Canad. J. Math. Vol. 70 (4), 2018 pp. 898–924
http://dx.doi.org/10.4153/CJM-2017-013-2

Spherical Fundamental Lemma for Metaplectic Groups

Caihua Luo

Abstract. In this paper, we prove the spherical fundamental lemma for metaplectic group Mp_{2n} based on the formalism of endoscopy theory by J. Adams, D. Renard, and W.-W. Li.

1 Introduction

Let F be a non-archimedean field of characteristic 0 with residue characteristic $p \neq 2$. Fix a non-trivial character ψ of F with conduct $\mathcal{O}_F$, let $\widetilde{\mathrm{Sp}}_\psi(W)$ be Weil's metaplectic 8-th cover of $\mathrm{Sp}(W)$, which is a pushout of the metaplectic cover $Mp_\psi(W)$ via $\mu_2 \hookrightarrow \mu_8$, *i.e.*,

$$\begin{tikzcd}
\mu_2 \arrow[r, hook] \arrow[d, hook] & Mp_\psi(W) \arrow[r, two heads, "p"] \arrow[d] & \mathrm{Sp}(W) \arrow[d, equal] \\
\mu_8 \arrow[r, hook] & \widetilde{\mathrm{Sp}}_\psi(W) \arrow[r, two heads, "p"] & \mathrm{Sp}(W).
\end{tikzcd}$$

Notice that W.-W. Li has built up the endoscopy theory for $\widetilde{\mathrm{Sp}}_\psi(W)$ in [Li11] and stabilized the elliptic trace formula of $\widetilde{\mathrm{Sp}}_\psi(W)$ in [Li15]. He has also proved the transfer conjecture and fundamental lemma of units for large p in [Li11]. But the spherical fundamental lemma has not yet been proved, and it is a necessary theorem for applying global arguments to prove the expected endoscopic character identities of $\widetilde{\mathrm{Sp}}_\psi(W)$. The purpose of this article is to adapt Clozel and Hales' ideas in [Clo90, Hal95] to prove the desired spherical fundamental lemma for $\widetilde{\mathrm{Sp}}_\psi(W)$. Herein we should mention that for standard endoscopy in linear groups, analogous "spherical fundamental lemma" was proved by Hales [Hal95], and for twisted endoscopy, it was recently shown by Lemaire–Moeglin–Waldspurger [LMW15] and Lemaire-Waldspurger [LW15].

Let $G = \mathrm{Sp}(W), \widetilde{G} = \widetilde{\mathrm{Sp}}_\psi(W)$. As shown by J. Adams, D. Renard, and W.-W. Li among others, the elliptic endoscopic groups of $\widetilde{G}$ are the split orthogonal groups

$$H = H_{n',n''} = H' \times H'' := SO(2n'+1) \times SO(2n''+1) \text{ with } n' + n'' = n,$$

and a standard norm correspondence is defined in Section 2.2.3 as

$$\mathcal{N}: H(F)_{ss}/\sim_{\mathrm{geo}} \longrightarrow G(F)_{ss}/\sim_{\mathrm{geo}}.$$

Received by the editors November 9, 2016; revised April 4, 2017.
Published electronically September 1, 2017.
AMS subject classification: 22E35.
Keywords: metaplectic group, endoscopic group, elliptic stable trace formula, fundamental lemma.

Associated with the norm correspondence $\mathcal{N}$, the transfer factors for norm pairs $(\gamma, \widetilde{\delta})$ are defined in Section 2.2.4 as follows:

$$\Delta(\gamma, \widetilde{\delta}) := \frac{\Theta'_\psi}{|\Theta'_\psi|}(-\widetilde{\delta}') \cdot \frac{\Theta''_\psi}{|\Theta''_\psi|}\big(\widetilde{\delta}''\big) \cdot \mathrm{sgn}_{K''/K''^{\#}}\big(P_{a'}(-a'')(a'')^{-n'} \det(\delta'+1)\big),$$

where Θ'_ψ (resp. Θ''_ψ) is the Harish–Chandra character of the Weil representation ω'_ψ (resp. ω''_ψ) of $\widetilde{\mathrm{Sp}}_\psi(W')$ (resp. $\widetilde{\mathrm{Sp}}_\psi(W'')$), and $P_{a'} \in F[T]$ is the characteristic polynomial of $a' \in K'^{\times}$. Let $K = G(\mathcal{O}_F)$ and $K_H = H(\mathcal{O}_F)$; we define the associated spherical (anti-genuine) Hecke algebras

$$\mathcal{H}_K(\widetilde{G})_{--} := C_c^\infty\big(\widetilde{G}//G(\mathcal{O}_F)\big)_{--}, \qquad \mathcal{H}_{K_H}(H(F)) := C_c^\infty\big(H(F)//H(\mathcal{O}_F)\big).$$

Let

$$b: \mathcal{H}_K(\widetilde{G})_{--} \longrightarrow \mathcal{H}_{K_H}(H(F))$$

be the conjectural transfer map defined by the norm map $\mathcal{N}$ and Satake isomorphisms, *i.e.*, Theorem 3.1.1. For $\phi \in \mathcal{H}_K(\widetilde{G})_{--}$ and $f \in \mathcal{H}_{K_H}(H(F))$, set $O_{\widetilde{\delta}}(\phi)$ as the normalized orbital integral associated with $\widetilde{\delta} \in \widetilde{G}$ and $SO_\gamma(f)$ as the normalized stable orbital integral associated with $\gamma \in H(F)$. Denote

$$\Lambda(\gamma, \phi) := \sum_{\delta} \Delta(\gamma, \widetilde{\delta}) O_{\widetilde{\delta}}(\phi) - SO_\gamma(b(\phi)).$$

Main Theorem *Assume the (stable) orbital integrals are compatibly normalized as in* [Li12b]. *Let* $\phi \in \mathcal{H}_K(\widetilde{G})_{--}$, $f = b(\phi) \in \mathcal{H}_{K_H}(H(F))$. *Then* $\Lambda(\gamma, \phi) = 0$ *for any* $\gamma \in H(F)_{G-\mathrm{reg}}$.

As is well known, Clozel [Clo90] and Hales [Hal95] have a standard way to tackle this problem. So naturally we will try to adapt their arguments to prove our theorem. Notice that the main ingredients of Clozel and Hales' arguments are as follows.

- Howe's finiteness theorem,
- Vignéras' characterization of orbital integrals,
- Clozel and Waldspurger's theorem concerning compact trace,
- Keys' reducibility theorem of unramified unitary principal series.

So basically we should extend these results to $\widetilde{\mathrm{Sp}}_\psi(W)$. Note that Vignéras' characterization of orbital integrals has also been established for finite central covering groups in [Vig82]. Instead of using Vigneras' characterization indirectly, we will use W.-W. Li's isomorphism theorem which comes much more directly. Therefore, our ingredients in some sense have been built up as follows.

- Howe's finiteness theorem for covering groups [Luoar],
- W.-W. Li's isomorphism theorem concerning transfer maps [Li16],
- Clozel and Waldspurger's theorem for covering groups [Luoar],
- Irreducibility theorem of unramified unitary principal series of $\widetilde{\mathrm{Sp}}_\psi(W)$ [Szp13].

The outline of this article is as follows. In Section 2, we recall in greater detail the theory of endoscopy for $\widetilde{\mathrm{Sp}}_\psi(W)$ established by J. Adams, D. Renard, and W.-W. Li. In Section 3 we prove some key results about genuine spherical representations that will play an essential role in the last section, while the last section is devoted to the proof of the Main Theorem.

2 Endoscopy and Trace Formula

For the convenience of the reader, we summarize W.-W. Li's work as follows [Li11, Li15, Li16].

2.1 Notations and Facts

2.1.1 Local Case

• Let F be a non-archimedean field of characteristic 0 with residue characteristic $p \neq 2$, and let $\psi_F: F \to S^1$ be a non-trivial character of conductor $\mathcal{O}_F$.

• Let $(W, \langle \cdot, \cdot \rangle)$ be the non-degenerate symplectic F-space of $2n$-dimension associated with the symplectic form $\begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$ and let $G = \mathrm{Sp}(W)$ be the associated symplectic group. Fix a selfdual lattice $L = \mathcal{O}_F^{2n} \subset W$; we define $K = \mathrm{Stab}_G(L) = G(\mathcal{O}_F)$.

• For the Heisenberg group $H(W) = W \oplus F$, the multiplication is defined as follows.

$$(w_1, t_1)(w_2, t_2) = \Big(w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2 \rangle \Big).$$

We have $Z(H(W)) = 0 \oplus F$. Notice that G acts naturally on $H(W)$ by $g.(w,t) = (g.w, t)$, trivially on $Z(H(W))$. By Stone–Von Neumann theorem, this action defines a projective representation of G, *i.e.*, $G \to GL(S_{\psi_F})/\mathbb{C}^\times$, where $(\rho_{\psi_F}, S_{\psi_F})$ is the unique smooth irreducible representation of $H(W)$ with central character ψ_F.

• $\overline{\mathrm{Sp}}(W)_{\psi_F} := \{(g, A(g)) \in \mathrm{Sp}(W) \times GL(S_{\psi_F}) : \rho_{\psi_F}^g = A(g)^{-1} \circ \rho_{\psi_F} \circ A(g)\}$. This is a central extension of $\mathrm{Sp}(W)$ by $\mathbb{C}^\times$ such that A can be lifted to a representation ω_{ψ_F} of $\overline{\mathrm{Sp}}(W)_{\psi_F}$: $\omega_{\psi_F}(g, A(g)) := A(g)$. This is the so-called *Weil representation.*

2.1.2 Global Case

• Let F^* be a number field, $\mathcal{O}$ the associated ring of integers, and $\mathbb{A}$ the associated Adèle ring. Fix a non-trivial automorphic character $\psi: F^* \backslash \mathbb{A} \to S^1$, $\psi = \otimes_v \psi_v$.

• Let $(W, \langle \cdot, \cdot \rangle)$ be a non-degenerate symplectic F^* vector space of dimension $2n$, $(W_v, \langle \cdot, \cdot \rangle) := (W, \langle \cdot, \cdot \rangle) \otimes_{F^*} F_v^*$, and $\mathrm{Sp}(W, \mathbb{A}) := \prod_v' \mathrm{Sp}(W_v)$.

• For Adèlic Heisenberg group $H(W, \mathbb{A}) := \prod_v' H(W_v)$, $\mathrm{Sp}(W, \mathbb{A})$ acts naturally on $H(W, \mathbb{A})$. Again, by the Stone–Von Neumann theorem, we get a central extension $\overline{\mathrm{Sp}}(W, \mathbb{A})_\psi$ of $\mathrm{Sp}(W, \mathbb{A})$ by $\mathbb{C}^\times$ and its associated global Weil representation $\omega_\psi := \bigotimes w_{\psi_v}$.

2.1.3 Schrödinger Model

Associated with a complete polarization $W = X + Y$, fix a Haar measure on Y and a self-dual Haar measure on W with respect to $\psi_F(\langle \cdot, \cdot \rangle)$, one can then construct a model of the representation $(\rho_{\psi_F}, S_{\psi_F})$ as follows:

$$S_Y = \mathrm{Ind}_{H(Y)}^{H(W)} \psi_F,$$

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

where, by abuse of notation, ψ_F is the unique extension of the character ψ_F of $Z(H(W))$ to $H(Y)$ given by $\psi_F(y,t) = \psi_F(t)$. Here the induced representation means smooth induced representation, and the action of $H(W)$ is given by right translation.

Associated with two complete polarizations $W = X_1 + Y_1$ and $W = X_2 + Y_2$, Lion and Perrin [LP81] have defined a canonical intertwining operator $\mathcal{I}_{Y_1,Y_2}: S_{Y_1} \to S_{Y_2}$ as follows. For $f_1 \in S_{Y_1}$, $Y_{12} := Y_1 \cap Y_2$, the integral

$$I_{Y_1,Y_2}(f(h)) = \int_{Y_{12}\backslash Y_2} f((y,0)h)dy$$

is absolutely convergent and not identically zero, and defines an element of S_{Y_2}. Note that the isomorphism I_{Y_1,Y_2} depends on a choice of the Haar measure on $Y_{12}\backslash Y_2$. On the other hand, for any $g \in \mathrm{Sp}(W)$, one can define a natural isomorphism

$$A^0(g): S_Y \longrightarrow S_{Y g^{-1}}$$
$$(A^0(g)f)(h) = f(h^g)$$

that satisfies

$$\rho_{\psi_F}(h)A^0(g) = A^0(g)\rho_{\psi_F}(h^g).$$

Put these isomorphisms together, and one can then define an action of $g \in \mathrm{Sp}(W)$ on S_Y as

$$A_Y(g) := I_{Yg^{-1},Y} \circ A^0(g).$$

It is easy to check that $(g, A_Y(g)) \in \overline{\mathrm{Sp}}(W)_{\psi_F}$. We define the above section of $p: \overline{\mathrm{Sp}}(W)_{\psi_F} \to \mathrm{Sp}(W)$ as $\sigma_Y: g \mapsto (g, A_Y(g))$, and it is well known that σ_Y splits over $P_Y = \mathrm{Stab}_{\mathrm{Sp}(W)}(Y)$.

2.1.4 Structure

- $\overline{\mathrm{Sp}}(W)_{\psi_F}$, for various choices of ψ_F, are canonically isomorphic in the category of central extensions of $\mathrm{Sp}(W)$ by $\mathbb{C}^\times$. From now on, we will omit the subscript ψ_F.
- $\overline{\mathrm{Sp}}(W) \simeq \widetilde{\mathrm{Sp}}^{(2)}(W) \times_{\mu_2} \mathbb{C}^\times$, where $\widetilde{\mathrm{Sp}}^{(2)}(W)$ is the unique non-trivial two-fold cover of $\mathrm{Sp}(W)$. Denote

$$\widetilde{G} = \widetilde{\mathrm{Sp}}^{(8)}(W) := \widetilde{\mathrm{Sp}}^{(2)}(W) \times_{\mu_2} \mu_8.$$

- Let P_Y be the Siegel parabolic associated with a maximal isotropic subspace Y of W; then P_Y splits in $\widetilde{G}$ via the composite of σ_Y and the projection map of $\overline{\mathrm{Sp}}(W) \to \widetilde{G}$, *i.e.*,

$$\begin{tikzcd} & \widetilde{G} \arrow[d, "p"] \\ P_Y \arrow[ur, "\sigma_Y"] \arrow[r, "i"'] & G. \end{tikzcd}$$

Note that such σ_Y does not depend on the choices of Haar measures. Conventionally, one writes -1 for $\sigma_Y(-1)$ in $\widetilde{G}$, which does not depend on the choice of Y.

- For K hyperspecial compact subgroup of $\mathrm{Sp}(W)$, K splits in $\widetilde{G}$. Globally, $\mathrm{Sp}(W)$ splits in $\overline{\mathrm{Sp}}(W, \mathbb{A})$.

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

- For $\widetilde{x}, \widetilde{y} \in \widetilde{G}$, $\widetilde{x}$ and $\widetilde{y}$ commute if and only if x and y commute.
- For a maximal split torus T of G, let $\widetilde{T}$ be the preimage of the projection map p; one can then define a genuine $W^G(T)$-invariant character χ_{ψ_F} of $\widetilde{T}$ that is compatible with the local theta correspondence associated to ψ_F in [Kud96] as follows:

$$\chi_{\psi_F}: \left(\begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix}, \epsilon \right) \longmapsto \epsilon\gamma(\det(a), \psi_F)^{-1},$$

where $\gamma(\cdot, \psi_F)$ is the relative Weil index. Note that this is also compatible with the natural splitting of T in $\widetilde{T}$ given by σ_Y in [Kud96, Theorem 4.5]

$$\sigma_Y: g = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} \longmapsto (g, \gamma(\det(a), \psi_F)),$$

i.e., $\chi_{\psi_F} \circ \sigma_Y = \mathrm{id}$.
- Denote

$$N := \left\{ (\epsilon_v) \in \bigoplus \mathrm{Ker}(p_v) = \bigoplus \mu_2 : \prod \epsilon_v = 1 \right\},$$

$$\widetilde{\mathrm{Sp}}^{(2)}(W, \mathbb{A}) := \prod_v{}' \widetilde{\mathrm{Sp}}^{(2)}(W_v)/N.$$

We have $\overline{\mathrm{Sp}}(W, \mathbb{A})_\psi \simeq \widetilde{\mathrm{Sp}}^{(2)}(W, \mathbb{A}) \times_{\mu_2} \mathbb{C}^\times$. Similarly, we define the μ_8 cover $\widetilde{\mathrm{Sp}}^{(8)}(W, \mathbb{A})$ of $\mathrm{Sp}(W, \mathbb{A})$ to be the pushout of the double cover as defined before.

2.2 Endoscopy

2.2.1 Regular Semisimple Conjugacy Classes

- $\mathrm{Sp}(W)$ with dim $W = 2n$: the regular semisimple conjugacy classes are parametrized by the following data $\mathcal{O}(K/K^\#, x, c)$:
  - (K, τ) $2n$-dimensional étale F-algebra with involution τ. Denote by $K^\#$ the τ-fixed étale subalgebra of K.
  - $x \in K^\times$ such that $\tau(x) = x^{-1}$ and $K = F[x]$.
  - $c \in K^\times$ with $\tau(c) = -c$.
- $SO(V, q)$ splits with dim V=2m+1: the strongly regular semisimple conjugacy classes are parametrized by the following data $\mathcal{O}(K/K^\#, x, c)$:
  - (K, τ) $2m$-dimensional étale F-algebra with involution τ. Denote $K^\#$ the τ-fixed étale subalgebra of K.
  - $x \in K^\times$ such that $\tau(x) = x^{-1}$ and $K = F[x]$.
  - $c \in K^\times$ such that $\tau(c) = c$.
- $\mathcal{O}(K_1/K_1^\#, x_1, c_1)$ and $\mathcal{O}(K_2/K_2^\#, x_2, c_2)$ are equivalent if and only if there exists an F-algebra isomorphism and involution $\sigma: (K_1, \tau_1) \xrightarrow{\sim} (K_2, \tau_2)$ such that $\sigma(x_1) = x_2$ and $\sigma(c_1)c_2^{-1} \in N_{K_2/K_2^\#}(K_2^\times)$.

2.2.2 Endoscopic Groups and Stable Conjugacy

- Let $G = \mathrm{Sp}(W)$, $\widetilde{G} = \widetilde{\mathrm{Sp}}^{(8)}(W)$, the elliptic endoscopic groups of $\widetilde{G}$ are $H = H_{n',n''} = H' \times H'' := SO(V', q') \times SO(V'', q'')$ with $n' + n'' = n$, where the quadratic forms are

$$q'(x_{-n'}, \dots, x_0, \dots, x_{n'}) = 2\sum_{i=1}^{n'} x_{-i}x_i + x_0^2$$

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

and

$$q''(x_{-n''},\dots,x_0,\dots,x_{n''}) = 2\sum_{i=1}^{n''} x_{-i}x_i + x_0^2.$$

For simplicity, we write the above special orthogonal groups in the form $SO(2m+1)$.

- $\widetilde{\delta}_1, \widetilde{\delta}_2 \in \widetilde{G}_{\mathrm{reg}}$ are stably conjugate if δ_1 and δ_2 are stably conjugate in G, and $\Theta_{\psi_F}(-\widetilde{\delta}_1) = \Theta_{\psi_F}(-\widetilde{\delta}_2)$, where Θ_{ψ_F} is the Harish–Chandra character of the Weil representation ω_{ψ_F}. Thus, the conjugacy classes within a stable conjugacy class in $\widetilde{G}_{\mathrm{reg}}$ are parametrized by $H^1(F,T)$.

2.2.3 Norm Correspondence

Fix F-split tori $S' \in H'$, $S'' \in H''$ and let $S = S' \times S''$. Also fix F-split torus $T' \in G' = \mathrm{Sp}(2n')$, $T'' \in G'' = \mathrm{Sp}(2n'')$, and $T \in G = \mathrm{Sp}(W)$. Let $W^G(T)$ be the Weyl group associated with T in G, similarly for other groups. There are natural F-isomorphisms:

$$\mu'\colon S' \xrightarrow{\sim} T', \quad \mu''\colon S'' \xrightarrow{\sim} T'', \quad \nu\colon T'\times T'' \xrightarrow{\sim} T$$

and homomorphisms compatible with the above isomorphisms:

$$\begin{aligned} W^{H'}(S') &\xrightarrow{\sim} W^{G'}(T'),\\ W^{H''}(S'') &\xrightarrow{\sim} W^{G''}(T''),\\ W^{G'}(T')\times W^{G''}(T'') &\hookrightarrow W^G(T). \end{aligned}$$

Thus, we obtain F-homomorphisms:

$$\begin{aligned} \mu'\colon S'/W^{H'}(S') &\xrightarrow{\sim} T'/W^{G'}(T'),\\ \mu''\colon S''/W^{H''}(S'') &\xrightarrow{\sim} T''/W^{G''}(T''),\\ \nu\colon T'/W^{G'}(T')\times T''/W^{G''}(T'') &\longrightarrow T/W^G(T). \end{aligned}$$

Let

$$\mu = \mu_{n',n''} \coloneqq \nu\circ(\mathrm{id},-\mathrm{id})\circ(\mu',\mu'')\colon S/W^H(S) \longrightarrow T/W^G(T).$$

Thus, we get the "norm map"

$$\mathcal{N}\colon H(F)_{ss}/\!\sim_{\mathrm{geo}} \longrightarrow G(F)_{ss}/\!\sim_{\mathrm{geo}}.$$

Explicitly in terms of parameters:

$\mathcal{O}(K'/K'^{\#},x',c')\times\mathcal{O}(K''/K''^{\#},x'',c'') \leftrightarrow \mathcal{O}(K/K^{\#},(x',-x''),c)$ with $K = K'\times K''$.

We say $\widetilde{\delta}\in\widetilde{G}$ and $\gamma\in H$ are *correspondent* if $\mathcal{N}(\{\gamma\}) = \{\delta\}$.

2.2.4 Transfer Factor

Recall that W.-W. Li has constructed the transfer factors for the above endoscopic groups [Li11, §5.3]. Suppose that $\gamma'\in\mathcal{O}(K'/K'^{\#},a',c')$ and $\gamma''\in\mathcal{O}(K''/K''^{\#},a'',c'')$, for $\widetilde{\delta}\in\widetilde{G}$ and $\gamma = (\gamma',\gamma'')\in H_{G-\mathrm{reg}}(F)$ such that δ and γ are correspondent. Then we have a unique orthogonal decomposition of $W = W'\oplus W''$, stable under the action of δ such that $\delta'\in\mathcal{O}(K'/K'^{\#},a',c')$ and $\delta''\in\mathcal{O}(K''/K''^{\#},-a'',c'')$, where $\delta' \coloneqq \delta|_{W'}$ and $\delta'' \coloneqq \delta|_{W''}$. This gives rise to a canonical homomorphism $j\colon \widetilde{\mathrm{Sp}}(W')\times\widetilde{\mathrm{Sp}}(W'')\to$

$\widetilde{\mathrm{Sp}}(W)$; we can then take $\widetilde{\delta'} \in \widetilde{\mathrm{Sp}}(W')$, $\widetilde{\delta''} \in \widetilde{\mathrm{Sp}}(W'')$ such that $j(\widetilde{\delta'}, \widetilde{\delta''}) = \widetilde{\delta}$. Note that the pair (δ', δ'') is unique and belongs to $\mathrm{Sp}(W')_{\mathrm{reg}} \times \mathrm{Sp}(W'')_{\mathrm{reg}}$. On the other hand, $(\widetilde{\delta'}, \widetilde{\delta''})$ is unique up to multiplication by $(\epsilon, \epsilon^{-1})$, where $\epsilon \in \mathrm{Ker}(p)$.

Definition Under the above assumptions, we can define the transfer factor for $(\gamma, \widetilde{\delta})$ as

$$\Delta(\gamma, \widetilde{\delta}) := \frac{\Theta'_\psi}{|\Theta'_\psi|}(-\widetilde{\delta'}) \cdot \frac{\Theta''_\psi}{|\Theta''_\psi|}(\widetilde{\delta''}) \cdot \mathrm{sgn}_{K''/K''^{\#}}\big(P_{a'}(-a'')(a'')^{-n'} \det(\delta' + 1)\big),$$

where Θ'_ψ (resp. Θ''_ψ) is the Harish–Chandra character of the Weil representation ω'_ψ(resp. ω''_ψ) of $\widetilde{\mathrm{Sp}}(W')$ (resp. $\widetilde{\mathrm{Sp}}(W'')$), and $P_{a'} \in F[T]$ is the characteristic polynomial of $a' \in K'^\times$. Conventionally, $\Delta(\gamma, \widetilde{\delta}) := 0$ if $(\gamma, \widetilde{\delta})$ is not a norm correspondence pair.

The transfer factor defined above has the following known properties:

- (Genuine) $\Delta(\gamma, \epsilon\widetilde{\delta}) = \epsilon\Delta(\gamma, \widetilde{\delta})$ for $\epsilon \in \mathrm{Ker}(p)$.
- (Cocycle property) If $\widetilde{\delta}$ and $\widetilde{\delta_1}$ are stably conjugate,

  $$\Delta(\gamma, \widetilde{\delta_1}) = \langle \kappa, \mathrm{inv}(\delta, \delta_1)\rangle \Delta(\gamma, \widetilde{\delta}),$$

  where $\mathrm{inv}(\delta, \delta_1)$ is the associated cohomology class in $H^1(F, G_\delta)$, and the endoscopic character κ is defined as follows:

  As $H^1(F, G_\delta) = H^1(F, T') \times H^1(F, T'')$ associated with the decomposition of $\delta = (\delta', \delta'')$, which is determined by the decomposition of γ, κ is defined as the composites of the projection to the second H^1 and the product map of $H^1(F, T'') \simeq \mu_2^{I''^*}$ to μ_2, *i.e.*,

  $$\begin{aligned} \kappa\colon \mu_2^{I'^*} \times \mu_2^{I''^*} &\longrightarrow \mu_2 \\ \big((t'_i)_{i\in I'}, (t''_i)_{i\in I''}\big) &\longmapsto \prod_{i\in I''} t''_i, \end{aligned}$$

  where (I', I'^*) and (I'', I''^*) are determined by the decomposition of the parameters K' and K'' as follows. If

  $$K' = \prod_{i\in I'} K'_i, \quad K'' = \prod_{i\in I''} K''_i,$$

  then I'^* is the cardinality of the set of quadratic extensions $K'_i/K'^{\#}_i$ for $i \in I'$, similar for I''^*.
- (Symmetric) $\Delta_{n',n''}((\gamma', \gamma''), \widetilde{\delta}) = \Delta_{n'',n'}((\gamma'', \gamma'), -\widetilde{\delta})$.
- (Normalization à la Waldspurger) For the hyperspecial subgroup pair $(K, K_H) = (G(\mathcal{O}_F), H(\mathcal{O}_F))$, and the norm correspondence pair $(\gamma, \delta) \in K \times K_H$, $\Delta(\gamma, \delta) = 1$ provided (γ, δ) are of regular reduction.
- (Product formula) Suppose $(\gamma, \delta) \in H_{G-\mathrm{reg}}(F^*) \times G(F^*)$ is a norm correspondence pair, and $\delta = (\widetilde{\delta_v})_v$ in $\widetilde{G}(\mathbb{A})$; then
  - $\Delta_v(\gamma, \widetilde{\delta_v}) = 1$ for almost all place v;
  - $\prod_v \Delta(\gamma, \widetilde{\delta_v}) = 1$.

- (Parabolic descent) If a norm pair (γ,δ) lies in Levi subgroups $M_H \times M$ with
$$M_H = \prod_{i\in I}(GL(n_i') \times GL(n_i'')) \times H^b,\ M = \prod_{i\in I} GL(n_i) \times G^b, \text{ and } n_i = n_i' + n_i'',$$
then (γ^b,δ^b) is also a norm pair in $H^b \times G^b$. Denote by $\Delta_{H,\widetilde{G}}$ and $\Delta_{H^b,\widetilde{G}^b}$ the transfer factors associated with (H,G) and (H^b,G^b), respectively; then
$$\Delta_{H,\widetilde{G}}(\gamma,\widetilde{\delta}) = \Delta_{H^b,\widetilde{G}^b}(\gamma^b,\widetilde{\delta}^b),$$
where $\widetilde{\delta}^b$ is given by the relation: $j(\sigma_{GL}(\delta_{GL}),\widetilde{\delta}^b) = \widetilde{\delta}$, with $\delta = \delta_{GL} \times \delta^b$ and σ_{GL} the natural splitting defined in Section 2.1.4.

2.2.5 Transfer Conjecture and Fundamental Lemma

For $x \in G$, $G_x := C_G(x)^o$, let
$$D_G(x) = |\det(1 - \mathrm{Ad}(x)|_{\mathrm{Lie}\, G/\mathrm{Lie}\, G_x})|^{1/2},$$
$C_c^\infty(\widetilde{G})_{--}$ be the anti-genuine subspace of $C_c^\infty(\widetilde{G})$, *i.e.*, $\phi(\epsilon\widetilde{x}) = \epsilon^{-1}\phi(\widetilde{x})$, similar notion for other groups and function spaces. We define the normalized (stable) orbital integral on $\gamma \in H$ for $f \in C_c^\infty(H(F))$ as

$$\text{(H)} \qquad O_\gamma(f) = D_H(\gamma)\int_{H_\gamma(F)\backslash H(F)} f(h^{-1}\gamma h)\,dh,$$
$$SO_\gamma(f) = D_H(\gamma)\int_{(H_\gamma\backslash H)(F)} f(h^{-1}\gamma h)\,dh.$$

Similarly, for $\widetilde{\delta} \in \widetilde{G}_{\mathrm{reg}}$, and $\phi \in C_c^\infty(\widetilde{G})_{--}$,

$$\text{(G)} \qquad O_{\widetilde{\delta}}(\phi) = D_G(\delta)\int_{G_\delta(F)\backslash G(F)} \phi(\widetilde{g}^{-1}\widetilde{\delta}\widetilde{g})\,dg.$$

Remark 1 The compatible Haar measures for (H) and (G) are defined via the canonical isomorphisms between the centralizers of regular elements.

As in [Li16], we set
$$\mathcal{I}(\widetilde{G})_{--} := \{O_?(\phi) : \phi \in C_c^\infty(\widetilde{G})\},$$
$$S\mathcal{I}(H_{n',n''}) := \{SO_?(f) : f \in C_c^\infty(H_{n',n''}(F))\};$$

$\mathcal{I}_{\mathrm{cusp}}(\widetilde{G})_{--}$:= the subspace in $\mathcal{I}(\widetilde{G})_{--}$ of elements supported on the elliptic set,

$S\mathcal{I}_{\mathrm{cusp}}(H_{n',n''}(F))$:=
the subspace in $S\mathcal{I}(H_{n',n''}(F))$ of elements supported on the elliptic set.

Theorem 2.2.1 (Transfer theorem [Li11, Proposition 5.20]) *Fix compatible Haar measures on $G(F)$ and $H(F)$ as in* [Li11, Proposition 5.20]. *Suppose $\phi \in C_c^\infty(\widetilde{G})_{--}$, then there exists $f \in C_c^\infty(H(F))$ such that*
$$\sum_\delta \Delta(\gamma,\widetilde{\delta})O_{\widetilde{\delta}}(\phi) = SO_\gamma(f)$$
for all $\gamma \in H_{G-\mathrm{reg}}(F)$. We say that (ϕ, f) is a transfer pair for $(\widetilde{G}, H(F))$.

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

If F is archimedean, for anti-genuine Schwartz function $\phi \in \mathcal{S}(\widetilde{G})_{--}$, we can take Schwartz function $f \in \mathcal{S}(H(F))$ such that the above transfer identity holds.

One can collect all the possible transfer maps defined in Theorem 2.2.1, and then define a "collective" transfer $\mathcal{T}^{\mathcal{E}}$ as follows:

$$\begin{aligned}\mathcal{T}^{\mathcal{E}}: \mathcal{I}(\widetilde{G})_{--} &\longrightarrow \bigoplus_{\substack{n'+n''=n\\ H:=H_{n',n''}}} S\mathcal{I}(H(F)),\\ O_?(\phi) &\longmapsto \sum_{\substack{n'+n''=n\\ H:=H_{n',n''}}} SO_?(f^H).\end{aligned}$$

Theorem 2.2.2 (Fundamental lemma for units [Li11, Theorem 5.23]) *Suppose the residue characteristic p of F is large enough (*cf. [Li11, Theorem 5.23]*for the explicit requirement), $K = G(\mathcal{O}_F)$ is a hyperspecial compact subgroup of $G(F)$ and $K_H = H(\mathcal{O}_F)$ the associated hyperspecial subgroup of $H(F)$; we define $\mu_K(\epsilon x) := \epsilon^{-1}$ if $x \in K$, otherwise 0. Then $(\mu_K, 1_{K_H})$ is a transfer pair provided*

$$\operatorname{meas}(\widetilde{K}) = \operatorname{meas}(K) = \operatorname{meas}(K_H) = 1.$$

Theorem 2.2.3 (Isomorphism theorem [Li16, Theorem 5.3.1]) *The collective transfer map*

$$\mathcal{T}^{\mathcal{E}}: \mathcal{I}_{\text{cusp}}(\widetilde{G})_{--} \longrightarrow \bigoplus_{H_{n',n''}} S\mathcal{I}_{\text{cusp}}(H_{n',n''})$$

is an isomorphism.

2.3 Trace Formula

2.3.1 Stable Trace Formula: Elliptic Regular Terms

Note that W.-W. Li has built up the stable trace formula for elliptic terms [Li15], but we only need to use elliptic regular part. For simplicity, we herein just state the easy part. Let $\Gamma_{\text{rel}}(G(F^*))$ be the set of representatives for the elliptic regular semisimple conjugacy classes in $G(F^*)$, $\Sigma_{G-\text{rel}}(H(F^*))$ the set of representatives for the elliptic G-regular semisimple stable conjugacy classes in $H(F^*)$, similarly for other groups.

Definition 2.3.1 For $\phi \in C_c^\infty(\widetilde{G}(\mathbb{A}))_{--}$, we define the elliptic regular part of the trace formula in [Li15] for ϕ by $T_{\text{rel}}^{\widetilde{G}}(\phi)$ as follows:

$$T_{\text{rel}}^{\widetilde{G}} := \sum_{\delta \in \Gamma_{\text{rel}}(G(F^*))} \tau(G_\delta) O_\delta(\phi),$$

where $O_\delta(\phi) = \prod_v O_{\widetilde{\delta}_v}(\phi_v)$ with $\delta = (\widetilde{\delta}_v)_v$ in $\widetilde{G}(\mathbb{A})$, for $\phi = \prod_v \phi_v$, and $\tau(G_\delta)$ is the associated Tamagawa measure that equals 1.

Suppose H is an endoscopic group of $\widetilde{G}$. For $f^H \in C_c^\infty(H(\mathbb{A}))$, we define the stable analogue $ST_{G-\text{rel}}^H(f^H)$ for H as follows.

$$ST_{G-\text{rel}}^H(f^H) := \tau(H) \sum_{\gamma \in \Sigma_{G-\text{rel}}(H(F^*))} SO_\gamma^H(f^H),$$

where $SO_\gamma^H(f^H) = \prod_v SO_\gamma^H(f_v^H)$ for $f^H = \prod_v f_v^H$, and $\tau(H)$ is the associated Tamagawa measure.

Lemma 2.3.2 ([Li15, Lemma 5.2.1]) *There exists a canonical bijection between the sets*

$$\left\{(\delta,\kappa) : \delta \in \Sigma_{\mathrm{rel}}(G), \kappa \in \pi_0(Z(G_\delta^\vee))^\Gamma\right\},$$
$$\left\{((n',n''),\gamma) : n' + n'' = n, \gamma \in \Sigma_{G-\mathrm{rel}}(H)\right\},$$

where $H := H_{n',n''}$, and $\Gamma := Gal(\overline{F}^*/F^*)$. The bijection is characterized by the following conditions:

(i) (γ,δ) *is a G-regular norm correspondence pair with respect to (n',n'');*

(ii) $\kappa\colon H^1(F^*, G_\delta(\overline{\mathbb{A}})/G_\delta(\overline{F}^*)) \to \mathbb{C}^\times$ *is the endoscopic character associated with $((n',n''),\gamma)$.*

Theorem 2.3.3 (Stable trace formula: elliptic regular terms) *Suppose $\phi = \prod_v \phi_v \in C_c^\infty(\widetilde{G})_{--}$ and an adélic transfer function $f^H = \prod_v f_v^H \in C_c^\infty(H(\mathbb{A}))$ is chosen for each given elliptic endoscopic group $H := H_{n',n''}$. Then we have*

$$T_{\mathrm{rel}}^{\widetilde{G}}(\phi) = \sum_{\substack{H := H_{n',n''} \\ n'+n''=n}} \iota(\widetilde{G},H) ST_{G-\mathrm{rel}}^H(f^H),$$

where $\iota(\widetilde{G},H) = 1/2$ if one of n' and n'' is zero, and $\iota(\widetilde{G},H) = 1/4$ otherwise.

Proof Even though this is an easy corollary of Wen-wei Li's stable trace formula, we would provide a sketch of his proof for the convenience of the readers.

Step 1: Let

$$\mathcal{D}(G_\delta, G; F^*) := \mathrm{Ker}\Big(H^1(F^*, G_\delta) \to H^1(F^*, G)\Big) = H^1(F^*, G_\delta)$$

as $H^1(F^*, G) = 1$, by the decomposition of the conjugacy classes in a stable conjugacy class, we have

$$T_{\mathrm{rel}}^{\widetilde{G}}(\phi) = \sum_{\delta \in \Sigma_{\mathrm{rel}}(G)} \sum_{x \in \mathcal{D}(G_\delta, G; F^*)} O_{x^{-1}\delta x}(\phi).$$

Step 2: Fix $\delta \in \Sigma_{\mathrm{rel}}(G)$. For

$$\kappa \in \pi_0(Z(G_\delta^\vee)^\Gamma) \overset{\text{Duality}}{\simeq} H^1\Big(F^*, G_\delta(\overline{\mathbb{A}})/G_\delta(\overline{F}^*)\Big)^D$$

which results from Tate–Nakayama duality [Kot86], by Poisson summation formula [Lab01, Theorem 3.9] or [Kot86, Theorem 6.6], we get a further expansion:

$$T_{\mathrm{rel}}^{\widetilde{G}}(\phi) = \sum_{\delta \in \Sigma_{\mathrm{rel}}(G)} \sum_{\kappa \in \pi_0(Z(G_\delta^\vee)^\Gamma)} O_\delta^\kappa(\phi),$$

where

$$O_\delta^\kappa(\phi) := \int_{\mathcal{D}(G_\delta, G; \mathbb{A})} \kappa(y) O_{y^{-1}\delta y}(\phi) dy$$

with

$$\mathcal{D}(G_\delta, G; \mathbb{A}) := \mathrm{Ker}(H^1(\mathbb{A}, G_\delta) \longrightarrow H^1(\mathbb{A}, G)) = H^1(\mathbb{A}, G_\delta).$$

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

Step 3: By the bijection correspondence in the above lemma, denote $((n', n''), \gamma)$ the triple corresponded to (δ, κ); it suffices to prove that for $H := H_{n',n''}$:

$$O_{\delta}^{\kappa}(\phi) = SO_{\gamma}^{H}(f^{H}).$$

Step 4: (Product formulas)

$$O_{\delta}^{\kappa}(\phi) = \prod_{v} \int_{(G_{\delta}\backslash G)(F_v^*)} \kappa_v(x_v)\phi_v(x_v^{-1}\delta x_v)dx_v,$$
$$SO_{\gamma}^{H}(f^{H}) = \prod_{v} SO_{\gamma}^{H_v}(f_v^{H}).$$

By the product formulas, it suffices reduces to prove that the local parts involved are equal to each other. Note that the local part is just the transfer identity proved in [Li11, Proposition 5.20] or Theorem 2.2.1, whence the theorem holds. ■

2.3.2 Simple Trace Formulas

Before stating Arthur's simple trace formula, we introduce some notions on test functions. Say $\phi = \sum \otimes_v \phi_v' \in C_c^{\infty}(\widetilde{G}(\mathbb{A}))_{--}$ is supercuspidal at finite place v_0 if trace $\widetilde{\pi}_{v_0}(\phi_{v_0}') = 0$ for all irreducible genuine unitary non-supercuspidal representations $\widetilde{\pi}$, and define the similar notion for test functions in $C_c^{\infty}(H(\mathbb{A}))$.

Arthur's simple trace formula ([Art88, Corollary 7.3 & 7.4]) *Consider such test functions $f = \sum \otimes_v f_v' \in C_c^{\infty}(H(\mathbb{A}))$ that satisfies the following conditions:*

(i) *at some finite place v_0, f is supercuspidal;*
(ii) *at another finite place v_1, $O_{\gamma}(f_{v_1}') = 0$ for all $\gamma \in H(F_{v_1})$, which is not semisimple and F_{v_1}-elliptic.*

Then

$$\sum_{\gamma \in H(F^*)_{\mathrm{ell}}/\,\mathrm{conj}} \mathrm{vol}(H_{\gamma}(F^*)\backslash H_{\gamma}(\mathbb{A})) \int_{H_{\gamma}(\mathbb{A})\backslash H(\mathbb{A})} f(x^{-1}\gamma x)dx = \sum_{\pi \subset L^2_{\mathrm{disc}}(H(F^*)\backslash H(\mathbb{A}))} m(\pi)\,\mathrm{trace}\,\pi(f),$$

where $m(\pi)$ is the multiplicity of π in $L^2_{\mathrm{disc}}(H(F^*)\backslash H(\mathbb{A}))$.

Simple trace formula for $\widetilde{G}$ ([Li14b, Theorem 6.7]) *Consider such anti-genuine test functions $\phi = \sum \otimes_v \phi_v' \in C_c^{\infty}(\widetilde{G}(\mathbb{A}))_{--}$, which satisfies the following conditions:*

(i) *at some finite place v_0, ϕ is supercuspidal;*
(ii) *at another finite place v_1, $O_{\widetilde{\delta}}(\phi_{v_1}') = 0$ for all $\widetilde{\delta} \in \widetilde{G}(F_{v_1})$, which is not semisimple and F_{v_1}-elliptic.*

Then

$$\sum_{\delta \in G(F^*)^{bon}_{\mathrm{ell}}/\,\mathrm{conj}} \mathrm{vol}(G_{\delta}(F^*)\backslash G_{\delta}(\mathbb{A})) \int_{G_{\delta}(\mathbb{A})\backslash G(\mathbb{A})} \phi(x^{-1}\delta x)dx = \sum_{\widetilde{\pi} \subset L^2_{\mathrm{disc}}(G(F^*)\backslash \widetilde{G}(\mathbb{A}))} m(\pi)\,\mathrm{trace}\,\widetilde{\pi}(\phi),$$

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

where $m(\widetilde{\pi})$ is the multiplicity of $\widetilde{\pi}$ in $L^2_{\mathrm{disc}}(G(F^*)\backslash\widetilde{G}(\mathbb{A}))$.

3 Representation Theory

Before going to the proof part of the Main Theorem, we recall some well-known facts that will play an important role later on.

3.1 Unramified Representations and Spherical Functions

Recall the Satake transfer map

$$\begin{aligned} S\colon \mathcal{H}_K(\widetilde{G})_{--} &\longrightarrow \mathcal{H}_{K_T}(\widetilde{T})^{W^G(T)} \\ f &\longmapsto S(f)(\widetilde{t}) = f^\wedge(\widetilde{t}) := \delta(t)^{1/2}\int_N f(\widetilde{t}n)dn. \end{aligned}$$

We have similar notions for H and other groups. Then we have the following well-known properties of the Satake transfer maps that will be used later on.

Theorem 3.1.1 (Satake isomorphism) *The Satake transfer maps S are isomorphisms as follows:*

$$\begin{aligned} \mathcal{H}_K(\widetilde{G})_{--} \simeq \mathcal{H}_{K_T}(\widetilde{T})^{W^G(T)}_{--} &\overset{\chi_\psi}{\xrightarrow{\sim}} \mathcal{H}_{K_T}(T)^{W^G(T)} \\ &\simeq \mathbb{C}[X^*(T^\vee)]^{W^G(T)} \simeq \mathbb{C}[T^\vee]^{W^G(T)}, \\ \mathcal{H}_{K_H}(H(F)) \simeq \mathcal{H}_{K^H_S}(S)^{W^H(S)} &\simeq \mathbb{C}[X^*(S^\vee)]^{W^H(S)} \simeq \mathbb{C}[S^\vee]^{W^H(S)}. \end{aligned}$$

Thus, we can define a natural transfer map $b\colon \mathcal{H}_K(\widetilde{G})_{--} \to \mathcal{H}_{K_H}(H(F))$ via the dual of $\mu\colon S/W^H(S) \to T/W^G(T)$ defined in Section 2.2.3.

Proof See [Li14a, Section 3.2] and [Wei14, Theorem 3.8] for a discussion and proof in general. ∎

For parabolic subgroup $P = MN \subset G$, we write $M = M_{GL} \times M_{\mathrm{Sp}}$ with M_{GL} the associated GL-part and M_{Sp} the Sp-part of M. Denote by $\overline{f}^{(P)}$ the constant term of a test function $f \in C_c^\infty(\widetilde{G})$ along P:

$$\overline{f}^{(P)}(\widetilde{m}) = \delta_P(m)^{1/2}\int_N \overline{f}(\widetilde{m}n)dn,$$

where

$$\overline{f}(\widetilde{g}) = \int_K f(k\widetilde{g}k^{-1})dk, \qquad \delta_P(m) = |\det(\mathrm{Ad}(m)|_{\mathrm{Lie}\,N})|.$$

Recall the natural splitting

$$j\colon M_{GL} \times \widetilde{M}_{\mathrm{Sp}} \xrightarrow{\sigma_{GL}\times 1} \widetilde{M}$$

defined by σ_{GL} on the GL-part in Section 2.1.4.

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

Lemma 3.1.2 (Parabolic descent) *For a Levi subgroup M of G and the associated Levi subgroup M_H of H, we have the following diagram:*

$$\begin{tikzcd}
\mathcal{H}_K(\widetilde{G})_{--} \arrow[rrr, "S_\psi"] \arrow[dr, "\overline{(\cdot)}^{(P)}"'] \arrow[ddd, "b"] & & & \mathcal{H}_{K_T}(T)^{W^G(T)} \arrow[dl] \arrow[ddd, "b"] \\
& \mathcal{H}_{K_M}(\widetilde{M})_{--} \arrow[r, "S_\psi^M"] \arrow[d, "b^M"] & \mathcal{H}_{K_T}(T)^{W^M(T)} \arrow[d, "b^M"] & \\
& \mathcal{H}_{K_{M_H}}(M_H) \arrow[r, "S^{M_H}"] & \mathcal{H}_{K_S^H}(S)^{W^{M_H}(S)} & \\
\mathcal{H}_{K_H}(H) \arrow[ur, "\overline{(\cdot)}^{(P_H)}"'] \arrow[rrr, "S"] & & & \mathcal{H}_{K_S^H}(S)^{W^H(S)}. \arrow[ul]
\end{tikzcd}$$

This gives rise to the commutativity of the left side of the diagram. On the other hand, the natural splitting of T in Section 2.1.4:

$$\begin{tikzcd}
M_{GL} \times \widetilde{M}_{\mathrm{Sp}} \arrow[r, "\sigma_{GL}\times 1"] & \widetilde{M} \\
T \arrow[u, "1\times\sigma_{\mathrm{Sp}}"] \arrow[r, "\sigma"] \arrow[ur, phantom, "\circlearrowright"] & \widetilde{T}. \arrow[u]
\end{tikzcd}$$

gives rise to the following commutative diagram:

$$\begin{tikzcd}
\mathcal{H}_{K_M}(\widetilde{M})_{--} \arrow[r, "S_\psi^M"] \arrow[d, "j^*"] & \mathcal{H}_{K_T}(T)^{W^M(T)} \arrow[d, equal] \\
\mathcal{H}_{K_M}(M_{GL}\times\widetilde{M}_{\mathrm{Sp}})_{--} \arrow[r, "S_\psi^{M_{\mathrm{Sp}}}"] & \mathcal{H}_{K_T}(T)^{W^M(T)}
\end{tikzcd}$$

Combined with the previous commutative diagram, we have the parabolic descent diagram:

$$(\star) \qquad \begin{tikzcd}
\mathcal{H}_K(\widetilde{G})_{--} \arrow[r, "j^*\circ\overline{(\cdot)}^{(P)}"] \arrow[d, "b"] & \mathcal{H}_{K_M}(M_{GL}\times\widetilde{M}_{\mathrm{Sp}})_{--} \arrow[d, "1\times b^{M_{\mathrm{Sp}}}"] \\
\mathcal{H}_{K_H}(H) \arrow[r, "\overline{(\cdot)}^{(P_H)}"] & \mathcal{H}_{K_{M_H}}(M_H).
\end{tikzcd}$$

Proof This follows from a routine check. ■

Recall the spherical local Langlands correspondence for $\widetilde{G}$ and H as follows. Let $\Pi_{\mathrm{sph},--}(\widetilde{G})$ be the set of equivalent classes of irreducible admissible genuine K-spherical representations of $\widetilde{G}$, similar notion for H. Let W_F be the Weil group associated with F. Note that the dual group of $\widetilde{G}$ is $\mathrm{Sp}(2n, \mathbb{C})$ as suggested by local theta correspondence [GS12] (see also [GGar, Wei15]).

Theorem 3.1.3 *The unramified local Langlands correspondence*

$$\Pi_{\mathrm{sph},--}(\widetilde{G}) \xrightarrow[L_\psi]{1-1} \{\phi\colon W_F \to \mathrm{Sp}(2n, \mathbb{C}) \mid \phi \textit{ factors through } \mathbb{Z}\}/\operatorname{conj}$$

is given as follows:

$$\begin{aligned} \Pi_{\mathrm{sph},--}(\widetilde{G}) \simeq \operatorname{Irr}(\mathcal{H}_K(\widetilde{G})_{--}) &\overset{S_\psi}{\simeq} (\mathbb{C}[X^*(T^\vee)]^{W^G(T)})^D = T^\vee(\mathbb{C})/W^G(T) \\ \widetilde{\pi} &\longmapsto (f \longmapsto Tr\,\widetilde{\pi}(f)), \end{aligned}$$

which is compatible with the local theta correspondence.

Proof The compatibility results from the consistence of the splitting of T in $\widetilde{T}$ and the construction of local theta correspondence as follows.

$$\begin{tikzcd} \mathcal{H}_K(\widetilde{G})_{--} \arrow[rr, "f \mapsto Tr\,\mathrm{Ind}(\chi_\psi \chi)(f)"] \arrow[d, "f \mapsto S(f)"] \arrow[drr, phantom, "\circlearrowleft"] & & \mathbb{C} \\ \mathcal{H}_{K_T}(\widetilde{T})_{--}^{W^G(T)} \arrow[rr, "\chi_\psi"] & & \mathcal{H}_{K_T}(T)^{W^G(T)} \arrow[u, "f \mapsto \frac{1}{|W^G(T)|} Tr\ \chi(f)"'] \end{tikzcd}$$

■

Recall that for $K = G(\mathcal{O})$, the Schwartz space of anti-genuine K-bi-invariant functions on $\widetilde{G}$ is

$$\begin{aligned} \mathcal{S}_K(\widetilde{G})_{--} := \Big\{ f\colon \widetilde{G} \to \mathbb{C} \mid f \text{ anti-genuine } K\text{-bi-invariant}, \\ |f(\widetilde{x})| \le C\Xi(x)(1 + \|x\|)^{-r} \text{ for all } r > 0 \Big\}, \end{aligned}$$

C being a positive constant which depends on f and r. Here Ξ is the elementary spherical function on $\widetilde{G}(F)$, and $\|x\|$ the distance function (*cf.* [Wal03], [Sil79, P.174]). The vector space $\mathcal{S}_K(\widetilde{G})_{--}$ is topologized by means of the set of the following seminorms:

$$v_r(f) := \sup\Big\{ |f(\widetilde{x})\Xi(x)^{-1}(1 + \|x\|)^r| : \widetilde{x} \in \widetilde{G} \Big\}, \text{ for } r > 0.$$

Then $\mathcal{S}_K(\widetilde{G})_{--}$ is a Fréchet space. Note that for $f \in \mathcal{S}_K(\widetilde{G})_{--}$ and $\widetilde{\pi}$ genuine tempered representation of $\widetilde{G}$, trace $\widetilde{\pi}(f)$ exists (*cf.* [Wal03]).

Let $T_u^\vee$ denote the maximal compact subgroup of $T^\vee$; then Satake isomorphism tells us that $T_u^\vee / W^G(T)$ parametrizes the genuine tempered, unramified principal series representations of $\widetilde{G}$. Let $C^\infty(T_u^\vee)$ be the Fréchet space of all infinitely differentiable complex-valued functions on $T_u^\vee$ with the Schwartz topology defined by the following seminorms:

$$p_n(\phi) := \max\{|(D_n\phi)(t)| : D_n \text{ degree } n \text{ differential operators, } t \in T_u^\vee\}, \text{ for } n \in \mathbb{Z}_+.$$

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

Lemma 3.1.4 ([Clo90, Lemma 5.1]) *Consider the Fourier transform*

$$\mathfrak{F}_\psi\colon \mathcal{G}_K(\widetilde{G})_{--} \longrightarrow C^\infty(T_u^\vee)^{W^G(T)}$$

given by $f \mapsto f^\vee\colon f^\vee(t) = \operatorname{trace}\widetilde{\pi}_t(f)$, *where* $\widetilde{\pi}_t$ *is the genuine unramified representation associated to* t. *Then the following diagram commutes*

$$\begin{CD} \mathcal{H}_K(\widetilde{G})_{--} @>{\sim}>{S_\psi}> \mathbb{C}[T^\vee]^{W^G(T)} \\ @VVV @VVV \\ \mathcal{G}_K(\widetilde{G})_{--} @>{\sim}>{\mathfrak{F}_\psi}> C^\infty(T_u^\vee)^{W^G(T)}, \end{CD}$$

and the Fourier transform map $\mathfrak{F}_\psi$ *is a topological isomorphism.*

Proof The commutative diagram follows from the construction of the unramified local Langlands correspondence in Theorem 3.1.3. For the isomorphism part, note that the linear case was proved by Tadić in [Tad83]. The nonlinear case follows easily from the covering Satake isomorphism and the covering Plancherel theorem [Li12a, Theorem 2.6.4]. ■

Now we can state one of the main results that will play an important role in the proof of Main Theorem later on.

Lemma 3.1.5 ([Clo90, Lemma 5.5]) *Let* t_i $(i = 1, \dots, N)$ *be distinct elements of* $T^\vee/W$. *Assume that the linear form*

$$\phi \longmapsto \sum_i c_i \phi^\vee(t_i) \quad (c_i \neq 0)$$

on $\mathcal{H}_K(\widetilde{G})_{--}$ *extends continuously to* $\mathcal{G}_K(\widetilde{G})_{--}$. *Then* $t_i \in T_u^\vee/W$ *for all* i.

Proof For the convenience of the reader, we reproduce the proof from [Clo90]. Consider the diagram in Lemma 3.1.4 or [Clo90, Lemma 5.1]:

$$\begin{CD} \mathcal{H}_K(\widetilde{G})_{--} @>{\sim}>{S_\psi}> \mathbb{C}[T^\vee]^{W^G(T)} \\ @VVV @VVV \\ \mathcal{G}_K(\widetilde{G})_{--} @>{\sim}>{\mathfrak{F}_\psi}> C^\infty(T_u^\vee)^{W^G(T)}. \end{CD}$$

Via the bottom isomorphism, the linear form in the lemma extends continuously to $C^\infty(T_u^\vee)^{W^G(T)}$. By the definition of the Schwartz topology, we have

$$|\sum c_i \phi^\vee(t_i)| \leq \sum_j C_j \max_{t \in T_u^\vee} |D_j \phi^\vee(t)| \tag{3.1}$$

where $C_j > 0$ and the set of j is finite, and D_j is a differential operator with constant coefficients on $T_u^\vee$. We write

$$\phi^\vee(t) = \sum_{\lambda \in X^*(T^\vee)} \phi(\lambda) t^\lambda. \tag{3.2}$$

Substituting formula (3.2) back to inequality (3.1) yields:

$$\Big|\sum_{\lambda}\phi(\lambda)\sum_i c_i t_i^{\lambda}\Big| \le \sum_j C_j \max_{t\in T_u^\vee}\Big|\sum_{\lambda}P_j(\lambda)\phi(\lambda)t^{\lambda}\Big|,$$

where P_j is a polynomial on $X^*(T^\vee)$, the Fourier transform of D_j. Now take $\phi = \sum_{w\in W^G(T)}\delta_{w\lambda_0}$ for $\lambda_0 \in X^*(T^\vee)$, which is the unique spherical function such that $\phi^\vee(t) = \sum_{w\in W^G(T)} t^{w\lambda_0}$. Then

$$\begin{aligned}\Big|\sum_{i,w} c_i t_i^{w\lambda_0}\Big| &\le \sum_j C_j \max_{t\in T_u^\vee}\Big|\sum_w P_i(w\lambda_0)t^{w\lambda_0}\Big|\\ &\le \sum_j C_j \sum_w |P_i(w\lambda_0)|.\end{aligned}$$

Varying λ_0 and using the fact that the t_i^w are distinct, analyzing the exponentials shows that the t_i must be unitary. ■

3.2 Representation Theory of Complex Groups

Let L be a connected reductive complex group. Fix a maximal compact subgroup K_L of L and let $C_c^\infty(L,K_L)$ denote the space of compactly supported C^∞ functions that are right and left K_L-finite. Let B be a Borel subgroup with Langlands decomposition $B = MAN$, W the Weyl group of $T = MA$ in L. Fix a character σ_0 of M, let $\pi_{\sigma_0,\lambda} = \mathrm{Ind}(\sigma_0\otimes\lambda)$ for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $\mathfrak{a} = \mathrm{Lie}(A)$, and $\mathfrak{a}_{\mathbb{C}} = \mathfrak{a}\otimes\mathbb{C}$. Let $C_c^\infty(L,K_L;\sigma_0)$ be the subspace of $C_c^\infty(L,K_L)$ satisfying the condition

$$\langle \mathrm{trace}\,\pi_{\sigma,\lambda}, f\rangle = 0,$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and all $\sigma \notin \{W\cdot\sigma_0\}$. We sum up some properties that will be used later on as follows.

- All tempered irreducible representations are full unitary induced representations from Borel subgroups [Duf75].
- The Grothendieck group of admissible representations has a basis consisting of full induced representations from Borel subgroup.
- Clozel and Delorme's invariant Paley–Wiener theorem [CD84]: the vector space of functions

$$F_f(\lambda) = F_f(\sigma_0,\lambda) := \langle \mathrm{trace}\,\pi_{\sigma_0,\lambda}, f\rangle$$

for $f \in C_c^\infty(L,K_L;\sigma_0)$, consists of all functions in the Paley–Wiener space on the complex vector space $\mathfrak{a}_{\mathbb{C}}^*$ and that

$$F_f(w\lambda) = \langle \mathrm{trace}\,\pi_{w^{-1}\sigma_0,\lambda}, f\rangle$$

for $w \in W$.
- (Vanishing property)Let $W_{\sigma_0} = \{w \in W : w.\sigma_0 = \sigma_0\}$. If an absolutely convergent sum

$$\sum_{\lambda\in i\mathfrak{a}^*/W_{\sigma_0}} a(\lambda)F_f(\lambda) = 0 \qquad (3.3)$$

for all $f \in C_c^\infty(L,K_L;\sigma_0)$, then $a(\lambda)F_f(\lambda) = 0$ for all λ and f [Hal95, P.991].

Proof Otherwise, there exist λ_0 and f such that $c = |a(\lambda_0)F_f(\lambda_0)|$ is nonzero. The sum (3.3) can then be broken into the term $a(\lambda_0)F_f(\lambda_0)$, a sum over a large enough W_{σ_0}-invariant finite set $S_0 \subset i\mathfrak{a}^*$, and a sum over the remaining terms, such that

$$\sum_{\lambda \in i\mathfrak{a}^* \setminus S_0 \cup \{W_{\sigma_0}.\lambda_0\}} |a(\lambda)F_f(\lambda)| < c.$$

Pick a Paley–Wiener function h on $\mathfrak{a}^*_{\mathbb{C}}$ such that $h(\lambda_0) = 1$, $h(\lambda) = 0$ for $\lambda \in S_0$, and $|h(\lambda)| \leq 1$ for all $\lambda \in i\mathfrak{a}^*$. Notice that $h(\lambda)F_f(\lambda)$ is also a Paley-Wiener function, so there exists $f' \in C_c^\infty(L, K_L; \sigma_0)$ such that

$$F_{f'}(\lambda) = h(\lambda)F_f(\lambda) = \langle \operatorname{trace} \pi_{\sigma_0,\lambda}, f' \rangle.$$

Apply equation (3.3) to f' to conclude that

$$\sum a(\lambda)h(\lambda)F_f(\lambda) = 0.$$

This gives rise to the contradiction

$$c = |a(\lambda_0)h(\lambda_0)F_f(\lambda_0)| = \Big| \sum_{i\mathfrak{a}^* \setminus \{W_{\sigma_0}.\lambda_0\}} a(\lambda)h(\lambda)F_f(\lambda) \Big| < c. \qquad \blacksquare$$

Recall that $\widetilde{G}(\mathbb{C}) = G(\mathbb{C}) \times \mu_8$, $H(\mathbb{C}) = SO_{2n'+1}(\mathbb{C}) \times SO_{2n''+1}(\mathbb{C})$. The endoscopy theory for $(\widetilde{G}(\mathbb{C}), H(\mathbb{C}))$ shown by W.-W. Li gives (see [Li16] for details).

- The transfer factor $\Delta(\gamma,(t,\delta)) = t$ for all $t \in \mu_8$ and norm pair (γ,δ) with $\gamma \in H(\mathbb{C})$, $\delta \in G(\mathbb{C})$. This means that one can identify $C_c^\infty(\widetilde{G}(\mathbb{C}))_{--}$ with $C_c^\infty(G(\mathbb{C}))$ via $f \mapsto f(1,\cdot)$.
- (Transfer map) For $\phi \in C_c^\infty(G(\mathbb{C}))$, there exists $f \in C_c^\infty(H(\mathbb{C}))$ such that

$$O_\gamma(f) = O_\delta(\phi)$$

for all norm pairs $(\gamma,\delta) \in H_{G-\mathrm{reg}} \times G_{\mathrm{reg}}$ with $\gamma = (\gamma',\gamma'') \in H_{G-\mathrm{reg}}$.

- (Endoscopic character identity) For matching pairs of representations of the torus, $\big((\sigma,\lambda),((\sigma',\lambda'),(\sigma'',\lambda''))\big)$, *i.e.*, $\sigma = \sigma' \otimes \sigma''$ and $\lambda = \lambda' \otimes \lambda''$, and matching pairs (ϕ, f) of test functions as above, we have

$$\sigma''(-1)\lambda''(-1)F_\phi(\sigma,\lambda) = F_f\big((\sigma',\sigma''),(\lambda',\lambda'')\big).$$

3.3 Clozel, Waldspurger Theorem

Here we recall Clozel and Waldspurger's theorem, which says that the compact trace of $\widetilde{G}$ can be expressed as a linear combination of traces of its Levi subgroups. Denote $\operatorname{trace}_c \widetilde{\pi}(\phi)$ to be the compact trace that is defined in [Clo89]; in brief, this is equal to $\operatorname{trace} \widetilde{\pi}(1_c\phi)$ with 1_c the characteristic function of the compact elements in $\widetilde{G}(F)$. For admissible representation $\widetilde{\pi}$ of $\widetilde{G}$, we denote $\widetilde{\pi}_N$ to be the associated unnormalized Jacquet module with respect to parabolic subgroup $\widetilde{P} = \widetilde{M}N \subset \widetilde{G}$. Let $\widehat{\tau}_P^G$ be the characteristic function of the obtuse Weyl chamber associated with P (see [Art05, p. 29]), H be the Harish-Chandra map $\widetilde{M} \to \mathfrak{a}_M$, and $\widehat{\chi}_N = \widehat{\tau}_P^G \circ H$. Let a_P be the dimension of $\mathfrak{a}_M$.

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

Lemma 3.3.1 (Clozel, Waldspurger [Clo90, p .259])

$$\langle \operatorname{trace} \widetilde{\pi}, \phi \rangle_c = \sum_{P \in \mathcal{P}} (-1)^{a_P - a_G} \Big\langle \operatorname{trace} \delta_P^{-1/2} \widetilde{\pi}_N, \widehat{\chi}_N \overline{\phi}^{(P)} \Big\rangle_{\widetilde{M}}.$$

Proof See [Luoar, Corollary 2]. ∎

4 Main Theorem and its Proof

Recall that $G = \mathrm{Sp}(W)$, $\widetilde{G} = \widetilde{\mathrm{Sp}}^{(8)}(W)$, the elliptic endoscopic groups of $\widetilde{G}$ are the split orthogonal groups

$$H = H_{n',n''} = H' \times H'' := SO(2n'+1) \times SO(2n''+1) \text{ with } n' + n'' = n.$$

Recall $K = G(\mathcal{O}_F)$, $K_H = H(\mathcal{O}_F)$, and the associated spherical (anti-genuine) Hecke algebras

$$\mathcal{H}_K(\widetilde{G})_{--} := C_c^\infty(\widetilde{G}//G(\mathcal{O}_F))_{--}, \qquad \mathcal{H}_{K_H}(H(F)) := C_c^\infty(H(F)//H(\mathcal{O}_F)).$$

Let

$$b : \mathcal{H}_K(\widetilde{G})_{--} \longrightarrow \mathcal{H}_{K_H}(H(F))$$

be the conjectured transfer map defined by the Satake isomorphisms in Theorem 3.1.1. Denote

$$\Lambda(\gamma, \phi) := \sum_{\delta} \Delta(\gamma, \widetilde{\delta}) O_{\widetilde{\delta}}(\phi) - SO_\gamma(b(\phi)).$$

Then we can state our spherical fundamental lemma for $\widetilde{G}$ as follows.

Main Theorem *Assume the (stable) orbital integrals are compatibly normalized as in Section 2.2.5 or* [Li11, Proposition 5.20]. *Let* $\phi \in \mathcal{H}_K(\widetilde{G})_{--}$, $f = b(\phi) \in \mathcal{H}_{K_H}(H(F))$. *Then* $\Lambda(\gamma, \phi) = 0$ *for all* $\gamma \in H(F)_{G\text{-reg}}$.

We are now ready to prove the Main Theorem. Roughly, our proof consists of three steps. The first step is to reduce $\Lambda(\gamma, \phi) = 0$ to the case of elliptic G-regular elements by induction. The second step is then to apply stable trace formulas to get an obstruction for $\Lambda(\gamma, \phi) = 0$, *i.e.*, an equivalent criterion in Abstract Proposition. The last step is to use the unitarity criterion in Lemma 3.1.5 to show the triviality of the obstruction, *i.e.,* Abstract Lemma.

4.1 Reduction

We follow the standard argument to first reduce to the case of elliptic elements. By induction, we can assume that the spherical fundamental lemma holds for all metaplectic groups of rank smaller than that of $\widetilde{G}$. Assume that $\widetilde{\delta} \in \widetilde{G}$ is regular, and let $\widetilde{T}$ be the preimage of $T = C_G(\delta)$ in $\widetilde{G}$. Assume $\widetilde{T}$ is contained in a Levi subgroup $\widetilde{M} \subset \widetilde{G}$ over F.

For a maximal torus T_H of H, recall that for regular elements of T_H, the conjugacy classes in a stable conjugacy class is parametrized by $\mathfrak{D}(T_H/F) = \mathrm{Ker}(H^1(F, T_H) \to$

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

$H^1(F,H))$. As $H^1(F,M_H)\to H^1(F,H)$ is injective (see [DG, Exposé XXVI, Corollaire 5.2]), this set remains the same when we consider T_H as a torus in M_H. Analogously, for regular elements of $\widetilde{T}$, the conjugacy classes in a stable conjugacy class is parametrized by $\mathfrak{D}(T/F) = H^1(F,T)$ (*cf.* [Li11, lemma 5.7]). As $H^1(F,M) = 1$, this set remains the same when we consider $\widetilde{T}$ as a covering torus in $\widetilde{M}$.

This implies that the Main Theorem can be reduced to the case where $\delta \in G$ is elliptic as follows. In fact, if $\widetilde{\delta} \in \widetilde{M} \subset \widetilde{G}$ is not a norm in $\widetilde{G}$, it is also not a norm in $\widetilde{M}$. If $\phi \in \mathcal{H}_K(\widetilde{G})_{--}$, and $\overline{\phi}^{(P)} \in \mathcal{H}_{K_M}(\widetilde{M})_{--}$ is its constant term along $\widetilde{M}$, the orbital integral of ϕ at elements of $\widetilde{M}$ can be computed by descent formula from $\overline{\phi}^{(P)}$ (*cf.* [Clo85, Lemma 1]). Similarly, the same is true for stable orbital integrals. Write $M = M_{GL}\times M_{\mathrm{Sp}}$, and $M_H = M^H(GL)\times M^H_{SO}$. By the parabolic descent property of transfer factors in Section 2.2.4 and the parabolic descent property of Satake isomorphisms *i.e.,* Lemma 3.1.2, we have

$$\begin{aligned}
\Lambda^{(\widetilde{G},H)}(\gamma,\phi) &= \Lambda^{(\widetilde{M},M_H)}(\gamma,\overline{\phi}^{(P)})\\
&= \sum_{\delta_{\mathrm{Sp}}} \Delta^{(\widetilde{M}_{\mathrm{Sp}},M^H_{SO})}(\gamma_{SO},\widetilde{\delta}_{\mathrm{Sp}}) O^M_{j(\gamma_{GL}\times\widetilde{\delta}_{\mathrm{Sp}})}(\overline{\phi}^{(P)}) - SO^{M_H}_{\gamma_{GL}\times\gamma_{SO}}(\overline{b(\phi)}^{(P_H)})\\
&= \sum_{\delta_{\mathrm{Sp}}} \Delta^{(\widetilde{M}_{\mathrm{Sp}},M^H_{SO})}(\gamma_{SO},\widetilde{\delta}_{\mathrm{Sp}}) O^M_{\gamma_{GL}\times\widetilde{\delta}_{\mathrm{Sp}}}(j^*\overline{\phi}^{(P)}) - SO^{M_H}_{\gamma_{GL}\times\gamma_{SO}}(\overline{b(\phi)}^{(P_H)})\\
&\overset{(\star)}{=} 0
\end{aligned}$$

for non-elliptic $\gamma \in H(F)$ via the induction hypothesis.

4.2 Obstruction

In what follows, we adapt the global argument of [Hal95] to get an obstruction for the truth of the spherical fundamental lemma. We first construct a global situation that specializes to give our local data $(F,\widetilde{G},H,\psi_F,\langle\cdot,\cdot\rangle,\Lambda(\cdot,\phi))$ at some finite place (see [Li16, Proposition 8.4.1]). We choose a totally complex global field F^* and a place ω_0 of F^* such that $F^*_{\omega_0} = F$. Since G splits, there is a split G^*/F^* such that $G^*\times_{F^*}F \simeq G$. Similarly for H, there exists a split H^*/F^* such that $H^*\times_{F^*}F\simeq H$. By a variant of a result of Sansuc (*cf.* [KR00, Lemma 1]), $G^*(F^*)$ and $H^*(F^*)$ have dense image in $G^*(F^*_S)$ and $H^*(F^*_S))$ for the completion F^*_S at any finite set S of places of F^*. For simplicity, we now write G instead of G^* and H instead of H^*. Notice that G and H satisfy the Hasse principle for H^1.

Fix a G-regular elliptic element $\gamma_H \in H(F)$. We select a strongly regular semisimple element $\gamma \in H(F^*)$ approximating γ_H at ω_0. More specifically, we demand that $\Lambda(\gamma_{\omega_0},\phi) = \Lambda(\gamma_H,\phi)$ for all $\phi \in \mathcal{H}_K(\widetilde{G})_{--}$. Such elements exist by weak approximation and Howe finiteness conjecture (*cf.* [Luoar]). We can also assume that γ belongs to an anisotropic unramified Cartan subgroup at some place $v_0 \neq \omega_0$, and that γ_ω lies in a given open set U (to be specified below) for every archimedean place ω. Let T be the centralizer of γ. The Cartan subgroup T is anisotropic and unramified at v_0, and so by the Tchebotarev density theorem, it is anisotropic and unramified at infinitely many places.

In what follows, we would like to specialize some local test functions to simplify the elliptic stable trace formulas of H and $\widetilde{G}$, *i.e.,*

$$T_{\mathrm{ell}}^{\widetilde{G}}(f) = \iota(\widetilde{G}, H) ST_{\text{équi,ell}}^{H}(f^H) \qquad \textit{and} \qquad T_{\mathrm{ell}}^{H}(f^H) = ST_{\text{équi,ell}}^{H}(f^H).$$

We now set aside six finite places v_1, v_2, v_3, ω_1, ω_2, and ω_3 of F^* such that T is anisotropic and unramified at those places. We also choose a finite set of finite places, say S, containing the distinguished place ω_0 and all ramified places v, *i.e.,* $G \times_{F^*} F_v^*$ or $H \times_{F^*} F_v^*$ is ramified, such that the fundamental lemma for units holds for any finite place $v \notin S' := S \cup \{v_1, v_2, v_3, \omega_1, \omega_2, \omega_3\}$.

We define a set $\sum_{S'} = \{(\phi, f)\}$ of test functions $\phi = \otimes_v \phi_v$, $f = \otimes_v f_v$ on $\widetilde{G}(\mathbb{A}_{F^*})$ *and* $H(\mathbb{A}_{\mathbb{F}^*})$ respectively such that the following hold.

• ϕ_{ω_1} is a anti-genuine test function supported on the regular elliptic set. By the transfer conjecture proved by Wen-Wei Li (Theorem 2.2.1), there exists $f_{\omega_1} \in C_c^\infty(H)$ such that $(\phi_{\omega_1}, f_{\omega_1})$ is a transfer pair.

• f_{ω_2} is a test function supported on the regular elliptic set. By the transfer isomorphism theorem (Theorem 2.2.3), there exists $\phi_{\omega_2} \in C_{c,--}^\infty(\widetilde{G}_{\omega_2})$ such that $(\phi_{\omega_2}, f_{\omega_2})$ is a transfer pair.

• ϕ_{v_1} is a linear combination of matrix coefficients of a genuine supercuspidal representation of $\widetilde{G}$, and f_{v_1} is the associated transfer function (*cf.* Theorem 2.2.1).

• f_{v_2} is a linear combination of matrix coefficients of a supercuspidal representation of H. By the transfer isomorphism theorem (Theorem 2.2.3), there exists $\phi_{v_2} \in C_{c,--}^\infty(\widetilde{G}_{v_2})$ such that (ϕ_{v_2}, f_{v_2}) is a transfer pair.

• ϕ_{ω_3} is a anti-genuine cuspidal function, and f_{ω_3} is the associated transfer function (*cf.* Theorem 2.2.1). Further, by the transfer isomorphism theorem (Theorem 2.2.3), we can select ϕ_{ω_3} such that only H is involved.

• f_{v_3} is a cuspidal function such that only H itself is involved in the elliptic stable trace formula of H. By the transfer isomorphism theorem (Theorem 2.2.3), there exists ϕ_{v_3} such that (ϕ_{v_3}, f_{v_3}) is a transfer pair.

• At the archimedean places ∞, (ϕ_∞, f_∞) is a transfer pair such that $\phi_\infty \in C_c^\infty(G(\mathbb{C}), K; \sigma_0)$ for some regular character $\sigma_0 \in \widehat{M}$ such that $F_\phi(\sigma_0, \lambda)$ is not identically zero. This gives rise to an open set $U \subset G(\mathbb{C})_{\mathrm{reg}}$ on which the orbital integrals $O_\delta(\phi)$ of $\delta \in U$ are nonzero.

• $(\phi^{S'}, f^{S'})$ are unit elements in the spherical Hecke algebras that are transfer pairs outside S' guaranteed by W.-W. Li's fundamental lemma for units (Theorem 2.2.2).

• At ω_0, take $f_{\omega_0} = b(\phi_{w_0})$, with $\phi_{\omega_0} \in \mathcal{H}_K(\widetilde{G})_{--}$.

• At the remaining places v, we may take arbitrary transfer pairs (ϕ_v, f_v) guaranteed by the transfer conjecture (Theorem 2.2.1).

Suppose $(\phi_{\omega_0}, b(\phi_{\omega_0}))$ is a transfer pair at ω_0. By Kottwitz's elliptic stable trace formula for H and W.-W. Li's elliptic stable trace formula for $\widetilde{G}$ (Theorem 2.3.3), we have

$$T_{\mathrm{ell}}^{\widetilde{G}}(\phi) = \iota(\widetilde{G}, H) T_{\mathrm{ell}}^{H}(f).$$

Viewed as an identity on ϕ_∞ and f_∞, Arthur's simple trace formulas for H and $\widetilde{G}$ in Section 2.3.2 tell us that the spectral side of the identity $T_{\mathrm{ell}}^{\widetilde{G}}(\phi) = \iota(\widetilde{G}, H) T_{\mathrm{ell}}^{H}(f)$

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

takes the form as follows:

$$\sum_{\widetilde{\pi}\subset L^2_{\mathrm{disc}}(G(F^*)\backslash\widetilde{G}(\mathbb{A}))} m(\widetilde{\pi})\,\mathrm{trace}\,\widetilde{\pi}^{\infty}(\phi^{\infty})\,\mathrm{trace}\,\widetilde{\pi}_{\infty}(\phi_{\infty}) = \iota(\widetilde{G},H)\sum_{\pi\subset L^2_{\mathrm{disc}}(H(F^*)\backslash H(\mathbb{A}))} m(\pi)\,\mathrm{trace}\,\pi^{\infty}(f^{\infty})\,\mathrm{trace}\,\pi_{\infty}(f_{\infty}).$$

By the structure of the Grothendieck group of admissible representations in Section [3.2](#), *i.e.*,

$$[\widetilde{\pi}_{\infty}] = \sum_{\sigma,\lambda} b^G(\widetilde{\pi}_{\infty},\sigma,\lambda)[\mathrm{Ind}^{\widetilde{G}}(\widetilde{\sigma}\otimes\lambda)],$$
$$[\pi_{\infty}] = \sum_{\sigma,\lambda} b^H(\pi_{\infty},\sigma,\lambda)[\mathrm{Ind}^{H}(\sigma\otimes\lambda)],$$

for some integer coefficients $b^G(\widetilde{\pi}_{\infty},\sigma,\lambda)$ and $b^H(\pi_{\infty},\sigma,\lambda)$. We then have

$$\sum_{\sigma,\lambda} c^G(\sigma,\lambda,\phi^{\infty})F_{\phi_{\infty}}(\sigma,\lambda) = \sum_{(\sigma',\sigma''),(\lambda',\lambda'')} c^H((\sigma',\sigma''),(\lambda',\lambda''),f^{\infty})F_{f_{\infty}}((\sigma',\sigma''),(\lambda',\lambda'')),$$

where

$$c^G(\sigma,\lambda,\phi^{\infty}) = \sum_{\widetilde{\pi}\subset L^2_{\mathrm{disc}}} m(\widetilde{\pi})b^G(\widetilde{\pi}_{\infty},\sigma,\lambda)\,\mathrm{trace}\,\widetilde{\pi}^{\infty}(\phi^{\infty}),$$

and

$$c^H((\sigma',\sigma''),(\lambda',\lambda''),f^{\infty}) = \iota(\widetilde{G},H)\sum_{\pi\subset L^2_{\mathrm{disc}}} m(\pi)b^H(\pi_{\infty},(\sigma',\sigma''),(\lambda',\lambda''))\,\mathrm{trace}\,\pi^{\infty}(f^{\infty}).$$

Further by the endoscopy theory for complex groups in Section [3.2](#) and the choices of test function ϕ_{∞}, we then have

$$\sum_{\lambda\in\mathfrak{a}^*_{\mathbb{C}}} a(\lambda,\phi^{\infty})F_{\phi_{\infty}}(\sigma_0,\lambda) = 0, \tag{4.1}$$

where

$$a(\lambda,\phi^{\infty}) = c^G(\sigma_0,\lambda,\phi^{\infty}) - \sigma_0''(-1)\lambda''(-1)c^H\big((\sigma_0',\sigma_0''),(\lambda',\lambda''),f^{\infty}\big),$$

with $\sigma_0 = \sigma_0'\otimes\sigma_0''$ and $\lambda = \lambda'\otimes\lambda''$. Notice that the involved representations are unitary, so the summands in equation ([4.1](#)) should be over $i\mathfrak{a}^*$. Thus, we have $a(\lambda,\phi^{\infty})F_{\phi_{\infty}}(\sigma_0,\lambda) = 0$ for all $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ by the vanishing property in Section [3.2](#). Each term $a(\lambda,\phi^{\infty})F_{\phi_{\infty}}(\sigma_0,\lambda)$, viewed as a function of ϕ_{ω_0} in the Hecke algebra of $\widetilde{G}_{\omega_0}$, is linear. By Harish-Chandra's finiteness theorem for both $\widetilde{G}$ and H (*cf.* [HC59]), each identity $a(\lambda,\phi^{\infty})F_{\phi_{\infty}}(\sigma_0,\lambda) = 0$ is a finite sum of the form as follows:

$$\sum_{\widetilde{\pi}_{\omega_0}\ \mathrm{spherical}} a^G(\widetilde{\pi}_{\omega_0},\lambda,\phi^{\omega_0})\,\mathrm{trace}\,\widetilde{\pi}_{\omega_0}(\phi_{\omega_0}) = \sum_{\pi_{\omega_0}\ \mathrm{spherical}} a^H(\pi_{\omega_0},\lambda,\phi^{\omega_0})\,\mathrm{trace}\,\pi_{\omega_0}(b(\phi_{\omega_0})) \tag{4.2}$$

where

$$a^G(\widetilde{\pi}_{\omega_0}, \lambda, \phi^{\omega_0}) = F_{\phi_\infty}(\sigma_0, \lambda) \sum_{\substack{\widetilde{\pi}' \subset L^2_{\mathrm{disc}} \\ \widetilde{\pi}'_{\omega_0} = \widetilde{\pi}_{\omega_0}}} m(\widetilde{\pi}') b^G(\widetilde{\pi}'_\infty, \sigma_0, \lambda) \operatorname{trace} \widetilde{\pi}'^{\infty,\omega_0}(\phi^{\infty,\omega_0})$$

and

$$a^H(\pi_{\omega_0}, \lambda, \phi^{\omega_0}) = \sigma_0''(-1)\lambda''(-1)\iota(\widetilde{G}, H) F_{\phi_\infty}(\sigma_0, \lambda) \times \sum_{\substack{\pi' \subset L^2_{\mathrm{disc}} \\ \pi'_{\omega_0} = \pi_{\omega_0}}} m(\pi') b^H(\pi'_\infty, \sigma_0, \lambda) \operatorname{trace} \pi'^{\infty,\omega_0}(b(\phi^{\infty,\omega_0})).$$

In conclusion, we have deduced the necessary condition of the following equivalent criterion for the truth of spherical fundamental lemma.

Abstract Proposition *Spherical fundamental lemma holds* ⇔ (4.2) *holds for all pairs* (ϕ, f) *in* $\sum_{S'}$.

Proof We have already shown (⇒). For the converse, if the character identities $a(\lambda, \phi^{\omega_0}) F_{\phi_\infty}(\sigma_0, \lambda) = 0$ hold for all ϕ_{ω_0} and λ; then we have an equality on the spectral side of the trace formulas. The identity $T^{\widetilde{G}}_{\mathrm{ell}}(\phi) = \iota(\widetilde{G}, H) T^H_{\mathrm{ell}}(f)$ then holds. On the other hand, the choice of test function f_{v_3} gives rise to $T^H_{\mathrm{ell}}(f) = ST^H_{\mathrm{ell}}(f)$, and then $T^{\widetilde{G}}_{\mathrm{ell}}(\phi) = \iota(\widetilde{G}, H) ST^H_{\mathrm{ell}}(f)$, *i.e.*,

$$\sum_{\delta \in \Sigma_{\mathrm{rel}}(G(F^*))} \sum_{\kappa \in \pi_0(Z(G^\vee_\delta)^\Gamma)} O^\kappa_\delta(\phi) = \iota(\widetilde{G}, H) \tau(H) \sum_{\gamma \in \Sigma_{G-\mathrm{rel}}(H(F^*))} SO_\gamma(f) = \sum_{\gamma \in \Sigma_{G-\mathrm{rel}}(H(F^*))} SO_\gamma(f).$$

Thus, the identity $\Lambda(\gamma, \phi) = 0$ results from the induction assumption and the following choices of test functions:

• Let f^H_e be the function obtained from f by replacing f_{w_0} with the characteristic function of a compact set that meets all elliptic conjugacy classes in H_{w_0}. Shrink the support of the function f_v at some place v so that the only $H(\mathbb{A})$-conjugacy classes in $H(\mathbb{A})$ intersecting the support of f^H_e come from γ. This is possible because of Kottwitz's finiteness theorem [Kot86, 8.2]. The transfer of T to $\widetilde{G}$ gives a corresponding global element $\delta \in T(F^*) \subset G(F^*)$. Every $G(\mathbb{A})$-conjugacy class in $\widetilde{G}(\mathbb{A})$ that comes from a global element other than $\delta \in G(F^*)$ and that is elliptic at ω_0 has vanishing κ-orbital integrals at some place other than ω_0.

• Further arrange that the κ-orbital integrals of ϕ on δ are nonzero at all non-archimedean places except possibly ω_0. This is possible by the choices made above [Hal95, Lemma 5.1]. ■

4.3 Triviality

To prove the triviality of the obstruction *i.e.*, (4.2), we follow Hales's argument in [Hal95] using Howe's finiteness conjecture. From now on, we omit the subscript v that indicates the finite place v. By Theorem 3.1.3, for each spherical representation π

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

of H, there exists a genuine spherical representation $\widetilde{\pi}$ of G and a parameter $s \in T^\vee$ such that

$$\operatorname{trace} \pi(b(\phi)) = \operatorname{trace} \widetilde{\pi}(\phi) = (\chi_\psi \phi^\wedge)(s).$$

This allows us to rewrite the desired identity of the obstruction as $A(\phi) = 0$, where $A(\phi)$ is a finite sum of the form

$$A(\phi) = \sum_{s \in T^\vee(\mathbb{C})/W^G(T)} a(s)(\chi_\psi \phi^\wedge)(s)$$

for certain functions $a(s)$ on $T^\vee$.

Recall that $\operatorname{trace}_c \widetilde{\pi}(\phi)$ is the compact trace defined in Section 3.3. There are similar notions for π on H.

Proposition 4.3.1 ([Hal95]) *Assume the fundamental lemma holds for all proper Levi subgroups of $\widetilde{G}$ and the associated endoscopic groups obtained by descent from H; then the linear functional $\phi \mapsto A(\phi)$ on the anti-genuine Hecke algebra is a finite combination of the linear functionals $\Lambda(\gamma_H, \cdot)$, for $\gamma_H \in H(F)_{\widetilde{G}-\mathrm{reg}}$. The linear functional $A(\phi)$ is also a finite linear combination of linear functionals of the form*

$$\phi \longmapsto \operatorname{trace}_c \widetilde{\pi}(\phi) \quad \textit{and} \quad \phi \longmapsto \operatorname{trace}_c \pi(b(\phi)), \textit{ for } \widetilde{\pi} \textit{ and } \pi \textit{ spherical.}$$

Proof By hypothesis, we can assume that $\Lambda(\gamma_H, \phi) = 0$ for non-elliptic γ_H. Thus, the expansion to be produced in the proposition will only involve functionals $\Lambda(\gamma_H, \cdot)$ for γ_H elliptic.

For the first statement, applying Howe's finiteness conjecture (*cf.* [Luoar]) for $\gamma_H \in H(F)_{\widetilde{G}-\mathrm{reg}}$, the space of distributions $\phi \mapsto \Lambda(\gamma_H, \phi)$ on the anti-genuine spherical Hecke algebra of $\widetilde{G}$ is finite dimensional. Thus, the vanishing of $\Lambda(\gamma, \phi)$ for all $\gamma_H \in H(F)_{\widetilde{G}-\mathrm{reg}}$ can be replaced by the finitely many vanishing conditions, *i.e.,*

$$\Lambda(\gamma_j, \phi), \text{ for } j = 1, \ldots, k,$$

for an appropriate finite collection $\{\gamma_j\}$ of strongly $\widetilde{G}$-regular semisimple elements in H. When these vanishing conditions hold, the spherical fundamental lemma holds. Then by the implication $(\Rightarrow)$ in the Abstract Proposition, $A(\phi) = 0$. This means that the functional A is a linear combination of the functionals $\Lambda(\gamma_j, \cdot)$.

For the second statement, applying Howe's finiteness conjecture for H, we find that there is a finite set of tempered representations such that their compact traces form a basis for the span of $\{\operatorname{trace}_c \pi : \pi \text{ irreducible tempered}\}$ on $\mathcal{H}_{K_H}(H(F))$. Note that for $f \in \mathcal{H}_{K_H}(H(F))$, $\operatorname{trace}_c(f) = \operatorname{trace}(1_c f)$, and for any elliptic $\gamma \in H(F)_{\mathrm{reg}}$, $O_\gamma(1_c f) = O_\gamma(f)$. Then by Kazhdan's density theorem, $O_\gamma(f)$ has an expansion of the sort given in the proposition. Similarly for any elliptic $\delta \in G(F)$ and $\phi \in \mathcal{H}_K(\widetilde{G})_{--}$, $O_{\widetilde{\delta}}(\phi)$ has an expansion as given in the proposition, whence the proposition holds. ■

Now we can turn to the proof of the triviality of the obstruction. Actually, the two expressions in Proposition 4.3.1 will deduce a contradiction if the triviality does not hold. Note that orbital integrals are tempered distributions (*cf.* [Clo90, Lemma 5.5]), so the finite set of parameters s, for which $a(s) \neq 0$ consists of unitary parameters. This results from Lemma 3.1.5 as $A(\phi)$ has a finite expression of orbital integrals in

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

Proposition 4.3.1. Given this, we can sum up the triviality as an abstract lemma as follows, which is the key point of Clozel and Hales's arguments in [Clo90, Hal95].

Abstract Lemma *If $A: \mathcal{H}_K(\widetilde{G})_{--} \to \mathbb{C}$ is a linear functional that satisfies:*

(i) *A is a finite linear combination of characters $Tr(\widetilde{\pi})$ for $\widetilde{\pi}$ irreducible genuine tempered unramified representations,*

(ii) *A is a finite linear combination of compact traces $Tr_c(\widetilde{\pi})$ and $Tr_c(\pi) \circ b$ for $\widetilde{\pi}$ and π unramified representations,*

then A is zero.

Proof To prove this, we adapt Hales' argument as follows, which is almost the same as in [Hal95, p. 986]:

• (Unitary parameter) As tempered unramified representations are parameterized by unitary elements in $T^\vee(\mathbb{C})$ by Lemma 3.1.5, one can write $A(\phi)$ as a finite sum of the form

$$A(\phi) = \sum_{s \in T_u^\vee(\mathbb{C})/W^G(T)} a(s)(\chi_\psi \phi^\wedge)(s).$$

• (Nonunitary parameter) First, we can assume that each of the representation $\widetilde{\pi}$ and π in the second expression comes from a nonunitary parameter in the spectrum. This results from the following facts:

– A spherical tempered representation of $\widetilde{G}$ is a full induced unitary principal series [Szp13, Theorem C].

– A spherical tempered representation on the adjoint group H is a full induced unitary principal series [Key82].

– All spherical principal series representations have the same compact trace.

Second, let $c(\phi, \lambda)$, for $\lambda \in X^*(T^\vee)$, be the coefficients in

$$(\chi_\psi \phi^\wedge)(t) = \sum_\lambda c(\phi, t)\lambda(t).$$

Associated with $\lambda \in X^*(T^\vee)$, there is a anti-genuine spherical Hecke function ϕ_λ determined by the condition that $c(\phi_\lambda, \cdot)$ is the characteristic function of the $W^G(T)$-orbit of λ. The functions ϕ_λ form a linear basis of the anti-genuine spherical Hecke algebra. Note that for those compact traces parametrized by $\{\lambda\}$, we have the following lemma.

Lemma 4.3.2 *Fix $\widetilde{\pi}$ genuine unramified representation of $\widetilde{G}$. There exists a nonempty open cone $C \subset X^*(T^\vee)$, such that for any $\lambda \in C$ and ϕ_λ the associated spherical function as above, the compact trace $Tr_c(\widetilde{\pi})(\phi_\lambda)$ can be expressed as the form $\sum_j e_j \lambda(z_j)$ with the coefficients $\{e_j\}_j$ do not depend on λ.*

Proof By Clozel and Waldspurger's Theorem (Lemma 3.3.1), the compact trace is a linear combination of forms $(\widehat{\chi}_N \phi_\lambda^{(P)})^\wedge(z)$. The function $\widehat{\chi}_N$ factors as a composite of three maps

$$\widetilde{M}(F) \longrightarrow \mathfrak{a}_M \longrightarrow \mathfrak{a}_{M_0} \longrightarrow \mathbb{R},$$

where the first map is the Harish-Chandra map $H_M: \widetilde{M} \to \mathfrak{a}_M$, the second map is a natural identification of $\mathfrak{a}_M$ with a subspace of $\mathfrak{a}_{M_0}$ (*cf.* [Art78]), the third map is the

https://doi.org/10.4153/CJM-2017-013-2 Published online by Cambridge University Press

characteristic function $\widehat{\tau}_P^G$ of the *obtuse* Weyl chamber associated to P (*cf.* [Art78, p. 936]). In particular, for $\lambda \in X^*(T^\vee)$ and $m \in M(F)$, we have $\widehat{\chi}_N(m\varpi^\lambda m^{-1}) = \widehat{\tau}_P^G(\lambda)$, where we have identified $X^*(T^\vee)$ with a lattice in $\mathfrak{a}_{M_0}$. Denote by $B_M = TN_M$ the Borel subgroup of M, $B = TN_G$ the Borel subgroup of G and $P = MN_P$. Then

$$\begin{aligned}(\widehat{\chi}_N\phi_\lambda^{(P)})^\wedge(\widetilde{t}) &= \delta_{B_M}^{1/2}(t)\int_{N_M}\widehat{\chi}_N(\widetilde{t}n)\phi_\lambda^{(P)}(\widetilde{t}n)dn\\ &= \delta_{B_M}^{1/2}(t)\int_{N_M}\widehat{\chi}_N(\widetilde{t}n_1)\delta_P^{1/2}(t)\int_{N_P}\phi_\lambda(\widetilde{t}n_1n_2)dn_2dn_1\\ &= \delta_B^{1/2}(t)\int_{N_G}\widehat{\chi}_N(\widetilde{t})\phi_\lambda(\widetilde{t}n)dn.\end{aligned}$$

Further, based on the natural isomorphisms in Theorem 3.1.1, it is an easy calculation to deduce (*cf.* [Hal95, p. 986])

$$(\chi_\psi(\widehat{\chi}_N\phi_\lambda^{(P)})^\wedge)(z) = \sum_{w\in W^G(T)}\widehat{\tau}_P^G({}^w\lambda){}^w\lambda(z),$$

where ${}^w\lambda = w\cdot\lambda$. There are finitely many hyperplanes $X_1,\dots,X_r$ through the origin of $X^*(T^\vee)\otimes\mathbb{R}$ such that $\widehat{\tau}_P^G({}^w\lambda) = \widehat{\tau}_P^G({}^w\lambda')$ for all P and all $w \in W^G(T)$, whenever λ and λ' belong to the same component of $X^*(T^\vee)\otimes\mathbb{R}\backslash(X_1\cup\cdots\cup X_r)$. Fix such a component C. Then

$$(\chi_\psi(\widehat{\chi}_N\phi_\lambda^{(P)})^\wedge)(z) = \sum_{w\in W'}\lambda(w\cdot z)$$

for $\lambda \in C$, for some subset $W' \subset W^G(T)$ that depends on C, but not on $\lambda \in C$. ■

Returning to the proof of the Abstract Lemma, note that the terms $\mathrm{trace}_c\,\pi(b(\phi))$ can be treated similarly as in Lemma 4.3.2. Also notice that the transfer map b sends nonunitary parameters of H to nonunitary parameters of G. Then $\mathrm{Trace}_c\pi(b(\phi))$ may be expressed as a linear combination of terms $\lambda(z)$, again for lattice points λ in a suitable open cone of $X^*(T^\vee)$. By passing to a smaller open cone $C' \subset C$, if necessary, to accommodate the terms $\mathrm{trace}_c\,\pi(b(\phi))$, then $A(\phi_\lambda)$ has the form $\sum_j b_j\lambda(z_j)$ for $\lambda \in C'$.

By the above argument, the identity has two finite sum expressions as follows.

$$\sum_s a(s)(\chi_\psi\phi_\lambda^\wedge)(s) = \sum_i a_i\lambda(s_i) = \sum_j b_j\lambda(z_j),$$

for $\lambda \in C'$. We can assume these s_i, z_j are linear independent, *i.e.*, distinct. Thus, Lemma 4.3.3 says that $a_i = b_j = 0$ for all i, j, whence $A = 0$. ■

Lemma 4.3.3 (Hales lemma *cf.* [Hal95]) *Consider a function $B(\lambda) = \sum_{i=1}^r c_i\lambda(t_i)$ on $\lambda \in X^*(\mathbb{C}^{\times n}) = \mathbb{Z}^n$, with $c_1,\dots,c_r \in \mathbb{C}$ and distinct $t_1,\dots,t_r \in \mathbb{C}^{\times n}$. If $B(\lambda) = 0$ for all $\lambda \in C \cap X^*(\mathbb{C}^{\times n})$, where $C \subset X^*(\mathbb{C}^{\times n})\otimes\mathbb{R}$ is an open cone, then $c_i = 0$ for all i.*

Proof The proof results from the following two steps.

• Step 1: There exists a much smaller open cone $C' \subset C$ such that $\lambda(t_i) \neq \lambda(t_j)$ for any $1 \le i \neq j \le r$ and $\lambda \in C'$;

• Step 2: There exists an arithmetic sequence of lattice points in $C' \cap X^*(\mathbb{C}^{\times n})$ of length r, like $\{m(a_1,\dots,a_r) : m = 1,\dots,r\}$.

Note that the associated Vandermonde determinant is nonzero guaranteed by Step 1, we then know $(c_1, \ldots, c_r) = 0$. ∎

Acknowledgments We are much indebted to Professor Wee Teck Gan for his guidance and numerous discussions on various topics. Thanks are also due to the referee for his/her detailed comments.

References

[Art78] J. G. Arthur, *A trace formula for reductive groups. I. Terms associated to classes in* $G(\mathbb{Q})$. Duke Math. J. **45**(1978), no. 4, 911–952. http://dx.doi.org/10.1215/S0012-7094-78-04542-8

[Art88] ———, *The invariant trace formula. II. Global theory*. J. Amer. Math. Soc. **1**(1988), no. 3, 501–554. http://dx.doi.org/10.1090/S0894-0347-1988-0939691-8

[Art05] ———, *An introduction to the trace formula*. In: Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., 4, American Mathematical Society, Providence, RI, 2005, pp. 1–263.

[CD84] L. Clozel and P. Delorme, *Le théoreme de Paley-Wiener invariant pour les groupes de Lie réductifs*. Invent. Math. **77**(1984), no. 3, 427–453. http://dx.doi.org/10.1007/BF01388832

[Clo85] L. Clozel, *Sur une conjecture de Howe. I*. Compositio Math. **56**(1985), no. 1, 87–110.

[Clo89] ———, *Orbital integrals on p-adic groups: a proof of the Howe conjecture*. Ann. of Math. **129**(1989), no. 2, 237–251. http://dx.doi.org/10.2307/1971447

[Clo90] ———, *The fundamental lemma for stable base change*. Duke Math. J. **61**(1990), no. 1, 255–302. http://dx.doi.org/10.1215/S0012-7094-90-06112-5

[DG] M. Demazure and Alexandre Grothendieck, *Schémas en groupes, Tome III: Structure des schémas en groupes réductifs*. Lecture Notes in Mathematics, 153, Springer, New York, 1970, pp. 1962–1964.

[Duf75] M. Duflo, *Représentations irréductibles des groupes semi-simples complexes*. In: Analyse harmonique sur les groupes de Lie (Sém., Nancy-Strasbourg, 1973–75), Lectures Notes in Math., 497, Springer, Berlin, 1975, pp. 26–88.

[GGar] W. T. Gan and F. Gao, *The Langlands-Weissman program for Brylinski-Deligne extensions*. Asterisque, to appear.

[GS12] W. T. Gan and G. Savin, *Representations of metaplectic groups I: epsilon dichotomy and local Langlands correspondence*. Compos. Math. **148**(2012), no. 06, 1655–1694. http://dx.doi.org/10.1112/S0010437X12000486

[Hal95] T. C. Hales, *On the fundamental lemma for standard endoscopy: reduction to unit elements*. Canad. J. Math. **47**(1995), no. 5, 974–994. http://dx.doi.org/10.4153/CJM-1995-051-5

[HC59] Harish-Chandra, *Automorphic forms on a semisimple Lie group*. Proc. Nat. Acad. Sci. U. S. A. **45**(1959), 570–573. http://dx.doi.org/10.1073/pnas.45.4.570

[Key82] D. Keys, *Reducibility of unramified unitary principal series representations of p-adic groups and class-1 representations*. Math. Ann. **260**(1982), no. 4, 397–402. http://dx.doi.org/10.1007/BF01457019

[Kot86] R. E. Kottwitz, *Stable trace formula: elliptic singular terms*. Math. Ann. **275**(1986), no. 3, 365–399. http://dx.doi.org/10.1007/BF01458611

[KR00] R. E. Kottwitz and J. D. Rogawski, *The distributions in the invariant trace formula are supported on characters*. Canad. J. Math. **52**(2000), no. 4, 804–814. http://dx.doi.org/10.4153/CJM-2000-034-6

[Kud96] S. Kudla, *Notes on the local theta correspondence*. 1996. http://www.math.toronto.edu/skudla/castle.pdf

[Lab01] J.-P. Labesse, *Nombres de Tamagawa des groupes réductifs quasi-connexes*. Manuscripta Math. **104**(2001), no. 4, 407–430. http://dx.doi.org/10.1007/s002290170016

[Li11] W.-W. Li, *Transfert des intégrales orbitales pour le groupe métaplectique*. Compos. Math. **147**(2011), no. 2, 524–590. http://dx.doi.org/10.1112/S0010437X10004963

[Li12a] ———, *La formule des traces pour les revêtements de groupes réductifs connexes. II. Analyse harmonique locale*. Ann. Sci. Éc. Norm. Supér. **45**(2012), no. 5, 787–859. http://dx.doi.org/10.24033/asens.2178

[Li12b] ———, *Le lemme fondamental pondéré pour le groupe métaplectique*. Canad. J. Math. **64**(2012), no. 3, 497–543. http://dx.doi.org/10.4153/CJM-2011-088-1

[Li14a] ———, *La formule des traces pour les revêtements de groupes réductifs connexes. I. Le développement géométrique fin*. J. Reine Angew. Math. **686**(2014), 37–109. http://dx.doi.org/10.1515/crelle-2012-0015
[Li14b] ———, *La formule des traces pour les revêtements de groupes réductifs connexes. IV. Distributions invariantes*. Ann. Inst. Fourier (Grenoble) **64**(2014), no. 6, 2379–2448. http://dx.doi.org/10.5802/aif.2915
[Li15] ———, *La formule des traces stable pour le groupe métaplectique: les termes elliptiques*. Invent. Math. **202**(2015), no. 2, 743–838. http://dx.doi.org/10.1007/s00222-015-0577-9
[Li16] ———, *Spectral transfer for metaplectic groups. I. Local character relations*. J. Inst. Math. Jussieu, 1–99. http://dx.doi.org/10.1017/S1474748016000384
[LMW15] B. Lemaire, C. Mœglin, and J.-L. Waldspurger, *Le lemme fondamental pour l'endoscopie tordue: réduction aux éléments unités*. arxiv:1506.03383
[LW15] B. Lemaire and J.-L. Waldspurger, *Le lemme fondamental pour l'endoscopie tordue: le cas où le groupe endoscopique non ramifié est un tore*. arxiv:1511.08606
[LP81] G. Lion and P. Perrin, *Extension des representations de groupes unipotents p-adiques Calculs d'obstructions*. In: Noncommutative harmonic analysis and Lie groups (Marseille, 1980), Lecture Notes in Math., 880, Springer, Berlin-New York, 1981, pp. 337–356.
[Luoar] C. Luo, *Howe finiteness conjecture for covering groups*. Math. Res. Lett., to appear.
[Sil79] A. J. Silberger, *Introduction to harmonic analysis on reductive p-adic groups. Based on lectures by Harish-Chandra at The Institute for Advanced Study, 1971-73*. Mathematical Notes, 23, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1979.
[Szp13] D. Szpruch, *Some irreducibility theorems of parabolic induction on the metaplectic group via the Langlands-Shahidi method*. Israel J. Math. **195**(2013), no. 2, 897–971. http://dx.doi.org/10.1007/s11856-012-0140-y
[Tad83] M. Tadić, *Harmonic analysis of spherical functions on reductive groups over p-adic fields*. Pacific J. Math. **109**(1983), no. 1, 215–235. http://dx.doi.org/10.2140/pjm.1983.109.215
[Vig82] M.-F. Vignéras, *Caractérisation des intégrales orbitales sur un groupe réductif p-adique*. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28**(1982), 945–961.
[Wal03] J.-L. Waldspurger, *La formule de Plancherel pour les groupes p-adiques. D'apres Harish-Chandra*. J. Inst. Math. Jussieu **2**(2003), no. 2, 235–333. http://dx.doi.org/10.1017/S1474748003000082
[Wei14] M. H. Weissman, *Covers of tori over local and global fields*. Amer. J. Math. **138**(2016), no. 6, 1533–1573. http://dx.doi.org/10.1353/ajm.2016.0046
[Wei15] ———, *L-groups and parameters for covering groups*. arxiv:1507.01042

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076
e-mail: cluo@u.nus.edu