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DISCOUNTED OPTIMAL STOPPING PROBLEMS IN FIRST-PASSAGE TIME MODELS WITH RANDOM THRESHOLDS

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Abstract

We derive closed-form solutions to some discounted optimal stopping problems related to the perpetual American cancellable dividend-paying put and call option pricing problems in an extension of the Black–Merton–Scholes model. The cancellation times are assumed to occur when the underlying risky asset price process hits some unobservable random thresholds. The optimal stopping times are shown to be the first times at which the asset price reaches stochastic boundaries depending on the current values of its running maximum and minimum processes. The proof is based on the reduction of the original optimal stopping problems to the associated free-boundary problems and the solution of the latter problems by means of the smooth-fit and modified normal-reflection conditions. We show that the optimal stopping boundaries are characterised as the maximal and minimal solutions of certain first-order nonlinear ordinary differential equations.

Keywords: Discounted optimal stopping problem; geometric Brownian motion; first passage times; running maximum and minimum processes; free-boundary problem; smooth fit and normal reflection; a change-of-variable formula with local time on surfaces; perpetual American options; random dividends

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1. Introduction

The main aim of this paper is to present closed-form solutions to the discounted optimal stopping problems with the values

$$V_1^* = \sup_\tau \mathbb{E}\bigg[\mathrm{e}^{-r\tau} (K_1 - X_\tau)\, I(\tau < \theta_1) + \mathrm{e}^{-r\theta_1} (\alpha_1 + \beta_1 X_{\theta_1})\, I(\theta_1 \le \tau) + \frac{\nu_1}{r}\Big(1 - \mathrm{e}^{-r(\tau \wedge \theta_1)}\Big) \bigg] \tag{1.1}$$

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and

$$V_2^* = \sup_\tau \mathbb{E}\bigg[e^{-r\tau}(X_\tau - K_2)\, I(\tau < \theta_2) \\ - e^{-r\theta_2}(\alpha_2 + \beta_2 X_{\theta_2})\, I(\theta_2 \le \tau) + \frac{\nu_2}{r}\big(1 - e^{-r(\tau \wedge \theta_2)}\big)\bigg] \tag{1.2}$$

for some given constants $K_i > 0$, $\alpha_i \ge 0$, $\beta_i \ge 0$, and $\nu_i \ge 0$, for every $i = 1, 2$, where $I(\cdot)$ denotes the indicator function. Here, for a precise formulation of the problem, we consider a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with a standard Brownian motion $B = (B_t)_{t\ge 0}$ and strictly positive integrable random variables η and ξ, which have strictly increasing continuously differentiable cumulative distribution functions $F_i(x)$ such that $F_i(0) \equiv 1 - F_i(\infty) = 0$ and $0 < F_i(x) < 1$ as well as $F_i'(x) > 0$, for all $x > 0$ and $i = 1, 2$ (B and η or ξ are assumed to be independent under the probability measure $\mathbb{P}$). We assume that the process $X = (X_t)_{t\ge 0}$ is given by

$$X_t = x\, \exp\big((r - \delta - \sigma^2/2)t + \sigma B_t\big) \tag{1.3}$$

so that it solves the stochastic differential equation

$$dX_t = (r - \delta)\, X_t\, dt + \sigma\, X_t\, dB_t \quad (X_0 = x), \tag{1.4}$$

where $r > 0$, $\delta > 0$, and $\sigma > 0$ are some given constants and $x > 0$ is fixed. Suppose that the process X describes the price of a risky asset in a financial market, where r is the riskless interest rate, δ is the dividend rate paid to the asset holders, and σ is the volatility rate. Here K_i, for $i = 1, 2$, are the strike prices, $\alpha_1 + \beta_1 X$ is a (linear) recovery (in the put option case), and $\alpha_2 + \beta_2 X$ is a (linear) penalty (in the call option case), while ν_i, for $i = 1, 2$, are the promised rates of continuously paid dividends of certain contingent claims. We also define the random times θ_i, for $i = 1, 2$, by

$$\theta_1 = \inf\{t \ge 0 \mid X_t \ge \eta\} \quad \text{and} \quad \theta_2 = \inf\{t \ge 0 \mid X_t \le \xi\} \tag{1.5}$$

and assume that cancellations of these dividend-paying contingent claims are announced by the issuers of those products at these times, which are based on the market price of the underlying risky asset. In particular, these properties mean that the holders of such contingent claims may impose some prior (Bayesian) distribution on the unknown and, to them, unobservable (random) cancellation thresholds η and ξ chosen by the issuers. Note that European-type defaultable contingent claims with fixed finite-time horizon which have similar payoff and dividend structure were described in Bielecki and Rutkowski [7, Section 2.1] and Linetsky [29], among others (see also the related references therein).

Suppose that the suprema in (1.1) and (1.2) are taken over all stopping times τ with respect to the natural filtration $(\mathcal{F}_t)_{t\ge 0}$ of the process X, and the expectations there are taken with respect to the risk-neutral probability measure $\mathbb{P}$. In this view, the values V_1^* and V_2^* in (1.1) and (1.2) can be interpreted as the rational (or no-arbitrage) ex-dividend prices of the perpetual American cancellable dividend-paying put and call options in an extension of the Black–Merton–Scholes model (see e.g. [43, Chapter VII, Section 3g]). Observe that the structure of the reward functionals in (1.1) and (1.2) allows us to describe the associated contracts as standard game (or Israeli) contingent claims introduced by Kifer [25]. Such contacts enable their issuers to exercise their right to withdraw the contracts prematurely, by paying some penalties agreed in advance. Further developments of the Israeli options and the associated zero-sum

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optimal stopping (Dynkin) games were provided by Kyprianou [27], Kühn and Kyprianou [26], Kallsen and Kühn [24], Baurdoux and Kyprianou [3, 4, 5], Ekström and Villeneuve [12], and Baurdoux, Kyprianou, and Pardo [6], among others. In contrast to the concept of game contingent claims mentioned above, in the present paper we study the cancellable perpetual American options in which the exogenous terminations of the contracts occur at the first times when the underlying risky asset price processes reach certain random thresholds, which are unknown and unobservable to the holders of the claims. We assume that these thresholds are independent of the geometric Brownian motion describing the underlying risky asset price. Some extensive overviews of the perpetual American options in diffusion models of financial markets and other related results in the area are provided in Shiryaev [43, Chapter VIII, Section 2a], Peskir and Shiryaev [37, Chapter VII, Section 25], and Detemple [10], among others. Note that other applications of the concept described above include the consideration of perpetual American dividend-paying options with credit risk which are defaulted at the times when the underlying risky asset price processes reach such random thresholds. Other perpetual American defaultable and withdrawable dividend-paying options were recently considered in [14] and [15] in some other diffusion-type models of financial markets with full and partial information.

We further study the problems of (1.1) and (1.2) as the associated optimal stopping problems of (2.4) and (2.5) for the two-dimensional continuous Markov processes having the underlying risky asset price X and its running maximum S or minimum Q as their state space components. The resulting problems turn out to be necessarily two-dimensional in the sense that they cannot be reduced to optimal stopping problems for one-dimensional Markov processes. Note that the reward functionals of the optimal stopping problems in (2.4) and (2.5) contain complicated stochastic integrals with respect to the running maximum and minimum processes. This feature initiates further developments of techniques to determine the structure of the associated continuation and stopping regions as well as appropriate modifications of the normal-reflection conditions in the equivalent free-boundary problems. Discounted optimal stopping problems for the running maxima and minima of the initial continuous (diffusion-type) processes were initiated by Shepp and Shiryaev [40, 41, 42] and further developed by Pedersen [32], Guo and Shepp [22], Gapeev [13], Guo and Zervos [23], Peskir [35, 36], Glover, Hulley, and Peskir [20], Gapeev and Rodosthenous [16, 17, 18], Rodosthenous and Zervos [39], and Gapeev, Kort, and Lavrutich [19], among others. It was shown, by means of the maximality principle established by Peskir [33] for solutions of optimal stopping problems, which is equivalent to the superharmonic characterisation of payoff functions, that the optimal stopping boundaries are given by the appropriate extremal solutions of certain (systems of) first-order nonlinear ordinary differential equations. More complicated optimal stopping problems in models with spectrally negative Lévy processes and their running maxima were studied by Asmussen, Avram, and Pistorius [1], Avram, Kyprianou, and Pistorius [2], Ott [31], and Kyprianou and Ott [28], among others.

The rest of the paper is organised as follows. In Section 2 we embed the original problems of (1.1) and (1.2) into the optimal stopping problems of (2.4) and (2.5) for the two-dimensional continuous Markov processes (X, S) and (X, Q) defined in (1.3) and (2.1). It is shown that the optimal stopping times τ_1^* and τ_2^* are the first times at which the process X reaches some lower or upper boundaries $a^*(S)$ and $b^*(Q)$ depending on the current values of the processes S and Q, respectively. In Section 3 we derive closed-form expressions for the associated value functions $V_1^*(x, s)$ and $V_2^*(x, q)$ as solutions to the equivalent free-boundary problems, and apply the modified normal-reflection conditions at the edges of the two-dimensional state spaces for

(X, S) and (X, Q) to characterise the optimal stopping boundaries $a^*(S)$ and $b^*(Q)$ as the maximal and minimal solutions to the resulting first-order nonlinear ordinary differential equations, respectively. In Section 4, by using the change-of-variable formula with local time on surfaces from Peskir [34], we verify that the solutions of the free-boundary problems provide the solutions of the original optimal stopping problems. The main results of the paper are stated in Lemma 2.1 and Theorem 4.1.

2. Preliminaries

In this section we introduce the setting and notation of two-dimensional optimal stopping problems, which are related to the pricing of perpetual American cancellable dividend-paying put and call options. We then formulate the equivalent free-boundary problems.

2.1. The optimal stopping problems

Let us now define the *running maximum* and *minimum* processes $S = (S_t)_{t \geq 0}$ and $Q = (Q_t)_{t \geq 0}$, associated with X, by

$$S_t = s \vee \Big(\max_{0 \leq u \leq t} X_u \Big) \quad \text{and} \quad Q_t = q \wedge \Big(\min_{0 \leq u \leq t} X_u \Big) \tag{2.1}$$

for some arbitrary $s \geq x \geq q > 0$. Then the conditional probabilities of the events that cancellation occurs before any time $t \geq 0$ take the form

$$\mathbb{P}(\theta_1 \leq t \mid \mathcal{F}_t) = \mathbb{P}(S_t \geq \eta \mid \mathcal{F}_t) = F_1(S_t)$$

and

$$\mathbb{P}(\theta_2 \leq t \mid \mathcal{F}_t) = \mathbb{P}(Q_t \leq \xi \mid \mathcal{F}_t) = G_2(Q_t),$$

where $F_i(x)$, for $i = 1, 2$, are the cumulative distribution functions of η and ξ, respectively, and we set $G_i(x) = 1 - F_i(x)$, for all $x > 0$ and every $i = 1, 2$. Thus, by virtue of the assumptions made above, we have $G_i(0) = 1 - G_i(\infty) = 1$ and $0 < G_i(x) < 1$ as well as $G_i'(x) < 0$, for all $x > 0$ and every $i = 1, 2$. In this case the values of (1.1) and (1.2) admit the representations

$$\begin{aligned} V_1^* = \sup_\tau \mathbb{E}\bigg[& \mathrm{e}^{-r\tau}(K_1 - X_\tau) G_1(S_\tau) \\ & + \int_0^\tau \mathrm{e}^{-rt}(\alpha_1 + \beta_1 X_t)\, \mathrm{d}F_1(S_t) + \int_0^\tau \mathrm{e}^{-rt}\, \nu_1 G_1(S_t)\, \mathrm{d}t \bigg] \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} V_2^* = \sup_\tau \mathbb{E}\bigg[& \mathrm{e}^{-r\tau}(X_\tau - K_2) F_2(Q_\tau) \\ & - \int_0^\tau \mathrm{e}^{-rt}(\alpha_2 + \beta_2 X_t)\, \mathrm{d}G_2(Q_t) + \int_0^\tau \mathrm{e}^{-rt}\, \nu_2 F_2(Q_t)\, \mathrm{d}t \bigg], \end{aligned} \tag{2.3}$$

where the suprema are taken over all stopping times of τ with respect to $(\mathcal{F}_t)_{t \geq 0}$. In this case, taking into account the fact that the processes S and Q may change their values only when $X_t = S_t$ and $X_t = Q_t$, for $t \geq 0$, respectively, we see that the problems in (2.2) and (2.3) can be naturally embedded into the optimal stopping problems for the (time-homogeneous strong) Markov processes $(X, S) = (X_t, S_t)_{t\geq 0}$ and $(X, Q) = (X_t, Q_t)_{t\geq 0}$ with the value functions

$$V_1^*(x, s) = \sup_\tau \mathbb{E}_{x,s}\Big[e^{-r\tau}(K_1 - X_\tau)G_1(S_\tau) + \int_0^\tau e^{-rt}(\alpha_1 + \beta_1 S_t)\, F_1'(S_t)\, dS_t + \int_0^\tau e^{-rt}\, \nu_1\, G_1(S_t)\, dt\Big] \tag{2.4}$$

and

$$V_2^*(x, q) = \sup_\tau \mathbb{E}_{x,q}\Big[e^{-r\tau}(X_\tau - K_2)F_2(Q_\tau) - \int_0^\tau e^{-rt}(\alpha_2 + \beta_2\, Q_t)G_2'(Q_t)\, dQ_t + \int_0^\tau e^{-rt}\, \nu_2\, F_2(Q_t)\, dt\Big], \tag{2.5}$$

where $\mathbb{E}_{x,s}$ and $\mathbb{E}_{x,q}$ denote the expectations with respect to the probability measures $\mathbb{P}_{x,s}$ and $\mathbb{P}_{x,q}$ under which the two-dimensional Markov processes (X, S) and (X, Q) defined in (1.3) and (2.1) start at $(x, s) \in E_1 = \big\{(x, s) \in \mathbb{R}^2 \mid 0 < x \leq s\big\}$ and $(x, q) \in E_2 = \big\{(x, q) \in \mathbb{R}^2 \mid 0 < q \leq x\big\}$, respectively. It follows from the results of [9, Theorem 4.1] based on the solutions of the associated (doubly) reflected backward stochastic differential equations that the optimal stopping problems of (2.4) and (2.5) have values. We further obtain closed-form solutions to the optimal stopping problems in (2.4) and (2.5) and verify in Theorem 4.1 below that the value functions $V_1^*(x, s)$ and $V_2^*(x, q)$ are the solutions of the problems in (2.2) and (2.3), and thus of the original problems in (1.1) and (1.2) under $s = x$ and $q = x$, respectively.

2.2. The structure of optimal stopping times

Let us now determine the structure of the optimal stopping times at which the holders should exercise the contracts.

(i) By means of standard applications of Itô's formula (see e.g. [30, Theorem 4.4] or [38, Chapter II, Theorem 3.2]) to the processes $e^{-rt}(K_1 - X_t)G_1(S_t)$ and $e^{-rt}(X_t - K_2)F_2(Q_t)$, we obtain the representations

$$e^{-rt}(K_1 - X_t)\, G_1(S_t) = (K_1 - x)\, G_1(s) + N_t^1 + \int_0^t e^{-ru}\, (\delta\, X_u - r\, K_1)\, G_1(S_u)\, du + \int_0^t e^{-ru}(K_1 - X_u)\, I(X_u = S_u)\, G_1'(S_u)\, dS_u$$

and

$$e^{-rt}(X_t - K_2)\, F_2(Q_t) = (x - K_2)\, F_2(q) + N_t^2 + \int_0^t e^{-ru}\, (r\, K_2 - \delta\, X_u)\, F_2(Q_u)\, du + \int_0^t e^{-ru}(X_u - K_2)\, I(X_u = Q_u)\, F_2'(Q_u)\, dQ_u$$

for all $t \geq 0$. Here the processes $N^i = (N_t^i)_{t\geq 0}$, $i = 1, 2$, defined by

$$N_t^1 = -\int_0^t e^{-ru}\, \sigma\, X_u\, G_1(S_u)\, dB_u \quad \text{and} \quad N_t^2 = \int_0^t e^{-ru}\, \sigma\, X_u\, F_2(Q_u)\, dB_u,$$

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are continuous uniformly integrable martingales under the probability measures $\mathbb{P}_{x,s}$ and $\mathbb{P}_{x,q}$, for each $(x, s) \in E_1$ and $(x, q) \in E_2$, respectively. Then, by applying Doob's optional sampling theorem (see e.g. [30, Chapter III, Theorem 3.6] or [38, Chapter II, Theorem 3.2]), we obtain that the expected rewards from (2.4) and (2.5) admit the representations

$$\begin{aligned}
&\mathbb{E}_{x,s}\bigg[e^{-r\tau}(K_1 - X_\tau)\, G_1(S_\tau) \\
&\qquad + \int_0^\tau e^{-rt}\,(\alpha_1 + \beta_1\, S_t)\, F_1'(S_t)\, dS_t + \int_0^\tau e^{-rt}\, \nu_1\, G_1(S_t)\, dt\bigg] \\
&= (K_1 - x)\, G_1(s) + \mathbb{E}_{x,s}\bigg[\int_0^\tau e^{-rt}\,(\delta\, X_t - r\, K_1 + \nu_1)\, G_1(S_t)\, dt \\
&\qquad + \int_0^\tau e^{-rt}(K_1 - \alpha_1 - (1+\beta_1)\, S_t)\, I(X_t = S_t)\, G_1'(S_t)\, dS_t\bigg]
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
&\mathbb{E}_{x,q}\bigg[e^{-r\tau}(X_\tau - K_2)\, F_2(Q_\tau) \\
&\qquad - \int_0^\tau e^{-rt}\,(\alpha_2 + \beta_2\, Q_t)\, G_2'(Q_t)\, dQ_t + \int_0^\tau e^{-rt}\, \nu_2\, F_2(Q_t)\, dt\bigg] \\
&= (x - K_2)\, F_2(q) + \mathbb{E}_{x,q}\bigg[\int_0^\tau e^{-rt}\,(r\, K_2 + \nu_2 - \delta\, X_t)\, F_2(Q_t)\, dt \\
&\qquad + \int_0^\tau e^{-rt}\,((1+\beta_2)\, Q_t - K_2 + \alpha_2)\, I(X_t = Q_t)\, F_2'(Q_t)\, dQ_t\bigg]
\end{aligned} \tag{2.7}$$

for $(x, s) \in E_1$ and $(x, q) \in E_2$, for any stopping time τ of the process (X, S) or (X, Q), respectively. Observe from the structure of the integrands and the fact that $0 < G_i(x) < 1$ and $0 < F_i(x) < 1$, for all $x > 0$ and every $i = 1, 2$, that the expectations of the integrals in the second lines of the formulas in (2.6) and (2.7) are finite. Moreover, by virtue of the assumed integrability of the random variables η and ξ, it is seen that the expectations of the integrals in the third lines of the formulas in (2.6) and (2.7) are finite too.

We now recall the assumptions that $0 < F_i(x) < 1$ and $F_i'(x) > 0$, so that $0 < G_i(x) < 1$ and $G_i'(x) < 0$ holds, for all $x > 0$ and every $i = 1, 2$. Then, according to the properties that $0 < G_i(S_t) < 1$ and $G_i'(S_t) < 0$, for any $t \geq 0$ and every $i = 1, 2$, by virtue of the fact that the process S is positive and increasing, it is seen from the structure of the integrands in (2.6) that the optimal stopping time τ_1^* is infinite, whenever $K_1 \leq \nu_1/r$ holds. Furthermore, by virtue of the properties that $0 < G_i(S_t) < 1$ and $0 < F_i(Q_t) < 1$, for any $t \geq 0$ and every $i = 1, 2$, it follows from the structure of the first integrands in (2.6) and (2.7) that it is not optimal to exercise the cancellable put option when $\overline{a} \leq X_t < S_t$ with $\overline{a} = (rK_1 - \nu_1)/\delta$ under $K_1 > \nu_1/r$, while it is not optimal to exercise the cancellable call option when $Q_t < X_t \leq \underline{b}$ with $\underline{b} = (rK_2 + \nu_2)/\delta$, for any $t \geq 0$, respectively. In other words, these facts mean that the set $\{(x, s) \in E_1 \mid \overline{a} \leq x < s\}$ under $K_1 > \nu_1/r$ belongs to the continuation region C_1^* that has the form

$$C_1^* = \{(x, s) \in E_1 \mid V_1^*(x, s) > (K_1 - x)\, G_1(s)\}, \tag{2.8}$$

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while the set $\{(x, q) \in E_2 \mid q < x \le \underline{b}\}$ belongs to the continuation region C_2^* given by

$$C_2^* = \{(x, q) \in E_2 \mid V_2^*(x, q) > (x - K_2)\, F_2(q)\} \tag{2.9}$$

(see e.g. [37, Chapter I, Section 2.2]).

(ii) Note that by virtue of properties of the running maximum S and minimum Q from (2.1) of the geometric Brownian motion X from (1.3)–(1.4) (see e.g. [11, Section 3.3] for similar arguments applied to the running maxima of the Bessel processes), it is seen that, for any $s > 0$ and $q > 0$ fixed and an infinitesimally small deterministic time interval Δ, we have

$$S_\Delta = s \vee \max_{0 \le u \le \Delta} X_u = s \vee (s + \Delta X) + \mathrm{o}(\Delta) \quad \text{as } \Delta \downarrow 0$$

and

$$Q_\Delta = q \wedge \min_{0 \le u \le \Delta} X_u = q \wedge (q + \Delta X) + \mathrm{o}(\Delta) \quad \text{as } \Delta \downarrow 0,$$

where we set $\Delta X = X_\Delta - s$ and $\Delta X = X_\Delta - q$, respectively. Observe that $\Delta S = \mathrm{o}(\Delta)$ when $\Delta X \le 0$, $\Delta S = \Delta X + \mathrm{o}(\Delta)$ when $\Delta X > 0$, $\Delta Q = \mathrm{o}(\Delta)$ when $\Delta X \ge 0$, and $\Delta Q = \Delta X + \mathrm{o}(\Delta)$ when $\Delta X < 0$, where we set $\Delta S = S_\Delta - s$ and $\Delta Q = Q_\Delta - q$, and recall that $\mathrm{o}(\Delta)$ denotes a random function satisfying $\mathrm{o}(\Delta)/\Delta \to 0$ as $\Delta \downarrow 0$ ($\mathbb{P}$-a.s.). In this case, using the asymptotic formulas

$$\mathbb{E}_{s,s}[\Delta X;\, \Delta X > 0] \equiv \mathbb{E}_{s,s}[\Delta X\, I(\Delta X > 0)] \sim s\, \sigma \sqrt{\frac{\Delta}{2\pi}} \quad \text{as } \Delta \downarrow 0$$

and

$$\mathbb{E}_{q,q}[\Delta X;\, \Delta X < 0] \equiv \mathbb{E}_{q,q}[\Delta X\, I(\Delta X < 0)] \sim -q\, \sigma \sqrt{\frac{\Delta}{2\pi}} \quad \text{as } \Delta \downarrow 0,$$

as well as applying the representations in (2.6) and (2.7), we get

$$\begin{aligned} &\mathbb{E}_{s,s}\big[\mathrm{e}^{-r\Delta}\, (\delta\, s - r\, K_1 + \nu_1)\, G_1(s)\, \Delta + \mathrm{e}^{-r\Delta}\, (K_1 - \alpha_1 - (1 + \beta_1)\, s)\, G_1'(s)\, \Delta S\big] \\ &\quad \sim \mathrm{e}^{-r\Delta}\, (\delta\, s - r\, K_1 + \nu_1)\, G_1(s)\, \Delta + \mathrm{e}^{-r\Delta}\, (K_1 - \alpha_1 - (1 + \beta_1)\, s)\, G_1'(s)\, s\, \sigma \sqrt{\frac{\Delta}{2\pi}} \\ &\quad \text{as } \Delta \downarrow 0 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\mathbb{E}_{q,q}\big[\mathrm{e}^{-r\Delta}\, (r\, K_2 + \nu_2 - \delta\, q)\, F_2(q)\, \Delta + \mathrm{e}^{-r\Delta}\, ((1 + \beta_2)\, q - K_2 + \alpha_2)\, F_2'(q)\, \Delta Q\big] \\ &\quad \sim \mathrm{e}^{-r\Delta}\, (r\, K_2 + \nu_2 - \delta\, q)\, F_2(q)\, \Delta - \mathrm{e}^{-r\Delta}\, ((1 + \beta_2)\, q - K_2 + \alpha_2)\, F_2'(q)\, q\, \sigma \sqrt{\frac{\Delta}{2\pi}} \\ &\quad \text{as } \Delta \downarrow 0 \end{aligned} \tag{2.11}$$

for each $s > 0$ and $q > 0$ fixed. Since we have $G_i'(s) < 0$ and $F_i'(q) > 0$, for all $s > 0$, $q > 0$, and every $i = 1, 2$, we see that the resulting coefficients by the terms of order $\sqrt{\Delta}$ in (2.10) and (2.11) are strictly positive, when $s > s'$ with $s' = (K_1 - \alpha_1)/(1 + \beta_1)$ under $K_1 > \alpha_1$ (or when $s > 0$ under $K_1 \le \alpha_1$), and $q < q'$ with $q' = (K_2 - \alpha_2)/(1 + \beta_2)$ under $K_2 > \alpha_2$. Hence, taking

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into account the fact that the process S is positive and increasing and the process Q is positive and decreasing, by virtue of the properties that $G_i'(S_t) < 0$ and $F_i'(Q_t) > 0$, for any $t \geq 0$ and every $i = 1, 2$, we may therefore conclude from the structure of the second integrands in (2.6) and (2.7), as well as the heuristic arguments presented in (2.10) and (2.11) above, that it is not optimal to exercise the cancellable put option when $s' < S_t = X_t$ with $s' = (K_1 - \alpha_1)/(1 + \beta_1)$ under $K_1 > \alpha_1$ (or when $0 < S_t = X_t$ under $K_1 \leq \alpha_1$), while it is not optimal to exercise the cancellable call option when $X_t = Q_t < q'$ with $q' = (K_2 - \alpha_2)/(1 + \beta_2)$ under $K_2 > \alpha_2$, for any $t \geq 0$, respectively. In other words these facts mean that the sets $d_1' = \{(x, s) \in E_1 \mid x = s > s'\}$ under $K_1 > \alpha_1$ (which becomes the whole diagonal $d_1 = \{(x, s) \in E_1 \mid x = s\}$ under $K_1 \leq \alpha_1$) and $d_2' = \{(x, q) \in E_2 \mid x = q < q'\}$ under $K_2 > \alpha_2$ (which becomes an empty set under $K_2 \leq \alpha_2$) surely belong to the continuation regions C_1^* and C_2^* in (2.8) and (2.9) above. For simplicity of presentation, we further assume that the inequalities $K_1 > \alpha_1 \vee (\nu_1/r)$ and $K_2 > \alpha_2$ hold.

(iii) On the other hand, it follows from the definition of the processes (X, S) and (X, Q) in (1.3) and (2.1) and the structure of the rewards in (2.4) and (2.5) with the representations in (2.6) and (2.7) that for each $s > 0$ fixed there exists a sufficiently small $x > 0$ such that the point (x, s) belongs to the stopping region D_1^*, which has the form

$$D_1^* = \{(x, s) \in E_1 \mid V_1^*(x, s) = (K_1 - x)\, G_1(s)\}, \tag{2.12}$$

while for each $q > 0$ fixed there exists a sufficiently large $x > 0$ such that the point (x, q) belongs to the stopping region D_2^*, which is given by

$$D_2^* = \{(x, q) \in E_2 \mid V_2^*(x, q) = (x - K_2)\, F_2(q)\} \tag{2.13}$$

(see e.g. [37, Chapter I, Section 2.2]). According to arguments similar to those applied in [11, Section 3.3] and [33, Section 3.3], the latter properties can be explained by the fact that the costs of waiting until the process X coming from such a small $x > 0$ increases the current value of the running maximum process S, and that the costs of waiting until the process X coming from such a large $x > 0$ decreases the current value of the running minimum process Q may be too large, due to the presence of the discounting factor in the reward functionals of (2.4) and (2.5). It is seen from the results of Theorem 4.1 proved below that the value functions $V_1^*(x, s)$ and $V_2^*(x, q)$ are continuous, so that the sets C_1^* and C_2^* in (2.8) and (2.9) are open while the sets D_1^* and D_2^* in (2.12) and (2.13) are closed.

Observe that if we take some $(x, s) \in D_1^*$ from (2.12) and use the fact that the process (X, S) started at some (x_1, s) such that $x_1 < x$ passes through the point (x, s) before hitting the diagonal $d_1 = \{(x, s) \in E_1 \mid x = s\}$, then (2.4) and (2.6) imply

$$V_1^*(x_1, s) - (K_1 - x_1)G_1(s) \leq V_1^*(x, s) - (K_1 - x)G_1(s) = 0,$$

so that $(x_1, s) \in D_1^*$. Moreover, if we take some $(x, q) \in D_2^*$ from (2.13) and use the fact that the process (X, Q) started at some (x_2, q) such that $x_2 > x$ passes through the point (x, q) before hitting the diagonal $d_2 = \{(x, q) \in E_2 \mid x = q\}$, then (2.5) and (2.7) imply

$$V_2^*(x_2, q) - (x_2 - K_2)F_2(q) \leq V_2^*(x, q) - (x - K_2)F_2(q) = 0,$$

so that $(x_2, q) \in D_2^*$. On the other hand, if we take some $(x, s) \in C_1^*$ from (2.8) and use the fact that the process (X, S) started at (x, s) passes through some point (x_1', s) such that $x_1' > x$ before hitting the diagonal d_1, then (2.4) and (2.6) yield

$$V_1^*(x_1', s) - (K_1 - x_1')G_1(s) \geq V_1^*(x, s) - (K_1 - x)G_1(s) > 0,$$

so that $(x_1', s) \in C_1^*$. Moreover, if we take some $(x, q) \in C_2^*$ from (2.9) and use the fact that the process (X, Q) started at (x, q) passes through some point (x_2', q) such that $x_2' < x$ before hitting

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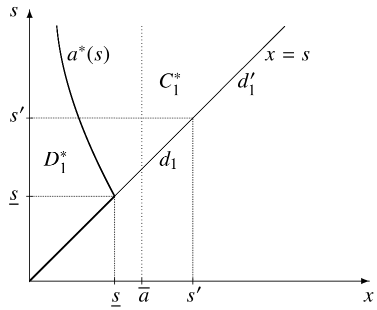

FIGURE 1. The optimal exercise boundary $a^*(s)$.

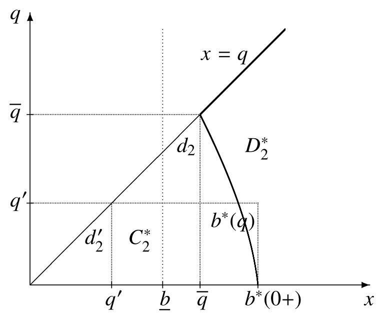

FIGURE 2. The optimal exercise boundary $b^*(q)$.

the diagonal d_2, then (2.5) and (2.7) yield

$$V_2^*(x_2', q) - (x_2' - K_2)F_2(q) \geq V_2^*(x, q) - (x - K_2)F_2(q) > 0,$$

so that $(x_2', q) \in C_2^*$. Hence, combining these arguments with the comments in [11, Section 3.3] and [33, Section 3.3] and recalling the fact that the sets $d_1' = \{(x, s) \in E_1 \mid x = s > s'\}$ and $d_2' = \{(x, q) \in E_2 \mid x = q < q'\}$ surely belong to the continuation regions C_1^* and C_2^* in (2.8) and (2.9), respectively, we may conclude that there exist functions $a^*(s)$ and $b^*(q)$ satisfying the inequalities $a^*(s) < s \wedge \overline{a}$ with $\overline{a} = (rK_1 - \nu_1)/\delta$ and $b^*(q) > q \vee \underline{b}$ with $\underline{b} = (rK_2 + \nu_2)/\delta$, for all $s > \underline{s}$ and $q < \overline{q}$ and some $0 \leq \underline{s} \leq s' \wedge \overline{a}$ and $\overline{q} \geq q' \vee \underline{b}$ fixed, as well as the equalities $a^*(s) = s$ and $b^*(q) = q$, for all $s \leq \underline{s}$ and $q \geq \overline{q}$, such that the continuation regions C_1^* and C_2^* in (2.8) and (2.9) have the form

$$C_1^* = \{(x, s) \in E_1 \mid a^*(s) < x \leq s\} \quad \text{and} \quad C_2^* = \{(x, q) \in E_2 \mid q \leq x < b^*(q)\}, \tag{2.14}$$

while the stopping regions D_1^* and D_2^* in (2.12)–(2.13) are given by

$$D_1^* = \{(x, s) \in E_1 \mid x \leq a^*(s)\} \quad \text{and} \quad D_2^* = \{(x, q) \in E_2 \mid x \geq b^*(q)\} \tag{2.15}$$

under $K_1 > \alpha_1 \vee (\nu_1/r)$ and $K_2 > \alpha_2$, respectively (see Figures 1 and 2 for depictions of the optimal exercise boundaries $a^*(s)$ and $b^*(q)$).

We summarise the arguments shown above in the following assertion.

Lemma 2.1. *Let the processes* (X, S) *and* (X, Q) *be given by* (1.3) *and* (2.1), *with some* $r > 0$, $\delta > 0$, *and* $\sigma > 0$ *fixed, and suppose the inequalities* $K_1 > \alpha_1 \vee (\nu_1/r)$ *and* $K_2 > \alpha_2$ *hold, for*

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some $\alpha_i \geq 0$, $\beta_i \geq 0$, and $\nu_i \geq 0$, for $i = 1, 2$ fixed. Suppose that the random times θ_i, for $i = 1, 2$, are defined in (1.5) for strictly positive continuous integrable random variables η and ξ with a strictly increasing continuously differentiable cumulative distribution function $F_i(x) \equiv 1 - G_i(x)$ such that $F_i(0) = 1 - F_i(\infty) = 0$ and $0 < F_i(x) < 1$ as well as $F_i'(x) > 0$, for all $x > 0$ and every $i = 1, 2$. Then the optimal stopping times in the problems of (2.4) and (2.5) have the structure

$$\tau_1^* = \inf\{t \geq 0 \mid X_t \leq a^*(S_t)\} \quad \text{and} \quad \tau_2^* = \inf\{t \geq 0 \mid X_t \geq b^*(Q_t)\} \tag{2.16}$$

for some functions $a^*(s)$ and $b^*(q)$ satisfying the inequalities $a^*(s) < s \wedge \overline{a}$ with $\overline{a} = (rK_1 - \nu_1)/\delta$ and $b^*(q) > q \vee \underline{b}$ with $\underline{b} = (rK_2 + \nu_2)/\delta$, for all $s > \underline{s}$ and $q < \overline{q}$ and some $0 \leq \underline{s} \leq s' \wedge \overline{a}$ and $\overline{q} \geq q' \vee \underline{b}$ fixed, where $s' = (K_1 - \alpha_1)/(1 + \beta_1)$ and $q' = (K_2 - \alpha_2)/(1 + \beta_2)$, as well as the equalities $a^*(s) = s$ and $b^*(q) = q$, for all $s \leq \underline{s}$ and $q \geq \overline{q}$, respectively.

2.3. The free-boundary problems

By means of standard arguments based on the application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}$ of the process (X, S) or (X, Q) from (1.4) and (2.1) takes the form

$$\mathbb{L} = (r - \delta)\, x\, \partial_x + \frac{\sigma^2 x^2}{2}\, \partial_{xx} \quad \text{in } 0 < x < s \text{ or } 0 < q < x$$

$$\partial_s = 0 \quad \text{at } 0 < x = s \quad \text{or} \quad \partial_q = 0 \quad \text{at } 0 < x = q$$

(see e.g. [33, Section 3.1]). In order to find analytic expressions for the unknown value functions $V_1^*(x, s)$ and $V_2^*(x, q)$ in (2.4) and (2.5) and the unknown boundaries $a^*(s)$ and $b^*(q)$ from (2.16), we apply the results of general theory for solving optimal stopping problems for Markov processes presented in [37, Chapter IV, Section 8], among others (see also [37, Chapter V, Sections 15–20] for optimal stopping problems for maxima processes and other related references). More precisely, for the original optimal stopping problems in (2.4) and (2.5), we formulate the associated free-boundary problems (see e.g. [37, Chapter IV, Section 8]) and then verify in Theorem 4.1 below that the appropriate candidate solutions of the latter problems coincide with the solutions of the original problems. In other words we reduce the optimal stopping problems of (2.4) and (2.5) to the following equivalent free-boundary problems:

$$(\mathbb{L}V_1 - rV_1)(x, s) = -\nu_1\, G_1(s) \quad \text{for } (x, s) \in C_1 \setminus \{(x, s) \in E_1 \mid x = s\}, \tag{2.17}$$

$$(\mathbb{L}V_2 - rV_2)(x, q) = -\nu_2\, F_2(q) \quad \text{for } (x, q) \in C_2 \setminus \{(x, q) \in E_2 \mid x = q\}, \tag{2.18}$$

$$V_1(x, s)\big|_{x=a(s)+} = (K_1 - a(s))\, G_1(s), \quad V_2(x, q)\big|_{x=b(q)-} = (b(q) - K_2)\, F_2(q), \tag{2.19}$$

$$\partial_x V_1(x, s)\big|_{x=a(s)+} = -G_1(s), \quad \partial_x V_2(x, q)\big|_{x=b(q)-} = F_2(q), \tag{2.20}$$

$$\partial_s V_1(x, s)\big|_{x=s-} = -(\alpha_1 + \beta_1\, s)\, F_1'(s), \quad \partial_q V_2(x, q)\big|_{x=q+} = (\alpha_2 + \beta_2\, q)\, G_2'(q), \tag{2.21}$$

$$V_1(x, s) = (K_1 - x)\, G_1(s) \quad \text{for } (x, s) \in D_1, \tag{2.22}$$

$$V_2(x, q) = (x - K_2)\, F_2(q) \quad \text{for } (x, q) \in D_2, \tag{2.23}$$

$$V_1(x, s) > (K_1 - x)\, G_1(s) \quad \text{for } (x, s) \in C_1, \tag{2.24}$$

$$V_2(x, q) > (x - K_2)\, F_2(q) \quad \text{for } (x, q) \in C_2, \tag{2.25}$$

$$(\mathbb{L}V_1 - rV_1)(x, s) < -\nu_1\, G_1(s) \quad \text{for } (x, s) \in D_1, \tag{2.26}$$

$$(\mathbb{L}V_2 - rV_2)(x, q) < -\nu_2\, F_2(q) \quad \text{for } (x, q) \in D_2, \tag{2.27}$$

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where C_i and D_i, $i = 1, 2$, are defined as C_i^* and D_i^*, $i = 1, 2$, in (2.14) and (2.15) with $a(s)$ and $b(q)$ instead of $a^*(s)$ and $b^*(q)$, respectively. Here the instantaneous stopping as well as the smooth-fit and modified normal-reflection conditions of (2.19)–(2.21) are satisfied, for all $s > \underline{s}$ and $q < \overline{q}$, respectively. Observe that the superharmonic characterisation of the value function (see e.g. [37, Chapter IV, Section 9]) implies that $V_1^*(x, s)$ and $V_2^*(x, q)$ are the smallest functions satisfying (2.17)–(2.19) and (2.22)–(2.25) with the boundaries $a^*(s)$ and $b^*(q)$, respectively. Note that (2.26)–(2.27) follow directly from the assertion of Lemma 2.1 proved in part (i) of Section 2.2 above.

3. Solutions to the free-boundary problems

In this section we obtain solutions to the free-boundary problems in (2.17)–(2.27) and derive first-order nonlinear ordinary differential equations for the candidate optimal stopping boundaries.

3.1. The candidate value functions

It is shown that the second-order ordinary differential equations in (2.17)–(2.18) have the general solutions

$$V_1(x, s) = C_{1,1}(s)\, x^{\gamma_1} + C_{1,2}(s)\, x^{\gamma_2} + \nu_1\, G_1(s)/r \tag{3.1}$$

and

$$V_2(x, q) = C_{2,1}(q)\, x^{\gamma_1} + C_{2,2}(q)\, x^{\gamma_2} + \nu_2\, F_2(q)/r,$$

where $C_{1,j}(s)$ and $C_{2,j}(q)$, $j = 1, 2$, are some arbitrary (continuously differentiable) functions, and γ_j, $j = 1, 2$, are given by

$$\gamma_j = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - (-1)^j \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}},$$

so that $\gamma_2 < 0 < 1 < \gamma_1$ holds. Then, by applying the conditions of (2.19)–(2.21) to the functions in (3.1), we obtain the equalities

$$C_{1,1}(s)\, a^{\gamma_1}(s) + C_{1,2}(s)\, a^{\gamma_2}(s) + \nu_1\, G_1(s)/r = (K_1 - a(s))\, G_1(s), \tag{3.2}$$
$$\gamma_1\, C_{1,1}(s)\, a^{\gamma_1}(s) + \gamma_2\, C_{1,2}(s)\, a^{\gamma_2}(s) = -a(s)\, G_1(s), \tag{3.3}$$
$$C'_{1,1}(s)\, s^{\gamma_1} + C'_{1,2}(s)\, s^{\gamma_2} + \nu_1\, G'_1(s)/r = -(\alpha_1 + \beta_1\, s)\, F'_1(s) \tag{3.4}$$

for all $s > \underline{s}$, and

$$C_{2,1}(q)\, b^{\gamma_1}(q) + C_{2,2}(q)\, b^{\gamma_2}(q) + \nu_2\, F_2(q)/r = (b(q) - K_2)\, F_2(q), \tag{3.5}$$
$$\gamma_1\, C_{2,1}(q)\, b^{\gamma_1}(q) + \gamma_2\, C_{2,2}(q)\, b^{\gamma_2}(q) = b(q)\, F_2(q), \tag{3.6}$$
$$C'_{2,1}(q)\, q^{\gamma_1} + C'_{2,2}(q)\, q^{\gamma_2} + \nu_2\, F'_2(q)/r = (\alpha_2 + \beta_2\, q)\, G'_2(q) \tag{3.7}$$

for all $q < \overline{q}$, respectively. Hence, by solving the systems of equations in (3.2)–(3.3) and (3.5)–(3.6), we obtain that the candidate value functions admit the representations

$$V_1(x, s; a(s)) = C_{1,1}(s; a(s))\, x^{\gamma_1} + C_{1,2}(s; a(s))\, x^{\gamma_2} + \nu_1\, G_1(s)/r \tag{3.8}$$

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for $a(s) < x \le s$, with

$$C_{1,j}(s; a(s)) = \frac{(\gamma_{3-j}(K_1 - \nu_1/r) - (\gamma_{3-j} - 1)a(s))G_1(s)}{(\gamma_{3-j} - \gamma_j)a^{\gamma_j}(s)} \tag{3.9}$$

for $j = 1, 2$, and

$$V_2(x, q; b(q)) = C_{2,1}(q; b(q))\, x^{\gamma_1} + C_{2,2}(q; b(q))\, x^{\gamma_2} + \nu_2\, F_2(q)/r \tag{3.10}$$

for $q \le x < b(q)$, with

$$C_{2,j}(q; b(q)) = \frac{((\gamma_{3-j} - 1)b(q) - \gamma_{3-j}(K_2 + \nu_2/r))F_2(q)}{(\gamma_{3-j} - \gamma_j)b^{\gamma_j}(q)} \tag{3.11}$$

for $j = 1, 2$, respectively. Moreover, by means of straightforward computations, it can be deduced from (3.8) and (3.10) that the first-order and second-order partial derivatives $\partial_x V_1(x, s; a(s))$ and $\partial_{xx} V_1(x, s; a(s))$ of the function $V_1(x, s; a(s))$ take the form

$$\partial_x V_1(x, s; a(s)) = C_{1,1}(s; a(s))\, \gamma_1\, x^{\gamma_1 - 1} + C_{1,2}(s; a(s))\, \gamma_2\, x^{\gamma_2 - 1}$$

and

$$\partial_{xx} V_1(x, s; a(s)) = C_{1,1}(s; a(s))\, \gamma_1(\gamma_1 - 1)\, x^{\gamma_1 - 2} + C_{1,2}(s; a(s))\, \gamma_2(\gamma_2 - 1)\, x^{\gamma_2 - 2}$$

on the interval $a(s) < x \le s$, for each $s > \underline{s}$, while the first-order and second-order partial derivatives $\partial_x V_2(x, q; b(q))$ and $\partial_{xx} V_2(x, q; b(q))$ of the function $V_2(x, q; b(q))$ take the form

$$\partial_x V_2(x, q; b(q)) = C_{2,1}(q; b(q))\, \gamma_1\, x^{\gamma_1 - 1} + C_{2,2}(q; b(q))\, \gamma_2\, x^{\gamma_2 - 1} \tag{3.12}$$

and

$$\partial_{xx} V_2(x, q; b(q)) = C_{2,1}(q; b(q))\, \gamma_1(\gamma_1 - 1)\, x^{\gamma_1 - 2} + C_{2,2}(q; b(q))\, \gamma_2(\gamma_2 - 1)\, x^{\gamma_2 - 2} \tag{3.13}$$

on the interval $q \le x < b(q)$, for each $q < \overline{q}$ fixed.

3.2. The candidate stopping boundaries

By applying the conditions of (3.4) and (3.7) to the functions in (3.9) and (3.11), we conclude that the candidate boundaries satisfy the first-order nonlinear ordinary differential equations

$$a'(s) = \frac{\Psi_{1,1}(s, a(s))s^{\gamma_1} + \Psi_{1,2}(s, a(s))s^{\gamma_2} + (\alpha_1 + \beta_1 s - \nu_1/r)G_1'(s)}{\Phi_{1,1}(s, a(s))s^{\gamma_1} + \Phi_{1,2}(s, a(s))s^{\gamma_2}} \tag{3.14}$$

for $s > \underline{s}$, and

$$b'(q) = \frac{\Psi_{2,1}(q, b(q))q^{\gamma_1} + \Psi_{2,2}(q, b(q))q^{\gamma_2} + (\alpha_2 + \beta_2 q + \nu_2/r)F_2'(q)}{\Phi_{2,1}(q, b(q))q^{\gamma_1} + \Phi_{2,2}(q, b(q))q^{\gamma_2}} \tag{3.15}$$

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for $q < \overline{q}$, respectively. Here the functions $\Phi_{1,j}(s, a(s))$, $\Psi_{1,j}(s, a(s))$ and $\Phi_{2,j}(q, b(q))$, $\Psi_{2,j}(q, b(q))$ are defined by

$$\Phi_{1,j}(s, a(s)) = \frac{((\gamma_1 - 1)(\gamma_2 - 1) - \gamma_1\gamma_2(K_1 - \nu_1/r)/a(s))G_1(s)}{(\gamma_{3-j} - \gamma_j)a^{\gamma_j}(s)}, \tag{3.16}$$

$$\Psi_{1,j}(s, a(s)) = \frac{((\gamma_{3-j} - 1)a(s) - \gamma_{3-j}(K_1 - \nu_1/r))G_1'(s)}{(\gamma_{3-j} - \gamma_j)a^{\gamma_j}(s)} \tag{3.17}$$

for $s > 0$, and

$$\Phi_{2,j}(q, b(q)) = \frac{((\gamma_1 - 1)(\gamma_2 - 1) - \gamma_1\gamma_2(K_2 + \nu_2/r)/b(q))F_2(q)}{(\gamma_{3-j} - \gamma_j)\, b^{\gamma_j}(q)}, \tag{3.18}$$

$$\Psi_{2,j}(q, b(q)) = \frac{((\gamma_{3-j} - 1)b(q) - \gamma_{3-j}(K_2 + \nu_2/r))F_2'(q)}{(\gamma_{3-j} - \gamma_j)b^{\gamma_j}(q)} \tag{3.19}$$

for $q > 0$ and every $j = 1, 2$. We have also used the obvious facts that $F_i'(s) = -G_i'(s)$, for all $s > 0$, and $G_i'(q) = -F_i'(q)$, for all $q > 0$, by virtue of the definition of the function $G_i(x) = 1 - F_i(x)$, for all $x > 0$, and every $i = 1, 2$.

3.3. The maximal and minimal admissible solutions $a^*(s)$ and $b^*(q)$

We further consider the *maximal and minimal admissible* solutions of first-order nonlinear ordinary differential equations as the largest and smallest possible solutions $a^*(s)$ and $b^*(q)$ of the equations in (3.14) and (3.15) with (3.16)–(3.17) and (3.18)–(3.19) which satisfy the inequalities $a^*(s) < s \wedge \overline{a}$ and $b^*(q) > q \vee \underline{b}$ with $\overline{a} = (rK_1 - \nu_1)/\delta$ and $\underline{b} = (rK_2 + \nu_2)/\delta$, for all $s > \underline{s}$ and $q < \overline{q}$ and some $0 \le \underline{s} \le s'$ and $\overline{q} \ge q'$ fixed, with $s' = (K_1 - \alpha_1)/(1 + \beta_1)$ under $K_1 > \alpha_1$ and $q' = (K_2 - \alpha_2)/(1 + \beta_2)$ under $K_2 > \alpha_2$, respectively. By virtue of the classical results on the existence and uniqueness of solutions for first-order nonlinear ordinary differential equations, we may conclude that these equations admit (locally) unique solutions, in view of the fact that the right-hand sides in (3.14) and (3.15) with (3.16)–(3.17) and (3.18)–(3.19) are (locally) continuous in $(s, a(s))$ and $(q, b(q))$ and (locally) Lipschitz in $a(s)$ and $b(q)$, for each $s > \underline{s}$ and $q < \overline{q}$ fixed (see also [33, Section 3.9] for similar arguments based on the analysis of other first-order nonlinear ordinary differential equations). Then it is shown by means of technical arguments based on Picard's method of successive approximations that there exist unique solutions $a(s)$ and $b(q)$ to the equations in (3.10) and (3.11) with (3.12)–(3.2) and (3.13)–(3.2), for $s > \underline{s}$ and $q < \overline{q}$, started at some points (s_0, s_0) and (q_0, q_0) such that $s_0 > \underline{s}$ and $q_0 < \overline{q}$ (see also [21, Section 3.2] and [33, Example 4.4] for similar arguments based on the analysis of other first-order nonlinear ordinary differential equations).

Hence, in order to construct the appropriate functions $a^*(s)$ and $b^*(q)$ which satisfy (3.14) and (3.15) and which stay strictly above and below the appropriate diagonal, for $s > \underline{s}$ and $q < \overline{q}$, respectively, we can follow the arguments from [36, Section 3.5] (among others). These are based on the construction of sequences of the so-called bad–good solutions that intersect the diagonals. For this purpose, for any sequences $(s_l)_{l\in\mathbb{N}}$ and $(q_l)_{l\in\mathbb{N}}$ such that $s_l > \underline{s}$ and $q_l < \overline{q}$ as well as $s_l \uparrow \infty$ and $q_l \downarrow 0$ as $l \to \infty$, we can construct the sequence of solutions $a_l(s)$ and $b_l(q)$, $l \in \mathbb{N}$, to the equations (3.14) and (3.15), for all $s > \underline{s}$ and $q < \overline{q}$ such that $a_l(s_l) = s_l$ and $b_l(q_l) = q_l$ hold, for each $l \in \mathbb{N}$. It follows from the structure of the equations in (3.14) and (3.15) as well as the functions in (3.16)–(3.17) and (3.18)–(3.19) that the properties $a_l'(s_l) < 1$ and $b_l'(q_l) < 1$ hold, for each $l \in \mathbb{N}$ (see also [32, pp. 979–982] for the analysis of solutions of another first-order nonlinear differential equation). Observe that by virtue of the uniqueness of solutions mentioned above, we know that each pair of curves $s \mapsto a_l(s)$ and $s \mapsto a_m(s)$ as well as $q \mapsto b_l(q)$ and $q \mapsto b_m(q)$ cannot intersect, for $l, m \in \mathbb{N}$, $l \neq m$, and thus we see that the sequence $(a_l(s))_{l\in\mathbb{N}}$ is increasing and the sequence $(b_l(q))_{l\in\mathbb{N}}$ is decreasing, so that the limits $a^*(s) = \lim_{l\to\infty} a_l(s)$ and $b^*(q) = \lim_{l\to\infty} b_l(q)$ exist, for each $s > \underline{s}$ and $q < \overline{q}$, respectively. We may therefore conclude that $a^*(s)$ and $b^*(q)$ provide the maximal and minimal solutions to the equations in (3.14) and (3.15) such that $a^*(s) < s \wedge \overline{a}$ and $b^*(q) > q \vee \underline{b}$ hold, for all $s > \underline{s}$ and $q < \overline{q}$.

Moreover, since the right-hand sides of the first-order nonlinear ordinary differential equations in (3.14) and (3.15) with (3.16)–(3.17) and (3.18)–(3.19) are (locally) Lipschitz in s and q, respectively, one can deduce by means of Gronwall's inequality that the functions $a_l(s)$ and $b_l(q)$, $l \in \mathbb{N}$, are continuous, so that the functions $a^*(s)$ and $b^*(q)$ are continuous too. The appropriate *maximal admissible* solutions of first-order nonlinear ordinary differential equations and the associated maximality principle for solutions of optimal stopping problems, which is equivalent to the superharmonic characterisation of the payoff functions, were established in [33] and further developed in [5], [13], [16, 17, 18], [19], [20], [21], [22], [23], [28], [31], [32], [35, 36], and [39], among other subsequent papers (see also [37, Chapter I; Chapter V, Section 17] for other references).

4. Main results and proofs

In this section, based on the expressions computed above, we formulate and prove the main results of the paper, which are based on the verification of the fact that the solution of the free-boundary problems in (2.17)–(2.27) provides the solutions of the optimal stopping problems of (2.4) and (2.5). Note that it also follows from the maximality principle established in [33, Theorem 3.1] (see also [33, Corollary 3.2] for the case of positive processes) that the solutions of the systems in (2.17)–(2.18) and (2.19)–(2.23) associated with the maximal and minimal admissible solutions of the first-order nonlinear ordinary differential equations in (3.14) and (3.15) satisfy (2.24)–(2.25) and (2.26)–(2.27). Recall that the existence of solutions of the optimal stopping problems in (2.4) and (2.5) follows from the results of [9, Theorem 4.1], based on the solutions of the associated (doubly) reflected backward stochastic differential equations.

Theorem 4.1. *Suppose that the assumptions of Lemma 2.1 are satisfied. Then the value functions of the perpetual American cancellable put and call option optimal stopping problems in (2.4) and (2.5) are given by*

$$V_1^*(x, s) = \begin{cases} V_1(x, s; a^*(s)) & \text{if } a^*(s) < x \leq s \text{ and } s > \underline{s}, \\ (K_1 - x)G_1(s) & \text{if } 0 < x \leq a^*(s) \text{ and } s > \underline{s}, \\ (K_1 - x)G_1(s) & \text{if } 0 < x \leq s \leq \underline{s}, \end{cases}$$

and

$$V_2^*(x, q) = \begin{cases} V_2(x, q; b^*(q)) & \text{if } q \leq x < b^*(q) \text{ and } 0 < q < \overline{q}, \\ (x - K_2)F_2(q) & \text{if } x \geq b^*(q) \text{ and } 0 < q < \overline{q}, \\ (x - K_2)F_2(q) & \text{if } x \geq q \geq \overline{q}, \end{cases} \tag{4.1}$$

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and the optimal exercise times have the form (2.16)*. Here the functions* $V_1(x, s; a(s))$ *and* $V_2(x, q; b(q))$ *are given by* (3.8) *and* (3.10) *with* (3.9) *and* (3.11)*, and the optimal exercise boundaries* $a^*(s)$ *and* $b^*(q)$ *provide the maximal and minimal solutions of the first-order nonlinear ordinary differential equations in* (3.14) *and* (3.15) *with* (3.16)–(3.17) *and* (3.18)–(3.19)*, satisfying the inequalities* $a^*(s) < s \wedge \overline{a}$ *with* $\overline{a} = (rK_1 - \nu_1)/\delta$ *and* $b^*(q) > q \vee \underline{b}$ *with* $\underline{b} = (rK_2 + \nu_2)/\delta$*, for all* $s > \underline{s}$ *and* $q < \overline{q}$ *and some* $0 \le \underline{s} \le s' \wedge \overline{a}$ *and* $\overline{q} \ge q' \vee \underline{b}$ *fixed, with* $s' = (K_1 - \alpha_1)/(1 + \beta_1)$ *and* $q' = (K_2 - \alpha_2)/(1 + \beta_2)$*. Further, the equalities* $a^*(s) = s$ *and* $b^*(q) = q$ *hold, for all* $s \le \underline{s}$ *and* $q \ge \overline{q}$*, respectively.*

Since the two assertions stated above are proved using similar arguments, we only give a proof for the case of the two-dimensional optimal stopping problem of (2.5) related to the dividend-paying perpetual American cancellable call option. Observe that we can put $s = x$ and $q = x$ to obtain the values of the original perpetual American cancellable option pricing problems of (2.2) and (2.3) from the values of the optimal stopping problems of (2.4) and (2.5).

Proof. In order to verify the assertion stated above, it remains for us to show that the function defined in (4.1) coincides with the value function in (2.5), and that the stopping time τ_2^* in (2.16) is optimal with the boundary $b^*(q)$ specified above. For this purpose, let $b(q)$ be any solution of the ordinary differential equation in (3.15) satisfying the inequality $b(q) > q \vee \underline{b}$, for all $q < \overline{q}$ and some $\overline{q} \ge q' \vee \underline{b}$ fixed. Also, let $V_2^b(x, q)$ denote the right-hand side of (4.1) associated with $b(q)$. Then it is shown by means of straightforward calculations from the previous section that the function $V_2^b(x, q)$ solves the system of (2.18) with the right-hand sides of (2.22)–(2.25) and (2.27) and satisfies the right-hand conditions of (2.19)–(2.21). Recall that the function $V_2^b(x, q)$ is $C^{2,1}$ on the closure $\overline{C}_2$ of C_2 and is equal to $(x - K_2)F_2(q)$ on D_2, which are defined as $\overline{C}_2^*$, C_2^* and D_2^* in (2.14) and (2.15) with $b(q)$ instead of $b^*(q)$, respectively. Hence, taking into account the assumption that the boundary $b(q)$ is continuously differentiable, for all $q < \overline{q}$, by applying the change-of-variable formula from [34, Theorem 3.1] to the process $\mathrm{e}^{-rt} V_2^b(X_t, Q_t)$ (see also [37, Chapter II, Section 3.5] for a summary of the related results and further references), we obtain

$$\begin{aligned} \mathrm{e}^{-rt}\, V_2^b(X_t, Q_t) &= V_2^b(x, q) \\ &+ \int_0^t \mathrm{e}^{-ru}\, \big(\mathbb{L}V_2^b - rV_2^b\big)(X_u, Q_u)\, I(X_u \neq b(Q_u), X_u \neq Q_u)\, \mathrm{d}u \\ &+ \int_0^t \mathrm{e}^{-ru}\, \partial_q V_2^b(X_u, Q_u)\, I(X_u = Q_u)\, \mathrm{d}Q_u + M_t^2 \end{aligned} \tag{4.2}$$

for all $t \ge 0$. Here the process $M^2 = (M_t^2)_{t \ge 0}$, defined by

$$M_t^2 = \int_0^t \mathrm{e}^{-ru}\, \partial_x V_2^b(X_u, Q_u)\, I(X_u \neq Q_u)\, \sigma\, X_u\, \mathrm{d}B_u, \tag{4.3}$$

is a continuous local martingale with respect to the probability measure $\mathbb{P}_{x,q}$. Note that since the time spent by the process (X, Q) at the boundary surface $\partial C_2 = \{(x, q) \in E_2 \mid x = b(q)\}$, as well as at the diagonal $d_2 = \{(x, q) \in E_2 \mid x = q\}$, is of the Lebesgue measure zero (see e.g. [8, Chapter II, Section 1]), the indicators in the first line of the formula in (4.2) as well as in (4.3) can be ignored. Moreover, since the component Q decreases only when the process

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(X, Q) is located on the diagonal $d_2 = \{(x, q) \in E_2 \mid x = q\}$, the indicator in the second line of (4.2) and the one in (4.3) can also be set equal to one. Observe that the integral in the second line of (4.2) will actually be compensated accordingly, due to the fact that the candidate value function $V_2^b(x, q)$ satisfies the modified normal-reflection condition from the right-hand side of (2.21) at the diagonal d_2.

It follows from straightforward calculations and the arguments of the previous section that the function $V_2^b(x, q)$ satisfies the second-order ordinary differential equation in (2.18) which, together with the conditions of (2.19)–(2.20) and (2.22) as well as the fact that inequality (2.27) holds, implies that the inequality $(\mathbb{L}V_2^b - rV_2^b)(x, q) \le -\nu_2 F_2(q)$ is satisfied, for all $0 < q < x$ such that $q < \overline{q}$ and $x \neq b(q)$. Moreover, we observe directly from (3.10) with (3.11) as well as (3.12) and (3.13) that the function $V_2^b(x, q) - (x - K_2)F_2(q)$ is convex and decreases to zero, because its first-order partial derivative $\partial_x V_2^b(x, q) - F_2(q)$ is negative and increases to zero, while its second-order partial derivative $\partial_{xx} V_2^b(x, q)$ is positive, on the interval $q \le x < b(q)$, for each $q < \overline{q}$ fixed. Thus we may conclude that the right-hand inequality in (2.25) holds, which together with the right-hand conditions of (2.19)–(2.20) and (2.22) implies that the inequality $V_2^b(x, q) \ge (x - K_2)F_2(q)$ is satisfied, for all $(x, q) \in E_2$. Let $(\varkappa_n)_{n\in\mathbb{N}}$ be the localising sequence of stopping times for the process M^2 from (4.3) such that $\varkappa_n = \inf\{t \ge 0 \mid |M_t^2| \ge n\}$, for each $n \in \mathbb{N}$. It therefore follows from (4.2) that we have the inequalities

$$
\begin{aligned}
&e^{-r(\tau\wedge\varkappa_n)}\,(X_{\tau\wedge\varkappa_n} - K_2)\,F_2(Q_{\tau\wedge\varkappa_n}) \\
&\quad - \int_0^{\tau\wedge\varkappa_n} e^{-ru}\,(\alpha_2 + \beta_2\,Q_u)\,G_2'(Q_u)\,dQ_u + \int_0^{\tau\wedge\varkappa_n} e^{-ru}\,\nu_2\,F_2(Q_u)\,du \\
&\le e^{-r(\tau\wedge\varkappa_n)}\,V_2(X_{\tau\wedge\varkappa_n}, Q_{\tau\wedge\varkappa_n}) \\
&\quad - \int_0^{\tau\wedge\varkappa_n} e^{-ru}\,(\alpha_2 + \beta_2\,Q_u)\,G_2'(Q_u)\,dQ_u + \int_0^{\tau\wedge\varkappa_n} e^{-ru}\,\nu_2\,F_2(Q_u)\,du \\
&\le V_2^b(x, q) + M^2_{\tau\wedge\varkappa_n}
\end{aligned}
\tag{4.4}
$$

for any stopping time τ of the process X and each $n \in \mathbb{N}$ fixed. Then, taking the expectation with respect to $\mathbb{P}_{x,q}$ in (4.4), by means of Doob's optional sampling theorem, we get

$$
\begin{aligned}
&\mathbb{E}_{x,q}\bigg[e^{-r(\tau\wedge\varkappa_n)}\,(X_{\tau\wedge\varkappa_n} - K_2)\,F_2(Q_{\tau\wedge\varkappa_n}) \\
&\quad - \int_0^{\tau\wedge\varkappa_n} e^{-ru}\,(\alpha_2 + \beta_2\,Q_u)\,G_2'(Q_u)\,dQ_u + \int_0^{\tau\wedge\varkappa_n} e^{-ru}\,\nu_2\,F_2(Q_u)\,du\bigg] \\
&\le \mathbb{E}_{x,q}\bigg[e^{-r(\tau\wedge\varkappa_n)}\,V_2^b(X_{\tau\wedge\varkappa_n}, Q_{\tau\wedge\varkappa_n}) \\
&\qquad - \int_0^{\tau\wedge\varkappa_n} e^{-ru}\,(\alpha_2 + \beta_2\,Q_u)\,G_2'(Q_u)\,dQ_u + \int_0^{\tau\wedge\varkappa_n} e^{-ru}\,\nu_2\,F_2(Q_u)\,du\bigg] \\
&\le V_2^b(x, q) + \mathbb{E}_{x,q}\big[M^2_{\tau\wedge\varkappa_n}\big] \\
&= V_2^b(x, q)
\end{aligned}
\tag{4.5}
$$

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for all $0 < q \le x$ such that $q < \overline{q}$, and each $n \in \mathbb{N}$. Hence, letting n go to infinity and using Fatou's lemma, we obtain from (4.5) the inequalities

$$\begin{aligned}
&\mathbb{E}_{x,q}\bigg[\mathrm{e}^{-r\tau}\,(X_\tau - K_2)\,F_2(Q_\tau) \\
&\qquad - \int_0^\tau \mathrm{e}^{-ru}\,(\alpha_2 + \beta_2\,Q_u)\,G_2'(Q_u)\,\mathrm{d}Q_u + \int_0^\tau \mathrm{e}^{-ru}\,\nu_2\,F_2(Q_u)\,\mathrm{d}u\bigg] \\
&\le \mathbb{E}_{x,q}\bigg[\mathrm{e}^{-r\tau}\,V_2^b(X_\tau, Q_\tau) \\
&\qquad - \int_0^\tau \mathrm{e}^{-ru}\,(\alpha_2 + \beta_2\,Q_u)\,G_2'(Q_u)\,\mathrm{d}Q_u + \int_0^\tau \mathrm{e}^{-ru}\,\nu_2\,F_2(Q_u)\,\mathrm{d}u\bigg] \\
&\le V_2^b(x, q)
\end{aligned} \tag{4.6}$$

for any stopping time τ, and all $0 < q \le x$ such that $q < \overline{q}$. Thus, taking the supremum over all stopping times τ and then the infimum over all boundaries b in (4.6), we obtain the inequalities

$$\begin{aligned}
&\sup_\tau \mathbb{E}_{x,q}\bigg[\mathrm{e}^{-r\tau}\,(X_\tau - K_2)\,F_2(Q_\tau) - \int_0^\tau \mathrm{e}^{-ru}\,(\alpha_2 + \beta_2\,Q_u)\,G_2'(Q_u)\,\mathrm{d}Q_u \\
&\qquad + \int_0^\tau \mathrm{e}^{-ru}\,\nu_2\,F_2(Q_u)\,\mathrm{d}u\bigg] \le \inf_b V_2^b(x, q) = V_2^{b^*}(x, q)
\end{aligned} \tag{4.7}$$

for all $0 < q \le x$ such that $q < \overline{q}$, where $b^*(q)$ is the minimal solution of the ordinary differential equation in (3.15) and satisfies the inequality $b^*(q) > q \vee \underline{b}$, for all $q < \overline{q}$. By using the fact that the function $V_2^b(x, q)$ is (strictly) increasing in the value $b(q)$, for each $q < \overline{q}$ fixed, we see that the infimum in (4.7) is attained over any sequence of solutions $(b_m(q))_{m\in\mathbb{N}}$ to (3.15) satisfying the inequality $b_m(q) > q \vee \underline{b}$, for all $q < \overline{q}$, for each $m \in \mathbb{N}$, and such that $b_m(q) \downarrow b^*(q)$ as $m \to \infty$, for each $q < \overline{q}$ fixed. It follows from the (local) uniqueness of the solutions to the first-order (nonlinear) ordinary differential equation in (3.15) that no distinct solutions intersect, so that the sequence $(b_m(q))_{m\in\mathbb{N}}$ is decreasing and the limit $b^*(q) = \lim_{m\to\infty} b_m(q)$ exists, for each $q < \overline{q}$ fixed. Since the inequalities in (4.6) hold for $b^*(q)$ too, we see that (4.7) holds for $b^*(q)$ and $(x, q) \in E_2$ as well. We also note from inequality (4.5) that the function $V_2^b(x, q)$ is superharmonic for the Markov process (X, Q) on E_2. Hence, taking into account the fact that $V_2^b(x, q)$ is increasing in $b(q) > q \vee \underline{b}$, for all $q < \overline{q}$, and the inequality $V_2^b(x, q) \ge (x - K_2)F_2(q)$ holds, for all $(x, q) \in E_2$, we observe that the selection of the minimal solution $b^*(q)$ that stays strictly above the diagonal $d_2 = \{(x, q) \in E_2 \mid x = q\}$ and the level $x = \underline{b}$ is equivalent to the implementation of the superharmonic characterisation of the value function as the smallest superharmonic function dominating the payoff function (see [33] or [37, Chapter I; Chapter V, Section 17]).

In order to prove the fact that the boundary $b^*(q)$ is optimal, we consider the sequence of stopping times τ_m, $m \in \mathbb{N}$, defined as in the right-hand part of (2.16) with $b_m(q)$ instead of $b^*(q)$, where $b_m(q)$ is a solution to the first-order ordinary differential equation in (3.15), and such that $b_m(q) \downarrow b^*(q)$ as $m \to \infty$, for each $q < \overline{q}$ fixed. Then, by virtue of the fact that the function $V_2^{b_m}(x, q)$ from the right-hand side of (4.1) associated with the boundary $b_m(q)$ satisfies the equation of (2.18) and the condition of (2.19), and taking into account the structure of τ_2^* in

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(2.16), it follows from an expression equivalent to (4.2) that we have

$$\begin{aligned}
&e^{-r(\tau_m\wedge\varkappa_n)}\,(X_{\tau_m\wedge\varkappa_n}-K_2)\,F_2(Q_{\tau_m\wedge\varkappa_n})\\
&\quad-\int_0^{\tau_m\wedge\varkappa_n} e^{-ru}\,(\alpha_2+\beta_2\,Q_u)\,G_2'(Q_u)\,dQ_u+\int_0^{\tau_m\wedge\varkappa_n} e^{-ru}\,\nu_2\,F_2(Q_u)\,du\\
&=e^{-r(\tau_m\wedge\varkappa_n)}\,V_2^{b_m}(X_{\tau_m\wedge\varkappa_n},Q_{\tau_m\wedge\varkappa_n})\\
&=-\int_0^{\tau_m\wedge\varkappa_n} e^{-ru}\,(\alpha_2+\beta_2\,Q_u)\,G_2'(Q_u)\,dQ_u+\int_0^{\tau_m\wedge\varkappa_n} e^{-ru}\,\nu_2\,F_2(Q_u)\,du\\
&=V_2^{b_m}(x,q)+M^2_{\tau_m\wedge\varkappa_n}
\end{aligned}\tag{4.8}$$

for all $0<q\le x$ such that $q<\overline{q}$ and each $n, m\in\mathbb{N}$. Observe that by virtue of the arguments from [43, Chapter VIII, Section 2a], we have the property

$$\begin{aligned}
&\mathbb{E}_{x,q}\Bigg[\sup_{t\ge0}\bigg(e^{-r(\tau_2^*\wedge t)}\,\big(X_{\tau_2^*\wedge t}-K_2\big)\,F_2\big(Q_{\tau_2^*\wedge t}\big)\\
&\quad-\int_0^{\tau_2^*\wedge t} e^{-ru}\,(\alpha_2+\beta_2\,Q_u)\,G_2'(Q_u)\,dQ_u+\int_0^{\tau_2^*\wedge t} e^{-ru}\,\nu_2\,F_2(Q_u)\,du\bigg)\Bigg]<\infty
\end{aligned}$$

for all $(x,q)\in E_2$, and the variable $e^{-r\tau_2^*}(X_{\tau_2^*}-K_2)F_2(Q_{\tau_2^*})$ is equal to zero on the event $\{\tau_2^*=\infty\}$ ($\mathbb{P}_{x,q}$-a.s.), because the value $b^*(0+)$ is finite. Hence, letting m and n go to infinity and using the condition of (2.19) together with the property $\tau_m\downarrow\tau_2^*$ ($\mathbb{P}_{x,q}$-a.s.) as $m\to\infty$, we can apply the Lebesgue-dominated convergence theorem to the appropriate (diagonal) subsequence in (4.8) to obtain the equality

$$\begin{aligned}
&\mathbb{E}_{x,q}\Bigg[e^{-r\tau_2^*}\,\big(X_{\tau_2^*}-K_2\big)\,F_2\big(Q_{\tau_2^*}\big)\\
&\quad-\int_0^{\tau_2^*} e^{-ru}\,(\alpha_2+\beta_2\,Q_u)\,G_2'(Q_u)\,dQ_u+\int_0^{\tau_2^*} e^{-ru}\,\nu_2\,F_2(Q_u)\,du\Bigg]=V_2^{b^*}(x,q)
\end{aligned}$$

for all $0<x\le q$ such that $q<\overline{q}$, which together with (4.7) directly implies the desired assertion. □

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References

[1] ASMUSSEN, S., AVRAM, F. AND PISTORIUS, M. (2003). Russian and American put options under exponential phase-type Lévy models. *Stoch. Process. Appl.* **109**, 79–111.
[2] AVRAM, F., KYPRIANOU, A. E. AND PISTORIUS, M. (2004). Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Ann. Appl. Prob.* **14**, 215–238.
[3] BAURDOUX, E. J. AND KYPRIANOU, A. E. (2004). Further calculations for Israeli options. *Stochastics* **76**, 549–569.
[4] BAURDOUX, E. J. AND KYPRIANOU, A. E. (2008). The McKean stochastic game driven by a spectrally negative Lévy process. *Electronic J. Prob.* **8**, 173–197.
[5] BAURDOUX, E. J. AND KYPRIANOU, A. E. (2009). The Shepp–Shiryaev stochastic game driven by a spectrally negative Lévy process. *Theory Prob. Appl.* **53**, 481–499.
[6] BAURDOUX, E. J., KYPRIANOU, A. E. AND PARDO, J. C. (2011). The Gapeev–Kühn stochastic game driven by a spectrally positive Lévy process. *Stoch. Process. Appl.* **121**, 1266–1289.
[7] BIELECKI, T. R. AND RUTKOWSKI, M. (2004). *Credit Risk: Modeling, Valuation and Hedging*, 2nd edn. Springer, Berlin.
[8] BORODIN, A. N. AND SALMINEN, P. (2002). *Handbook of Brownian Motion*, 2nd edn. Birkhäuser, Basel.
[9] CVITANIĆ, J. AND KARATZAS, I. (1996). Backward stochastic differential equations with reflection and Dynkin games. *Ann. Prob.* **24**, 2024–2056.
[10] DETEMPLE, J. (2006). *American-Style Derivatives: Valuation and Computation*. Chapman and Hall/CRC, Boca Raton.
[11] DUBINS, L., SHEPP, L. A. AND SHIRYAEV, A. N. (1993). Optimal stopping rules and maximal inequalities for Bessel processes. *Theory Prob. Appl.* **38**, 226–261.
[12] EKSTRÖM, E. AND VILLENEUVE, S. (2006). On the value of optimal stopping games. *Ann. Appl. Prob.* **16**, 1576–1596.
[13] GAPEEV, P. V. (2007). Discounted optimal stopping for maxima of some jump-diffusion processes. *J. Appl. Prob.* **44**, 713–731.
[14] GAPEEV, P. V. AND LI, L. (2022). Optimal stopping problems for maxima and minima in models with asymmetric information. *Stochastics* **94**, 602–628.
[15] GAPEEV, P. V. AND AL MOTAIRI, H. (2018). Perpetual American defaultable options in models with random dividends and partial information. *Risks* **6**, 127.
[16] GAPEEV, P. V. AND RODOSTHENOUS, N. (2014). Optimal stopping problems in diffusion-type models with running maxima and drawdowns. *J. Appl. Prob.* **51**, 799–817.
[17] GAPEEV, P. V. AND RODOSTHENOUS, N. (2016). On the drawdowns and drawups in diffusion-type models with running maxima and minima. *J. Math. Anal. Appl.* **434**, 413–431.
[18] GAPEEV, P. V. AND RODOSTHENOUS, N. (2016). Perpetual American options in diffusion-type models with running maxima and drawdowns. *Stoch. Process. Appl.* **126**, 2038–2061.
[19] GAPEEV, P. V., KORT, P. M. AND LAVRUTICH, M. N. (2021). Discounted optimal stopping problems for maxima of geometric Brownian motions with switching payoffs. *Adv. Appl. Prob.* **53**, 189–219.
[20] GLOVER, K., HULLEY, H. AND PESKIR, G. (2013). Three-dimensional Brownian motion and the golden ratio rule. *Ann. Appl. Prob.* **23**, 895–922.
[21] GRAVERSEN, S. E. AND PESKIR, G. (1998). Optimal stopping and maximal inequalities for geometric Brownian motion. *J. Appl. Prob.* **35**, 856–872.
[22] GUO, X. AND SHEPP, L. A. (2001). Some optimal stopping problems with nontrivial boundaries for pricing exotic options. *J. Appl. Prob.* **38**, 647–658.
[23] GUO, X. AND ZERVOS, M. (2010). π options. *Stoch. Process. Appl.* **120**, 1033–1059.
[24] KALLSEN, J. AND KÜHN, C. (2004). Pricing derivatives of American and game type in incomplete markets. *Finance Stoch.* **8**, 261–284.
[25] KIFER, Y. (2000). Game options. *Finance Stoch.* **4**, 443–463.
[26] KÜHN, C. AND KYPRIANOU, A. E. (2007). Callable puts as composite exotic options. *Math. Finance* **17**, 487–502.
[27] KYPRIANOU, A. E. (2004). Some calculations for Israeli options. *Finance Stoch.* **8**, 73–86.
[28] KYPRIANOU, A. E. AND OTT, C. (2014). A capped optimal stopping problem for the maximum process. *Acta Appl. Math.* **129**, 147–174.
[29] LINETSKY, V. (2006). Pricing equity derivatives subject to bankruptcy. *Math. Finance* **16**, 255–282.

[30] LIPTSER, R. S. AND SHIRYAEV, A. N. (2001). *Statistics of Random Processes I*, 2nd edn. Springer, Berlin.
[31] OTT, C. (2013). Optimal stopping problems for the maximum process with upper and lower caps. *Ann. Appl. Prob.* **23**, 2327–2356.
[32] PEDERSEN, J. L. (2000). Discounted optimal stopping problems for the maximum process. *J. Appl. Prob.* **37**, 972–983.
[33] PESKIR, G. (1998). Optimal stopping of the maximum process: the maximality principle. *Ann. Prob.* **26**, 1614–1640.
[34] PESKIR, G. (2007). A change-of-variable formula with local time on surfaces. In *Séminaire de Probabilités XL* (Lecture Notes Math. **1899**), pp. 69–96. Springer.
[35] PESKIR, G. (2012). Optimal detection of a hidden target: the median rule. *Stoch. Process. Appl.* **122**, 2249–2263.
[36] PESKIR, G. (2014). Quickest detection of a hidden target and extremal surfaces. *Ann. Appl. Prob.* **24**, 2340–2370.
[37] PESKIR, G. AND SHIRYAEV, A. N. (2006). *Optimal Stopping and Free-Boundary Problems*. Birkhäuser, Basel.
[38] REVUZ, D. AND YOR, M. (1999). *Continuous Martingales and Brownian Motion*. Springer, Berlin.
[39] RODOSTHENOUS, N. AND ZERVOS, M. (2017). Watermark options. *Finance Stoch.* **21**, 157–186.
[40] SHEPP, L. A. AND SHIRYAEV, A. N. (1993). The Russian option: reduced regret. *Ann. Appl. Prob.* **3**, 631–640.
[41] SHEPP, L. A. AND SHIRYAEV, A. N. (1994). A new look at the pricing of Russian options. *Theory Prob. Appl.* **39**, 103–119.
[42] SHEPP, L. A. AND SHIRYAEV, A. N. (1996). A dual Russian option for selling short. In *Probability Theory and Mathematical Statistics,* eds I. A. Ibragimov and A. Yu. Zaitsev, pp. 209–218. Gordon and Breach, Amsterdam.
[43] SHIRYAEV, A. N. (1999). *Essentials of Stochastic Finance*. World Scientific, Singapore.