This shows that the constructions of Kirillov’s orbit method give rise to a map from a certain class of Poisson primitive ideals of $S(W)$ to a certain class of primitive ideals of $U(W)$, which are known in the literature [CM07, SW16] as kernels of maps to the localised Weyl algebra. We believe this map extends to a surjection from Poisson primitive ideals of $S(\mathfrak{g})$ to primitive ideals of $U(\mathfrak{g})$; this is the subject of ongoing research.

Throughout this section, we write the localised Weyl algebra as $A = \mathbb{k}[t, t^{-1}, \partial]$, with $\partial t = t\partial + 1$.

We first describe a polarisation for $\chi_{x;\alpha,\gamma}$. Let $x, \alpha, \gamma \in \mathbb{k}$ with $x \neq 0, (\alpha, \gamma) \neq (0, 0)$ and let $\chi = \chi_{x;\alpha,\gamma}$. Recall the computation of W^χ in Lemma 2.2.1, and consider the Lie subalgebra $W(t-x)$ of W , which contains W^χ . We have

$$\dim W/W^\chi = 2 \implies W(t-x) = \mathbb{k}(t-x)\partial \oplus W^\chi.$$

As $B_\chi((t-x)\partial, (t-x)\partial) = 0$, thus $W(t-x)$ is a totally isotropic subspace of (W, B_χ) ; by dimension count it is maximal totally isotropic. Thus, $W(t-x)$ is a polarisation of W at χ , as in [Dix96, 1.12.8]. Further, $W(t-x)$ is the unique polarisation of W at χ : because any polarisation of W at χ must be a codimension-one subalgebra of W , by Corollary 6.0.3 it must be equal to some $W(t-y)$ and it is easy to see that we must have $y = x$.

Note that χ is a character of $W(t-x)$; let $\mathbb{k}m_{x;\gamma}$ be the corresponding one-dimensional representation of $W(t-x)$, with basis element $m_{x;\gamma}$. (The restriction $\chi|_{W(t-x)}$, which sends $p\partial \mapsto \gamma p'(x)$, depends only on x and γ : this is the reason to omit α in the notation $\mathbb{k}m_{x;\gamma}$.) Put

$$M_{x;\gamma} := U(W) \otimes_{U(W(t-x))} \mathbb{k}m_{x;\gamma}.$$

As $e_{-1} = \partial \notin W(t-x)$, thus $W = \mathbb{k}e_{-1} \oplus W(t-x)$ and by the Poincaré–Birkhoff–Witt theorem the set $\{e_{-1}^k m_{x;\gamma} | k \in \mathbb{Z}_{\geq 0}\}$ is a basis for $M_{x;\gamma}$.

We now give an alternative construction of $M_{x;\gamma}$. Set

$$N_x := \mathbb{k}[t, t^{-1}, (t-x)^{-1}] / \mathbb{k}[t, t^{-1}].$$

For every x , N_x is a simple (faithful) left A -module.

Remark 7.1.1. The space N_x can be thought of as a space of distributions on \mathbb{k}^\times ; for, setting $\delta_x = (t-x)^{-1} \in N_x$ we have $(t-x)\delta_x = 0$ so δ_x behaves like a δ -function at x . The elements $\partial^k \delta_x$ form a basis of N_x .

Recall that, for any $\gamma \in \mathbb{k}$, the map

$$\pi_\gamma : W \rightarrow A, \quad f\partial \mapsto f\partial + \gamma f' \tag{7.1.2}$$

is a Lie algebra homomorphism; see [CM07]. Thus, π_γ extends to define a ring homomorphism $U(W) \rightarrow A$. Note that the map p_γ defined in (2.2.2) is *not* the associated graded map attached to π_γ even though they are clearly analogous.

The images of π_γ have been computed in [CM07, Lemma 2.1], and we give them here.

LEMMA 7.1.3. *We have:*

- (a) $\text{im}(\pi_0) = \mathbb{k} \oplus A\partial$, and $\text{im} \pi_1 = \mathbb{k} \oplus \partial A$;
- (b) $\text{im}(\pi_\gamma) = A$ if $\gamma \neq 0, 1$.

Remark 7.1.4. The restriction of π_γ to $U(W_{\geq 1})$ was considered, under slightly different notation, in [SW16]; see [SW16, Remark 3.14]. It was shown there that the ideal $\ker \pi_0|_{W_{\geq 1}} = \ker \pi_1|_{W_{\geq 1}}$ is not finitely generated as a left or right ideal of $U(W_{\geq 1})$.

For every $\gamma \in \mathbb{k}$ the map π_γ from (7.1.2) induces the structure of a W -module on N_x ; we denote the space N_x with the corresponding W -module structure \cdot_γ by N_x^γ .

PROPOSITION 7.1.5. *Let $x \neq 0, \alpha, \gamma \in \mathbb{k}$. Then $N_x^\gamma \cong M_{x;\gamma-1}$. Moreover, N_x^γ is a simple W -module if and only if $\gamma \neq 1$. There is an exact sequence*

$$0 \rightarrow N_x^0 \rightarrow N_x^1 \rightarrow \mathbb{k} \rightarrow 0.$$

For all γ the annihilator of N_x^γ is equal to $\ker \pi_\gamma$, which is primitive.

Proof. Let $p \in \mathbb{k}[t, t^{-1}]$ with $(t - x)|p$. By taking the Taylor expansion of p about x one may verify that

$$(p\partial) \cdot_{\gamma} \delta_x = (\gamma - 1)p'(x)\delta_x. \tag{7.1.6}$$

This immediately implies that there exists a surjective module homomorphism $M_{x;\gamma-1} \rightarrow N_x^{\gamma}$ which sends $m_{x;\gamma-1} \mapsto \delta_x$. The basis element $e_{-1}^k m_{x;\gamma-1}$ maps to $\partial^k \delta_x$, so this map is an isomorphism.

Assume $\gamma \neq 0, 1$. In this case π_{γ} is surjective by Lemma 7.1.3, so simplicity of N_x^{γ} follows from simplicity of N_x as an A -module.

We claim that N_x^0 is simple. To see this, let $0 \neq n \in N_x$, which we recall is a simple A -module. By construction of N_x there is some $0 \neq f(t) \in \mathbb{k}[t, t^{-1}]$ so that $fn = 0$ and, thus, $\partial n \neq 0$ as $A\partial + Af(t) = A$. Thus, $A\partial n = N_x$, proving the claim because by Lemma 7.1.3 $A\partial \subseteq \pi_0(U(W))$.

Finally, we consider N_x^1 . Lemma 7.1.3 implies $\partial A \triangleleft \text{im } \pi_1 = \mathbb{k} \oplus \partial A$ and, thus, $\tilde{N} := \partial AN_x = \partial N_x$ is a submodule of N_x^1 , with $N_x^1/\tilde{N} \cong \mathbb{k}$. We claim that \tilde{N} is simple. Let $0 \neq n \in \tilde{N}$. As N_x is a simple A -module, $\partial An = \tilde{N}$ and so $U(W) \cdot_1 n = (\mathbb{k} \oplus \partial A)n = \tilde{N}$, as needed. The reader may verify that as in (7.1.6)

$$p\partial \cdot_1 \partial \delta_x = -p'(x)\partial \delta_x,$$

and so $\tilde{N} \cong M_{x;-1} \cong N_x^0$. The claim about annihilators follows from the fact that N_x is a faithful module over the (simple) ring A . That $\ker \pi_{\gamma}$ is primitive is immediate for $\gamma \neq 1$; for $\gamma = 1$ it follows from the fact that $\ker \pi_1 = \ker \pi_0$. □

Remark 7.1.7. (a) Note that the primitive ideal $\ker \pi_{\gamma}$ is completely prime. We do not know of a primitive (or prime) ideal of $U(W)$ which is not completely prime.

(b) We believe that the ideals $\ker \pi_{\gamma}$ above are all of the primitive ideals of $U(W)$ of Gelfand–Kirillov codimension two.

(c) By Remark 7.1.4, $\ker \pi_0 = \ker \pi_1$. However, $\ker p_0 \neq \ker p_1$. To see this, note that if $\ker p_0 = \ker p_1$ then $\chi_{1;1,0} \in V(\ker p_1) = \overline{\mathbb{O}(\chi_{1;1,1})}$. Thus, either $\mathbb{O}(\chi_{1;1,0}) = \mathbb{O}(\chi_{1;1,1})$ or $\dim \mathbb{O}(\chi_{1;1,0}) < \dim \mathbb{O}(\chi_{1;1,1})$. Neither is true.

Thus, the orbit method does not give a bijection from Poisson primitive ideals of $S(W)$ to primitive ideals of $U(W)$.

7.2 Conjectures for $U(\mathfrak{g})$

We have focused almost entirely on the symmetric algebra of \mathfrak{g} , where \mathfrak{g} is one of Vir , W or several related Lie algebras. However, our results are at a minimum suggestive for the enveloping algebras of these Lie algebras. In this final subsection, we make several conjectures for $U(\mathfrak{g})$. Broadly speaking, these are instances of the meta-conjecture.

The ideal structure of $U(\mathfrak{g})$ is closely analogous to the Poisson structure of $S(\mathfrak{g})$.

For each conjecture, we give the Poisson result which suggested it to us.

CONJECTURE 7.2.1 (Cf. Corollary 3.3.5). If $\zeta \neq 0$, then $U(Vir)/(z - \zeta)$ is simple.

CONJECTURE 7.2.2 (Cf. Corollaries 4.3.11 and 5.2.14). Restriction gives a bijection between primitive (respectively, prime) ideals of $U(W)$ and $U(W_{\geq -1})$, and a homeomorphism

$$\text{PSpec}_{\text{prim}} U(W) \xrightarrow{\sim} \text{PSpec}_{\text{prim}} U(W_{\geq -1}).$$

CONJECTURE 7.2.3 (Cf. Proposition 5.4.1). Every proper prime ideal of $U(W)$ contains a proper primitive ideal.

CONJECTURE 7.2.4 (Cf. Proposition 4.3.14). The $\ker \pi_\gamma$ are all of the prime ideals of $U(W)$ of co-GK-dimension two.

CONJECTURE 7.2.5 (cf. §7.1). Kirillov’s orbit method, i.e. the assignment

a local function $\chi \rightarrow$ a polarization of $\chi \rightarrow$ the annihilator of the induced module

always produces a primitive ideal, is independent of polarisation, and depends only on $\mathbb{O}(\chi)$. There is thus an induced map $\text{PSpec}_{\text{prim}} S(\mathfrak{g}) \rightarrow \text{Spec}_{\text{prim}} U(\mathfrak{g})$ for $\mathfrak{g} = \text{Vir}, W, W_{\geq -1}$. This map is surjective onto $\text{Spec}_{\text{prim}} U(\mathfrak{g})$.

These conjectures are the subject of ongoing research.

Index of notation

$(a, b)_f$	947	$Loc_{\mathfrak{g}}^\lambda$	965
$a(\mathfrak{k})$	975	$Loc_{\mathfrak{g}}^{\leq n}, \widetilde{Loc}_{\mathfrak{g}}^{\leq n}, Loc_x, Loc_x^{\leq n}$	950
$\mathbb{A}(\lambda)$	959, 966	Local function	944
$B_\chi(x, y)$	939	\mathfrak{m}_χ	934, 937
$\chi_x; \alpha_0, \dots, \alpha_n$	943	$\text{MSpec } A$	937
$d(\mathfrak{k})$	975	$\overline{\mathbb{O}}(\chi)$, orbit closure of χ	962
D_χ	940	Order of a local function	944
$D(\lambda(\chi))$	965	$p_\gamma : S(W) \rightarrow \mathbb{k}[t, t^{-1}, y]$	942
Dil_x	951	$\pi_{\mathfrak{g}}^{\leq n}, \phi_{\mathfrak{g}}^{\leq n}$	951
$DLoc_x$	951	$P(\chi)$, Poisson core	934, 938
$DLoc_x^{\leq n}$	951	Poisson morphism	940
$DLoc_x^{k+}, DLoc_x^{\leq n, k+}$	953	$\text{PSpec}_{\text{prim}} A, \text{PSpec } A$	938
\widehat{DLoc}^λ	967	Pseudo-orbit, $\mathbb{O}(\chi)$	938
$\widehat{DLoc}^{\leq n}, \widehat{DLoc}$	952	$\text{rad}(f)$	948
\widehat{DLoc}^λ -orbit quotient	967	\mathfrak{S}_λ	959
$D(u_1, \dots, u_n; v_1, \dots, v_n)$	939	$\Sigma_{\mathfrak{g}}^\lambda$	965
$e_i(x), e_i(x)^*$	953	$Shift_z, Shifts$	952
$\text{End}_{t \rightarrow s}(\cdot)$	951	Support of a local function	944
ev_χ	937	Support of a subalgebra, $\text{supp}(\mathfrak{k})$	975
$f_{\mathfrak{k}}$	974	$u_\chi, \mathfrak{u}_\chi$	940
gaps(\mathfrak{k}), gaps of a subalgebra	975	Vir	937
\mathfrak{g}^χ	939	$V_{\mathfrak{g}}^\lambda$	959, 965
$I(n)$	946	$V(N) \subseteq \text{MSpec}(A)$	938
$I(X) \subseteq A$	938	$V_P(N) \subseteq \text{PSpec } A$	972
$J_{\mathfrak{g}}^\lambda := I(V_{\mathfrak{g}}^\lambda)$	968	W	937
$\lambda(\chi)$, order partition of χ	944	$W(g)$	941, 947
$\lambda(Q)$, generic order partition of Q	968	$W_{\geq -1}$	937
$\text{ldeg}(\mathfrak{k})$, leading degrees of a subalgebra	975	$W_{\geq -1}(f)$	945
$\text{lie}_x^{\leq n}$	951	$W_{\geq 1}$	937
$\widehat{\text{lie}}^{\leq n}$	952	$W_{x;\alpha}^{2;1}, W_{x;\alpha}^{2;2}$	976
		$W_{x,y;\alpha,\beta}^{3A}, W_{x,y;\alpha}^{3B1}$, etc.	978
		$Y(Q)$	968

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