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PERFECT 1-FACTORISATIONS OF K_{16}

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Abstract

We report the results of a computer enumeration that found that there are 3155 perfect 1-factorisations (P1Fs) of the complete graph K_{16}. Of these, 89 have a nontrivial automorphism group (correcting an earlier claim of 88 by Meszka and Rosa ['Perfect 1-factorisations of K_{16} with nontrivial automorphism group', *J. Combin. Math. Combin. Comput.* **47** (2003), 97–111]). We also (i) describe a new invariant which distinguishes between the P1Fs of K_{16}, (ii) observe that the new P1Fs produce no atomic Latin squares of order 15 and (iii) record P1Fs for a number of large orders that exceed prime powers by one.

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1. Introduction

A k-factor in a graph is a k-regular spanning subgraph. In particular, a 1-factor (also called a perfect matching) is a set of edges in a graph which cover every vertex exactly once. A 1-factorisation is a collection of 1-factors which partitions the edges of the graph. Suppose that we have a 1-factorisation $\mathcal{F}$ of some graph. The union of any two distinct 1-factors in $\mathcal{F}$ is a 2-factor, that is, a collection of cycles. If, regardless of which two 1-factors we choose in $\mathcal{F}$, their union is a single cycle, then we say that $\mathcal{F}$ is perfect. Throughout, we will use the abbreviation P1F for *perfect* 1-*factorisation*. For background reading on P1Fs, including definitions of terms not included here, we refer to [8–10].

The P1Fs of complete graphs up to K_{14} have been known for some time [3]. The main purpose of this note is to report on a computer enumeration of the next case, namely P1Fs of K_{16}. We also check the Latin squares associated with the new P1Fs (Section 3) and find that they are not atomic, discuss invariants which distinguish nonisomorphic P1Fs (Section 4) and record a number of new P1Fs for large orders that are one more than a prime power (Section 5).

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Kotzig's P1F conjecture famously asserts that P1Fs of K_n exist for all even n. Only three infinite families are known [1]. They cover all orders of the form $n = p + 1$ or $n = 2p$, where p is an odd prime. Sporadic constructions including those reported in Section 5, together with [7, 9, 11, 14], demonstrate existence for $n \le 56$ and the following orders:

$$126, 170, 244, 344, 530, 730, 1332, 1370, 1850, 2198, 2810, 3126, 4490, 6860, 6890, 11450, 11882, 12168, 15626, 16808, 22202, 24390, 24650, 26570, 29792, 29930, 32042, 38810, 44522, 50654, 51530, 52442, 63002, 72362, 76730, 78126, 79508, 103824, 148878, 161052, 205380, 226982, 300764, 357912, 371294, 493040, 571788, 1030302, 1092728, 1225044, 1295030, 2048384, 2248092, 2476100, 2685620, 3307950, 3442952, 4330748, 4657464, 5735340, 6436344, 6967872, 7880600, 9393932, 11089568, 11697084, 13651920, 15813252, 18191448, 19902512, 22665188.$$

Kotzig's P1F conjecture remains open for other orders. The smallest unsolved cases are $\{64, 66, 70, 76, 78, 88, 92, 96, 100\}$.

2. The catalogue

We labelled the 16 vertices of our complete graph `a,b, ..., p` and assumed without loss of generality that every P1F contained the two factors

$$F_1 = \{\mathtt{ab}, \mathtt{cd}, \mathtt{ef}, \mathtt{gh}, \mathtt{ij}, \mathtt{kl}, \mathtt{mn}, \mathtt{op}\}, \quad F_2 = \{\mathtt{ap}, \mathtt{bc}, \mathtt{de}, \mathtt{fg}, \mathtt{hi}, \mathtt{jk}, \mathtt{lm}, \mathtt{no}\}. \tag{2.1}$$

Next, we generated all 1-factors of K_{16} containing the edge `ac`. Up to isomorphism, this gave us 1647 options for the initial three 1-factors F_1, F_2, F_3. In each isomorphism class we chose F_3 to be the lexicographically least option. Then for each of these isomorphism representatives we did the following. First, we generated and stored the set $\mathcal{T}$ of all 1-factors that were compatible with F_1, F_2, F_3, in the sense that $F_i \cup F$ is a Hamilton cycle for all $F \in \mathcal{T}$ and $i \in \{1, 2, 3\}$. The number of 1-factors we needed to store at this stage depended on the choice of F_1, F_2, F_3 and ranged from 56 816 to 59 312. We also computed (and stored) which pairs of these 1-factors in $\mathcal{T}$ were compatible (made a Hamilton cycle). A list of 'active' 1-factors was maintained, which initially consisted of $\mathcal{T}$.

We then used backtracking to add one 1-factor at a time from the active list. Each new factor was forced to include a particular edge that was judged to be the best option of the edges remaining to be used. The criterion was to choose the edge that was contained in the fewest active 1-factors, thereby limiting the branching that our search undertook. Having chosen a new 1-factor, we checked that no relabelling of the current partial 1-factorisation would give a lexicographically smaller set of three initial factors than the one currently being processed. Assuming that to be the case, we updated the list of active 1-factors by removing any which were incompatible with the new

1-factor and continued the search. Any P1Fs that we found were canonically labelled (see Section 4) and then output. The result was the following theorem.

THEOREM 2.1. *There are* 3155 *P1Fs of* K_{16} *up to isomorphism. Of these,* 89 *have a nontrivial automorphism group.*

The 3155 P1Fs can be downloaded from [12]. Note that the same catalogue has been independently and concurrently generated by Meszka [5]. The P1Fs with nontrivial automorphism groups all had cyclic automorphism groups, with generators as follows:

- a unique starter induced P1F, with automorphism group generated by a permutation with cycle type $15^1 1^1$;
- a unique even-starter induced P1F, with automorphism group generated by a permutation with cycle type $14^1 1^2$;
- four P1Fs with automorphism group generated by a permutation with cycle type $7^2 1^2$;
- five P1Fs with automorphism group generated by a permutation with cycle type $5^3 1^1$;
- 19 P1Fs with automorphism group generated by a permutation with cycle type $3^5 1^1$;
- 59 P1Fs with automorphism group generated by a permutation with cycle type $2^7 1^2$.

These numbers agree with an earlier enumeration by Meszka and Rosa [6], *except* that we found one extra P1F with automorphism group of order 7. That P1F can be obtained by developing the following two 1-factors under the permutation (abcdefg)(hijklmn):

$$\{\mathtt{ab}, \mathtt{cg}, \mathtt{do}, \mathtt{em}, \mathtt{fi}, \mathtt{hp}, \mathtt{jl}, \mathtt{kn}\}, \ \{\mathtt{ac}, \mathtt{bk}, \mathtt{dj}, \mathtt{ei}, \mathtt{fp}, \mathtt{gl}, \mathtt{ho}, \mathtt{mn}\}, \tag{2.2}$$

together with the 1-factor {ah, bi, cj, dk, el, fm, gn, op}. (Note that we have relabelled this P1F to display its symmetry and it does not contain (2.1).)

3. Connections with Latin squares

In this section we explore whether the newly discovered P1Fs of K_{16} produce interesting Latin squares. First we need to give some definitions.

A *Latin square* of order n is an $n \times n$ matrix in which each row and column is a permutation of some (fixed) symbol set of size n, say $\{1, 2, \ldots, n\}$. It is often useful to think of a Latin square of order n as a set of n^2 triples (i, j, k), where k is the symbol that appears in cell (i, j). Each Latin square then has six *conjugate* squares obtained by (uniformly) permuting the coordinates of each triple. Writing Latin squares of order n as sets of triples also provides for a natural action of $\mathcal{S}_n \wr \mathcal{S}_3$ on the Latin squares. The *species* (sometimes called *main class*) of a Latin square is its orbit under this action.

A *Latin subrectangle* is a rectangular submatrix of a Latin square L in which the same symbols occur in each row. If R is a $2 \times \ell$ Latin subrectangle of L and R is

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minimal in that it contains no $2 \times \ell'$ Latin subrectangle for $2 \leq \ell' < \ell$, then we say that R is a *row cycle of length* ℓ. Column cycles and symbol cycles can be defined similarly, and the operations of conjugacy on L interchange these objects.

A Latin square of order n is *row-Hamiltonian* if every row cycle has length n, *symbol-Hamiltonian* if every symbol cycle has length n and *column-Hamiltonian* if every column cycle has length n. These three types of square are related by conjugacy. A Latin square is *atomic* if it is row-Hamiltonian, symbol-Hamiltonian and column-Hamiltonian. In other words, a square is atomic if all its conjugates are row-Hamiltonian.

A 1-factorisation $\mathcal{F}$ of K_n is equivalent to a symmetric unipotent Latin square $\mathcal{U}(\mathcal{F})$, where we define $\mathcal{U}(\mathcal{F})[i, j] = k$ if the edge ij appears in the kth factor of $\mathcal{F}$, and $\mathcal{U}(\mathcal{F})[i, i] = n$ for all i. From $\mathcal{U}(\mathcal{F})$, we can form n idempotent symmetric Latin squares of order $n - 1$, by 'folding' a row and the corresponding column onto the main diagonal. To be precise, we form $\mathcal{I}(\mathcal{F}, j)$ from $\mathcal{U}(\mathcal{F})$ by replacing the entry in cell (i, i) by $\mathcal{U}(\mathcal{F})[i, j]$ for each i, and then deleting row j and column j. The 1-factorisation $\mathcal{F}$ is perfect if and only if for each j the corresponding Latin square $\mathcal{I}(\mathcal{F}, j)$ is symbol-Hamiltonian [13]. There is some interest in whether $\mathcal{I}(\mathcal{F}, j)$ has the stronger property of being atomic. Atomic Latin squares are known to exist for some composite orders, but no example is known of an atomic Latin square whose order is not a prime power [11]. This makes order 15, the smallest case where existence of an atomic Latin square is open, particularly interesting.

Each P1F $\mathcal{F}$ corresponds to k species of Latin squares $\mathcal{I}(\mathcal{F}, j)$, where k is the number of orbits of the automorphism group of $\mathcal{F}$. Hence, P1Fs with automorphism groups generated by permutations of cycle type $15^1 1^1, 14^1 1^2, 7^2 1^2, 5^3 1^1, 3^5 1^1, 2^7 1^2, 1^{16}$, respectively, produce $2, 3, 4, 4, 6, 9, 16$ species of Latin squares. Overall, our 3155 P1Fs of K_{16} produce 49 742 species containing symmetric symbol-Hamiltonian Latin squares. Checking a representative of each species, we established the following result.

Theorem 3.1. *There exists no symmetric atomic Latin square of order* 15.

The nearest that we got to an atomic Latin square was N_{15}, displayed in Figure 1. This square is derived from the unique even-starter induced P1F of K_{16}. It is one of the two species of Latin squares derived from that P1F that inherit an automorphism group of order 14. Indeed, it is clear that N_{15} has an automorphism applying the cycle $(1, 2, \ldots, 14)$ simultaneously to rows, columns and symbols. It is also clear that N_{15} is not atomic, because it has a row-cycle of length 3 (highlighted). However, of the $\binom{15}{2} = 105$ pairs of rows in N_{15}, there are 91 that produce a Hamilton row-cycle. The only pairs which fail are the 14 images under the automorphism group of the pair of rows containing the highlighted row-cycle.

4. Invariants

The literature contains several invariants that can be used for distinguishing nonisomorphic P1Fs. The best known of these are the train and tri-colour vector [10].

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$$\begin{bmatrix}
1 & 12 & 2 & 7 & 10 & \colorbox{gray}{8} & 5 & 15 & \colorbox{gray}{13} & \colorbox{gray}{3} & 6 & 4 & 14 & 11 & 9 \\
12 & 2 & 13 & 3 & 8 & 11 & 9 & 6 & 15 & 14 & 4 & 7 & 5 & 1 & 10 \\
2 & 13 & 3 & 14 & 4 & 9 & 12 & 10 & 7 & 15 & 1 & 5 & 8 & 6 & 11 \\
7 & 3 & 14 & 4 & 1 & 5 & 10 & 13 & 11 & 8 & 15 & 2 & 6 & 9 & 12 \\
10 & 8 & 4 & 1 & 5 & 2 & 6 & 11 & 14 & 12 & 9 & 15 & 3 & 7 & 13 \\
8 & 11 & 9 & 5 & 2 & 6 & 3 & 7 & 12 & 1 & 13 & 10 & 15 & 4 & 14 \\
5 & 9 & 12 & 10 & 6 & \colorbox{gray}{3} & 7 & 4 & \colorbox{gray}{8} & \colorbox{gray}{13} & 2 & 14 & 11 & 15 & 1 \\
15 & 6 & 10 & 13 & 11 & 7 & 4 & 8 & 5 & 9 & 14 & 3 & 1 & 12 & 2 \\
13 & 15 & 7 & 11 & 14 & 12 & 8 & 5 & 9 & 6 & 10 & 1 & 4 & 2 & 3 \\
3 & 14 & 15 & 8 & 12 & 1 & 13 & 9 & 6 & 10 & 7 & 11 & 2 & 5 & 4 \\
6 & 4 & 1 & 15 & 9 & 13 & 2 & 14 & 10 & 7 & 11 & 8 & 12 & 3 & 5 \\
4 & 7 & 5 & 2 & 15 & 10 & 14 & 3 & 1 & 11 & 8 & 12 & 9 & 13 & 6 \\
14 & 5 & 8 & 6 & 3 & 15 & 11 & 1 & 4 & 2 & 12 & 9 & 13 & 10 & 7 \\
11 & 1 & 6 & 9 & 7 & 4 & 15 & 12 & 2 & 5 & 3 & 13 & 10 & 14 & 8 \\
9 & 10 & 11 & 12 & 13 & 14 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 15
\end{bmatrix}.$$

FIGURE 1. The Latin square N_{15}.

In our discussion we will mention several *complete invariants* for P1Fs of K_{16}. That is, invariants which coincide for two P1Fs if and only if the P1Fs are isomorphic.

The train itself is a complete invariant for the P1Fs of K_{16}. However, the indegree sequence of the train is not; it partitions the 3155 P1Fs into 3104 equivalence classes. The P1Fs that do not have unique indegree sequences fall into 47 pairs and two triples. We now explicitly give the two triples. The format is similar to that used in (2.2), except that we compress factors by omitting spaces and punctuation, and use only a space to separate factors. The first triple shares the indegree sequence $[573, 784, 336, 86, 19, 2]$:

```
abcdefghijklmnop acbedgfihljmkonp adblcheofngmikjp aebnckdjfoglhpim afbicpdmeghojlkn
agbdcnemfjhkiolp ahbpcodkelfmgijn aibmcedpflgkhnjo ajbgcidnehfklomp akbfcmdoepgnhjil
albhcgdfejinkpmo ambjcfdhekgoipln anbocjdleifhgpkm aobkcldienfpgjhm apbcdefghijklmno

abcdefghijklmnop acbedgfihkjnlpmo adbkcienfhgpjmlo aebncgdfhlikjomp afbhcjdmeoglipkn
agbfckdoejhpimln ahblcfdkemgoinjp aibmcpdnehfogkjl ajbdclepfmgihnko akbjcndielfpgmho
albgcodjeifkhmnp ambpchdlekfngjio anbocmdhegfjilkp aobicedpflgnhjkm apbcdefghijklmno

abcdefghijklmnop acbedgfihljnkpmo adbfchejgmioknlp aebjcpdlfmgohnik afblckdoeghjinmp
agbncmdkelfjhoip ahbpcndfeigjkmlo aibmcjdpenfhglko ajbkcgdiemfohpln akbocedhfngpiljm
albicfdmeognhkjp ambdclehfkgijonp anbhcodjepflgkim aobgcidnekfphmjl apbcdefghijklmno
```

Meanwhile, the second triple shares the indegree sequence $[584, 765, 338, 94, 18, 1]$:

```
abcdefghijklmnop acbedgfihmjokpln adbicpeoflgnhjkm aebgcldfhpiojmkn afbhckdnejgoimlp
agbjcodkeifmhlnp ahbpcgdienfkjlmo aiblcndoepfjgmhk ajbnchdmegfpilko akbfcjdpemgihnlo
albocmdjehfngkip ambkcedhfoglinjp anbdcfelgjhoikmp aobmcidlekfhgpjn apbcdefghijklmno

abcdefghijklmnop acbedgfihljnkpmo adbmcjehfpgkioln aebhckdofngijlmp afbjcndleogmhpik
agbfcedmhkiljonp ahbncgdfeijpkmlo aibdcmejfhgoknlp ajbocidnepfkglhm akbpcodjegflhnim
albgcpdkemfohjin amblchdienfjgpko anbicldpekfmgjho aobkcfdhelgnipjm apbcdefghijklmno

abcdefghijklmnop acbedgfihkjnlpmo adblciemfhgojpkn aebjcfdngmhoilkp afbdcoeigkhpjmln
agbpcndkehflimjo ahbmcjdpenfkgilo aibkcldheofjgnmp ajbgcedmfphlinko akbhcmdjelfngpio
albockdiepfmgjhn ambfchdoejgliknp anbicpdlegfohjkm aobncgdfekhmipjl apbcdefghijklmno
```

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Among the pairs of P1Fs whose train indegree sequences coincide, there is (only) one that involves two P1Fs whose automorphism groups have different orders. Specifically, the following rigid P1F has indegree sequence $[598, 748, 332, 102, 18, 2]$, which coincides with one of the P1Fs with automorphism group of order 2 reported in [6]:

```
abcdefghijklmnop acbedgfjhlinkomp adbmcpeiflgohkjn aebdclfogihmjpkn afbjcndkelgmhpio
agblcjdhekfnipmo ahbpcodnejfkglim aibhcfdoemgnjlkp ajbgcedmfhiklonp akbicgdpeofmhjln
albocmdjepfigkhn ambkchdfengpiljo anbfckdieghojmlp aobncidlehfpgjkm apbcdefghijklmno
```

The train is a complete invariant for P1Fs of K_{16}, but comparing trains is an instance of digraph isomorphism, which is not known to be possible in polynomial time. That makes it desirable to identify features of the train that are easily computed (like the indegree sequence), but still have enough information to be a complete invariant (unlike the indegree sequence). One such property is the following. For each vertex v of the train, define $p(v)$ to be the length of the shortest directed path from v to any vertex w that is in a directed cycle. Since every vertex has outdegree 1, it is a simple matter to construct a path starting at v and follow the unique outgoing arc until we reach a vertex w that we have previously seen on the path. The distance from v to the first occurrence of w is, by definition, $p(v)$. This also shows that $p(v)$ is well defined. Note that if v itself is in a cycle, then $p(v) = 0$. Also, for a general 1-factorisation, the cycle involving w may be a loop. However, for a P1F of K_n (where $n > 2$), this cannot happen, because loops only occur on vertices that are otherwise isolated. To see this, consider two vertices $u = (\{a, b\}, F)$ and $v = (\{c, d\}, G)$ of the train, where F, G are 1-factors and a, b, c, d are vertices of the graph being factorised. If there was an arc uv as well as a loop on v that would mean (up to swapping c and d) that $ac, bd \in F$ and $ab, cd \in G$, which produces a 4-cycle in $F \cup G$.

We found that for $i = 0, 1, 2, 3, 4, 5$, the counts of how many vertices have $p(v) = i$ together form a complete invariant for the P1Fs of K_{16}. Counting only up to $i = 4$ does not suffice, since there is a single pair of (rigid) P1Fs whose first difference occurs at $i = 5$. They are

```
abcdefghijklmnop acbedgfihmjokpln adbmcfeigjhlkonp aebdcnflgkhjiomp afbicjdkenglhpmo
agblcodhejfnipkm ahbkcpdjeofmgnil aibncldoepfjgmhk ajbfckdnemgiholp akbjcedlfpgohnim
albpchdiegfojmkn ambhcgdfekinjplo anbocidmehfkgpjl aobgcmdpelfhikjn apbcdefghijklmno
```

which has counts $[139, 19, 15, 14, 17, 17]$ and

```
abcdefghijklmnop acbedgfihljokmnp adbicefmgjhoknlp aebkcndhfogpiljm afbpcjdlekgihnmo
agbjchdpemfkinlo ahbockdmeiflgnjp aibgcmdoepfjhkln ajbfcldnehgoikmp akbmcidfeoglhpjn
albhcpdiejfngmko amblcgdjenfhiokp anbdcoelfpgkhjim aobncfdkeghmipjl apbcdefghijklmno
```

which has counts $[139, 19, 15, 14, 17, 22]$.

Another well-studied invariant of P1Fs is the tri-colour vector [10]. It was already reported in [6] that the tri-colour vector does not fully distinguish the P1Fs of K_{16} with nontrivial automorphism group. We found that it partitions the set of all P1Fs of K_{16} into 2320 equivalence classes.

Next we propose a new invariant that performs better than either the train indegree sequence or the tri-colour vector on the P1Fs of K_{16}. It is based on the vertex cycles which were used for switchings among 1-factorisations in [4]. These cycles correspond exactly to the row-cycles in $\mathcal{U}(\mathcal{F})$ other than the row-cycles of length 2 that include entries on the main diagonal. Finding the length of all row-cycles in $\mathcal{U}(\mathcal{F})$ is easily done in cubic time. The vector of tallies of how many row-cycles there are of each length is a complete invariant for P1Fs of K_{16}.

Another invariant (that is presumably more discriminating in general) is to count for each row of $\mathcal{U}(\mathcal{F})$ how many row-cycles of each length involve that row. The resulting list of n vectors should then be sorted lexicographically to accommodate relabelling of the vertices of $\mathcal{F}$. We found that for the P1Fs of K_{16} this invariant was complete even if we only counted cycles of lengths 3 and 4. Counting only the cycles of length 3 was not a complete invariant, but did separate the 3155 P1Fs into 3102 equivalence classes.

The invariant involving $p(v)$ as described above can be calculated in polynomial time, but not obviously in cubic time. By comparison, the indegree sequence of the train, tri-colour vector and our new invariant based on vertex-cycle lengths are all easily computed in time cubic in the order of the graph being factorised. The indegree sequence and tri-colour vector are not complete invariants for P1Fs of K_{16} and it seems likely that the vertex-cycle lengths and path lengths will not be a complete invariant when n gets larger. However, it is possible in polynomial time to canonically label a P1F, which does provide a complete invariant for every order. We may define our canonical form to contain the factors F_1 and F_2 from (2.1) and to be the lexicographically least possibility under that assumption. Starting with any P1F, there are only quadratically many choices for the factors that will become F_1 and F_2. For each of those choices, there are only linearly many ways that the two factors can map to F_1 and F_2 (given that their union is a single cycle). Thus, we can check all relabellings in polynomial time and choose the canonical one. Once we have a canonical relabelling, isomorphism checking is a triviality. The fact that isomorphism testing for P1Fs can by done in polynomial time has long been known [2, 6]. The P1Fs given explicitly above, except the one given in (2.2), have been given in the canonical form just described. All P1Fs in the catalogue [12] are also in this form.

5. New orders of P1Fs

The paper [11] gave constructions for many new orders of P1Fs using the quotient coset starter method. Subsequent to publishing that paper, the author discovered many more examples and put them on his web page, where they have been visible for more than 12 years. In the interests of committing them to a more permanent part of the literature, we take this opportunity to record them here. The P1Fs of orders 1092728 and 1225044 were apparently previously found by Volker Leck, but we are not aware of them having been published.

The following produce P1Fs for complete graphs of orders $q + 1 = p^3 + 1$, where p is prime. They can be considered as extra entries for [11, Table 6] and that paper should be consulted for the meaning of the notation.

$p = 101$, $q = 1030301$, $\zeta(x) = x^3 + x + 3$, $\tilde{c} = [813092, 759910, 233271, 3]$,
$p = 103$, $q = 1092727$, $\zeta(x) = x^3 + x + 4$, $\tilde{c} = [828376, 896]$,
$p = 107$, $q = 1225043$, $\zeta(x) = x^3 + x + 9$, $\tilde{c} = [1107573, 151]$,
$p = 109$, $q = 1295029$, $\zeta(x) = x^3 + x + 6$, $\tilde{c} = [271574, 645911, 1082655, 4]$,
$p = 127$, $q = 2048383$, $\zeta(x) = x^3 + x + 15$, $\tilde{c} = [840749, 23]$,
$p = 131$, $q = 2248091$, $\zeta(x) = x^3 + x + 3$, $\tilde{c} = [2096100, 298]$,
$p = 139$, $q = 2685619$, $\zeta(x) = x^3 + x + 7$, $\tilde{c} = [436598, 2118]$,
$p = 149$, $q = 3307949$, $\zeta(x) = x^3 + x + 14$, $\tilde{c} = [1861398, 3141536, 1357853, 1]$,
$p = 151$, $q = 3442951$, $\zeta(x) = x^3 + x + 5$, $\tilde{c} = [1492322, 66]$,
$p = 163$, $q = 4330747$, $\zeta(x) = x^3 + x + 4$, $\tilde{c} = [2015256, 4602]$,
$p = 167$, $q = 4657463$, $\zeta(x) = x^3 + x + 3$, $\tilde{c} = [3183263, 109]$,
$p = 179$, $q = 5735339$, $\zeta(x) = x^3 + x + 4$, $\tilde{c} = [2740965, 1219]$,
$p = 191$, $q = 6967871$, $\zeta(x) = x^3 + x + 3$, $\tilde{c} = [4789910, 1160]$,
$p = 199$, $q = 7880599$, $\zeta(x) = x^3 + x + 13$, $\tilde{c} = [3457494, 2368]$,
$p = 211$, $q = 9393931$, $\zeta(x) = x^3 + x + 24$, $\tilde{c} = [5457264, 1168]$,
$p = 223$, $q = 11089567$, $\zeta(x) = x^3 + x + 9$, $\tilde{c} = [4722613, 4305]$,
$p = 227$, $q = 11697083$, $\zeta(x) = x^3 + x + 9$, $\tilde{c} = [9051956, 1442]$,
$p = 239$, $q = 13651919$, $\zeta(x) = x^3 + x + 11$, $\tilde{c} = [1597504, 5918]$,
$p = 251$, $q = 15813251$, $\zeta(x) = x^3 + x + 7$, $\tilde{c} = [9285089, 11965]$,
$p = 263$, $q = 18191447$, $\zeta(x) = x^3 + x + 8$, $\tilde{c} = [8313030, 2840]$,
$p = 271$, $q = 19902511$, $\zeta(x) = x^3 + x + 4$, $\tilde{c} = [6563520, 170]$,
$p = 283$, $q = 22665187$, $\zeta(x) = x^3 + x + 24$, $\tilde{c} = [2245440, 3574]$.

The following produce P1Fs for complete graphs of orders $q + 1 = p^5 + 1$, where p is prime. They can be considered as extra entries for [11, Table 7].
$p = 19$, $q = 2476099$, $\zeta(x) = x^5 + x + 9$, $\tilde{c} = [949007, 791]$,
$p = 23$, $q = 6436343$, $\zeta(x) = x^5 + x + 3$, $\tilde{c} = [1045440, 7580]$.

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