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ON THE CONTINUOUS AND SMOOTH FIT PRINCIPLE FOR OPTIMAL STOPPING PROBLEMS IN SPECTRALLY NEGATIVE LÉVY MODELS

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Abstract

We consider a class of infinite time horizon optimal stopping problems for spectrally negative Lévy processes. Focusing on strategies of threshold type, we write explicit expressions for the corresponding expected payoff via the scale function, and further pursue optimal candidate threshold levels. We obtain and show the equivalence of the continuous/smooth fit condition and the first-order condition for maximization over threshold levels. As examples of its applications, we give a short proof of the McKean optimal stopping problem (perpetual American put option) and solve an extension to Egami and Yamazaki (2013).

Keywords: Optimal stopping; spectrally negative Lévy process; scale function; continuous and smooth fit

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1. Introduction

Optimal stopping problems arise in various areas ranging from the classical sequential testing/change-point detection problems to applications in finance. Although all formulations reduce to the problem of maximizing/minimizing the expected payoff over a set of stopping times, the solution methods are mostly problem-specific; they depend significantly on the underlying process, payoff function, and time-horizon. This paper pursues a common tool for the class of *infinite time horizon* optimal stopping problems for *spectrally negative Lévy processes*, i.e. Lévy processes with only negative jumps.

By extending the classical continuous diffusion model to the Lévy model, one can achieve richer and more realistic models. In mathematical finance, the continuity of paths is empirically rejected and cannot explain, for example, the volatility smile and nonzero credit spreads for short-maturity corporate bonds. These issues can often be alleviated by introducing jumps; see, e.g. [17] and [29]. Recently, we have seen significant progress in the theory of optimal stopping for Lévy processes and other jump processes. The fluctuation theory, in particular, has played a key role in characterizing efficiently the value function and the optimal stopping time; see [14], [20], [39], and [45] among others.

In this paper, we revisit the optimal stopping problem for a general spectrally negative Lévy process, and pursue a solution in a rather straightforward way. Despite the aforementioned

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results, existing results under Lévy processes are still significantly more limited than the one-dimensional diffusion case as in [2], [10], [18], [22], [23], and [41]. Without the continuity of paths, the process can jump over a given threshold. For a general payoff function, we naturally need to take care of the overshoot distribution, which is generally a big hurdle that typically makes the problem intractable. However, thanks to the recent advances in the fluctuation theory of spectrally negative Lévy processes (see [11] and [31]), this can be handled by using the so-called *scale function*.

The objective of this paper is to pursue, with the help of the scale function, a common technique for the class of optimal stopping problems for spectrally negative Lévy processes. Focusing on the first time it down-crosses a fixed threshold, we express the corresponding expected payoff in terms of the scale function. This semi-explicit form enables us to differentiate and take limits thanks to the smoothness and asymptotic properties of the scale function as obtained, for example, in [15] and [31]. By differentiating the expected payoff with respect to the threshold level, we obtain the *first-order condition* as well as the candidate optimal level that makes it vanish. We also obtain the *continuous/smooth fit condition* when the process is of bounded variation or when it contains a diffusion component. These conditions are in fact equivalent and can be obtained generally under mild conditions.

The spectrally negative Lévy model has been drawing much attention recently as a generalization of the classical Black–Scholes model in mathematical finance and also as a generalization of the Cramér–Lundberg model in insurance. A number of authors have succeeded in extending the classical results to the spectrally negative Lévy model by way of scale functions. We refer the reader to [6] and [7] for stochastic games, [5], [8], [9], [32], and [37] for the optimal dividend problem, [1] and [4] for American and Russian options, [26], [33], and [36] for credit risk, and [49] for inventory models. In particular, Egami and Yamazaki [25] modeled and obtained the optimal timing of capital reinforcement. As an application of the results obtained in this paper, we give a short proof of the McKean optimal stopping (perpetual American put option) problem with additional running rewards, as well as an extension and its analytical solution to [25].

The rest of the paper is organized as follows. In Section 2, we review the optimal stopping problem for spectrally negative Lévy processes, and then express the expected value corresponding to the first down-crossing time in terms of the scale function. In Section 3, we obtain the first-order condition as well as the continuous/smooth fit condition and show their equivalence. In Section 4, we solve the McKean optimal stopping problem and an extension to [25]. We conclude the paper in Section 5.

2. The optimal stopping problem for spectrally negative Lévy processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space hosting a spectrally negative Lévy process $X = \{X_t : t \geq 0\}$ characterized uniquely by the *Laplace exponent*

$$\psi(\beta) := \mathbb{E}^0[\mathrm{e}^{\beta X_1}] = c\beta + \frac{1}{2}\sigma^2\beta^2 + \int_0^\infty (\mathrm{e}^{-\beta z} - 1 + \beta z\,\mathbf{1}_{\{0<z<1\}})\Pi(\mathrm{d}z), \qquad \beta \in \mathbb{R}, \quad (2.1)$$

where $c \in \mathbb{R}$, $\sigma \geq 0$, and Π is a measure on $(0, \infty)$ such that

$$\int_0^\infty (1 \wedge z^2)\Pi(\mathrm{d}z) < \infty. \qquad (2.2)$$

Here and throughout the paper, $\mathbb{P}^x$ is the conditional probability where $X_0 = x \in \mathbb{R}$ and $\mathbb{E}^x$ is its expectation (also $\mathbb{P} \equiv \mathbb{P}^0$ and $\mathbb{E} \equiv \mathbb{E}^0$). It is well known that ψ is 0 at the origin, convex on

$\mathbb{R}_+$, and has a right-continuous inverse

$$\Phi(q) := \sup\{\lambda \ge 0 \colon \psi(\lambda) = q\}, \qquad q \ge 0.$$

In particular, when

$$\int_0^\infty (1 \wedge z)\Pi(\mathrm{d}z) < \infty, \tag{2.3}$$

we can rewrite

$$\psi(\beta) = \mu\beta + \frac{1}{2}\sigma^2\beta^2 + \int_0^\infty (\mathrm{e}^{-\beta z} - 1)\Pi(\mathrm{d}z), \qquad \beta \in \mathbb{R},$$

where

$$\mu := c + \int_0^1 z\Pi(\mathrm{d}z).$$

The process has paths of bounded variation if and only if $\sigma = 0$ and (2.3) holds. It is also assumed that X is not a negative subordinator (decreasing almost surely (a.s.)). That is, we require μ to be strictly positive if $\sigma = 0$ and (2.3) holds.

Let $\mathbb{F} := (\mathcal{F}_t)_{t\ge 0}$ be the filtration generated by X and $\mathcal{S}$ a set of $\mathbb{F}$-stopping times. We shall consider a general optimal stopping problem of the form

$$u(x) := \sup_{\tau\in\mathcal{S}} \mathbb{E}^x\bigg[\mathrm{e}^{-q\tau} g(X_\tau)\,\mathbf{1}_{\{\tau<\infty\}} + \int_0^\tau \mathrm{e}^{-qt} h(X_t)\,\mathrm{d}t\bigg], \qquad x \in \mathbb{R} \tag{2.4}$$

for some discount factor $q > 0$ and locally-bounded measurable functions $g, h \colon \mathbb{R} \to \mathbb{R}$ which represent, respectively, the payoff received at a given stopping time τ and the running reward up to τ. It is assumed here that the expectation is well defined; we shall give further assumptions on g and h in order to guarantee that it is indeed so. See Assumption 2.2 and the assumptions in Lemma 2.3 below.

Typically, the optimal stopping time is given by the *first down-crossing time* of the form

$$\tau_A := \inf\{t > 0 \colon X_t \le A\}, \qquad A \in \mathbb{R}, \tag{2.5}$$

with $\inf\varnothing = \infty$. Let us denote the corresponding expected payoff by

$$u_A(x) := \mathbb{E}^x\bigg[\mathrm{e}^{-q\tau_A} g(X_{\tau_A})\,\mathbf{1}_{\{\tau_A<\infty\}} + \int_0^{\tau_A} \mathrm{e}^{-qt} h(X_t)\,\mathrm{d}t\bigg], \qquad x, A \in \mathbb{R},$$

which can be decomposed into

$$u_A(x) = \begin{cases} \Gamma_1(x; A) + \Gamma_2(x; A) + \Gamma_3(x; A), & x > A, \\ g(x), & x \le A, \end{cases}$$

where, for every $x > A$,

$$\begin{aligned}
\Gamma_1(x; A) &:= g(A)\mathbb{E}^x[\mathrm{e}^{-q\tau_A}], \\
\Gamma_2(x; A) &:= \mathbb{E}^x[\mathrm{e}^{-q\tau_A}(g(X_{\tau_A}) - g(A))\,\mathbf{1}_{\{X_{\tau_A}<A,\tau_A<\infty\}}], \\
\Gamma_3(x; A) &:= \mathbb{E}^x\bigg[\int_0^{\tau_A} \mathrm{e}^{-qt} h(X_t)\,\mathrm{d}t\bigg].
\end{aligned} \tag{2.6}$$

In Subsection 2.2 we express each term via the scale function.

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Remark 2.1. This paper does not consider the *first up-crossing time*, which is defined by $\tau_B^+ := \inf\{t > 0 \colon X_t \geq B\}$, because, for the spectrally negative Lévy case, the process always creeps upward ($g(X_{\tau_B^+}) = g(B)$ a.s. on $\{\tau_B^+ < \infty\}$), and the expression of the expected value is much simplified. We focus on a more interesting and challenging case where the optimal stopping time is conjectured to be a first down-crossing time. We refer the reader to [20] and [44], among others, for related problems under a more general Markov process.

Remark 2.2. It is possible to reduce the problem so that the running reward part is 0 (i.e. $h \equiv 0$); see, e.g. [21] for the reduction technique. However, we decide to solve the problem in the current form because the running reward part can be handled more easily than the stopping payoff part, especially for the verification of optimality. In our examples in Section 4, the optimality holds under a mild monotonicity on the function h; this gives a more intuitive explanation about the problems and their optimal solutions.

2.1. Scale functions

In this subsection we give a brief review on the scale function that will be needed for our analysis. For a comprehensive account of the scale function, see [11], [12], [31], and [33]. See [24], [30], and [46] for numerical methods for computing the scale function.

Associated with every spectrally negative Lévy process, there exists a (q-)scale function

$$W^{(q)} \colon \mathbb{R} \to \mathbb{R}, \qquad q \geq 0,$$

that is continuous, strictly increasing on $[0, \infty)$, and uniquely determined by

$$\int_0^\infty \mathrm{e}^{-\beta x} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(\beta) - q}, \qquad \beta > \Phi(q).$$

Fix $a > x > 0$. If τ_a^+ is the first time the process goes above a and τ_0 is the first time it goes below 0 as a special case of (2.5), then we have

$$\mathbb{E}^x[\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+ < \tau_0,\, \tau_a^+ < \infty\}}] = \frac{W^{(q)}(x)}{W^{(q)}(a)},$$

$$\mathbb{E}^x[\mathrm{e}^{-q\tau_0}\mathbf{1}_{\{\tau_a^+ > \tau_0,\, \tau_0 < \infty\}}] = Z^{(q)}(x) - Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)},$$

where

$$Z^{(q)}(x) := 1 + q\int_0^x W^{(q)}(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}.$$

Here we have

$$W^{(q)}(x) = 0 \quad \text{on } (-\infty, 0) \quad \text{and} \quad Z^{(q)}(x) = 1 \quad \text{on } (-\infty, 0].$$

We also have

$$\mathbb{E}^x[\mathrm{e}^{-q\tau_0}] = Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x), \qquad x > 0. \tag{2.7}$$

In particular, $W^{(q)}$ is continuously differentiable on $(0, \infty)$ if Π does not have atoms and $W^{(q)}$ is twice differentiable on $(0, \infty)$ if $\sigma > 0$; see, e.g. [15]. Throughout this paper, we assume the former.

Assumption 2.1. *We assume that Π does not have atoms.*

Fix $q > 0$. The scale function increases exponentially;

$$W^{(q)}(x) \sim \frac{\mathrm{e}^{\Phi(q)x}}{\psi'(\Phi(q))} \quad \text{as } x \uparrow \infty. \tag{2.8}$$

There exists a (scaled) version of the scale function $W_{\Phi(q)} = \{W_{\Phi(q)}(x); x \in \mathbb{R}\}$ that satisfies

$$W_{\Phi(q)}(x) = \mathrm{e}^{-\Phi(q)x} W^{(q)}(x), \qquad x \in \mathbb{R} \tag{2.9}$$

and

$$\int_0^\infty \mathrm{e}^{-\beta x} W_{\Phi(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(\beta + \Phi(q)) - q}, \qquad \beta > 0.$$

Moreover, $W_{\Phi(q)}(x)$ is increasing, and as is clear from (2.8),

$$W_{\Phi(q)}(x) \uparrow \frac{1}{\psi'(\Phi(q))} \quad \text{as } x \uparrow \infty. \tag{2.10}$$

Regarding its behavior in the neighborhood of 0, it is known that

$$W^{(q)}(0) = \begin{cases} 0, & \text{unbounded variation,} \\ \dfrac{1}{\mu}, & \text{bounded variation,} \end{cases}$$

$$W^{(q)\prime}(0+) = \begin{cases} \dfrac{2}{\sigma^2}, & \sigma > 0, \\ \infty, & \sigma = 0 \text{ and } \Pi(0,\infty) = \infty, \\ \dfrac{q + \Pi(0,\infty)}{\mu^2}, & \sigma = 0 \text{ and } \Pi(0,\infty) < \infty,; \end{cases} \tag{2.11}$$

see Lemmas 4.3 and 4.4 of [33].

2.2. Expressing the expected payoff using the scale function

We now express (2.6) in terms of the scale function. For the rest of the paper, because $q > 0$, we must have $\Phi(q) > 0$.

First, the following is immediate by (2.7).

Lemma 2.1. *For every $x > A$, we have*

$$\Gamma_1(x; A) = g(A)\left[Z^{(q)}(x - A) - \frac{q}{\Phi(q)} W^{(q)}(x - A)\right].$$

For Γ_2 and Γ_3, we use the potential measure written in terms of the scale function. For the problem to be well defined, we assume the following throughout the paper, so that Γ_3 is finite. For a complete proof of Lemma 2.2 below, see [25].

Assumption 2.2. *We assume that $\int_0^\infty \mathrm{e}^{-\Phi(q)y} |h(y + A)|\,\mathrm{d}y < \infty$ for any $A \in \mathbb{R}$.*

Lemma 2.2. *For all $x > A$, we have*

$$\Gamma_3(x; A) = W^{(q)}(x - A) \int_0^\infty \mathrm{e}^{-\Phi(q)y} h(y + A)\,\mathrm{d}y - \int_A^x W^{(q)}(x - y) h(y)\,\mathrm{d}y.$$

For Γ_2, we first define, for every $A \in \mathbb{R}$,

$$
\begin{aligned}
\rho^{(q)}_{g,A} &:= \int_0^\infty \Pi(\mathrm{d}u) \int_0^u \mathrm{e}^{-\Phi(q)z}(g(z+A-u)-g(A))\,\mathrm{d}z \\
&\equiv \int_0^\infty \Pi(\mathrm{d}u) \int_A^{u+A} \mathrm{e}^{-\Phi(q)(y-A)}(g(y-u)-g(A))\,\mathrm{d}y, \\
\bar{\rho}^{(q)}_{g,A} &:= \int_0^\infty \Pi(\mathrm{d}u) \int_0^u \mathrm{e}^{-\Phi(q)z}|g(z+A-u)-g(A)|\,\mathrm{d}z \\
&\equiv \int_0^\infty \Pi(\mathrm{d}u) \int_A^{u+A} \mathrm{e}^{-\Phi(q)(y-A)}|g(y-u)-g(A)|\,\mathrm{d}y. \qquad (2.12)
\end{aligned}
$$

Lemma 2.3. *Fix $A \in \mathbb{R}$. Suppose that*

(1) *g is C^2 in a neighborhood of A and*

(2) *g satisfies*

$$\int_1^\infty \Pi(\mathrm{d}u) \max_{A-u\le\zeta\le A} |g(\zeta)-g(A)| < \infty, \qquad (2.13)$$

then $\bar{\rho}^{(q)}_{g,A} < \infty$.

Proof. See Appendix A.1.

For every $x > A$, we also define

$$
\begin{aligned}
\varphi^{(q)}_{g,A}(x) &:= \int_0^\infty \Pi(\mathrm{d}u) \int_0^{u\wedge(x-A)} W^{(q)}(x-z-A)(g(z+A-u)-g(A))\,\mathrm{d}z, \\
\bar{\varphi}^{(q)}_{g,A}(x) &:= \int_0^\infty \Pi(\mathrm{d}u) \int_0^{u\wedge(x-A)} W^{(q)}(x-z-A)|g(z+A-u)-g(A)|\,\mathrm{d}z.
\end{aligned}
$$

By (2.9) and (2.10),

$$
\begin{aligned}
\bar{\varphi}^{(q)}_{g,A}(x) &= \mathrm{e}^{\Phi(q)(x-A)} \int_0^\infty \Pi(\mathrm{d}u) \\
&\quad\times \int_0^{u\wedge(x-A)} \mathrm{e}^{-\Phi(q)z} W_{\Phi(q)}(x-z-A)|g(z+A-u)-g(A)|\,\mathrm{d}z \\
&\le \mathrm{e}^{\Phi(q)(x-A)} \frac{\bar{\rho}^{(q)}_{g,A}}{\psi'(\Phi(q))}, \qquad (2.14)
\end{aligned}
$$

and, hence, the finiteness of $\bar{\rho}^{(q)}_{g,A}$ also implies that of $\bar{\varphi}^{(q)}_{g,A}(x)$ for any $x > A$.

Using these notations, Lemma 2.2 together with the compensation formula shows the following.

Lemma 2.4. *If (1)–(2) of Lemma 2.3 hold for a given $A \in \mathbb{R}$, then*

$$\Gamma_2(x;A) = W^{(q)}(x-A)\rho^{(q)}_{g,A} - \varphi^{(q)}_{g,A}(x), \qquad x > A. \qquad (2.15)$$

Proof. Let $N(\cdot,\cdot)$ be the Poisson random measure associated with the jumps of $-X$ and $\underline{X}_t := \min_{0\le s\le t} X_s$ for all $t \ge 0$. We also let $x_\pm = \max(\pm x, 0)$ for any $x \in \mathbb{R}$. By the

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compensation formula (see, e.g. [31]),

$$
\begin{aligned}
&\mathbb{E}^x[\mathrm{e}^{-q\tau_A}(g(X_{\tau_A})-g(A))_+\mathbf{1}_{\{X_{\tau_A}<A,\,\tau_A<\infty\}}] \\
&\quad=\mathbb{E}^x\bigg[\int_0^\infty\int_0^\infty N(\mathrm{d}t,\,\mathrm{d}u)\mathrm{e}^{-qt}(g(X_{t-}-u)-g(A))_+\mathbf{1}_{\{X_{t-}-u\le A,\,\underline{X}_{t-}>A\}}\bigg] \\
&\quad=\mathbb{E}^x\bigg[\int_0^\infty \mathrm{e}^{-qt}\,\mathrm{d}t\int_0^\infty \Pi(\mathrm{d}u)(g(X_{t-}-u)-g(A))_+\mathbf{1}_{\{X_{t-}-u\le A,\,\underline{X}_{t-}>A\}}\bigg] \\
&\quad=\int_0^\infty \Pi(\mathrm{d}u)\mathbb{E}^x\bigg[\int_0^\infty \mathrm{e}^{-qt}(g(X_{t-}-u)-g(A))_+\mathbf{1}_{\{X_{t-}-u\le A,\,\underline{X}_{t-}>A\}}\,\mathrm{d}t\bigg] \\
&\quad=\int_0^\infty \Pi(\mathrm{d}u)\mathbb{E}^x\bigg[\int_0^{\tau_A} \mathrm{e}^{-qt}(g(X_{t-}-u)-g(A))_+\mathbf{1}_{\{X_{t-}\le A+u\}}\,\mathrm{d}t\bigg].
\end{aligned}
$$

By setting $h(y)\equiv(g(y-u)-g(A))_+\mathbf{1}_{\{y\le A+u\}}$ or, equivalently,

$$
h(y+A)\equiv(g(y+A-u)-g(A))_+\mathbf{1}_{\{y\le u\}},
$$

in Lemma 2.2,

$$
\begin{aligned}
&\mathbb{E}^x\bigg[\int_0^{\tau_A}\mathrm{e}^{-qt}(g(X_{t-}-u)-g(A))_+\mathbf{1}_{\{X_{t-}\le A+u\}}\,\mathrm{d}t\bigg] \\
&\qquad=W^{(q)}(x-A)\int_0^u \mathrm{e}^{-\Phi(q)y}(g(y+A-u)-g(A))_+\,\mathrm{d}y \\
&\qquad\quad-\int_A^x W^{(q)}(x-y)(g(y-u)-g(A))_+\mathbf{1}_{\{y\le A+u\}}\,\mathrm{d}y \\
&\qquad=W^{(q)}(x-A)\int_0^u \mathrm{e}^{-\Phi(q)y}(g(y+A-u)-g(A))_+\,\mathrm{d}y \\
&\qquad\quad-\int_0^{u\wedge(x-A)} W^{(q)}(x-z-A)(g(z+A-u)-g(A))_+\,\mathrm{d}z.
\end{aligned}
$$

By substituting this, we have

$$
\begin{aligned}
&\mathbb{E}^x[\mathrm{e}^{-q\tau_A}(g(X_{\tau_A})-g(A))_+\mathbf{1}_{\{X_{\tau_A}<A,\,\tau_A<\infty\}}] \\
&\quad=\int_0^\infty\Pi(\mathrm{d}u)\bigg[W^{(q)}(x-A)\int_0^u \mathrm{e}^{-\Phi(q)y}(g(y+A-u)-g(A))_+\,\mathrm{d}y \\
&\qquad\qquad-\int_0^{u\wedge(x-A)} W^{(q)}(x-z-A)(g(z+A-u)-g(A))_+\,\mathrm{d}z\bigg],
\end{aligned}
$$

which is finite by Lemma 2.3 and (2.14). Similarly, we can obtain

$$
\mathbb{E}^x[\mathrm{e}^{-q\tau_A}(g(X_{\tau_A})-g(A))_-\mathbf{1}_{\{X_{\tau_A}<A,\,\tau_A<\infty\}}]
$$

and (2.15) is immediate by taking the difference.

In view of (2.15), we can also write

$$
\begin{aligned}
W^{(q)}(x-A)\rho_{g,A}^{(q)}&=W_{\Phi(q)}(x-A)\mathrm{e}^{\Phi(q)x}\int_0^\infty\Pi(\mathrm{d}u)\int_A^{u+A}\mathrm{e}^{-\Phi(q)y}(g(y-u)-g(A))\,\mathrm{d}y, \\
\varphi_{g,A}^{(q)}(x)&=\int_0^\infty\Pi(\mathrm{d}u)\int_A^{(u+A)\wedge x}W^{(q)}(x-z)(g(z-u)-g(A))\,\mathrm{d}z. \qquad (2.16)
\end{aligned}
$$

3. First-order condition and continuous and smooth fit

The most common way of choosing the candidate threshold level is via the continuous and smooth fit principle. Define

$$u_A(A+) := \lim_{x\downarrow A} u_A(x) \quad \text{and} \quad u'_A(A+) := \lim_{x\downarrow A} u'_A(x), \qquad A\in\mathbb{R},$$

if these limits exist. The continuous and smooth fit chooses A such that $u_A(A+) = g(A)$ and $u'_A(A+) = g'(A)$, respectively. Alternatively, we can differentiate u_A with respect to A and obtain the first-order condition.

In this section, we pursue the candidate threshold level A^* in both ways. We first obtain, for a general case, the first derivative $\partial u_A(x)/\partial A$ and A that makes it vanish, and then the continuous fit condition for the case X is of bounded variation and the smooth fit condition for the case X has a diffusion component ($\sigma > 0$). We further discuss the equivalence of these conditions and how to obtain optimal strategies.

3.1. First-order condition

We shall obtain $\partial u_A(x)/\partial A$ for $x > A$. Let

$$\Psi(A) := -\frac{q}{\Phi(q)} g(A) + \rho^{(q)}_{g,A} + \int_0^\infty \mathrm{e}^{-\Phi(q)y} h(y+A)\,\mathrm{d}y, \qquad A\in\mathbb{R}.$$

Proposition 3.1. (Derivative of u_A with respect to A.) *For given $x > A$, suppose (1)–(2) of Lemma 2.3 hold and*

$$\int_1^\infty \Pi(\mathrm{d}u) \max_{0\le\xi\le\delta} |g(A+\xi) - g(A+\xi-u)| < \infty \tag{3.1}$$

for some $\delta > 0$. Then, we have

$$\frac{\partial}{\partial A} u_A(x) = -\Theta^{(q)}(x-A)\Big(\Psi(A) - \frac{\sigma^2}{2} g'(A)\Big),$$

where

$$\Theta^{(q)}(y) := \mathrm{e}^{\Phi(q)y} W'_{\Phi(q)}(y), \qquad y > 0.$$

Because $W_{\Phi(q)}$ is increasing, $\Theta^{(q)}$ is positive and, hence,

$$\Psi(A) - \frac{\sigma^2}{2} g'(A) \le (\ge) 0 \quad \Longrightarrow \quad \frac{\partial}{\partial A} u_A(x) \ge (\le) 0 \quad \text{for all } x > A. \tag{3.2}$$

If there exists A^* such that

$$\Psi(A^*) - \frac{\sigma^2}{2} g'(A^*) = 0 \tag{3.3}$$

then the stopping time τ_{A^*} naturally becomes a reasonable candidate for the optimal stopping time.

In order to show Proposition 3.1, we obtain the derivatives of Γ_i for $1 \le i \le 3$ with respect to A for any $x > A$. By applying straightforward differentiation in Lemma 2.1 and because $W^{(q)\prime}(x) = \Phi(q) W_{\Phi(q)}(x) + \Theta^{(q)}(x)$,

$$\frac{\partial}{\partial A}\Gamma_1(x;A) = g'(A)\Big[Z^{(q)}(x-A) - \frac{q}{\Phi(q)} W^{(q)}(x-A)\Big] + g(A)\frac{q}{\Phi(q)}\Theta^{(q)}(x-A). \tag{3.4}$$

For Γ_2, we first take the derivatives of (2.16) with respect to A.

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Lemma 3.1. *Fix $x > A$. Under the assumptions in Proposition 3.1,*

$$\frac{\partial}{\partial A}\int_0^\infty \Pi(\mathrm{d}u)\int_A^{u+A} \mathrm{e}^{-\Phi(q)y}(g(y-u)-g(A))\,\mathrm{d}y \\ = \mathrm{e}^{-\Phi(q)A}\int_0^\infty \Pi(\mathrm{d}u)\bigg[g(A)-g(A-u)-\frac{1-\mathrm{e}^{-\Phi(q)u}}{\Phi(q)}g'(A)\bigg], \tag{3.5}$$

and

$$\frac{\partial}{\partial A}\varphi_{g,A}^{(q)}(x) = \int_0^\infty \Pi(\mathrm{d}u)\bigg[W^{(q)}(x-A)(g(A)-g(A-u)) \\ - g'(A)\int_A^{(u+A)\wedge x} W^{(q)}(x-z)\,\mathrm{d}z\bigg]. \tag{3.6}$$

Proof. See Appendix A.2.

By applying Lemma 3.1 in (2.15)–(2.16), the derivative of Γ_2 with respect to A is immediately obtained.

Lemma 3.2. *Fix $x > A$. Under the assumptions in Proposition 3.1,*

$$\frac{\partial}{\partial A}\Gamma_2(x;A) \\ = -W'_{\Phi(q)}(x-A)\mathrm{e}^{\Phi(q)x}\int_0^\infty \Pi(\mathrm{d}u)\int_A^{u+A}\mathrm{e}^{-\Phi(q)y}(g(y-u)-g(A))\,\mathrm{d}y \\ + g'(A)\int_0^\infty \Pi(\mathrm{d}u)\bigg(\int_A^{(u+A)\wedge x} W^{(q)}(x-z)\,\mathrm{d}z - \frac{1-\mathrm{e}^{-\Phi(q)u}}{\Phi(q)}W^{(q)}(x-A)\bigg).$$

For Γ_3, as in the proof of Lemma 4.4 of [25], we have the following. Although the continuity of h is assumed throughout in [25], it is not required in the following lemma; this is clear from the proof of Lemma 4.4 of [25].

Lemma 3.3. *For every $x > A$,*

$$\frac{\partial}{\partial A}\Gamma_3(x;A) = -\Theta^{(q)}(x-A)\int_0^\infty \mathrm{e}^{-\Phi(q)y}h(y+A)\,\mathrm{d}y.$$

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. By combining (3.4) and Lemmas 3.2 and 3.3, we obtain

$$\frac{\partial}{\partial A}u_A(x) = -\Theta^{(q)}(x-A)\Psi(A) + g'(A)Q(x;A),$$

where, for $x > A$,

$$Q(x;A) := Z^{(q)}(x-A) - \frac{q}{\Phi(q)}W^{(q)}(x-A) \\ - \int_0^\infty \Pi(\mathrm{d}u)\bigg(W^{(q)}(x-A)\frac{1-\mathrm{e}^{-\Phi(q)u}}{\Phi(q)} - \int_A^{(u+A)\wedge x} W^{(q)}(x-z)\,\mathrm{d}z\bigg).$$

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By Lemma 2.1 and modifying Lemma 2.4, we can also write, for $x > A$,

$$Q(x; A) = \mathbb{E}^x[\mathrm{e}^{-q\tau_A}] - \mathbb{E}^x[\mathrm{e}^{-q\tau_A}\,\mathbf{1}_{\{X_{\tau_A} < A,\, \tau_A < \infty\}}] = \mathbb{E}^x[\mathrm{e}^{-q\tau_A}\,\mathbf{1}_{\{X_{\tau_A} = A,\, \tau_A < \infty\}}].$$

A spectrally negative Lévy process creeps downward if and only if there is a Gaussian component, i.e.

$$\mathbb{P}^x\{X_{\tau_A} = A, \tau_A < \infty\} > 0 \quad \text{for all } x > A \quad \Longleftrightarrow \quad \sigma > 0;$$

see Exercise 7.6 of [31]. Hence,

$$\sigma > 0 \quad \Longleftrightarrow \quad Q(x; A) > 0 \quad \text{for all } x > A.$$

This proves the desired result for the case $\sigma = 0$. For the case $\sigma > 0$, as in [13] and [42], we can also write

$$Q(x; A) = \frac{\sigma^2}{2}(W^{(q)\prime}(x - A) - \Phi(q)W^{(q)}(x - A)) = \frac{\sigma^2}{2}\Theta^{(q)}(x - A),$$

and, hence, the result also holds when $\sigma > 0$.

3.2. Continuous and smooth fit

We now pursue A^* such that $u_{A^*}(A^*+) = g(A^*)$ and $u'_{A^*}(A^*+) = g'(A^*)$, respectively, for the cases

- X is of bounded variation, and
- $\sigma > 0$.

We exclude the case where X is of unbounded variation with $\sigma = 0$ (in this case, $W^{(q)\prime}(0+) = \infty$ by (2.11) and, hence, the interchange of limits over integrals we conduct below may not be valid). However, this can be alleviated and the results hold generally for all spectrally negative Lévy processes when g is a constant in a neighborhood of A^*. Examples include [25], where $g(x) = 0$ on $(0, \infty)$, and [43], where $g(x) = 1$ on $(-\infty, 0]$ and $g(x) = 2$ on $(0, \infty)$; see Section 4.

For continuous fit, we need to obtain

$$\Gamma_1(A+; A) := \lim_{x \downarrow A} \Gamma_1(x; A), \qquad \Gamma_2(A+; A) := \lim_{x \downarrow A} \Gamma_2(x; A),$$
$$\Gamma_3(A+; A) := \lim_{x \downarrow A} \Gamma_3(x; A)$$

if these limits exist. Define also $\varphi^{(q)}_{g,A}(A+) := \lim_{x \downarrow A} \varphi^{(q)}_{g,A}(x)$, if it exists. It is easy to see that

$$\Gamma_1(A+; A) = g(A)\biggl(1 - \frac{q}{\Phi(q)} W^{(q)}(0)\biggr),$$
$$\Gamma_3(A+; A) = W^{(q)}(0) \int_0^\infty \mathrm{e}^{-\Phi(q)y} h(y + A)\,\mathrm{d}y. \tag{3.7}$$

The result for Γ_2 is immediate by the dominated convergence theorem thanks to Lemma 2.3 and (2.14)–(2.15).

Lemma 3.4. *Given (1)–(2) of Lemma 2.3 for a given $A \in \mathbb{R}$, we have*

(1) $\varphi^{(q)}_{g,A}(A+) = 0$,

(2) $\Gamma_2(A+; A) = W^{(q)}(0)\rho^{(q)}_{g,A}$.

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Now Lemma 3.4 and (3.7) show that

$$u_A(A+) = g(A) + W^{(q)}(0)\Psi(A). \tag{3.8}$$

This together with (2.11) shows the following.

Proposition 3.2. (Continuous fit.) *Fix $A \in \mathbb{R}$ and suppose (1)–(2) of Lemma 2.3 hold.*

(1) *If X is of bounded variation, the continuous fit condition $u_A(A+) = g(A)$ holds if and only if*
$$\Psi(A) = 0.$$

(2) *If X is of unbounded variation (including the case $\sigma = 0$), it is automatically satisfied.*

For the case where X is of unbounded variation with $\sigma > 0$, we shall pursue the smooth fit condition at $A \in \mathbb{R}$. The following lemma says in this case that the derivative can go into the integral sign and we can further interchange the limit.

Lemma 3.5. *Fix $A \in \mathbb{R}$. If $\sigma > 0$ and suppose (1)–(2) of Lemma 2.3 hold, then, for $x > A$,*

$$\varphi_{g,A}^{(q)\prime}(x) = \int_0^\infty \Pi(\mathrm{d}u) \int_0^{u\wedge(x-A)} W^{(q)\prime}(x-z-A)[g(z+A-u)-g(A)]\,\mathrm{d}z, \tag{3.9}$$

and

$$\varphi_{g,A}^{(q)\prime}(A+) = 0. \tag{3.10}$$

Proof. See Appendix A.3.

We are now ready to obtain $\Gamma_i'(A+; A)$ for $1 \le i \le 3$.

Lemma 3.6. *Fix $A \in \mathbb{R}$. Suppose $\sigma > 0$ and (1)–(2) of Lemma 2.3 hold. Then,*

(1) $\Gamma_1'(A+, A) = -W^{(q)\prime}(0+)g(A)q/\Phi(q)$,

(2) $\Gamma_2'(A+; A) = W^{(q)\prime}(0+)\rho_{g,A}^{(q)}$, *and*

(3) $\Gamma_3'(A+; A) = W^{(q)\prime}(0+)\int_0^\infty \mathrm{e}^{-\Phi(q)y}h(y+A)\,\mathrm{d}y$.

Proof. We see that (1) is immediate by Lemma 2.1. For (2), by (2.15),

$$\Gamma_2'(x; A) = W^{(q)\prime}(x-A)\rho_{g,A}^{(q)} - \varphi_{g,A}^{(q)\prime}(x), \qquad x > A.$$

By taking $x \downarrow A$ via (3.10), we have the claim. For (3), we have

$$\begin{aligned}\Gamma_3'(A+; A) &= \lim_{x\downarrow A}\left[W^{(q)\prime}(x-A)\int_0^\infty \mathrm{e}^{-\Phi(q)y}h(y+A)\,\mathrm{d}y - \int_A^x W^{(q)\prime}(x-y)h(y)\,\mathrm{d}y\right]\\ &= W^{(q)\prime}(0+)\int_0^\infty \mathrm{e}^{-\Phi(q)y}h(y+A)\,\mathrm{d}y.\end{aligned}$$

By the lemma above, we obtain

$$u_A'(A+) = W^{(q)\prime}(0+)\Psi(A)$$

or equivalently, by virtue of (2.11), the smooth fit condition at A^* is equivalent to (3.3).

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TABLE 1: Summary of continuous and smooth fit conditions.

	Continuous fit	Smooth fit
Bounded variation	$\Psi(A) = 0$	Not applicable
$\sigma > 0$	Automatically satisfied	$\Psi(A) = \sigma^2 g'(A)/2$

Proposition 3.3. (Smooth fit.) *Fix $A \in \mathbb{R}$. Suppose that $\sigma > 0$ and (1)–(2) of Lemma 2.3 hold. Then the smooth fit condition $u'_A(A+) = g'(A)$ holds if and only if*

$$\Psi(A) = \frac{\sigma^2}{2} g'(A).$$

We summarize the results obtained in Propositions 3.2 and 3.3 in Table 1. It is clear from Proposition 3.1 and Table 1 that the first-order condition and the continuous/smooth fit condition are indeed equivalent.

3.3. Obtaining optimal solutions

There are a number of examples where the optimality of the threshold strategy can be derived directly from the structure of the problem. See, e.g. [19] and [41] for examples of sufficient optimality conditions of the threshold strategy. In such cases, the problem reduces to solely computing A^*.

For a general problem where the optimality of the threshold strategy is not proven, we need to show that the value function satisfies the variational inequality:

(i) $u_{A^*}(x) \geq g(x)$ for all $x \in \mathbb{R}$;

(ii) $(\mathcal{L} - q)u_{A^*}(x) + h(x) = 0$ for all $x \in (A^*, \infty)$;

(iii) $(\mathcal{L} - q)u_{A^*}(x) + h(x) < 0$ for all $x \in (-\infty, A^*)$;

see, e.g. [40]. Here $\mathcal{L}$ is the infinitesimal generator associated with the process X applied to a sufficiently smooth function f

$$\mathcal{L} f(x) := cf'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_0^\infty [f(x-z) - f(x) + f'(x)z\,\mathbf{1}_{\{0<z<1\}}]\Pi(\mathrm{d}z).$$

As we shall show shortly below, conditions (i)–(ii) can be obtained upon some conditions. The proof of condition (iii) unfortunately relies on the structure of the problem. In order to complement this, in the next section we give examples where the optimality over all stopping times holds.

Lemma 3.7. *Suppose $A^* \in \mathbb{R}$ satisfies (3.3), $g \in C^2[A^*, \infty)$ and*

$$\Psi(A) - \frac{\sigma^2}{2} g'(A) > 0, \qquad A > A^*. \tag{3.11}$$

Then (i) is satisfied.

Proof. Because $g(x) = u_{A^*}(x)$ on $(-\infty, A^*]$, we only need to show (i) on (A^*, ∞). For any $x > A^*$, we obtain, by (3.2) and (3.8),

$$u_{A^*}(x) \geq \lim_{A \uparrow x} u_A(x) = g(x) + W^{(q)}(0)\Psi(x).$$

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For the unbounded variation case, because $W^{(q)}(0) = 0$, the result is immediate. For the bounded variation case (which necessarily means $\sigma = 0$), (3.11) implies that $\Psi(x) > 0$ and, hence, the result is also immediate.

Regarding condition (ii), integration by parts can be applied to obtain the following (see [25, Section A.5] for a complete proof).

Lemma 3.8. (Egami and Yamazaki [25].) *If h is continuous on (A^*, ∞), we have*

$$(\mathcal{L} - q)\bigg[\int_{A^*}^{x} W^{(q)}(x - y)h(y)\,\mathrm{d}y\bigg] = h(x), \qquad x > A^*.$$

By Lemma 3.8, we obtain the following.

Proposition 3.4. *Suppose, on (A^*, ∞), $f(x) := \mathbb{E}^x[\mathrm{e}^{-q\tau_{A^*}} g(X_{\tau_{A^*}})] = \Gamma_1(x; A^*) + \Gamma_2(x; A^*)$ is C^1 (C^2) for the case X is of bounded (respectively unbounded) variation, and h is continuous. Assume also that, when X is of unbounded variation with $\sigma = 0$, $W^{(q)}$ is C^2 on $(0, \infty)$. Then $(\mathcal{L} - q)u_{A^*}(x) + h(x) = 0$ for any $x > A^*$.*

Proof. See Appendix A.4.

4. Examples

In this section, we give examples to illustrate how we can apply the results obtained in the previous sections. We first consider, as a warm-up, a generalized version of the McKean optimal stopping problem with additional running rewards. We then extend Egami and Yamazaki [25] and obtain analytical solutions. We also give a brief review of Surya and Yamazaki [47] and Yamazaki [48], where the main results of this paper are directly applied.

4.1. The McKean optimal stopping

The classical McKean optimal stopping problem, also known as the pricing of a perpetual American put option, reduces to (2.4) with $g(x) = K - \mathrm{e}^x$ and $h \equiv 0$. Here, e^X models the stock price and $K > 0$ is the strike price; the option holder chooses a time to exercise so as to maximize the expected payoff. In particular, for the spectrally negative case, it has been shown (see [3]) that the optimal threshold level is given, when $\psi(1) \neq q$, by

$$A^* = \log\bigg(K \frac{q}{\Phi(q)} \frac{\Phi(q) - 1}{q - \psi(1)}\bigg). \tag{4.1}$$

We consider a more general case where h is any nondecreasing and continuous function and give a simple proof by directly using the results obtained in the previous sections. Here we assume that $\psi(1) \neq q$ (or $\Phi(q) \neq 1$); the case of $\psi(1) = q$ can be obtained by taking limits on the results described below (see [48] for details). Because

$$\begin{aligned}
&- g(A)\frac{q}{\Phi(q)} + \rho_{g,A}^{(q)} \\
&\quad = -\frac{q}{\Phi(q)}(K - \mathrm{e}^A) + \int_0^\infty \Pi(\mathrm{d}u) \int_0^u \mathrm{e}^{-\Phi(q)z}(\mathrm{e}^A - \mathrm{e}^{z+A-u})\,\mathrm{d}z \\
&\quad = -\frac{q}{\Phi(q)}K + \mathrm{e}^A\bigg[\frac{q}{\Phi(q)} + \int_0^\infty \Pi(\mathrm{d}u)\bigg(\frac{1 - \mathrm{e}^{-\Phi(q)u}}{\Phi(q)} - \mathrm{e}^{-u}\frac{1 - \mathrm{e}^{-(\Phi(q)-1)u}}{\Phi(q) - 1}\bigg)\bigg],
\end{aligned}$$

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we obtain

$$\Psi(A)-\frac{\sigma^2}{2}g'(A)=-\frac{q}{\Phi(q)}K+\frac{\mathrm{e}^A}{\Phi(q)}M_q+\int_0^\infty \mathrm{e}^{-\Phi(q)y}h(y+A)\,\mathrm{d}y, \tag{4.2}$$

where

$$M_q:=q+\frac{\sigma^2}{2}\Phi(q)+\int_0^\infty \Pi(\mathrm{d}u)\bigg[(1-\mathrm{e}^{-\Phi(q)u})-\mathrm{e}^{-u}(1-\mathrm{e}^{-(\Phi(q)-1)u})\frac{\Phi(q)}{\Phi(q)-1}\bigg].$$

Here, by the change of measure, M_q can be simplified.

Lemma 4.1. *We have* $M_q=(\Phi(q)/(\Phi(q)-1))(q-\psi(1))$.

Proof. By the definition of ψ and Φ, we rewrite M_q as

$$\begin{aligned}
&q+\frac{\sigma^2}{2}\Phi(q)+\int_0^\infty \Pi(\mathrm{d}u)\bigg[(1-\mathrm{e}^{-\Phi(q)u}-\Phi(q)u\,\mathbf{1}_{\{u\in(0,1)\}})\\
&\qquad\qquad\qquad -\mathrm{e}^{-u}(1-\mathrm{e}^{-(\Phi(q)-1)u})\frac{\Phi(q)}{\Phi(q)-1}+\Phi(q)u\,\mathbf{1}_{\{u\in(0,1)\}}\bigg]\\
&\quad=\bigg(c-\int_0^1 u(\mathrm{e}^{-u}-1)\Pi(\mathrm{d}u)\bigg)\Phi(q)+\frac{\sigma^2}{2}\Phi(q)(\Phi(q)+1)\\
&\qquad-\frac{\Phi(q)}{\Phi(q)-1}\int_0^\infty \Pi(\mathrm{d}u)\mathrm{e}^{-u}(1-\mathrm{e}^{-(\Phi(q)-1)u}+(\Phi(q)-1)u\,\mathbf{1}_{\{u\in(0,1)\}}).
\end{aligned}$$

Define, as the Laplace exponent of X under $\mathbb{P}_1$ with the change of measure

$$\frac{\mathrm{d}\mathbb{P}_1}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t}=\exp(X_t-\psi(1)t),\qquad t\ge 0,$$

$$\begin{aligned}
\psi_1(\beta)&:=\bigg(\sigma^2+c-\int_0^1 u(\mathrm{e}^{-u}-1)\Pi(\mathrm{d}u)\bigg)\beta+\frac{1}{2}\sigma^2\beta^2\\
&\qquad+\int_0^\infty(\mathrm{e}^{-\beta u}-1+\beta u\,\mathbf{1}_{\{u\in(0,1)\}})\mathrm{e}^{-u}\Pi(\mathrm{d}u).
\end{aligned}$$

Then, $\psi_1(\Phi(q)-1)=\psi(\Phi(q))-\psi(1)=q-\psi(1)$; see [31, p. 215]. Hence, simple algebra shows

$$\frac{\Phi(q)}{\Phi(q)-1}(q-\psi(1))=\frac{\Phi(q)}{\Phi(q)-1}\psi_1(\Phi(q)-1)=M_q,$$

as desired.

It is clear that $M_q>0$, and, hence, (4.2) is monotonically increasing in A. Recall also Assumption 2.2. Therefore, on condition that

$$\lim_{A\downarrow-\infty}\bigg[\Psi(A)-\frac{\sigma^2}{2}g'(A)\bigg]=-\frac{q}{\Phi(q)}K+\lim_{A\downarrow-\infty}\int_0^\infty \mathrm{e}^{-\Phi(q)y}h(y+A)\,\mathrm{d}y<0, \tag{4.3}$$

there exists a unique A^* such that (4.2) vanishes and, by (3.2),

$$\frac{\partial}{\partial A}u_A(x)\ge 0\quad\text{for all } x>A\qquad\Longleftrightarrow\qquad A\le A^*. \tag{4.4}$$

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This shows that τ_{A^*} is optimal among the set of all stopping times of threshold type. Note that in the special case where $h \equiv 0$, the optimal threshold A^* reduces to (4.1). Because the optimal stopping time is known to be of threshold type by [38], τ_{A^*} is indeed the optimal stopping time.

We now show that it is indeed optimal over all stopping times even when h is not 0. This reduces to showing (iii) (in Section 3.3) because (i) holds due to (4.4) and Lemma 3.7, and (ii) holds due to Proposition 3.4 and the smoothness of the value function given in Proposition 4.1 below. For this special case of g, we can simplify as in Exercise 8.7 (ii) and Corollary 9.3 of [31] for any $x, A \in \mathbb{R}$,

$$\mathbb{E}^x[\mathrm{e}^{-q\tau_A}(K - \mathrm{e}^{X_{\tau_A}})] = K\left(Z^{(q)}(x-A) - \frac{q}{\Phi(q)}W^{(q)}(x-A)\right) - \mathrm{e}^x\left(Z_1^{(q-\psi(1))}(x-A) - \frac{q-\psi(1)}{\Phi(q)-1}W_1^{(q-\psi(1))}(x-A)\right), \quad (4.5)$$

where W_1 and Z_1 are versions of W and Z associated with the measure $\mathbb{P}_1$ under the same change of measure as in the proof of Lemma 4.1. Here, note that, as in Lemmas 8.3 and 8.5 of [31], for each $x > 0$, the functions $q \mapsto W^{(q)}(x)$ and $q \mapsto Z^{(q)}(x)$ can be analytically extended to $q \in \mathbb{C}$. In particular, by Lemma 8.4 of [31],

$$\mathrm{e}^x W_1^{(q-\psi(1))}(x) = W^{(q)}(x), \qquad x \geq 0. \quad (4.6)$$

Proposition 4.1. *Suppose h is nondecreasing and continuous and satisfies Assumption 2.2 and (4.3). Then there exists a unique A^* such that (4.2) vanishes. Moreover, τ_{A^*} is an optimal stopping time and the optimal value function is given by*

$$u_{A^*}(x) = KZ^{(q)}(x-A^*) - \mathrm{e}^x Z_1^{(q-\psi(1))}(x-A^*) - \int_{A^*}^x W^{(q)}(x-y)h(y)\,\mathrm{d}y.$$

Proof. By Lemma 4.1 and the discussion above this proposition, there exists a unique A^* such that

$$0 = -qK + \mathrm{e}^{A^*}\frac{\Phi(q)}{\Phi(q)-1}(q-\psi(1)) + \Phi(q)\int_0^\infty \mathrm{e}^{-\Phi(q)y}h(y+A^*)\,\mathrm{d}y. \quad (4.7)$$

By (4.5)–(4.7),

$$\begin{aligned} u_{A^*}(x) &= KZ^{(q)}(x-A^*) - \mathrm{e}^x\left(Z_1^{(q-\psi(1))}(x-A^*) - W_1^{(q-\psi(1))}(x-A^*)\frac{q-\psi(1)}{\Phi(q)-1}\right) \\ &\quad - W^{(q)}(x-A^*)\mathrm{e}^{A^*}\frac{q-\psi(1)}{\Phi(q)-1} - \int_{A^*}^x W^{(q)}(x-y)h(y)\,\mathrm{d}y \\ &= KZ^{(q)}(x-A^*) - \mathrm{e}^x Z_1^{(q-\psi(1))}(x-A^*) - \int_{A^*}^x W^{(q)}(x-y)h(y)\,\mathrm{d}y. \end{aligned}$$

Notice that u_{A^*} is C^1 (C^2) on $\mathbb{R}\backslash\{A^*\}$ and C^0 (C^1) at A^* when X is of bounded (respectively unbounded) variation; see the proof of Lemma 4.5 of [25] for the transformation of the integral term.

We shall show (iii). By (2.1),

$$\mathscr{L}g(x) = -\mathrm{e}^x\left[c + \frac{1}{2}\sigma^2 + \int_0^\infty [\mathrm{e}^{-z} - 1 + z\,\mathbf{1}_{\{0<z<1\}}]\Pi(\mathrm{d}z)\right] = -\mathrm{e}^x\psi(1),$$

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and, hence,

$$(\mathcal{L} - q)g(x) + h(x) = -qK + \mathrm{e}^x(q - \psi(1)) + h(x). \tag{4.8}$$

Because h is nondecreasing and $x < A^*$,

$$\Phi(q) \int_0^\infty \mathrm{e}^{-\Phi(q)y} h(y + A^*)\,\mathrm{d}y \geq \Phi(q) \int_0^\infty \mathrm{e}^{-\Phi(q)y} h(x)\,\mathrm{d}y = h(x). \tag{4.9}$$

It is also easy to see that

$$\mathrm{e}^{A^*} \frac{\Phi(q)}{\Phi(q) - 1}(q - \psi(1)) \geq \mathrm{e}^x (q - \psi(1)). \tag{4.10}$$

Indeed, for the case where $q - \psi(1) > 0$, we must have $\Phi(q) - 1 > 0$, and, hence, (4.10) holds by $A^* > x$; for the case where $q - \psi(1) < 0$, the left-hand side is positive while the right-hand side is negative in (4.10). By (4.7)–(4.10), (iii) holds.

This, together with (i) and (ii), shows the optimality using a standard technique of optimal stopping; see [48] for the rest of the proof of optimality.

4.2. Generalization of Egami and Yamazaki [25]

We now solve an extension to [25], where we obtained an alarm system that determines when a bank needs to start enhancing its own capital ratio so as not to violate the capital adequacy requirements. Here, X models the bank's net worth or equity capital allocated to its loan/credit business. The problem is to strike the balance between minimizing the chance of violating the net capital requirement and the costs of premature undertaking (or the regret) measured, respectively, by

$$R_x^{(q)}(\tau) := \mathbb{E}^x[\mathrm{e}^{-q\theta}\, \mathbf{1}_{\{\tau \geq \theta, \theta < \infty\}}] \quad \text{and} \quad H_x^{(q,h)}(\tau) := \mathbb{E}^x\left[\mathbf{1}_{\{\tau < \infty\}} \int_\tau^\theta \mathrm{e}^{-qt} h(X_t)\,\mathrm{d}t\right],$$

where h is positive, continuous, and increasing, and $\theta := \inf\{t \geq 0\colon X_t \leq 0\}$ denotes the capital requirement violation time. We want to obtain, over the set of stopping times

$$\mathcal{S} := \{\tau \text{ stopping time}\colon \tau \leq \theta \text{ a.s.}\}, \tag{4.11}$$

an optimal stopping time that minimizes the linear combination of the two costs described above:

$$U_x^{(q,h)}(\tau, \gamma) := R_x^{(q)}(\tau) + \gamma H_x^{(q,h)}(\tau)$$

for some $\gamma > 0$. By taking advantage of the property of $\mathcal{S}$, the problem can be reduced to obtaining

$$\inf_{\tau \in \mathcal{S}} \mathbb{E}^x\left[\mathrm{e}^{-q\tau}\, \mathbf{1}_{\{X_\tau \leq 0,\, \tau < \infty\}} + \int_\tau^\theta \mathrm{e}^{-qt} h(X_t)\,\mathrm{d}t\right] = -u(x) + \mathbb{E}^x\left[\int_0^\theta \mathrm{e}^{-qt} h(X_t)\,\mathrm{d}t\right],$$

with

$$u(x) := \sup_{\tau \in \mathcal{S}} \mathbb{E}^x\left[-\mathrm{e}^{-q\tau}\, \mathbf{1}_{\{X_\tau \leq 0,\, \tau < \infty\}} + \int_0^\tau \mathrm{e}^{-qt} h(X_t)\,\mathrm{d}t\right].$$

In other words, the problem reduces to (2.4) with

$$g(x) = \begin{cases} 0, & x > 0, \\ -1, & x \leq 0, \end{cases}$$

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TABLE 2: Summary of continuous and smooth fit conditions.

	Continuous fit	Smooth fit
(i) Bounded variation	$\Psi(A) = 0$	Not applicable
(ii) Unbounded variation	Automatically satisfied	$\Psi(A) = 0$

and a special set of stopping times defined in (4.11). In [25], Egami and Yamazaki solved this problem for double exponential jump diffusion (see [28]), and for a general spectrally negative Lévy process.

We shall consider its extension for a more general g (or more general for $R_x^{(q)}(\tau) := -\mathbb{E}^x[\mathrm{e}^{-q\theta} g(X_\theta)\mathbf{1}_{\{\tau\geq\theta,\theta<\infty\}}]$) by assuming the following.

Assumption 4.1. (1) *g is negative and increasing on $(-\infty, 0]$ (and 0 on $(0, \infty)$) and satisfies, for $A > 0$, the assumptions in Proposition 3.1;*

(2) *h is positive, continuous and increasing and satisfies Assumption 2.2.*

The first assumption on g means that the penalty $|g(X_\theta)|$ increases as the overshoot $|X_\theta|$ increases. The second assumption on h is the same as in [25]; if a bank has a higher capital value, then it naturally has better access to high-quality assets.

In this problem, it can be conjectured that there exists a threshold level A^* such that τ_{A^*} is optimal. Here we can rewrite (2.12), for all $A > 0$,

$$\rho_{g,A}^{(q)} = \int_A^\infty \Pi(\mathrm{d}u) \int_0^{u-A} \mathrm{e}^{-\Phi(q)y} g(y + A - u)\,\mathrm{d}y,$$
$$\bar{\rho}_{g,A}^{(q)} = \int_A^\infty \Pi(\mathrm{d}u) \int_0^{u-A} \mathrm{e}^{-\Phi(q)y} |g(y + A - u)|\,\mathrm{d}y.$$

This avoids the integration of Π in the neighborhood of 0 and, hence, Lemma 3.5 also holds for the case of unbounded variation with $\sigma = 0$. Now, as a special case of Propositions 3.2–3.3 (noticing $g(A) = g'(A) = 0$ for all $A > 0$), we obtain the following.

Lemma 4.2. (Continuous and smooth fit.) *Suppose conditions (1)–(2) of Lemma 2.3 for a given $A > 0$.*

- *(Continuous fit.) If X is of bounded variation, the continuous fit condition $u_A(A+) = 0$ holds if and only if*

$$\Psi(A) = 0. \tag{4.12}$$

  If X is of unbounded variation, it is automatically satisfied.

- *(Smooth fit.) If X is of unbounded variation, the smooth fit condition $u'_A(A+) = 0$ holds if and only if (4.12) holds.*

See also Table 2 for a summary of conditions.

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Under Assumption 4.1, there exists at most one $A^* > 0$ that satisfies (4.12) because

$$\begin{aligned}\Psi'(A) = &\int_0^\infty \mathrm{e}^{-\Phi(q)y} h'(y+A)\,\mathrm{d}y + \int_A^\infty \Pi(\mathrm{d}u) \int_0^{u-A} \mathrm{e}^{-\Phi(q)y} g'(y+A-u)\,\mathrm{d}y \\ &- \int_A^\infty \Pi(\mathrm{d}u) \mathrm{e}^{-\Phi(q)(u-A)} g(0-) \\ &> 0. \end{aligned} \tag{4.13}$$

Verification of optimality: We let A^* be the unique root of $\Psi(A) = 0$ if it exists and set it to 0 otherwise. The optimality over $\mathcal{S}$ holds under the following assumption.

Assumption 4.2. *$W^{(q)}$ is C^2 on $(0, \infty)$ for the case X is of unbounded variation with $\sigma = 0$.*

As in the case of the McKean optimal stopping problem, we need only to show (iii) in Section 3.3 because (i) holds by (4.13) and Lemma 3.7, and (ii) holds by Proposition 3.4 and Assumption 4.2.

Lemma 4.3. *If $A^* > 0$, we have $(\mathcal{L} - q)g(x) + h(x) \le 0$ for every $x \in (0, A^*)$.*

Proof. Because $g(x) = g'(x) = 0$ for every $x > 0$,

$$(\mathcal{L} - q)g(x) + h(x) = \int_x^\infty \Pi(\mathrm{d}u) g(x-u) + h(x), \qquad x \in (0, A^*). \tag{4.14}$$

We shall show that this is negative. Because $A^* > 0$, we must have

$$\int_0^\infty \mathrm{e}^{-\Phi(q)y} h(y + A^*)\,\mathrm{d}y + \int_{A^*}^\infty \Pi(\mathrm{d}u) \int_0^{u-A^*} \mathrm{e}^{-\Phi(q)y} g(y + A^* - u)\,\mathrm{d}y = 0,$$

and hence

$$\begin{aligned} 0 &\ge h(x) \int_0^\infty \mathrm{e}^{-\Phi(q)y}\,\mathrm{d}y + \int_{A^*}^\infty \Pi(\mathrm{d}u) g(x-u) \int_0^{u-A^*} \mathrm{e}^{-\Phi(q)y}\,\mathrm{d}y \\ &\ge h(x) \int_0^\infty \mathrm{e}^{-\Phi(q)y}\,\mathrm{d}y + \int_{A^*}^\infty \Pi(\mathrm{d}u) g(x-u) \int_0^\infty \mathrm{e}^{-\Phi(q)y}\,\mathrm{d}y \\ &\ge h(x) \int_0^\infty \mathrm{e}^{-\Phi(q)y}\,\mathrm{d}y + \int_x^\infty \Pi(\mathrm{d}u) g(x-u) \int_0^\infty \mathrm{e}^{-\Phi(q)y}\,\mathrm{d}y \\ &= \left(h(x) + \int_x^\infty \Pi(\mathrm{d}u) g(x-u)\right) \int_0^\infty \mathrm{e}^{-\Phi(q)y}\,\mathrm{d}y, \end{aligned}$$

where the first inequality holds because g and h are increasing and $x < A^*$, the second holds because g is nonpositive, and the third holds because $x < A^*$ and g is nonpositive. This together with (4.14) shows the result.

Now the optimality holds by the martingale argument. For the rest of the proof, we refer the reader to the proof of Proposition 4.1 of [25].

Proposition 4.2. *If $A^* > 0$, then τ_{A^*} is the optimal stopping time and the value function is given by $u_{A^*}(x)$ for every $x > 0$. If $A^* = 0$, then the value function is given by $\lim_{A \downarrow 0} u_A(x)$ for every $x > 0$.*

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4.3. Other examples

The results of this paper are applicable to a wide range of optimal stopping problems. Here we give a brief review of two recent papers where these are used.

Surya and Yamazaki (see [47]) generalized the spectrally negative Lévy model (see [27] and [33]) of the optimal capital structure problem with endogenous bankruptcy, originally studied in [34] and [35]. The problem is to obtain an optimal bankruptcy level so as to maximize the company's equity value. This problem gives a classical framework for solving the tradeoff between minimizing bankruptcy costs and maximizing tax benefits in debt financing. Using the results of this paper (in particular, Propositions 3.1, 3.2, and 3.3), they have succeeded in incorporating the scale effects by generalizing the values of bankruptcy costs and tax benefits dependent on the firm's asset value. Their results reduce to those of [27] and [33] in the original setting where the bankruptcy cost is a fixed fraction of the asset value and the tax benefit rate is constant. Furthermore, a series of numerical results validate the optimality of their solutions, and, hence, also the main results of the current paper.

Yamazaki considered (see [48]) a multiple-stopping version of the problem discussed in Subsection 4.1 and also its generalization where g is a general decreasing and concave function. The optimal strategy is given by an increasing sequence of stopping times of threshold type as in (2.5), and these can be computed recursively without intricate computation. The numerical results confirm the optimality for the one-stage case (as discussed in Subsection 4.1 of the current paper) and also for the multiple-stage case.

5. Concluding remarks

We have discussed the optimal stopping problem for spectrally negative Lévy processes. By expressing the expected payoff via the scale function, we achieved the first-order condition as well as the continuous/smooth fit condition and showed their equivalence. The results obtained here can be applied to a wide range of optimal stopping problems for spectrally negative Lévy processes. As examples, we gave a short proof for the perpetual American option pricing problem and solved an extension to Egami and Yamazaki [25].

A natural direction for future research is to generalize it to the two-sided barrier case. In this case, it is easily conjectured that the additional smooth fit (or the first-order) condition at the upper barrier needs to be incorporated. This will give two equations for the lower and upper barrier levels; the solutions naturally become the optimal threshold levels. Applications include, for example, American strangles as in [16].

Another direction is to pursue similar results for optimal stopping games. Typically, the equilibrium strategies are given by stopping times of threshold type as in [6], [7], and [26]. Similar to the results obtained in this paper, the expected payoff admits expressions in terms of the scale function, and, hence, the first-order condition and the continuous/smooth fit can be obtained analytically.

Finally, the results can be extended to a general Lévy process with both positive and negative jumps. This can potentially be obtained in terms of the Wiener–Hopf factor as an alternative to the scale function.

Appendix A. Proofs

A.1. Proof of Lemma 2.3

By the assumption (1) and Taylor expansion, we can take $0 < \epsilon < 1$ such that, for any $0 < z < u < \epsilon$ and $\varrho_{A,\epsilon} := \max_{0 \le \xi \le \epsilon} |g''(A-\xi)| < \infty$,

$$|g(A-u+z) - g(A)| \le (u-z)|g'(A)| + \tfrac{1}{2}(u-z)^2 \varrho_{A,\epsilon} \le u|g'(A)| + \tfrac{1}{2}u^2 \varrho_{A,\epsilon}. \quad \text{(A.1)}$$

Therefore, by (2.2),

$$\int_0^\epsilon \Pi(\mathrm{d}u) \int_0^u \mathrm{e}^{-\Phi(q)z} |g(z+A-u) - g(A)| \,\mathrm{d}z \le \int_0^\epsilon \Pi(\mathrm{d}u) \Big(u^2 |g'(A)| + \frac{1}{2} u^3 \varrho_{A,\epsilon} \Big) < \infty.$$

On the other hand, by (2.13),

$$\begin{aligned} &\int_\epsilon^\infty \Pi(\mathrm{d}u) \int_0^u \mathrm{e}^{-\Phi(q)z} |g(z+A-u) - g(A)| \,\mathrm{d}z \\ &\quad \le \frac{1}{\Phi(q)} \int_\epsilon^\infty \Pi(\mathrm{d}u) \max_{A-u \le \zeta \le A} |g(\zeta) - g(A)| \\ &\quad < \infty. \end{aligned}$$

Combining the above, the proof is complete.

A.2. Proof of Lemma 3.1

(Proof of (3.5).) Define $\varrho(A) := \int_0^\infty \Pi(\mathrm{d}u) q(A;u)$ with

$$q(A;u) := \int_A^{u+A} \mathrm{e}^{-\Phi(q)y} [g(y-u) - g(A)] \,\mathrm{d}y, \qquad u \ge 0.$$

By assumption, we can choose $0 < \epsilon < 1$ such that g is C^2 on $[A-\epsilon, A+\epsilon]$.

We choose $0 < \delta < \epsilon$ that satisfies (3.1) and fix $0 < c < \delta$. By the mean value theorem, there exists $\xi \in (0, c)$ such that

$$q'(A+\xi; u) = \frac{q(A+c;u) - q(A;u)}{c}.$$

For every $z \in (A, A+c)$, we have

$$q'(z;u) = \mathrm{e}^{-\Phi(q)z} \Big(g(z) - g(z-u) - \frac{1 - \mathrm{e}^{-\Phi(q)u}}{\Phi(q)} g'(z) \Big). \quad \text{(A.2)}$$

The Taylor expansion implies that, for every $0 < u < \delta$,

$$\begin{aligned} \frac{|q(A+c;u) - q(A;u)|}{c} &= |q'(A+\xi;u)| \\ &\le \mathrm{e}^{-\Phi(q)(A+\xi)} \frac{u^2}{2} \Big[\max_{0 \le \zeta \le u} |g''(A+\xi-\zeta)| + \Phi(q)|g'(A+\xi)| \Big] \\ &\le \mathrm{e}^{-\Phi(q)A} \frac{u^2}{2} \Big[\max_{-\delta \le y \le \delta} |g''(A+y)| + \Phi(q) \max_{0 \le y \le \delta} |g'(A+y)| \Big], \end{aligned}$$

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uniformly in $c \in (0, \delta)$. The integral of the right-hand side over $(0, \delta)$ with respect to Π is, by (2.2),

$$\frac{1}{2}\mathrm{e}^{-\Phi(q)A}\Big[\max_{-\delta \le y \le \delta} |g''(A+y)| + \Phi(q) \max_{0 \le y \le \delta} |g'(A+y)|\Big] \int_0^\delta u^2 \Pi(\mathrm{d}u) < \infty.$$

On the other hand, for $u > \delta$, by (A.2),

$$\begin{aligned}&\frac{|q(A+c;u) - q(A;u)|}{c} \\ &\qquad \le \mathrm{e}^{-\Phi(q)A}\bigg(\max_{0 \le \zeta \le \delta} |g(A+\zeta) - g(A+\zeta-u)| + \max_{0 \le \zeta \le \delta} \frac{|g'(A+\zeta)|}{\Phi(q)}\bigg)\end{aligned}$$

whose integral over (δ, ∞) equals

$$\mathrm{e}^{-\Phi(q)A}\bigg(\int_\delta^\infty \max_{0 \le \zeta \le \delta} |(g(A+\zeta) - g(A+\zeta-u)|\Pi(\mathrm{d}u) + \max_{0 \le \zeta \le \delta} \frac{|g'(A+\zeta)|}{\Phi(q)}\Pi(\delta, \infty)\bigg),$$

which is finite by (3.1) and by how δ is chosen.

This allows us to apply the dominated convergence theorem, and we obtain

$$\begin{aligned}\lim_{c \downarrow 0} \frac{\varrho(A+c) - \varrho(A)}{c} &= \int_0^\infty \Pi(\mathrm{d}u) \lim_{c \downarrow 0} \frac{q(A+c;u) - q(A;u)}{c} \\ &= \int_0^\infty \Pi(\mathrm{d}u) q'(A;u) \\ &= \mathrm{e}^{-\Phi(q)A} \int_0^\infty \Pi(\mathrm{d}u)\bigg(g(A) - g(A-u) - g'(A)\frac{1-\mathrm{e}^{-\Phi(q)A}}{\Phi(q)}\bigg).\end{aligned}$$

The proof for the left derivative is similar, and this completes the proof of (3.5).

(Proof of (3.6).) Define

$$\tilde{q}(z;u,x) := \int_z^{(u+z)\wedge x} W^{(q)}(x-y)[g(y-u) - g(z)]\,\mathrm{d}y, \qquad z \in \mathbb{R},\ u > 0. \tag{A.3}$$

Then, by (2.16), we have $\varphi^{(q)}_{g,A}(x) = \int_0^\infty \Pi(\mathrm{d}u)\tilde{q}(A;u,x)$. We use the same $0 < \delta < \epsilon$ as in the proof of (3.5) above and fix c and ε such that

$$0 < c < \delta \wedge \frac{x-A}{4} =: \varepsilon.$$

For every fixed $0 < u < \varepsilon$, our assumptions imply that $\tilde{q}(\cdot;u,x)$ is C^2 on $(A, A+c)$. By the mean value theorem, there exists $\xi \in (0, c)$ such that

$$\tilde{q}'(A+\xi;u,x) = \frac{\tilde{q}(A+c;u,x) - \tilde{q}(A;u,x)}{c}. \tag{A.4}$$

Given z at which g is differentiable and satisfies $u + z < x$, differentiating (A.3) gives

$$\tilde{q}'(z;u,x) = W^{(q)}(x-z)(g(z) - g(z-u)) - g'(z)\int_z^{u+z} W^{(q)}(x-y)\,\mathrm{d}y.$$

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Because $x-u-A-\xi > (x-A)/4 > 0$ (and thus $u+(A+\xi) < x$) and g is differentiable at $A+\xi$,

$$\begin{aligned}
&\frac{|\tilde{q}(A+c;u,x)-\tilde{q}(A;u,x)|}{c}\\
&\quad= |\tilde{q}'(A+\xi;u,x)|\\
&\quad= \bigg|W^{(q)}(x-A-\xi)(g(A+\xi)-g(A+\xi-u))\\
&\qquad - g'(A+\xi)\int_{A+\xi}^{u+A+\xi} W^{(q)}(x-y)\,\mathrm{d}y\bigg|\\
&\quad\le W^{(q)}(x-A-\xi)|g(A+\xi)-g(A+\xi-u)-ug'(A+\xi)|\\
&\qquad + \bigg|g'(A+\xi)\int_{A+\xi}^{u+A+\xi}(W^{(q)}(x-A-\xi)-W^{(q)}(x-y))\,\mathrm{d}y\bigg|\\
&\quad\le W^{(q)}(x-A)|g(A+\xi)-g(A+\xi-u)-ug'(A+\xi)|\\
&\qquad + u|g'(A+\xi)||W^{(q)}(x-A-\xi)-W^{(q)}(x-u-A-\xi)|\\
&\quad\le f_1(A;u,x)+f_2(A;u,x),
\end{aligned}$$

where

$$f_1(A;u,x) := W^{(q)}(x-A)\max_{0\le\zeta\le\varepsilon}|g(A+\zeta)-g(A+\zeta-u)-ug'(A+\zeta)|,$$

$$f_2(A;u,x) := u\max_{0\le\zeta\le\varepsilon}|g'(A+\zeta)|\max_{0\le\zeta\le\varepsilon}|W^{(q)}(x-A-\zeta)-W^{(q)}(x-u-A-\zeta)|.$$

First, $\int_0^\varepsilon \Pi(\mathrm{d}u)f_1(A;u,x)$ is finite because, for every $u\le\varepsilon$, we have $u\le\delta$ and

$$\max_{0\le\zeta\le\varepsilon}|g(A+\zeta)-g(A+\zeta-u)-ug'(A+\zeta)| \le \frac{u^2}{2}\max_{A-\delta\le\zeta\le A+\delta}|g''(\zeta)|,$$

which is Π-integrable over $(0,\varepsilon)$ by (2.2). On the other hand, by (2.10) and because $0\le\zeta\le\varepsilon$ implies that $x-u-A-\zeta > (x-A)/4 > 0$, we have

$$\begin{aligned}
&|W^{(q)}(x-A-\zeta)-W^{(q)}(x-u-A-\zeta)|\\
&\quad= |\mathrm{e}^{\Phi(q)(x-A-\zeta)}W_{\Phi(q)}(x-A-\zeta)-\mathrm{e}^{\Phi(q)(x-u-A-\zeta)}W_{\Phi(q)}(x-u-A-\zeta)|\\
&\quad\le \bigg|\frac{\mathrm{e}^{\Phi(q)(x-A-\zeta)}-\mathrm{e}^{\Phi(q)(x-u-A-\zeta)}}{\psi'(\Phi(q))}\bigg|\\
&\qquad + \mathrm{e}^{\Phi(q)(x-u-A-\zeta)}|W_{\Phi(q)}(x-A-\zeta)-W_{\Phi(q)}(x-u-A-\zeta)|\\
&\quad\le \mathrm{e}^{\Phi(q)(x-A)}\bigg(\frac{1-\mathrm{e}^{-\Phi(q)u}}{\psi'(\Phi(q))}+u\max_{(x-A)/4\le y\le x-A}W'_{\Phi(q)}(y)\bigg),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\int_0^\varepsilon \Pi(\mathrm{d}u)f_2(A;u,x) &\le \max_{0\le\zeta\le\varepsilon}|g'(A+\zeta)|\mathrm{e}^{\Phi(q)(x-A)}\\
&\quad\times\int_0^\varepsilon u\bigg(\frac{1-\mathrm{e}^{-\Phi(q)u}}{\psi'(\Phi(q))}+u\max_{(x-A)/4\le y\le x-A}W'_{\Phi(q)}(y)\bigg)\Pi(\mathrm{d}u),
\end{aligned}$$

which is finite by (2.2).

We now fix $u > \varepsilon$ (which implies $u > c$). We have

$$\frac{|\tilde{q}(A+c; u, x) - \tilde{q}(A; u, x)|}{c} \le B_1(A, c; u, x) + B_2(A, c; u, x),$$

where

$$B_1(A, c; u, x) := \frac{1}{c}\bigg[\int_{(u+A)\wedge x}^{(u+A+c)\wedge x} W^{(q)}(x-y)|g(y-u) - g(A+c)|\,\mathrm{d}y + \int_A^{A+c} W^{(q)}(x-y)|g(y-u) - g(A)|\,\mathrm{d}y\bigg],$$

$$B_2(A, c; u, x) := \frac{|g(A+c) - g(A)|}{c}\int_{A+c}^{(u+A)\wedge x} W^{(q)}(x-y)\,\mathrm{d}y.$$

For the former, we have

$$\begin{aligned} &B_1(A, c; u, x)\\ &\quad\le 3W^{(q)}(x-A)\max_{A-u\le z\le A+c}|g(z) - g(A)|\\ &\quad\le 3W^{(q)}(x-A)\Big(\max_{A-u\le z\le A}|g(z) - g(A)| + \max_{0\le\zeta\le\delta}|g(A+\zeta) - g(A+\zeta-u)|\Big)\\ &\quad=: \bar{B}_1(A; u, x). \end{aligned}$$

Here, the first inequality holds because

$$|g(y-u) - g(A+c)| \le |g(y-u) - g(A)| + |g(A) - g(A+c)|$$

and, for $(u+A)\wedge x \le y \le (u+A+c)\wedge x$, it holds that $A-u \le A\wedge(x-u) \le y-u \le (A+c)\wedge(x-u) \le A+c$. For the second inequality, it holds trivially when the maximum is attained for some $A-u \le z \le A$. If it is attained at $z = A+l$ for some $0 < l \le c$, then, because $A-u \le A+l-u \le A$ (due to $c < u$) and $c < \delta$,

$$\begin{aligned} \max_{A-u\le z\le A+c}|g(z) - g(A)| &\le |g(A+l-u) - g(A)| + |g(A+l) - g(A+l-u)|\\ &\le \max_{A-u\le z\le A}|g(z) - g(A)| + \max_{0\le\zeta\le\delta}|g(A+\zeta) - g(A+\zeta-u)|. \end{aligned}$$

For the latter, by the C^2 property of g in the neighborhood of A, how δ is chosen, and $c < \delta$, we obtain

$$\begin{aligned} B_2(A, c; u, x) &\le \frac{|g(A+c) - g(A)|}{c}\int_{A+c}^{x} W^{(q)}(x-y)\,\mathrm{d}y\\ &\le \bigg(|g'(A)| + \frac{\delta}{2}\max_{A\le\zeta\le A+\delta}|g''(\zeta)|\bigg)\int_A^x \mathrm{e}^{\Phi(q)(x-y)}W_{\Phi(q)}(x-y)\,\mathrm{d}y\\ &\le \frac{1}{\Phi(q)\psi'(\Phi(q))}\bigg(|g'(A)| + \frac{\delta}{2}\max_{A\le\zeta\le A+\delta}|g''(\zeta)|\bigg)\mathrm{e}^{\Phi(q)(x-A)}\\ &=: \bar{B}_2(A; x). \end{aligned}$$

Combining these we have

$$\begin{aligned}
&\int_{\varepsilon}^{\infty} \Pi(\mathrm{d}u)(\bar{B}_1(A;u,x) + \bar{B}_2(A;x)) \\
&\quad\le 3W^{(q)}(x-A)\int_{\varepsilon}^{\infty} \Pi(\mathrm{d}u)\Big(\max_{A-u\le z\le A} |g(z)-g(A)| \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \max_{0\le\zeta\le\delta} |g(A+\zeta)-g(A+\zeta-u)|\Big) \\
&\qquad + \frac{1}{\Phi(q)\psi'(\Phi(q))}\bigg(|g'(A)| + \frac{\delta}{2}\max_{A\le\zeta\le A+\delta}|g''(\zeta)|\bigg)\mathrm{e}^{\Phi(q)(x-A)}\Pi(\varepsilon,\infty),
\end{aligned}$$

which is finite by (2.13) and (3.1).

In summary, (A.4) is bounded uniformly in $c \in (0,\varepsilon)$ by a function which is Π-integrable. Hence, by the dominated convergence theorem,

$$\begin{aligned}
\lim_{c\downarrow 0} \frac{\varphi^{(q)}_{g,A+c}(x) - \varphi^{(q)}_{g,A}(x)}{c} &= \int_0^{\infty} \Pi(\mathrm{d}u) \lim_{c\downarrow 0} \frac{\tilde{q}(A+c;u,x) - \tilde{q}(A;u,x)}{c} \\
&= \int_0^{\infty} \Pi(\mathrm{d}u)\tilde{q}'(A;u,x) \\
&= \int_0^{\infty} \Pi(\mathrm{d}u)\bigg[W^{(q)}(x-A)(g(A)-g(A-u)) \\
&\qquad\qquad - g'(A)\int_A^{(u+A)\wedge x} W^{(q)}(x-z)\,\mathrm{d}z\bigg].
\end{aligned}$$

The result for the left derivative can be proved in the same way.

A.3. Proof of Lemma 3.5

It is known (see [15]) that $\sigma > 0$ guarantees that $W_{\Phi(q)}$ is twice continuously differentiable and, hence, $W'_{\Phi(q)}$ is continuous on $(0,\infty)$. Furthermore, (2.11) implies that $W'_{\Phi(q)}(0+) = 2/\sigma^2 < \infty$ and (2.10) implies that $\lim_{x\uparrow\infty} W'_{\Phi(q)}(x) = 0$. Therefore, there exists $L < \infty$ such that

$$L := \sup_{x>0} W'_{\Phi(q)}(x).$$

For every fixed $0 < c < 1$ and $u > 0$,

$$\begin{aligned}
&\frac{1}{c}\bigg|\int_0^{u\wedge(x+c-A)} W^{(q)}(x+c-z-A)(g(z+A-u)-g(A))\,\mathrm{d}z \\
&\quad - \int_0^{u\wedge(x-A)} W^{(q)}(x-z-A)(g(z+A-u)-g(A))\,\mathrm{d}z\bigg| \\
&\quad \le l_1(x,A,c,u) + l_2(x,A,c,u),
\end{aligned} \tag{A.5}$$

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where

$$l_1(x, A, c, u) := \int_0^{u\wedge(x-A)} q(x, c, z, A)|g(z + A - u) - g(A)|\,\mathrm{d}z,$$

$$l_2(x, A, c, u) := \int_{u\wedge(x-A)}^{u\wedge(x+c-A)} \frac{W^{(q)}(x + c - z - A)}{c}|g(z + A - u) - g(A)|\,\mathrm{d}z,$$

$$q(x, c, z, A) := \frac{W^{(q)}(x + c - z - A) - W^{(q)}(x - z - A)}{c}.$$

Because

$$\begin{aligned} q(x, c, z, A) &= \frac{\mathrm{e}^{\Phi(q)(x+c-z-A)} W_{\Phi(q)}(x + c - z - A) - \mathrm{e}^{\Phi(q)(x-z-A)} W_{\Phi(q)}(x - z - A)}{c} \\ &= \mathrm{e}^{\Phi(q)(x-z-A)} \frac{1}{c}[(\mathrm{e}^{\Phi(q)c} - 1) W_{\Phi(q)}(x + c - z - A) \\ &\qquad + W_{\Phi(q)}(x + c - z - A) - W_{\Phi(q)}(x - z - A)] \\ &\le \mathrm{e}^{\Phi(q)(x-z-A)} \sup_{0<\delta<1} \left(\frac{\mathrm{e}^{\Phi(q)\delta} - 1}{\delta\psi'(\Phi(q))} + L\right), \end{aligned}$$

we have

$$\begin{aligned} l_1(x, A, c, u) &\le \sup_{0<\delta<1} \left(\frac{\mathrm{e}^{\Phi(q)\delta} - 1}{\delta\psi'(\Phi(q))} + L\right) \int_0^{u\wedge(x-A)} \mathrm{e}^{\Phi(q)(x-z-A)}|g(z + A - u) - g(A)|\,\mathrm{d}z \\ &=: \bar{l}_1(x, A, u), \end{aligned}$$

which is Π-integrable as

$$\int_0^\infty \Pi(\mathrm{d}u)\bar{l}_1(x, A, u) \le \mathrm{e}^{\Phi(q)(x-A)} \left(\frac{\mathrm{e}^{\Phi(q)c} - 1}{c\psi'(\Phi(q))} + L\right) \bar{\rho}^{(q)}_{g,A} < \infty.$$

On the other hand,

$$\begin{aligned} l_2(x, A, c, u) &\le \mathbf{1}_{\{u>x-A\}} W^{(q)}(x + 1 - A) \max_{A-u\le y\le A} |g(y) - g(A)| \\ &\le \mathbf{1}_{\{u>x-A\}} \frac{\mathrm{e}^{\Phi(q)(x+1-A)}}{\psi'(\Phi(q))} \max_{A-u\le y\le A} |g(y) - g(A)| \\ &=: \bar{l}_2(x, A, u), \end{aligned}$$

which is also Π-integrable by (2.13).

Now, by (A.5), the dominated convergence theorem applies, and noting $W^{(q)}(0) = 0$, we have

$$\begin{aligned} &\lim_{c\downarrow 0} \frac{\varphi^{(q)}_{g,A}(x + c) - \varphi^{(q)}_{g,A}(x)}{c} \\ &\quad = \int_0^\infty \Pi(\mathrm{d}u) \frac{\partial}{\partial x} \int_0^{u\wedge(x-A)} W^{(q)}(x - z - A)(g(z + A - u) - g(A))\,\mathrm{d}z \\ &\quad = \int_0^\infty \Pi(\mathrm{d}u) \int_0^{u\wedge(x-A)} W^{(q)\prime}(x - z - A)(g(z + A - u) - g(A))\,\mathrm{d}z. \end{aligned}$$

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The left derivative can be obtained in the same way. This proves (3.9).

For the proof of (3.10), let

$$k(x, A, u) := \int_0^{u \wedge (x-A)} W^{(q)\prime}(x - z - A)|g(z + A - u) - g(A)| \, \mathrm{d}z, \qquad x > A, \ u > 0.$$

Fix $A < x < A + 1$ and choose $0 < \epsilon < 1$ and $\varrho_{A,\epsilon}$ as in the proof of Lemma 2.3. By (A.1), for all $0 < u < \epsilon$,

$$\begin{aligned} k(x, A, u) &\leq \left(u|g'(A)| + \frac{1}{2}u^2 \varrho_{A,\epsilon}\right) \int_0^{u \wedge (x-A)} W^{(q)\prime}(x - z - A) \, \mathrm{d}z \\ &= \left(u|g'(A)| + \frac{1}{2}u^2 \varrho_{A,\epsilon}\right)(W^{(q)}(x - A) - W^{(q)}((x - A - u) \vee 0)) \\ &\leq \mathrm{e}^{\Phi(q)}\left(u|g'(A)| + \frac{1}{2}u^2 \varrho_{A,\epsilon}\right)\left(Lu + \frac{1 - \mathrm{e}^{-\Phi(q)u}}{\psi'(\Phi(q))}\right), \end{aligned} \tag{A.6}$$

where the last inequality holds because, by (2.10),

$$\begin{aligned} &W^{(q)}(x - A) - W^{(q)}((x - A - u) \vee 0) \\ &\quad = \mathrm{e}^{\Phi(q)(x-A)}[(W_{\Phi(q)}(x - A) - W_{\Phi(q)}((x - A - u) \vee 0)) \\ &\qquad\qquad + (1 - \mathrm{e}^{-\Phi(q)(u \wedge (x-A))})W_{\Phi(q)}((x - A - u) \vee 0)] \\ &\quad \leq \mathrm{e}^{\Phi(q)(x-A)}\left(Lu + \frac{1 - \mathrm{e}^{-\Phi(q)u}}{\psi'(\Phi(q))}\right) \\ &\quad \leq \mathrm{e}^{\Phi(q)}\left(Lu + \frac{1 - \mathrm{e}^{-\Phi(q)u}}{\psi'(\Phi(q))}\right). \end{aligned}$$

On the other hand, for $u \geq \epsilon$,

$$\begin{aligned} k(x, A, u) &\leq \max_{A-u \leq y \leq A} |g(y) - g(A)| \int_0^{u \wedge (x-A)} W^{(q)\prime}(x - z - A) \, \mathrm{d}z \\ &\leq W^{(q)}(1) \max_{A-u \leq y \leq A} |g(y) - g(A)|. \end{aligned} \tag{A.7}$$

If we define $\bar{k}(A, u)$ as the right-hand sides of (A.6) and (A.7) for $0 < u < \epsilon$ and for $u \geq \epsilon$, respectively, then

$$\begin{aligned} \int_0^\infty \Pi(\mathrm{d}u)\bar{k}(A, u) &= \int_0^\epsilon \Pi(\mathrm{d}u)\mathrm{e}^{\Phi(q)}\left(u|g'(A)| + \frac{1}{2}u^2 \varrho_{A,\epsilon}\right)\left(Lu + \frac{1 - \mathrm{e}^{-\Phi(q)u}}{\psi'(\Phi(q))}\right) \\ &\quad + \int_\epsilon^\infty \Pi(\mathrm{d}u)W^{(q)}(1) \max_{A-u \leq y \leq A} |g(y) - g(A)|, \end{aligned}$$

which is clearly finite by (2.2) and (2.13).

Now we can interchange the limit via the dominated convergence theorem as $x \downarrow A$ in (3.9) and obtain (3.10). This completes the proof.

A.4. Proof of Proposition 3.4

For all $B > x > A^*$, we have, by the strong Markov property,

$$\mathbb{E}^x[\mathrm{e}^{-q\tau_{A^*}} g(X_{\tau_{A^*}}) \mid \mathcal{F}_{t \wedge \tau_{A^*} \wedge \tau_B^+}] = \mathrm{e}^{-q(t \wedge \tau_{A^*} \wedge \tau_B^+)} f(X_{t \wedge \tau_{A^*} \wedge \tau_B^+}).$$

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Taking expectations on both sides we obtain

$$f(x) = \mathbb{E}^x[\mathrm{e}^{-q\tau_{A^*}} g(X_{\tau_{A^*}})] = \mathbb{E}^x[\mathrm{e}^{-q(t\wedge\tau_{A^*}\wedge\tau_B^+)} f(X_{t\wedge\tau_{A^*}\wedge\tau_B^+})].$$

Hence $\{\mathrm{e}^{-q(t\wedge\tau_{A^*}\wedge\tau_B^+)} f(X_{t\wedge\tau_{A^*}\wedge\tau_B^+});\ t \geq 0\}$ is a martingale and, because $B > A^*$ is arbitrary, $(\mathcal{L} - q)f(x) = (\mathcal{L} - q)(\Gamma_1(x; A^*) + \Gamma_2(x; A^*)) = 0$ on (A^*, ∞); see also Section 4 of [13] for a more rigorous proof.

On the other hand, it is known that

$$(\mathcal{L} - q)W^{(q)}(x) = (\mathcal{L} - q)Z^{(q)}(x) = 0, \qquad x > 0.$$

This, together with Lemma 3.8, gives

$$(\mathcal{L} - q)\Gamma_3(x; A^*) = -(\mathcal{L} - q)\bigg[\int_{A^*}^x W^{(q)}(x - y)h(y)\,\mathrm{d}y\bigg] = -h(x).$$

Summing up these results, for $\Gamma_i(x; A^*),\ i = 1, 2, 3$, completes the proof.

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