Journal of the Inst. of Math. Jussieu (2007) **6**(2), 279–315 © Cambridge University Press 279
doi:10.1017/S1474748006000090 Printed in the United Kingdom

SOME EXTREMELY AMENABLE GROUPS RELATED TO OPERATOR ALGEBRAS AND ERGODIC THEORY

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(Received 4 June 2004; accepted 27 August 2005)

Abstract A topological group G is called extremely amenable if every continuous action of G on a compact space has a fixed point. This concept is linked with geometry of high dimensions (concentration of measure). We show that a von Neumann algebra is approximately finite dimensional if and only if its unitary group with the strong topology is the product of an extremely amenable group with a compact group, which strengthens a result by de la Harpe. As a consequence, a C^*-algebra A is nuclear if and only if the unitary group $U(A)$ with the relative weak topology is strongly amenable in the sense of Glasner. We prove that the group of automorphisms of a Lebesgue space with a non-atomic measure is extremely amenable with the weak topology and establish a similar result for groups of non-singular transformations. As a consequence, we prove extreme amenability of the groups of isometries of $L^p(0,1)$, $1 \leqslant p < \infty$, extending a classical result of Gromov and Milman ($p = 2$). We show that a measure class preserving equivalence relation $\mathcal{R}$ on a standard Borel space is amenable if and only if the full group $[\mathcal{R}]$, equipped with the uniform topology, is extremely amenable. Finally, we give natural examples of concentration to a non-trivial space in the sense of Gromov occurring in the automorphism groups of injective factors of type III.

Keywords: extremely amenable groups; Lévy groups; concentration of measure; unitary groups of von Neumann algebras; groups of measure space transformations; full group

AMS 2000 *Mathematics subject classification:* Primary 22A05; 22F50; 37A15; 43A07; 46L05; 46L10

1. Introduction

Extreme amenability [**26**] (or the fixed point on compacta property [**46**]) is a relatively recent concept, useful for study of ‘large’ (non-locally compact) topological groups. Recall that a topological group G is *amenable* [**12**] if every continuous action of G by affine transformations on a convex compact subset of a topological vector space has a fixed point. A topological group G is *extremely amenable* if every continuous action of G on a compact space has a fixed point.

This is a strong nonlinear fixed point property, never observed in locally compact groups [**27**, **60**]. Beginning with the discovery by Gromov and Milman [**30**] that the unitary group of an infinite-dimensional Hilbert space with the strong operator topology is extremely amenable, a large number of concrete large groups of importance are now known to have this property.

The concept is interesting for several reasons. Extreme amenability is closely linked to asymptotic geometric analysis, and in many examples it is a manifestation of the phenomenon of concentration of measure on high-dimensional structures, captured in the concept of a Lévy group [**19**, **25**, **30**]. Extreme amenability of groups of automorphisms of various combinatorial structures is intimately related to Ramsey-type theorems for those structures [**40**, **48**]. While surely not all infinite-dimensional groups are extremely amenable, the fixed point on compacta property of subgroups can be used to make conclusions about the dynamical properties of larger groups, for instance, to describe their universal minimal compact flows [**20**, **48**]. And the recent work by Glasner, Tsirelson and Weiss [**23**] and Glasner and Weiss [**22**] links Lévy groups to the problem of (non)existence of spatial models for near actions.

No non-trivial locally compact group has the fixed point on compacta property, which is thus exclusively a property of 'large' topological groups. This was shown by Granirer and Lau [**27**]. Later Veech [**60**] has proved his well-known theorem stating that every locally compact group acts freely on a compact space. (Now there are at least four known proofs of this result, see [**40**] and references therein.)

The first example of a non-trivial extremely amenable group has been constructed by Herer and Christensen [**35**]. Gromov and Milman [**30**] have shown that the unitary group of an infinite-dimensional Hilbert space, equipped with the strong operator topology, is extremely amenable. The idea of the proof, using concentration of measure on high-dimensional structures, has turned out to be applicable in a great many situations. For example, Glasner, and also (independently, unpublished) Furstenberg and Weiss, have shown that the group $L^0(\mathbb{I}, U(1))$ of measurable maps from the standard Lebesgue space to the circle rotation group $U(1)$, equipped with the topology of convergence in measure, is extremely amenable (see [**25**]). The group $U(1)$ here can be replaced by any amenable locally compact group [**50**].

A direct link between extreme amenability and Ramsey theory has been established by one of the present authors, who has proved in [**48**] extreme amenability of the group $\mathrm{Aut}(\mathbb{Q}, \leqslant)$ of order-preserving bijections of the rationals, equipped with the topology of pointwise convergence on $\mathbb{Q}$ viewed as discrete. This statement is a reformulation of the classical Finite Ramsey Theorem. The recent paper by Kechris, Pestov and Todorcevic [**40**] explores this trend extensively, by establishing extreme amenability of groups of automorphisms of numerous countable Fraïssé structures. It should be noted that extreme amenability of a topological group G can be restated in terms of the so-called Ramsey–Dvoretzky–Milman property [**28**, Section 9.3] of transitive isometric actions of G on metric spaces, which is in turn linked to both discrete Ramsey theory and Ramsey-type properties in geometric functional analysis [**51**].

The universal minimal flow $M(G)$ of a topological group G is, in the case where G is locally compact and non-compact, a highly non-constructive object (for instance, non-metrizable [**40**, Appendix 2]). Of course, if G is an extremely amenable group, then $M(G) = \{*\}$ is a singleton. It appears that the first instance where the flow $M(G)$, different from a point, was described explicitly, was the case where $G = \mathrm{Homeo}_+(\mathbb{S}^1)$ is the group of orientation-preserving homeomorphisms of the circle with the compact-open topology. In this situation, $M(G)$ is the circle $\mathbb{S}^1$ itself, equipped with the canonical action of G [**48**]. The proof was using the existence of a large extremely amenable subgroup, namely $\mathrm{Homeo}_+(\mathbb{I})$. Glasner and Weiss have subsequently described in [**20**] the universal minimal flow of the infinite symmetric group with its (unique) Polish topology as the space of linear orders on the natural numbers, and then in [**21**] the universal minimal flow of the group of homeomorphisms of the Cantor set, C, as the space of maximal chains of closed subsets of C (a construction proposed by Uspenskij [**59**]). Numerous other examples of explicit computations of universal minimal flows of groups of automorphisms of countable structures can be found in the above cited paper [**40**].

Overall, the papers [**51**] and [**40**] will together provide an introduction to, a survey of, and bibliographical references to most of what is known to date about extremely amenable groups. An account of basic ideas of the theory is provided in the set of lecture notes by one of the present authors [**53**].

In this article we will establish extreme amenability of some concrete topological groups of importance using the concentration of measure, as well as connect the property with known concepts from operator algebras and ergodic theory. We investigate extreme amenability of topological groups of two types: unitary groups of von Neumann algebras and group of transformations of spaces with measure.

It was proved by de la Harpe [**13**] that a von Neumann algebra M is approximately finite dimensional if and only if its unitary group $U(M)$, equipped with the relative $s(M, M_*)$-topology, is amenable. We strengthen this result by showing that this is the case if and only if $U(M)$ is the product of a compact group and an extremely amenable group. Paterson [**47**] has deduced from de la Harpe's result the following: a C^*-algebra A is nuclear if and only if the unitary group $U(A)$, equipped with the relative weak (that is, $\sigma(A, A^*)$) topology, is amenable. We describe nuclear C^*-algebras as those C^*-algebras A whose unitary group $U(A)$ with the relative weak topology is strongly amenable in the sense of Glasner [**24**].

Let now (X, ν) be a standard non-atomic Borel probability space. We show that the group $\mathrm{Aut}(X, \nu)$ of all measure-preserving automorphisms of (X, ν), equipped with the weak topology, is extremely amenable. This result is a consequence of the Rokhlin–Kakutani lemma and the concentration of measure on finite symmetric groups discovered by Maurey [**43**]. The group $\mathrm{Aut}(X, \nu)$, equipped with the uniform topology, is no longer extremely amenable. We also establish the extreme amenability of the group $\mathrm{Aut}^*(X, \nu)$ of measure class preserving transformations of (X, ν), equipped with the weak topology. As a consequence of this result and a description of groups of isometries of spaces $L^p(X, \mu)$ belonging to Banach [**1**] and Lamperti [**41**], we prove that those groups, equipped with the strong operator topology, are extremely amenable for all $1 \leqslant p < \infty$, extending Gromov and Milman's classical result [**30**] (corresponding to $p = 2$).

We consider measure class preserving equivalence relations $\mathcal{R}$ on the standard Lebesgue measure space and prove that such a relation is amenable in the sense of Zimmer [**61**, **62**] if and only if the full group $[\mathcal{R}]$, equipped with the uniform topology, is extremely amenable. In order to obtain this and the previous result, we generalize Maurey's theorem to automorphism groups of measure spaces with finitely many points.

The concentration of measure phenomenon can be interpreted in terms of convergence of a family of spaces with metric and measure (mm-spaces) to a one-point space with respect to a suitable metric, as was shown by Gromov [**29**]. This leads to a more general concept of concentration to a non-trivial space. In the concluding part of the article, we show some natural examples of concentration of compact subgroups and other subobjects of the groups of automorphisms of injective factors of type III to non-trivial spaces.

Some of the above results have been announced by the present authors in [**19**].

2. Concentration and extreme amenability

In this section we will outline a general scheme of deducing fixed point theorems from the phenomenon of concentration of measure, introduced by Gromov and Milman [**30**]. We work in a slightly more general context and prove some new results, in particular a concentration of measure result for automorphism groups of finite measure spaces.

2.1. Concentration of measure and Lévy families

For a subset $A \subseteq X$ of a separated uniform space $X = (X, \mathcal{U})$ let $V[A] = \{x \in X : \exists a \in A,\ (x,a) \in V\}$ be the V-neighbourhood of A. We say that a net (μ_α) of probability measures on X has the *Lévy concentration property*, or simply *concentrates*, if for every family of Borel subsets $A_\alpha \subseteq X$ satisfying $\liminf_\alpha \mu_\alpha(A_\alpha) > 0$ and every entourage $V \in \mathcal{U}_X$ one has $\mu_\alpha(V[A_\alpha]) \to_\alpha 1$.

A triple (X, d, μ), where d is a metric on a set X and μ is a probability Borel measure on the metric space (X, d), is called a *metric space with measure*, or else an *mm-space* [**29**]. An infinite family (X_n, d_n, μ_n) of mm-spaces is a *Lévy family* if, whenever $A_n \subseteq X_n$ are Borel subsets satisfying $\liminf_n \mu_n(A_n) > 0$, one has for each $\varepsilon > 0$, $\lim_{n\to\infty} \mu_n((A_n)_\varepsilon) = 1$, where A_ε denotes the ε-neighbourhood of a set A.

The above property is equivalent to concentration of the family (μ_n) considered as probability measures supported on the disjoint union $\bigoplus_{n=1}^\infty X_n$, equipped with a metric d inducing the metrics d_n on each X_n and making X_n into an open and closed subset.

The next lemma first appears in [**30**, 2.2]. For a proof, see Proposition 3.8 in [**42**]. (The case of uniform spaces clearly reduces to that of mm-spaces.)

Lemma 2.1. *Let a net of probability measures (μ_α) (respectively, (ν_α)) on an uniform space X (respectively, Y) concentrate. Then the net of product measures $(\mu_\alpha \otimes \nu_\alpha)$ concentrates on the product space $X \times Y$.* □

The *concentration function*, α_X, of an mm-space [**30**] is defined for $\varepsilon \geqslant 0$ by

$$\alpha_X(\varepsilon) = \begin{cases} \frac{1}{2}, & \text{if } \varepsilon = 0, \\ 1 - \inf\{\mu(A_\varepsilon) : A \subseteq X \text{ is Borel},\ \mu(A) \geqslant \frac{1}{2}\}, & \text{if } \varepsilon > 0. \end{cases}$$

Definition 2.2. A family (X_n, d_n, μ_n) of mm-spaces is a *normal Lévy family* if for some constants $C_1, C_2 > 0$

$$\alpha_{X_n}(\varepsilon) \leqslant C_1 \exp(-C_2 n \varepsilon^2).$$

The following is simple and well-known.

Lemma 2.3. *Let (Y, ν) be a probability measure space, and let d, ρ be two measurable metrics on Y such that the identity map*

$$(Y, d, \nu) \to (Y, \rho, \nu)$$

is Lipschitz with constant L. Denote by α_d the concentration function of the mm-space (Y, d, ν), and similarly by α_ρ the concentration function of the mm-space (Y, ρ, ν). Then, for every $\varepsilon > 0$,

$$\alpha_\rho(\varepsilon) \leqslant \alpha_d(L^{-1}\varepsilon).$$

Proof. Indeed, the $L^{-1}\varepsilon$-neighbourhood (with respect to d) of an arbitrary measurable subset $A \subseteq Y$ having measure greater than or equal to $\frac{1}{2}$ is contained in the ε-neighbourhood (with respect to ρ) of A, and therefore the measure of the latter is at least as large as the measure of the former. □

A family (X_n, d_n, μ_n) is a Lévy family if and only if the concentration functions α_n of X_n converge to zero pointwise as $n \to \infty$ on the interval $(0, +\infty)$.

A well-known manifestation of concentration of measure (see [**42**, **44**, **45**]) is that on a highly concentrated space every uniformly continuous function is 'nearly constant nearly everywhere'. More precisely, we have the following lemma.

Lemma 2.4. *Let a family (μ_β) of probability measures on a uniform space X be Lévy. Then for every bounded uniformly continuous function f on X there exists a net of constants (c_β) such that for every $\varepsilon > 0$,*

$$\mu_\beta\{x \in X : |f(x) - c_\beta| > \varepsilon\} \to_\beta 0. \tag{2.1}$$

Proof, sketch. One can choose as c_β median values for f with respect to μ_β, that is, numbers with the property that sets

$$M_\beta^-(f) = \{x \in X : f(x) \leqslant c_\beta\} \quad \text{and} \quad M_\beta^+(f) = \{x \in X : f(x) \geqslant c_\beta\}$$

both have μ_β-measure at least $\frac{1}{2}$. Let $V = V(\varepsilon) \in \mathcal{U}_X$ be such that $|f(x) - f(y)| < \varepsilon$ whenever $(x, y) \in V$. The set in equation (2.1) is contained in the intersection of complements to V-neighbourhoods of $M_\beta^\pm$. Now the upper bound is obtained by applying the definition of a Lévy family of measures. □

Here are a few examples of Lévy families.

Example 2.5. The family of special unitary groups $SU(n)$, $n \in \mathbb{N}$, equipped with the normalized Haar measure and the Hilbert–Schmidt metric is a Lévy family [**30**].

Example 2.6. Let $X = (X, \mu)$ be a probability measure space. Introduce on X^n the product measure $\mu^{\otimes n}$ and the normalized Hamming distance

$$d(x, y) = (1/n)|\{i : x_i \neq y_i\}|.$$

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The family X^n, $n \in \mathbb{N}$, is then a Lévy family, with the concentration function α_n of X^n satisfying

$$\alpha_n(\varepsilon) \leqslant 2\exp(-\varepsilon^2 n) \tag{2.2}$$

(see, for example, [**57**, Proposition 2.1.1]).

Here is an immediate consequence. Let $X_1, X_2, \dots$ be a sequence of probability measure spaces. Equip the n-fold product $Y_n = X_1 \times \cdots \times X_n$ with the product measure and the normalized Hamming distance. Then the family $(Y_n)_{n=1}^{\infty}$ is Lévy, with the same concentration functions as in equation (2.2). (Apply Example 2.6 and Lemma 2.3 to the Lévy family $\mathbb{I}^n$ with Lebesgue measures and the normalized Hamming distance.)

Example 2.7. Symmetric groups $\mathfrak{S}_n$ of rank n, equipped with the *normalized Hamming distance*

$$d_n(\sigma,\tau) := (1/n)|\{i : \sigma(i) \neq \tau(i)\}|$$

and the normalized counting measure

$$\mu(A) := |A|/n!$$

form a normal Lévy family, with the concentration functions satisfying the estimate

$$\alpha_{\mathfrak{S}_n}(\varepsilon) \leqslant \exp(-\varepsilon^2 n/32) \tag{2.3}$$

(see [**43**] and also [**45**, Theorem 7.5] and [**42**, Corollary 4.3]).

There are many known techniques for establishing concentration inequalities. The following approach, based on martingales, allows one to prove Examples 2.6 and 2.7, as well as to obtain some useful variations and generalizations.

Following Milman and Schechtman [**45**], say that a finite metric space (Ω, d) is *of length at most* ℓ if there exist positive numbers $a_1, a_2, \dots, a_n$ with

$$\ell = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$$

and a refining sequence $(\Omega^k)_{k=0}^n$, $\Omega^k = \{A_i^k\}_{i=1}^{m_k}$ of partitions of Ω such that $\Omega^0 = \{\Omega\}$, $\Omega^n = \{\{x\} : x \in \Omega\}$, and for every $k = 1, \dots, n$ and every two elements A, B of the partition Ω^k that are both contained in some element of Ω^{k-1} there exists a bijection $\varphi : A \to B$ with $d(x, \varphi(x)) \leqslant a_k$ for all $x \in A$.

The following is Theorem 7.8 in [**45**], and the present improved constants are taken from Theorem 4.2 in [**42**].

Theorem 2.8. *Let (Ω, d) be a finite metric space of length at most ℓ, equipped with the normalized counting measure. Then*

$$\alpha_\Omega(\varepsilon) \leqslant \exp\left(-\frac{\varepsilon^2}{8\ell^2}\right). \qquad \square$$

The following is Theorem 7.12 in [**45**], and (with the present, better constants) Theorem 4.4 in [**42**].

Theorem 2.9. *Let G be a compact group equipped with a bi-invariant metric d, and let $\{e\} = H_0 < H_1 < \cdots < H_n = G$ be a sequence of closed subgroups. Equip every factor-space H_{i+1}/H_i with the factor-distance of d, and let d_i denote the diameter of H_{i+1}/H_i. Then the concentration function of the mm-space (G, d, μ), where μ is the normalized Haar measure, satisfies*

$$\alpha_G(\varepsilon) \leqslant \exp\left(-\frac{\varepsilon^2}{8\sum_{i=0}^{n-1} {d_i}^2}\right). \qquad \square$$

2.2. Two new examples of Lévy families

Here we apply Theorems 2.8 and 2.9 to deduce two new natural examples of Lévy families, which we will use below.

Corollary 2.10. *Let (X, μ) be a probability measure space with finitely many points, and let (K, d) be a compact metric group of diameter 1. The concentration function of the compact metric group $L^1(X, \mu; K)$, equipped with the normalized Haar measure and the $L^1(\mu)$-metric*

$$d^\dagger(f, g) = \int_X d(f(x), g(x))\,\mathrm{d}\mu(x),$$

satisfies

$$\alpha(\varepsilon) \leqslant \exp\left(-\frac{\varepsilon^2}{8\sum_{x\in X}\mu(\{x\})^2}\right).$$

Proof. If X is equal to $\{x_1, x_2, \dots, x_n\}$, then let H_i consist of all functions vanishing on $x_{i+1}, x_{i+2}, \dots, x_n$. The group H_{i+1}/H_i is isometrically isomorphic to the group $(K, \mu(\{x_{i+1}\})d)$, whose diameter equals $\mu(\{x_{i+1}\})$, and the result follows by Theorem 2.9. $\square$

Corollary 2.11. *Let (X_n, μ_n) be a sequence of probability spaces with finitely many points in each, having the property that the mass of the largest atom in X_n goes to zero as $n \to \infty$. Let (K, d) be a compact metric group of diameter 1. Then the family of compact groups $L^1(X_n, \mu_n; K)$, equipped with the normalized Haar measures and the $L^1(\mu)$-metrics, is Lévy, with*

$$\alpha_{X_n}(\varepsilon) \leqslant \exp\left(-\frac{\varepsilon^2}{8\max_{x\in X_n}\mu_n(\{x\})}\right).$$

Proof. Let $v_n = (\mu_n(\{x\}))_{x\in X_n}$. By Hölder's inequality,

$$\sum_{x\in X_n}\mu_n(\{x\})^2 = \langle v_n, v_n\rangle \leqslant \|v_n\|_1\|v_n\|_\infty = \max_{x\in X_n}\mu_n(\{x\}) \to 0 \quad \text{as } n \to \infty.$$

Corollary 2.10 now applies. $\square$

Let $X = (X, \mu)$ be a measure space. Denote by $\mathrm{Aut}^*(X, \mu)$ the group of all invertible measurable and non-singular transformations of X. In particular, if X has finitely many

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points and the measure has full support, then $\mathrm{Aut}^*(X,\mu)$ is, as an abstract group, just the symmetric group of rank $n = |X|$.

Define a left-invariant metric on $\mathrm{Aut}^*(X,\mu)$ (the uniform metric) as follows:

$$d_{\mathrm{unif}}(\tau,\sigma) = \mu\{x \in X : \tau(x) \neq \sigma(x)\}. \tag{2.4}$$

For instance, if a finite set X is equipped with a uniform measure, the uniform metric on $\mathrm{Aut}^*(X,\mu) \cong \mathfrak{S}_n$ coincides with the Hamming distance. For every measure space, the uniform metric generates a group topology, known as the *uniform topology*, which will play an important role later (cf. § 4.1).

Theorem 2.12. *Let $X = (X,\mu)$ be a probability space with finitely many points. The concentration function α of the automorphism group $\mathrm{Aut}^*(X,\mu)$, equipped with the uniform metric (2.4) and the normalized counting measure, satisfies*

$$\alpha(\varepsilon) \leqslant \exp\biggl(-\frac{\varepsilon^2}{32\sum_{x\in X}\mu(\{x\})^2}\biggr). \tag{2.5}$$

Proof. Let $X = \{x_1, x_2, \dots, x_n\}$, where

$$\mu(\{x_1\}) \geqslant \mu(\{x_2\}) \geqslant \cdots \geqslant \mu(\{x_{n-1}\}) \geqslant \mu(\{x_n\}).$$

For $k = 0,1,\dots,n$, let H_k be the subgroup stabilizing each element $x_1,\dots,x_k$. Thus,

$$\mathrm{Aut}^*(X,\mu) = H_0 > H_1 > H_2 > \cdots > H_n = \{e\}.$$

Let Ω^k be the partition of $G = \mathrm{Aut}^*(X,\mu)$ into left H_k-cosets σH_k, $\sigma \in G$.

Suppose that $A = \sigma H_k$ and $B = \tau H_k$ are contained in the same left H_{k-1}-coset. Then $\sigma(x_i) = \tau(x_i) = a_i$ for $i = 1,2,\dots,k-1$, while $a = \sigma(x_k)$ and $b = \tau(x_k)$ need not coincide. Thus, elements π of A are defined by the conditions

$$\pi(x_1) = a_1, \quad \pi(x_2) = a_2, \quad \dots, \quad \pi(x_{k-1}) = a_{k-1}, \quad \pi(x_k) = a,$$

while elements $\pi \in B$ are defined by the conditions

$$\pi(x_1) = a_1, \quad \pi(x_2) = a_2, \quad \dots, \quad \pi(x_{k-1}) = a_{k-1}, \quad \pi(x_k) = b.$$

Let $t_{a,b}$ denote the transposition of a and b. Consider the map

$$\varphi : \sigma H_k \ni j \mapsto t_{a,b} \circ j \in \tau H_k. \tag{2.6}$$

Clearly, φ is a bijection between $A = \sigma H_k$ and $B = \tau H_k$. The values of j and $t_{a,b} \circ j$ differ in at most two inputs, $x_k = j^{-1}(a)$ and $j^{-1}(b)$. Since $a, b \notin \{a_1, a_2, \dots, a_{k-1}\}$, it follows that $j^{-1}(a), j^{-1}(b) \notin \{x_1, x_2, \dots, x_{k-1}\}$ and $\mu(\{j^{-1}(b)\}) \leqslant \mu\{(x_k\})$. We conclude that, for every $j \in A = \sigma H_k$,

$$d_{\mathrm{unif}}(j, \varphi(j)) \leqslant 2\mu(\{x_k\}).$$

Consequently, the metric space $\mathrm{Aut}^*(X,\mu)$ has length at most $\ell = 2(\sum_{i=1}^n \mu(\{x_i\}^2))^{1/2}$, and Theorem 2.8 accomplishes the proof. □

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Corollary 2.13. *Let (X_n,μ_n) be a sequence of probability spaces with finitely many points in each, having the property that the mass of the largest atom in X_n goes to zero as $n\to\infty$. Then the family of automorphism groups $\mathrm{Aut}^*(X_n,\mu_n)$, $n\in\mathbb{N}$, equipped with the uniform metric and the normalized counting measure, is Lévy, with*

$$\alpha_{X_n}(\varepsilon)\leqslant \exp\biggl(-\frac{\varepsilon^2}{32\max_{x\in X_n}\mu_n(\{x\})}\biggr).$$

Proof. Same as the proof of Corollary 2.11. □

2.3. Fixed points

If $X=(X,\mathcal{U})$ is a uniform space, then σX will denote the *Samuel*, or *universal, compactification* of X, that is, the space of maximal ideals of the C^*-algebra $\mathrm{UCB}(X)$ of all uniformly continuous bounded complex-valued functions on X (see [**17**]).

If a group G acts on X by uniform isomorphisms, this action uniquely extends to an action of G on σX by homeomorphisms.

The following result is essentially Theorem 2.12 in [**50**], to which we give a sketch of a different proof.

Theorem 2.14. *Let a group G act by uniform isomorphisms on a uniform space X. Suppose a net (ν_α) of probability measures on X concentrates and has the property: $(*)$ for every $g\in G$, every $A\subseteq X$ and every $V\in\mathcal{U}_X$,*

$$\limsup_\alpha |\nu_\alpha(V[A])-\nu_\alpha(g\cdot V[A])|<1.$$

Then the Samuel compactification σX has a G-fixed point.

Proof, sketch. The net (ϕ_α) of means on $C(\sigma X)$, defined by $\phi_\alpha(f)=\int_X f(x)\,\mathrm{d}\nu_\alpha(x)$, has a weak* cluster point, say ϕ. By selecting a subnet, assume that $\phi_\alpha\to\phi$ in weak* topology. As a consequence of Lemma 2.4, for any bounded uniformly continuous function f on X, the $L_1(\nu_\alpha)$-distance between f and a suitable constant function (for instance, $M_\alpha(f)$, a ν_α-median of f) goes to zero as $\alpha\to\infty$. Consequently, the $L_1(\nu_\alpha)$-distance between $f\cdot g$ and $M_\alpha(f)\cdot M_\alpha(g)$ tends to zero as well, and

$$\biggl|\int fg\,\mathrm{d}\nu_\alpha-\biggl(\int f\,\mathrm{d}\nu_\alpha\biggr)\biggl(\int g\,\mathrm{d}\nu_\alpha\biggr)\biggr|\to_\alpha 0,$$

for all $f,g\in C(\sigma X)$, implying that the mean ϕ is multiplicative.

The condition $(*)$, jointly with concentration of (ν_α), easily implies that whenever $\liminf_\alpha\nu_\alpha(A)>0$, $g\in G$, and $V\in\mathcal{U}_X$, one has $\limsup_\alpha\nu_\alpha(VA\cap gVA)=1$. Consequently, if $f\in C(\sigma X)$, there is a subnet (β) with $M_\beta(f)-M_\beta({}^gf)\to 0$, where ${}^gf(x)=f(g^{-1}x)$. We conclude that $\phi_\beta(f)-\phi_\beta({}^gf)\to_\beta 0$, and as $\phi_\beta(f)\to_\beta\phi(f)$, it follows that ϕ is a G-invariant mean. □

Let X be a topological group G equipped with the right uniform structure, which has as a basis the entourages of the diagonal of the form $V_R=\{(x,y)\in G\times G\mid xy^{-1}\in V\}$,

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where V is a neighbourhood of e in G. Notice that for a subset A of G, the uniform neighbourhood $V_R[A]$ is just the product VA.

The corresponding Samuel compactification, denoted by $S(G)$, is the *greatest ambit* of G, that is, the maximal compact G-space containing a distinguished point with an everywhere dense orbit. The existence of a G-fixed point in $S(G)$ is therefore equivalent to extreme amenability of G.

The condition $(*)$ is a weak form of invariance of the measures ν_α. In many examples, one can assume that the measures ν_α are virtually invariant, that is, for every $g \in G$, one has $g * \nu_\alpha = \nu_\alpha$ for all $\alpha \geqslant \alpha_0$. A topological group G is called a *Lévy group* [**25**, **30**] if it contains a family $\mathcal{K}$ of compact subgroups, directed by inclusion, having everywhere dense union in G, and such that the corresponding normalized Haar measures, μ_K, on the groups $K \in \mathcal{K}$ concentrate in G_R.

The presently known examples of Lévy groups are numerous. The one of the greatest importance for us is the original example essentially obtained in [**30**].

Example 2.15 (Gromov and Milman). The group of all unitary operators of the form $\mathbb{I} + C$, where C is a compact operator of Schatten class 2, equipped with the Hilbert–Schmidt metric, is a Lévy group.

It follows at once from the fact that the increasing chain of subgroups $SU(n)$, $n \in \mathbb{N}$, forms a Lévy family with regard to the Hilbert–Schmidt metric [**30**, Example 3.4], and its union is well-known to be dense in the operator group in question.

We expand the concept of a Lévy group as follows.

Definition 2.16. Let us say that a topological group G is a *generalized Lévy group* if there is a net $(K_\alpha)_{\alpha \in A}$ of compact subgroups of G with the following properties.

(1) The family of normalized Haar measures μ_α on K_α concentrates in G.

(2) For every non-empty open subset $V \subseteq G$ there is an $\alpha \in A$ such that $V \cap K_\beta \neq \emptyset$ for all $\beta \geqslant \alpha$.

Remark 2.17. The second condition is equivalent to the following, formally stronger, condition: for every finite collection $g_1, g_2, \dots, g_N \in G$, $N \in \mathbb{N}$, and for every neighbourhood V of the identity in G there is an $\alpha \in A$ such that for all $\beta \geqslant \alpha$ and $i = 1, 2, \dots, N$, one has $Vg_i \cap K_\beta \neq \emptyset$.

Theorem 2.18. *Every generalized Lévy group is extremely amenable.* □

Proof. In view of Theorem 2.14, it is enough to verify the condition $(*)$. Assume it fails. Then for some $g \in G$, some (non-empty) $A \subset G$, and some symmetric neighbourhood of identity, V, one can construct a subnet (K_β) of groups with the properties $\mu_\beta(V^2 A) \to_\beta 1$ and $\mu_\beta(gV^2 A) \to_\beta 0$.

Because of condition (2) in the definition of a generalized Lévy group, one can assume without loss in generality that for all β, $gV \cap K_\beta \neq \emptyset$. Choose a net $g'_\beta \in K_\beta$ with

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$g = g'_\beta v$, where $v \in V$. Since $vV^2A \supseteq VA$, one has

$$\nu_\beta(gV^2A) \geqslant \nu_\beta(g'_\beta VA) = \nu_\beta(VA) \to_\beta 1,$$

meaning that $\nu_\beta(gV^2A) \to_\beta 1$ as well: a contradiction. □

Corollary 2.19 (Gromov and Milman). *Every Lévy group is extremely amenable.* □

To deduce Corollary 2.19 directly from Theorem 2.14, it is enough to note the extreme amenability of the everywhere dense subgroup $\cup\mathcal{K}$ of G, since both topological groups share the same greatest ambit.

There are still situations where one needs the condition $(*)$ in its full generality. For instance, this is how one proves the following Theorem 2.20.

Let G be a Hausdorff topological group and let (X, μ) be a Lebesgue space. Denote by $L(X, \mu; G)$, or simply $L(X, G)$ the group of equivalence classes modulo μ of all (strongly) measurable maps (cf. [**15**, 8.14.1(b)]) from (X, μ) to G, equipped with pointwise multiplication. The *topology of convergence in measure* on $L(X, G)$ has, as basic neighbourhoods of identity sets of the form

$$[V, \varepsilon] := \{g \in L(X, G) : \mu\{x \in X : g(x) \notin V\} < \varepsilon\},$$

where V is a neighbourhood of e_G in G and $\varepsilon > 0$. The topology of convergence in measure is a Hausdorff group topology on $L(X, G)$, and we will denote the resulting topological group by $L^0(X, \mu; G)$ or $L(X, G)$.

Theorem 2.20. *Let G be an amenable locally compact group and let X be a non-atomic Lebesgue measure space. Then the group $L^0(X, G)$ of all measurable maps from X to G, equipped with the topology of convergence in measure, is extremely amenable.*

Proof, sketch. By force of amenability of G, there is an invariant probability measure, ν, on the greatest ambit $S(G)$ of G. Choose a net (ν_α) of probability measures on G (for instance, with finite support), weak* converging to ν. For every measurable partition $\mathcal{W}$ of X define a measure $\nu_{\alpha,\mathcal{W}}$ on $L^0(X, G)$, supported on the subgroup of all $\mathcal{W}$-simple maps $X \to G$ (identified with the product $G^{|\mathcal{W}|}$), as the product measure, $\nu_{\alpha,\mathcal{W}} = \nu_\alpha^{\otimes|\mathcal{W}|}$. The net $(\nu_{\alpha,\mathcal{W}})$ has the desired properties: concentration follows, for example, from [**57**, p. 76 and Proposition 2.1.1], while the condition $(*)$ is checked directly. □

In the case where G is a compact group, the theorem was established by Glasner [**25**] and (independently, unpublished) Furstenberg and Weiss. The above version is Theorem 2.2 in [**50**]. It would be interesting to know if the result remains true for an arbitrary (non-locally compact) amenable topological group G. (Here of course one needs to replace measurable maps with either Borel measurable ones, or else strongly measurable, that is, measurable in the sense of Bourbaki (cf. [**18**, p. 357]).)

Remark 2.21. Notice that, for $G = \mathbb{R}$, the topological group $L^0(X) = L^0(X, \mathbb{R})$ is extremely amenable, but not a Lévy group, nor indeed a generalized Lévy group in the sense of our Definition 2.16, because it contains no non-trivial compact subgroups.

Let us finally state a simple and useful fact that belongs to the folklore.

Lemma 2.22. *Let G be a topological group, and let $\mathcal{G}$ be a family of topological subgroups of G with the property: if $H, K \in \mathcal{G}$, then $H \cup K \subseteq L$ for some $L \in \mathcal{G}$. Suppose that $\cup\{G : G \in \mathcal{G}\}$ is everywhere dense in G and that every $H \in \mathcal{G}$ is contained in a suitable extremely amenable subgroup of G. Then G is extremely amenable.*

Proof. Let G act continuously on a compact space X. For every $H \in \mathcal{G}$ denote by F_H the set of all H-fixed points in X. As H is contained in a suitable extremely amenable subgroup of G, F_H is non-empty and compact. As $\mathcal{G}$ is upward directed, the family $\{F_H : H \in \mathcal{G}\}$ is centred, that is, for every finite subset $H_1, \dots, H_k \in \mathcal{G}$, one has $\bigcap_{i=1}^n F_{H_i} \neq \varnothing$. Hence, $\bigcap_{H\in\mathcal{G}} F_H \neq \varnothing$, and every element of this intersection is a fixed point for G. □

Corollary 2.23. *Let G be a topological group containing a family of locally compact amenable subgroups directed by inclusion and having an everywhere dense union. Let X be a non-atomic Lebesgue measure space. Then the group $L^0(X, G)$ of all (equivalence classes) of Borel measurable maps from X to G, equipped with the topology of convergence in measure, is extremely amenable.*

Proof. Let $\mathcal{K}$ be a family of locally compact amenable subgroups of G as in the assumptions of the theorem. The group $L^0(X, \cup\mathcal{K})$ is everywhere dense in $L^0(X, G)$, and it is extremely amenable by Lemma 2.22, because each of the topological subgroups of the form $L^0(X, K)$, $K \in \mathcal{K}$ is extremely amenable by Theorem 2.20. □

3. Unitary groups of operator algebras

3.1. A characterization of approximately finite-dimensional von Neumann algebras

Let $\mathcal{H}$ be a Hilbert space, let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and let M_* be the predual of M. Recall that the *strong topology* on M, or the $s(M, M_*)$-topology, is the topology on M induced by the family of semi-norms

$$x \in M \mapsto \|x\|_\varphi = \varphi(x^*x)^{1/2}, \quad \varphi \in M_*^+,$$

and the *weak topology* on M is the $\sigma(M, M_*)$-topology. Then (see, for example, [**55**, Remark II.4.10]), the weak and the strong topologies coincide on the unitary group of M.

Let us mention two technical results that we will need.

Lemma 3.1. *Let A be an index set and let $(M_\alpha)_{\alpha\in A}$ be a family of von Neumann algebras. If M denotes the direct sum $\bigoplus_{\alpha\in A} M_\alpha$, then $U(M)$, endowed with the $s(M, M_*)$-topology, is isomorphic (as a topological group) to $\prod_{\alpha\in A} U(M_\alpha)$, where each $U(M_\alpha)$ is endowed with the $s(M_\alpha, (M_\alpha)_*)$-topology and $\prod_{\alpha\in A} U(M_\alpha)$ with the product topology.*

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Proof. As

$$M_* = \Big\{ (\varphi_\alpha)_{\alpha\in A} : \varphi_\alpha \in (M_\alpha)_*,\ \sum_{\alpha\in A} \|\varphi_\alpha\| < \infty \Big\},$$

the proof is clear. $\square$

Let $N \subset \mathcal{B}(\mathcal{H})$ be a von Neumann factor acting on a separable Hilbert space $\mathcal{H}$ and let $M = L^\infty(X,\mu) \otimes N$, where (X,μ) is a Lebesgue space.

Lemma 3.2. *With the above notation, the unitary group $U(M)$, with the $s(M, M_*)$-topology, is isomorphic to the group $L^0(X,\mu;U(N))$ of all measurable maps from X to $U(N)$, equipped with the topology of convergence in measure.*

Proof. Recall [**55**, Theorem IV.7.17] that $(L^\infty(X,\mu)\otimes N)_* = L^1_{N_*}(X,\mu)$ and that $\operatorname{span}\{f\otimes\varphi : f\in L^1(X,\mu),\ \varphi\in N_*\}$ is norm-dense in $L^1_{N_x}(X,\mu)$.

Let $(u_n)_{n\geqslant 1}$ be a sequence in $L^0(X,\mu;U(N))$ converging in measure to $u \in L^0(X,\mu;U(N))$. Then $u_n \to u$ weakly, as for any $\varphi \in N_*$ and $f \in L^1(X,\mu)$, $f\otimes\varphi(u_n) \to f\otimes\varphi(u)$. Indeed, if for $x\in X$, $g_n(x) = f(x)\varphi(u_n(x))$ and $g(x) = f(x)\varphi(u(x))$, then $(g_n)_{n\geqslant1}$, $g \in L^1(X,\mu)$ and

$$|g_n(x)| \leqslant |f(x)|\,\|\varphi\| \quad\text{and}\quad |g(x)| \leqslant |f(x)|\,\|\varphi\|, \quad \mu\text{-a.e.}$$

Hence

$$\begin{aligned}|f\otimes\varphi(u_n) - f\otimes\varphi(u)| &\leqslant \int |f(x)||\varphi(u_n(x)) - \varphi(u(x))|\,\mathrm{d}\mu(x)\\ &\leqslant \int |g_n(x) - g(x)|\,\mathrm{d}\mu(x) \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

Conversely, let $\varphi \in N_*$ be a state on N and let $\varepsilon > 0$. Let $(u_n)_{n\geqslant1}$ and u belong to $L^1(X,\mu;U(N))$. Then, for $x \in X$,

$$|\varphi(u_n(x)) - \varphi(u(x))| = |\varphi(u_n(x) - u(x))| \leqslant \|u_n(x) - u(x)\|_\varphi.$$

Hence (see [**8**, Proposition 3.1.5]),

$$\mu(\{x\in X : |\varphi(u_n(x)) - \varphi(u(x))| \geqslant \varepsilon\}) \leqslant \mu(\{x \in X : \|u_n(x) - u(x)\|_\varphi^2 \geqslant \varepsilon^2\}).$$

If $u_n \to u$ strongly, then

$$\|u_n - u\|_{f\otimes\varphi} = \Big(\int \|u_n(x) - u(x)\|_\varphi^2\,\mathrm{d}\mu(x)\Big)^{1/2} \xrightarrow[n\to\infty]{} 0.$$

By Chebyshev's inequality, $(u_n)_{n\geqslant1}$ converges in measure to u. $\square$

The main result of this section is the following.

Theorem 3.3. *If M is an injective von Neumann algebra, then its unitary group $U(M)$ with the $s(M, M_*)$-topology is the direct product of a compact group and an extremely amenable group.*

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Let M be an injective von Neumann algebra. By [**16**, Theorem 4], M is generated by an upwards directed collection $(M_\alpha)_{\alpha\in A}$ of countably generated injective von Neumann subalgebras. By [**13**, Lemma 4], $U(M)$ is the $s(M,M_*)$-closure of the upward directed collection of groups $U(M_\alpha)$. (The separability condition in [**13**] is not essential, as remarked by Haagerup in [**32**].) A countably generated injective von Neumann algebra is a direct sum of injective von Neumann algebras with separable predual. By Lemma 3.1, we can therefore assume in the proof of Theorem 3.3 that M_* is separable.

Moreover, M can be decomposed as a direct sum of finite von Neumann algebras of type I and type II, and of a properly infinite von Neumann algebra. Therefore, Theorem 3.3 will follow directly from Proposition 3.4, Corollary 3.6 and Proposition 3.7.

Proposition 3.4. *If M is a finite von Neumann algebra of type I, then $U(M)$, with the $s(M,M_*)$-topology, is the direct product of a family of compact groups and extremely amenable groups.*

Proof. By the structure theorem for finite type I von Neumann algebras (see [**55**, Theorem V.1.27]), M is isomorphic to a direct sum of algebras of the form $L^\infty(\Gamma,\mu)\otimes\mathcal{B}(\mathcal{H})$, where $\dim\mathcal{H}<\infty$ and Γ is a locally compact space, with a positive (finite) Radon measure μ.

If μ is atomic, then

$$U(L^\infty(\Gamma,\mu)\otimes\mathcal{B}(\mathcal{H}))\cong\prod_{\operatorname{supp}\mu}U(\mathcal{H})$$

and therefore is a compact group.

If μ is non-atomic, then the unitary group $U(L^\infty(\Gamma,\mu)\otimes\mathcal{B}(\mathcal{H}))$ is canonically isomorphic to $L^\infty(\Gamma,\mu;U(\mathcal{H}))$. It follows that $U(L^\infty(\Gamma,\mu)\otimes\mathcal{B}(\mathcal{H}))$ is extremely amenable by Lemma 3.2. By Lemma 3.1 and Corollary 2.23, the proposition is proved. □

Proposition 3.5. *Let M be a von Neumann algebra and let $(N_k)_{k\geqslant1}$ be an increasing sequence of finite-dimensional subfactors of M such that $N_\infty=\bigcup_{k\geqslant1}N_k$ is $s(M,M_*)$-dense in M. Then $U(M)$, with the $s(M,M_*)$-topology, is a Lévy group.*

Proof. For $k\geqslant1$, let $(e^k_{i,j})_{1\leqslant i,j\leqslant n_k}$ be a system of matrix units of N_k. If $u\in U(N_k)$, let $(u_{ij})\in U(n_k,\mathbb{C})$ denote the matrix defined by $u=\sum_{i,j=1}^k u_{ij}e^k_{ij}$. Let $SU(N_k)=\{u\in U(N_k):\det(u_{ij})=1\}$.

Let $\varphi\in M_*$ be a faithful state and let $\varepsilon>0$ be given. As $\varphi(1)=\sum_{i=1}^{n_k}\varphi(e^k_{ii})$, there exists K such that for $k\geqslant K$, $\varphi(e^k_{ii})\leqslant\frac14\varepsilon$ for some $1\leqslant i\leqslant n_k$.

Let $u\in U(N)$. By [**13**, Lemma 4] there exist $k\geqslant K$ and $u_k\in U(N_k)$ such that $\|u-u_k\|_\varphi\leqslant\frac12\varepsilon$. Let $1\leqslant i\leqslant n_k$ be such that $\varphi(e^k_{ii})\leqslant\frac14\varepsilon$ and

$$v_k=1-e^k_{ii}+\overline{\det(u^k_{ij})}e^k_{ii}\in N_k.$$

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Then $u_k v_k \in SU(N_k)$ and

$$\begin{aligned}\|u - u_k v_k\|_\varphi &\leqslant \|u - u_k\|_\varphi + \|u_k(1 - v_k)\|_\varphi \\ &\leqslant \tfrac{1}{2}\varepsilon + \|1 - v_k\|_\varphi \\ &\leqslant \tfrac{1}{2}\varepsilon + |1 - \det(u^k_{ij})|\varphi(e^k_{ii}) \\ &\leqslant \tfrac{1}{2}\varepsilon + 2 \times \tfrac{1}{4}\varepsilon = \varepsilon.\end{aligned}$$

Hence, $\bigcup_{k\geqslant 1} SU(N_k)$ is $s(M, M_*)$-dense in $U(M)$.

As $N_k \cong M_{n_k}(\mathbb{C})$, there exists $h_\varphi \in M_{n_k}(\mathbb{C})$, $h_\varphi > 0$ such that for

$$x = \sum_{i,j=1}^{n_k} x_{i,j} e^k_{i,j} \in N_k,$$

$\varphi(x) = \mathrm{Tr}(h_\varphi(x_{i,j}))$, where Tr denotes the trace on $M_{n_k}(\mathbb{C})$. As $\varphi(1) = \mathrm{Tr}(h_\varphi) = 1$, $\|h_\varphi\| \leqslant 1$ and for $x \in N_k$,

$$\|x\|_\varphi \leqslant \|(x_{ij})\|_2,$$

where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm. By [**30**, Example 3.4], $(SU(N_k))_{k\geqslant 1}$ forms a Lévy family with regard to the Hilbert–Schmidt metric, and consequently the $\|\cdot\|_\varphi$-metric, which ends the proof of the proposition. □

Let M be a properly infinite injective von Neumann algebra with separable predual. By [**16**], there exists an increasing sequence $(N_k)_{k\geqslant 1}$ of finite-dimensional subfactors whose union is $s(M, M_*)$-dense in M. By Proposition 3.5, we get the following corollary.

Corollary 3.6. *Let M be a properly infinite injective von Neumann algebra with separable predual. Then the unitary group $U(M)$, with the $s(M, M_*)$-topology, is extremely amenable.* □

Let M be a finite continuous injective von Neumann algebra with separable predual. By considering the standard representation of M, we can assume that M acts on a separable Hilbert space. Then by reduction theory and uniqueness of the injective factor of type II_1, M is isomorphic to $L^\infty(X, \mu) \otimes R$, where X is a standard Borel space, with a finite measure μ and R is the hyperfinite factor of type II_1.

By Proposition 3.5 and Corollary 2.19, $U(R)$, endowed with the $s(R, R_*)$-topology, is extremely amenable. By the same argument as in the proof of Proposition 3.4, if μ is an atomic measure, then $U(M)$ is isomorphic to $\prod_{\mathrm{supp}\,\mu} U(R)$, equipped with the product $s(R, R_*)$-topology. If μ is a non-atomic measure, then $U(M)$, with the $s(M, M_*)$-topology, is isomorphic to the group of all measurable maps from X to $U(R)$, equipped with the topology of convergence in measure. Then by Lemma 3.1 and Corollary 2.23 we get the following proposition.

Proposition 3.7. *If M is a finite continuous injective von Neumann algebra with separable predual, then $U(M)$, endowed with the $s(M, M_*)$-topology, is extremely amenable.* □

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3.2. A characterization of nuclear C^*-algebras

Recall that a continuous action of a topological group G on a compact space X is *proximal* if for every two points x, y and every entourage of the diagonal $V \subseteq X \times X$ there is a $g \in G$ with $(gx, gy) \in V$. A topological group G is called *strongly amenable* if every continuous proximal action of G on a compact space has a fixed point [**24**]. For instance, every compact group is strongly amenable simply because it has no non-trivial proximal actions on compacta. Every extremely amenable group is strongly amenable for obvious reasons. The class of strongly amenable groups is closed under direct products.

By the *completion* $\hat{G}$ of a topological group G we mean, as usual, the completion of G with regard to the two-sided uniform structure, that is, the supremum of the left and the right uniform structures on G. This is again a topological group (see, for example, Chapter 10 in [**54**]).

It is a standard fact, easily proved, that the completion, $\hat{G}$, of a topological group G is amenable (respectively, strongly amenable, extremely amenable) if and only if G is amenable (respectively, strongly amenable, extremely amenable).

The following corollary of our results strengthens a result due to Paterson [**47**, Theorem 2].

Theorem 3.8. *A C^*-algebra A is nuclear if and only if its unitary group $U(A)$, equipped with the topology $\sigma(A, A^*)$, is strongly amenable.*

Proof. Denote by M the universal von Neumann envelope of A. As is well known, M can be obtained as the completion of the C^*-algebra A with regard to the additive uniform structure determined by the relative weak topology [**14**, Corollary 12.1.3]). As a consequence, one can prove that the completion, $\widehat{U(A)}$, of the unitary group $U(A)$, equipped with the relative weak topology, is isomorphic to the unitary group of the von Neumann envelope M of A, equipped with the $s(M, M_*)$-topology.

Necessity (⇒): if A is nuclear, then the von Neumann envelope M is approximately finite dimensional, and therefore, by Theorem 3.3, the unitary group $U(M)$ with the topology $s(M, A^*)$ is the product of a compact group and an extremely amenable group. In particular, $U(M)$ is strongly amenable, and so is its everywhere dense topological subgroup $U(A)$.

Sufficiency (⇐): follows from de la Harpe's result in [**13**], as noted by Paterson [**47**]. If the topological group $U(A)$ is strongly amenable, then it is in particular amenable, and so is the topological group completion, $U(M)$. It follows by de la Harpe's result that M is approximately finite dimensional. □

3.3. Automorphism groups of von Neumann algebras

If M is a von Neumann algebra, let $\mathrm{Aut}(M)$ denote, as usual, the group of all $*$-automorphisms of M. We consider on $\mathrm{Aut}(M)$ the topology of norm pointwise convergence for the action of $\mathrm{Aut}(M)$ on M_*, given by

$$\alpha \in \mathrm{Aut}(M), \quad \varphi \in M_* \mapsto \varphi \circ \alpha^{-1} \in M_*.$$

With this topology, called the u-*topology*, $\mathrm{Aut}(M)$ is a topological group and it is a Polish group if M_* is separable.

If $u \in U(M)$, let $\mathrm{Ad}\, u$ denote the inner automorphism of M given by $\mathrm{Ad}\, u(x) = uxu^*$, $x \in M$, and let $\mathrm{Inn}(M)$ be the subgroup of all inner automorphisms of M.

If $U(M)$ is endowed with the $s(M, M_*)$-topology and $\mathrm{Inn}(M)$ with the u-topology, the canonical surjection $U(M) \to \mathrm{Inn}(M)$ is continuous, as for $\varphi \in M_*$, $u \in U(M)$,

$$\|\varphi \circ \mathrm{Ad}\, u - \varphi\| \leqslant 2\|\varphi\|\, \|u - 1\|_\varphi.$$

By Theorem 3.3, we therefore get the following proposition.

Proposition 3.9. *If M is an injective continuous von Neumann algebra, then* $\mathrm{Inn}(M)$, *with the topology of pointwise convergence in norm on M_*, is extremely amenable.* □

4. Groups of measure-preserving transformations

4.1. Weak and uniform topologies

For a standard non-atomic Lebesgue measure space $X = (X, \mu)$ denote by $\mathrm{Aut}(X, \mu)$ the group of (equivalence classes of) invertible measure-preserving transformations of X, and by $\mathrm{Aut}^*(X, \mu)$ the group of all invertible measurable and non-singular transformations of X.

The *weak topology* on $\mathrm{Aut}^*(X, \mu)$ (also-called the *coarse topology*) is induced by the strong operator topology on the isometry group of $L^p(X)$ under the quasi-regular representation of the group $\mathrm{Aut}^*(X, \mu)$ in the Banach space $L^p(X)$, where $1 \leqslant p < \infty$ is any (cf. [**6**, Theorem 8]).

The weak topology makes $\mathrm{Aut}^*(X, \mu)$ into a Polish topological group, while $\mathrm{Aut}(X, \mu)$ is a closed (therefore also Polish) subgroup (see [**39**, 17.46]). Notice that the weak topology makes perfect sense not just for a finite measure μ, but for sigma-finite one as well.

Define a left-invariant metric on $\mathrm{Aut}^*(X, \mu)$ as follows:

$$d_{\mathrm{unif}}(\tau, \sigma) = \mu\{x \in X : \tau(x) \neq \sigma(x)\}.$$

The topology induced on $\mathrm{Aut}^*(X, \mu)$ by the metric d is a group topology, known as the *uniform topology.* It is strictly finer than the weak topology [**58**, Proposition 3]. For example, the uniform topology makes $\mathrm{Aut}^*(X, \mu)$ into a (path-connected) non-separable group. When dealing with the uniform topology, we will always assume μ to be a finite measure.

Both the weak topology and the uniform topology only depend on the equivalence class of the measure μ, rather than on μ itself (see [**7**, Remark 3, p. 373]).

Sometimes we will indicate the uniform topology by the subscript 'u', as in $\mathrm{Aut}(X, \mu)_{\mathrm{u}}$. Similarly, for the weak topology the subscript 'w' will be used.

Note that the restriction of d to $\mathrm{Aut}(X, \mu)$ is bi-invariant, and so the topological group $\mathrm{Aut}(X, \mu)_{\mathrm{u}}$ is SIN (i.e. has small invariant neighbourhoods), that is, the left and the right uniform structures coincide.

4.2. Extreme amenability of $\mathbf{Aut}(X,\mu)$ with the weak topology

The following theorem (cf. [**33**, pp. 65–68]), belonging to the Kakutani–Rokhlin circle of results, is well-known. Identify the symmetric group $\mathfrak{S}_n$ with the subgroup of all measure-preserving automorphisms of $\mathbb{I}=[0,1]$ with the Lebesgue measure λ mapping each dyadic interval of rank n onto a dyadic interval of rank n via a translation (the interval exchange transformations).

Theorem 4.1 (weak approximation theorem). *The union of the subgroups $\mathfrak{S}_n$, $n\in\mathbb{N}$, is everywhere dense in $\mathrm{Aut}(\mathbb{I},\lambda)$ with respect to the weak topology.* □

Under the above embedding $\mathfrak{S}_n\hookrightarrow\mathrm{Aut}(\mathbb{I})$, the normalized Hamming distance d_n on $\mathfrak{S}_n$ (Example 2.7) coincides with the restriction of the metric d.

Theorem 4.2. *The group $\mathrm{Aut}(X,\mu)_{\mathrm{w}}$ of all measure-preserving automorphisms of a standard non-atomic finite or sigma-finite measure space, equipped with the weak topology, is a Lévy group and therefore extremely amenable.*

Proof. For μ finite, the group $\mathrm{Aut}(X,\mu)$ is topologically isomorphic with the group $\mathrm{Aut}(\mathbb{I},\lambda)$, where $\mathbb{I}=[0,1]$ is the unit interval with the Lebesgue measure. According to Maurey's theorem [**43**], the permutation groups $\mathfrak{S}_n$ of finite rank n, equipped with the normalized counting measure and the normalized Hamming distance, form a Lévy family (even with regard to the uniform topology). By Theorem 4.1, $\mathrm{Aut}(X,\mu)_{\mathrm{w}}$ forms a Lévy group, and Corollary 2.19 applies.

For μ sigma-finite it is enough to prove the result for the real line $\mathbb{R}$, equipped with the Lebesgue measure λ. For every $n\in\mathbb{N}$, let G_n denote the group of interval exchange transformations of the system of 8^{n+1} intervals of length 16^{-n-1} each, covering the interval $[-2^n,2^n]$ of length 2^{n+1}. A natural group isomorphism from the permutation group $\mathfrak{S}_{8^{n+1}}$ of rank 8^{n+1} equipped with the normalized Hamming distance onto G_n, equipped with the uniform metric, is Lipschitz with $L=2^{n+1}$. By Lemma 2.3 and Maurey's result (Example 2.7), the family (G_n) is a Lévy family, with

$$\begin{aligned}\alpha_{G_n}(\varepsilon)&\leqslant\alpha_{\mathfrak{S}_{8^{n+1}}}\left(\frac{\varepsilon}{2^{n+1}}\right)\\&\leqslant\exp\left(-\frac{\varepsilon^2}{4^{n+1}}\frac{8^{n+1}}{32}\right)\\&=\exp\left(-\frac{\varepsilon^2 2^{n+1}}{32}\right)\to 0\quad\text{as } n\to\infty.\end{aligned}$$

Since the groups (G_n) form an increasing chain with everywhere dense union in $\mathrm{Aut}(\mathbb{R},\lambda)$, we are done. □

4.3. Non-amenability of $\mathbf{Aut}(X,\mu)$ with uniform topology

Let π be a unitary representation of a group G in a Hilbert space $\mathcal{H}$, that is, $\mathcal{H}$ is a unitary G-module. One says that π (or the G-module $\mathcal{H}$) is *amenable in the sense of*

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Bekka [**2**] if there exists a state, ϕ, on the von Neumann algebra $\mathcal{B}(\mathcal{H})$ that is invariant under the action of G by conjugations: for all $T \in \mathcal{B}(\mathcal{H})$ and all $g \in G$,

$$\phi(T) = \phi(\pi_g^* T \pi_g).$$

Denote by $\mathbb{S}$ the unit sphere in the space $\mathcal{H}$. Notice that G acts on the space $\mathbb{C}^{\mathbb{S}}$ of all functions $\mathbb{S} \to \mathbb{C}$ by

$${}^g f(\xi) := f(\pi_g^*(\xi)) \equiv f(\pi_{g^{-1}}(\xi)),$$

where $g \in G$, $f \in \mathbb{C}^{\mathbb{S}}$, and $\xi \in \mathbb{S}$. This action leaves invariant the space $\mathrm{UCB}(\mathbb{S})$ of all uniformly continuous bounded complex-valued functions on the sphere. The following was proved by one of the present authors [**49**].

Theorem 4.3. *A unitary representation π of a group G is amenable if and only if there exists a G-invariant mean on the space* $\mathrm{UCB}(\mathbb{S})$*.* □

Let $\xi \in \mathbb{S}$. For every (complex-valued) function f on $\mathbb{S}$ denote by $\tilde{f}$ a function on G defined as follows:

$$\tilde{f}(g) := f(\pi_g \xi),$$

where $g \in G$. If $f \in \ell^\infty(\mathbb{S})$, then $\tilde{f} \in \ell^\infty(G)$. The latter is a Banach G-module with respect to the left regular representation of G:

$${}^g\varphi(x) := \varphi(g^{-1}x).$$

The mapping $f \mapsto \tilde{f}$ from $\ell^\infty(\mathbb{S})$ to $\ell^\infty(G)$ is G-equivariant, that is, commutes with the action of G:

$$\widetilde{{}^g f}(h) = {}^g f(\pi_h(\xi)) = f(\pi_g^* \pi_h(\xi)) = f(\pi_{g^{-1}h}(\xi)) = \tilde{f}(g^{-1}h) = {}^g(\tilde{f})(h).$$

Now assume that G is a topological group and the representation π is strongly continuous.

Lemma 4.4. *If $f \in \mathrm{UCB}(\mathbb{S})$, then $\tilde{f}$ is left uniformly continuous on G.*

Proof. Let $\varepsilon > 0$. Find a $\delta > 0$ such that, whenever $\|\xi - \zeta\| < \delta$, one has $|f(\xi) - f(\zeta)| < \varepsilon$. Due to the strong continuity of π, there is a neighbourhood of the identity, V, in G such that, whenever $g \in V$, one has $\|\xi - \pi_g \xi\| < \delta$. If now $g, h \in G$ are such that $g^{-1}h \in V$, then $|\pi_{g^{-1}h}(\xi) - \xi| < \delta$, that is, $|\pi_h(\xi) - \pi_g(\xi)| < \delta$, and

$$|\tilde{f}(g) - \tilde{f}(h)| = |f(\pi_g(\xi)) - f(\pi_h(\xi))| < \varepsilon.$$

□

Denote by $\mathrm{LUCB}(G)$ (respectively, $\mathrm{RUCB}(G)$) the C^*-algebra of all bounded left (respectively, right) uniformly continuous complex-valued functions on G. It is now easy to verify that the mapping $f \mapsto \tilde{f}$ is a positive linear operator from $\mathrm{UCB}(\mathbb{S})$ to $\mathrm{LUCB}(G)$, sending 1 to 1 and commuting with the action of G. Therefore, if there is an invariant mean on $\mathrm{LUCB}(G)$, by composing it with the mapping $f \mapsto \tilde{f}$, one gets an invariant mean on $\mathrm{UCB}(\mathbb{S})$. We obtain the following proposition.

Proposition 4.5. *If a topological group G admits a left-invariant mean on the space* LUCB(G) *of all left uniformly continuous bounded functions on G, then every strongly continuous unitary representation of G is amenable.* □

Recall that a topological group is called *amenable* if there exists a left-invariant mean on the space RUCB(G). If G is a SIN group, that is, the left and the right uniformities on G coincide, then LUCB(G) = RUCB(G) and we obtain the following.

Corollary 4.6. *Every strongly continuous unitary representation of an amenable SIN group is amenable.* □

Remark 4.7. In general, the condition that G is SIN cannot be dropped here, unless we presume that G is locally compact. For instance, the infinite unitary group $U(\ell_2)$ with the strong operator topology is (extremely) amenable, while the standard unitary representation of $U(\ell_2)$ is non-amenable (because this would clearly imply, for instance, amenability of every unitary representation of every countable group).

Theorem 4.8. *The topological group* $\mathrm{Aut}(X,\mu)_{\mathrm{u}}$ *is non-amenable.*

Proof. Take as $V = L^2_0(X)$ the closed unitary G-submodule of $L^2(X)$ consisting of all functions $f \in L^2(X)$ with

$$\int_X f(x)\,\mathrm{d}\mu(x) = 0.$$

According to Corollary 4.6, if $\mathrm{Aut}(X,\mu)_{\mathrm{u}}$ were amenable, then the regular representation γ of $\mathrm{Aut}(X,\mu)_{\mathrm{u}}$ in $L^2_0(X)$ would be amenable. (Recall that with regard to the uniform topology the group $\mathrm{Aut}(X,\mu)$ is a SIN group.) In particular, the restriction of γ to any subgroup G of $\mathrm{Aut}(X,\mu)_{\mathrm{u}}$ would be amenable as well. It is, therefore, enough to discover a group G acting on a standard non-atomic Lebesgue space X by measure-preserving transformations in such a manner that the associated regular representation of G in $L^2_0(X)$ is not amenable in the sense of Bekka.

The following is inspired by Bekka's work [**3**]. Let G be a semisimple real Lie group with finite centre and without compact factors, having property (T) (cf. [**3**]; e.g. $SL_3(\mathbb{R})$). Let Γ denote a lattice in G, that is, a discrete subgroup such that the Haar measure induces an invariant finite measure on G/Γ. Set $X = G/\Gamma$.

Assume that the standard representation π of G in $L^2_0(X) = L^2_0(G/\Gamma)$ is amenable. Since G is a Kazhdan group, according to a result by Bekka [**2**], π must have a finite-dimensional subrepresentation. (Later it was shown [**4**] that this property characterizes groups with property (T).)

Note that $L^2_0(X)$ has non-trivial G-invariant vectors: since the action of G on X is transitive, such vectors must be constant functions. Therefore, a finite-dimensional subrepresentation of π must be non-trivial. But this would result in a non-trivial continuous group homomorphism from G to a finite-dimensional unitary group, which is impossible by assumption. □

5. Full groups of amenable equivalence relations

5.1. The full group

Let $\mathcal{R}$ be a discrete measured equivalence relation, i.e. $\mathcal{R}$ is an equivalence relation with countable equivalence classes on a standard Borel space S, the graph $\mathcal{R} \subset S \times S$ is Borel, and μ is a quasi-invariant probability measure on S.

The full group $[\mathcal{R}]$ is defined as the group of all bimeasurable transformations σ of (S, μ) with

$$(s, \sigma(s)) \in \mathcal{R}, \quad \mu\text{-a.e.}$$

We will consider on $[\mathcal{R}]$ the uniform topology, determined by the measure μ. With this topology, $[\mathcal{R}]$ is a Polish group [**11**, **34**].

Following Zimmer [**61**], let G be a countable discrete group and let (S, μ) be a right free G-space. Then an element σ of the *full group* $[G]$ of G is determined by a measurable partition with the property

$$S = \coprod_{g \in G} S_g = \coprod_{g \in G} S_g g,$$

in such a way that $s\sigma = sg$ if $s \in S_g$. In particular, G is a subgroup of $[G]$, every element $h \in G$ being determined by the partition

$$S_g = \begin{cases} S, & \text{if } g = h, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Remark 5.1. With the measured dynamical system (S, μ, G) is associated a von Neumann algebra, the so-called Murray–von Neumann group measure space construction. One has the following unitary representations U, V of G and isometric $*$-representations M and N of $L^\infty(S)$ on $L^2(S \times G)$, given for $g \in G$, $f \in L^\infty(S, \mu)$, by

$$\begin{aligned} U_g \xi(x, h) &= r(x, g)^{1/2} \xi(xg, hg), & V_g \xi(x, h) &= \xi(x, g^{-1}h), \\ M_f \xi(x, h) &= f(x) \xi(x, h), & N_f \xi(x, h) &= f(xh^{-1}) \xi(x, h), \end{aligned}$$

where $r(x, g) = \mathrm{d}\mu(xg)/\mathrm{d}\mu(x)$ is the Radon–Nikodym cocycle of the action of G on (S, μ).

The von Neumann algebra generated by the operators V_g and N_f is denoted by L, while that generated by the operators U_g and M_f is denoted by R.

If $J \in U(L^2(S \times G))$ is given by

$$J\xi(x, g) = f(xg^{-1}, g^{-1}) r(x, g^{-1})^{1/2}, \quad (x, g) \in S \times G,$$

then $J^2 = 1$, and $JLJ = R$.

To any $\sigma \in [G]$, defined by the partitions $(S_g)_{g \in G}$ and $(S_g g)_{g \in G}$ of S, there are naturally associated unitaries $U_\sigma \in R$ and $V_\sigma \in L$ given by

$$U_\sigma = \sum_{g \in G} M_{f_g} U_g \in R \quad \text{and} \quad V_\sigma = \sum_{g \in G} N_{f_g} V_g \in L,$$

where $f_g = \chi_{S_g}$.

https://doi.org/10.1017/S1474748006000090 Published online by Cambridge University Press

If $U([G]) \subset U(R)$ (respectively, $V([G]) \subset U(L)$) is endowed with the strong operator topology and $[G]$ with the uniform topology, it is easy to check that U (respectively, V) is a topological isomorphism from $[G]$ onto its image.

5.2.

In this subsection, we will show that if $\mathcal{R}$ is an amenable equivalence relation, then the full group $[\mathcal{R}]$ with the uniform topology is extremely amenable. For a definition of an amenable equivalence relation, see [**61**, § 3].

By [**10**], we can assume that S is the compact abelian group $\prod\{0,1\}$, the equivalence relation $\mathcal{R}$ is induced by the action of the subgroup $G = \oplus\mathbb{Z}/2$, acting by addition: if $g = (g_k)_{k\geqslant 1} \in G$, and $x = (x_k)_{k\geqslant 1}$, then

$$xg = (x_k + g_k)_{k\geqslant 1} = x + g,$$

and μ is a G-ergodic non-atomic quasi-invariant measure on S.

Let us introduce some notation we will use in this section. For $n \geqslant 1$, let

$$S_n = \prod_{k=1}^{n}\{0,1\} \quad \text{and} \quad S^n = \prod_{k\geqslant n+1}\{0,1\}.$$

For $y \in S$ we will denote its tail by $y^n = (y_k)_{k\geqslant n+1} \in S^n$. For $x \in S_n$ let $C(x)$ denote the cylinder set $\{(y_k)_{k\geqslant 1} \in S : y_k = x_k,\ 1 \leqslant k \leqslant n\}$ of S. For every bijection $\sigma \in \mathrm{Bij}(S_n)$, let σ be the element of $[G]$ defined for $y = (x, y^n) \in C(x)$, $x \in S_n$, by $\sigma(y) = (\sigma(x), y^n)$, and let $\mathrm{Bij}(n)$ denote the finite subgroup of $[G]$ such transformations form.

For $n \geqslant 1$, if

$$G_n = \{g = (g_k)_{k\geqslant 1} \in G : g_k = 0\ \forall k \geqslant n+1\},$$

then $\mathrm{Bij}(n) \subset [G_n]$.

Lemma 5.2. *Keeping the above notation, $\bigcup_n \mathrm{Bij}(n)$ is uniformly dense in $[G]$.*

Proof. Let $(S_g)_{g\in G}$ and $(S_g g)_{g\in G}$ be a measurable partition of S and let $\alpha \in [G]$ be given by $\alpha(x) = xg$, $x \in S_g$. Let $0 < \varepsilon < 1$ be given and let N_1 be such that $\mu(\coprod_{g\in G_{N_1}} S_g) \geqslant 1 - \varepsilon$. Let us fix $0 < \delta < \frac{1}{3}\varepsilon \cdot 2^{2N_1}$.

The probability measure

$$\nu = \frac{1}{2^{N_1}} \sum_{g\in G_{N_1}} \mu \circ g$$

is G_{N_1}-invariant and equivalent to μ.

For $n \geqslant 1$, let E_n denote the conditional expectation with respect to ν and to the partition by the cylinders of S_n, i.e. for a measurable subset $B \subset S$,

$$E_n(\chi_B) = \sum_{x\in S_n} r_x^n(B)\chi_{C(x)},$$

where

$$r_x^n(B) = \frac{\nu(C(x)\cap B)}{\nu(C(x))}.$$

Let $N_2 \geqslant N_1$ be such that for all $B = S_g$ or $B = S_g g$, for $g \in G_{N_1}$,

$$\|E_{N_2}(\chi_B) - \chi_B\|_{1,\nu} = 2\sum_{x\in S_{N_2}} r_x^{N_2}(B)(1 - r_x^{N_2}(B))\nu(C(x)) < \delta^2. \tag{5.1}$$

If A is a measurable subset of S, let $S_{N_2}^A$ denote $\{x \in S_{N_2} : r_x^{N_2}(A) \geqslant 1-\delta\}$ and let $\tilde{A}$ denote the disjoint union $\coprod_{x\in S_{N_2}^A} C(x)$. Then the sets $(\tilde{S}_g)_{g\in G_{N_1}}$ satisfy, for $g, h \in G_{N_1}$, the following properties:

(i) $\|\chi_{S_g} - \chi_{\tilde{S}_g}\|_{1,\nu} \leqslant 3\delta$,

(ii) $\tilde{S}_g \cap \tilde{S}_h = \emptyset$, for $g \neq h$,

(iii) $\tilde{S}_g g \cap \tilde{S}_h h = \emptyset$, for $g \neq h$.

Indeed, for $g \in G_{N_1}$ we have by definition of $\tilde{S}_g$:

$$\begin{aligned}\|E_{N_2}(\chi_{S_g}) - \chi_{\tilde{S}_g}\|_{1,\nu} \leqslant \delta \sum &\{\nu(C(x)) : x \in S_{N_2},\ r_x^{N_2}(S_g) \leqslant \delta \text{ or } r_x^{N_2}(S_g) \geqslant 1-\delta\}\\ &+ \sum\{\nu(C(x)) : x \in S_{N_2},\ r_x^{N_2}(S_g)(1 - r_x^{N_2}(S_g)) > \delta(1-\delta)\}.\end{aligned}$$

By (5.1) we have

$$\begin{aligned}\|E_{N_2}(\chi_{S_g}) - \chi_{\tilde{S}_g}\|_{1,\nu} &\leqslant \delta + \frac{1}{2\delta(1-\delta)}\|E_{N_2}(\chi_{S_g}) - \chi_{S_g}\|_{1,\nu}\\ &\leqslant \delta + \frac{\delta}{2(1-\delta)} \leqslant \tfrac{5}{2}\delta.\end{aligned}$$

Then

$$\begin{aligned}\|\chi_{S_g} - \chi_{\tilde{S}_g}\|_{1,\nu} &\leqslant \|E_{N_2}(\chi_B) - \chi_B\|_{1,\nu} + \|E_{N_2}(\chi_{S_g}) - \chi_{\tilde{S}_g}\|_{1,\nu}\\ &\leqslant \delta^2 + \tfrac{5}{2}\delta \leqslant 3\delta,\end{aligned}$$

which proves (i).

For $x \in S_{N_2}$,

$$1 \geqslant \sum_{g\in G_{N_1}} \frac{\nu(C(x)\cap S_g)}{\nu(C(x))} = \sum_{g\in G_{N_1}} r_x^{N_2}(S_g) \geqslant 0.$$

Hence $S_{N_2}^{S_g} \cap S_{N_2}^{S_h} = \emptyset$ and $\tilde{S}_g \cap \tilde{S}_h = \emptyset$, which proves (ii).

For (iii), notice that for $x \in S_{N_2}$, $g \in G_{N_1}$,

$$r_{xg}^{N_2}(S_g) = \frac{\nu(S_g \cap C(xg))}{\nu(C(xg))} = \frac{\nu((S_g g \cap C(x))g)}{\nu(C(xg))} = r_x^{N_2}(S_g g),$$

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as ν is G_{N_1}-invariant. Hence $S_{N_2}^{S_g} g = S_{N_2}^{S_g g}$ and

$$\tilde{S}_g g = \coprod_{x \in S_{N_2}^{S_g}} C(xg) = \coprod_{x \in S_{N_2}^{S_g} g} C(x) = \coprod_{x \in S_{N_2}^{S_g g}} C(x).$$

If $g, h \in G_{N_1}$, $g \neq h$, then $S_g g \cap S_h h = \emptyset$ and (iii) holds.

We now define $\sigma \in \mathrm{Bij}(N_2)$ such that $d_\mu(\alpha, \sigma) < 2\varepsilon$. As $S_{N_2}^{S_g} g = S_{N_2}^{S_g g}$, for $g \in G_{N_1}$,

$$A = \Big\{ x \in S_{N_2} : x \notin \coprod_{g \in G_{N_1}} S_{N_2}^{S_g} \Big\} \quad \text{and} \quad B = \Big\{ x \in S_{N_2} : x \notin \coprod_{g \in G_{N_1}} (S_{N_2}^{S_g} g) \Big\}$$

have the same cardinality. Let $\tilde{\sigma}$ be an arbitrary bijection from A to B, and let σ denote the bijection of S_{N_2} defined by

$$\sigma(x) = \begin{cases} xg, & \text{if } x \in S_{N_2}^{S_g},\ g \in G_{N_1}, \\ \tilde{\sigma}(x), & \text{if } x \in A. \end{cases}$$

We shall also denote by σ the corresponding element of $\mathrm{Bij}(N_2)$. Then

$$\{x \in S : \sigma(x) \neq \alpha(x)\} \subseteq \Big(S \setminus \coprod_{g \in G_{N_1}} S_g \Big) \coprod \Big(\coprod_{g \in G_{N_1}} (S_g \setminus \tilde{S}_g) \Big)$$

and

$$d_\mu(\alpha, \sigma) \leqslant \mu\Big(S \setminus \coprod_{g \in G_{N_1}} S_g \Big) + \sum_{g \in G_{N_1}} \mu(S_g \Delta \tilde{S}_g).$$

As

$$\mu(S_g \Delta \tilde{S}_g) \leqslant 2^{N_1} \nu(S_g \Delta \tilde{S}_g) = 2^{N_1} \|\chi_{S_g} - \chi_{\tilde{S}_g}\|_{1,\nu} \leqslant 2^{N_1} 3\delta,$$

then $d_\mu(\alpha, \sigma) \leqslant \varepsilon + 2^{2N_1} 3\delta \leqslant 2\varepsilon$. □

As the size of the largest cylinder set determined by the first n coordinates clearly goes to zero as $n \to \infty$, Corollary 2.13 implies that the family of groups $\mathrm{Bij}(n)$ equipped with the uniform metric and the normalized counting measure is a Lévy family. Applying Lemma 5.2, we deduce the following proposition.

Proposition 5.3. *Let $\mathcal{R}$ be a discrete measured equivalence relation, acting ergodically on a Lebesgue space (S, μ). If $\mathcal{R}$ is an amenable equivalence relation, then the full group $[\mathcal{R}]$, endowed with the uniform topology, is a Lévy group (and in particular extremely amenable).* □

Remark 5.4. As pointed out to us by A. S. Kechris, extreme amenability of the full group $[G]$ in Proposition 5.3 can be alternatively deduced by applying an argument similar to that employed by one of the present authors to give a new proof of extreme amenability of the unitary group $U(\ell^2)_s$ in [**50**, § 4.5].

Namely, for $n \geqslant 1$, let $S^n = \prod_{k \geqslant n+1} \{0, 1\}$. To any $f \in L^0(S^n, S_{2^n})$ associate the element $\tilde{f} \in [G]$, defined for $x = (x_k)_{k \geqslant 1} \in S$, by $\tilde{f}(x) = (f(x^n)(x_1, \dots, x_n), x^n)$, where

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$x^n = (x_{n+1}, x_{n+1}, \dots) \in S^n$. Let L_n denote the subgroup of $[G]$ formed by $\{\tilde{f} \in [G] : f : S^n \to \mathfrak{S}_{2^n}$ is measurable$\}$. Let $\tilde{\mu}$ be the probability measure on S^n, given for $A \subset S^n$, A measurable, by

$$\tilde{\mu}(A) = \sum_{\bar{x} \in S_n} \mu(\{(\bar{x}, y) : y \in A\}).$$

For $f \in L^0(S^n, \mathfrak{S}_{2^n})$ we have

$$\begin{aligned} \mu(\{x \in S : \tilde{f}(x) \neq x\}) &= \sum_{\bar{x} \in S_n} \mu(\{(\bar{x}, y) : f(y)(\bar{x}) \neq \bar{x}\}) \\ &\subset \sum_{\bar{x} \in S_n} \mu(\{(\bar{x}, y) : f(y) \neq \mathrm{Id}\}) \\ &= \tilde{\mu}(\{y \in S^n : f(y) \neq \mathrm{Id}\}). \end{aligned}$$

Therefore, if $L^0(S^n, \mathfrak{S}_{2^n})$ is endowed with the topology of convergence in measure with respect to $\tilde{\mu}$ and $L_n \subset [G]$ with the uniform topology, then the group isomorphism $L^0(S^n, \mathfrak{S}_{2^n}) \ni f \mapsto \tilde{f} \in L_n$ is continuous.

By Theorem 2.20, $(L_n)_{n \geqslant 1}$ forms a sequence of extremely amenable subgroups of $[G]$, with $[G_n] \subset L_n$, $\forall n \geqslant 1$. As $([G_n])_{n \geqslant 1}$ forms an increasing sequence of subgroups of $[G]$, whose union is uniformly dense, we conclude by Lemma 2.22 that if G and X are as in Proposition 5.3, then the full group $[G]$, endowed with the uniform topology, is extremely amenable.

Notice, however, that the actual conclusion of our Proposition 5.3 is stronger, because there exist extremely amenable groups that are not Lévy groups (cf., for example, Remark 2.21).

5.3.

In this subsection we show that if $[\mathcal{R}]$ endowed with the uniform topology is amenable as a topological group, then $\mathcal{R}$ is an amenable equivalence relation.

If E is a separable Banach space, let $\mathrm{Iso}(E)$ denote the group of isometric isomorphisms of E, endowed with the Borel structure induced by the strong operator topology. As is well-known (see, for example, [**62**, Lemma 1.3]), the map from $\mathrm{Iso}(E)$ to $\mathrm{Homeo}(E_1^*)$, given by duality, is continuous, hence Borel, if we consider the topology of uniform convergence on $\mathrm{Homeo}(E_1^*)$.

Let $\alpha : \mathcal{R} \to \mathrm{Iso}(E)$ be a cocycle, that is, a measurable map such that

$$\alpha(x, y)\alpha(y, z) = \alpha(x, z) \quad \text{if } (x, y), (y, z) \in \mathcal{R}.$$

Let $\alpha^* : \mathcal{R} \to \mathrm{Homeo}(E_1^*)$ denote the cocycle given by

$$\alpha^*(x, y) = (\alpha(x, y)^{-1})^* = \alpha(y, x)^*, \quad (x, y) \in \mathcal{R}.$$

For $f \in L^1(S, E)$ and $\sigma \in [\mathcal{R}]$, let $T(\sigma)f$ denote the element of $L^1(S, E)$ given by

$$T(\sigma)f(s) = r(s, \sigma)\alpha(s, s\sigma)f(s\sigma), \quad s \in S,$$

https://doi.org/10.1017/S1474748006000090 Published online by Cambridge University Press

where $r(s,\sigma)$ denotes the Radon–Nikodym cocycle given for $s \in S$ and $\sigma \in [\mathcal{R}]$ by $\mathrm{d}\mu(s\sigma) = r(s,\sigma)\mathrm{d}\mu(s)$.

One checks easily that T defines a representation of $[\mathcal{R}]$ into $\mathrm{Iso}(L^1(S,E))$.

Lemma 5.5. *If $[\mathcal{R}]$ is endowed with the uniform topology, then T is strongly continuous.*

Proof. If $\xi \in E$ and $f \in L^1(S,\mathbb{C})$, let $\tilde{f}$ denote the element of $L^1(S,E)$ given by

$$\tilde{f}(x) = f(x)\xi.$$

As the set of all such functions $\tilde{f}$ topologically spans $L^1(S,E)$, it is enough to verify the continuity of every map of the form

$$[\mathcal{R}] \ni \sigma \mapsto T(\sigma)\tilde{f} \in L^1(S,E).$$

For $\sigma \in [\mathcal{R}]$ one has

$$\begin{aligned}
\|T(\sigma)\tilde{f} - \tilde{f}\| &= \int_S \|r(s,\sigma)\alpha(s,s\sigma)\tilde{f}(s\sigma) - \tilde{f}(s)\| \,\mathrm{d}\mu(s) \\
&= \int_{\mathrm{supp}(\sigma)} \|r(s,\sigma)f(s\sigma)\alpha(s,s\sigma)\xi - f(s)\xi\| \,\mathrm{d}\mu(s) \\
&\leqslant \|\xi\| \int_{\mathrm{supp}(\sigma)} (r(s,\sigma)|f(s\sigma)| + |f(s)|) \,\mathrm{d}\mu(s) \\
&\leqslant 2\|\xi\| \, \|f\|_\infty \mu(\mathrm{supp}(\sigma)) \\
&= 2\|\xi\| \, \|f\|_\infty d_{\mathrm{unif}}(\sigma, \mathrm{Id}),
\end{aligned}$$

where $\mathrm{supp}(\sigma) = \{s \in S : s\sigma \neq s\}$. The statement follows. □

5.4.

Let $\mathfrak{A} = \{A_s : s \in S\}$ be a Borel field of (non-empty) compact convex subsets of E^* such that for all $(x,y) \in \mathcal{R}$,

$$\alpha^*(x,y)(A_y) = A_x \quad \mu\text{-a.e.}$$

By [**62**, Proposition 2.2],

$$A = \{\lambda \in L^\infty(S,E^*) : \lambda(s) \in A_s \ \mu\text{-a.e.}\}$$

is a weak* compact convex non-empty subset of $L^\infty(S,E^*)_1$.

Moreover, if T^* denotes the dual representation of T in $L^\infty(S,E^*)$, given for $\sigma \in [\mathcal{R}]$ and $\lambda \in L^\infty(S,E^*)$ by

$$T^*(\sigma)\lambda(s) = \alpha^*(s,s\sigma)f(s\sigma), \quad s \in S,$$

then A is invariant under the (affine) action T^* of $[\mathcal{R}]$.

We have already noted that this action is continuous if A is equipped with the weak* topology. Therefore, if the full group $[\mathcal{R}]$, equipped with the uniform topology, is amenable as a topological group, then there exists a $[\mathcal{R}]$-invariant section of the Borel field $\mathfrak{A}$. In particular, this section is $\mathcal{R}$-invariant. Therefore, we have proved the following proposition.

Proposition 5.6. *Let $\mathcal{R}$ be a discrete measured equivalence relation, acting ergodically on a Lebesgue space (S, μ). If the full group $[\mathcal{R}]$, equipped with the uniform topology, is amenable as a topological group, then the equivalence relation $\mathcal{R}$ is amenable.*

We summarize the results of this section in the following theorem.

Theorem 5.7. *Let $\mathcal{R}$ be a discrete measured equivalence relation, acting ergodically on a Lebesgue space (S, μ). Then the following are equivalent.*

(1) *$\mathcal{R}$ is an amenable equivalence relation.*

(2) *$[\mathcal{R}]$ with the uniform topology is amenable.*

(3) *$[\mathcal{R}]$ with the uniform topology is extremely amenable.*

(4) *$[\mathcal{R}]$ with the uniform topology is a Lévy group.*

Proof. The implications (4) $\implies$ (3) $\implies$ (2) are trivial, (2) $\implies$ (1) is Proposition 5.6, and (1) $\implies$ (4) is Proposition 5.3. □

Remark 5.8. Let G be a discrete countable group acting freely and ergodically on a Lebesgue space (S, μ). Let $N[G]$ denote the normalizer of the full group. Any element of $N[G]$ (respectively, $[G]$) induces an automorphism (respectively, an inner automorphism) of the von Neumann factor R, normalizing $M(L^\infty(S, \mu))$ (we keep the notation of § 5.1). Then the u-topology (see § 3.3) on $N[G]$ coincides with the normal topology defined in § 3.1 of [**11**] and is coarser than the uniform topology. By Proposition 5.3, if (S, μ) is an amenable G-space, then $[G]$, endowed with the u-topology, is a Lévy group and in particular extremely amenable.

6. Groups of non-singular transformations

6.1. Extreme amenability of $\mathrm{Aut}^*(X, \mu)$ with the weak topology

In this subsection, we will prove the following result.

Theorem 6.1. $\mathrm{Aut}^*(X, \mu)_{\mathrm{w}}$ *with the weak topology is a Lévy group and therefore extremely amenable.*

If $(X, \mathcal{F}, \mu)$ is a non-atomic Lebesgue measure space, then it is isomorphic to the circle $\mathbb{S}^1$, with the Lebesgue measure λ and therefore the topological groups $\mathrm{Aut}^*(X, \mu)$ and $\mathrm{Aut}^*(\mathbb{S}^1, \lambda)$ are isomorphic. In [**38**], Katznelson shows (Theorem 4.1) that the set of all orientation preserving C^∞ diffeomorphisms of the circle $\mathbb{S}^1$ of type III_1 is dense in the group of all measure class preserving C^∞ diffeomorphisms of $\mathbb{S}^1$. In particular this insures the existence of ergodic type III_1 transformations of (X, μ).

Theorem 6.1 will therefore follow from Proposition 5.3 and the following proposition.

Proposition 6.2. *It T is an ergodic measure class preserving transformation of (X, μ) of type III_1, then the full group $[T]$ is weakly dense in* $\mathrm{Aut}^*(X, \mu)$.

In order to prove Proposition 6.2, we need to introduce a few concepts.

Following Tulcea [**58**], call a measurable mapping $f : X \to Y$ between two finite measure spaces a *Lebesgue mapping* if for every measurable $A \subseteq X$

$$\mu_Y(f(A)) = \frac{\mu_Y(Y)}{\mu_X(X)}\mu_X(A).$$

If X and Y are measurable subsets of a standard Lebesgue probability space, there is always a Lebesgue mapping from X onto Y.

Definition 6.3. Say that a non-singular transformation $g \in \mathrm{Aut}^*(X,\mu)$ is a *Lebesgue permutation* with respect to a finite measurable partition $\mathcal{P}$ of X, if

(1) for each element $A \in \mathcal{P}$ the image $g(A)$ also belongs to $\mathcal{P}$, and the restriction $g|_A$ is a Lebesgue mapping, and

(2) if $A \in \mathcal{P}$ and $g(A) = A$, then $g|_A = \mathrm{Id}_A$.

The following was deduced by Tulcea [**58**, Theorem 1] from a theorem, attributed by her to Linderholm.

Theorem 6.4. *Lebesgue permutations are everywhere dense in the uniform topology on the group* $\mathrm{Aut}^*(X,\mu)$*. Moreover, given a transformation* $\tau \in \mathrm{Aut}^*(X,\mu)$*, an* $\varepsilon > 0$*, and a finite measurable partition* $\mathcal{Q}$ *of* X*, there exists a finite partition* $\mathcal{Q}_1 \prec \mathcal{Q}$ *and a Lebesgue permutation* σ *with respect to* $\mathcal{Q}_1$ *such that* $d_{\mathrm{unif}}(\tau,\sigma) < \varepsilon$*.* □

Proof of Proposition 6.2. Let V be a weak neighbourhood of the identity of $\mathrm{Aut}^*(X,\mu)$, determined by an $\varepsilon > 0$ and a finite measurable partition $\mathcal{Q}$ of X, so that

$$V = \{\beta \in \mathrm{Aut}^*(X,\mu) : \|\chi_A - \beta\chi_A\|_1 < \varepsilon \text{ for each } A \in \mathcal{Q}\}.$$

By Theorem 6.4, it is enough, given a finite partition $\mathcal{P} = \{P_1, P_2, \dots, P_n\} \prec \mathcal{Q}$ of (X,μ) and a Lebesgue permutation τ of (X,μ) with respect to $\mathcal{P}$, to find a $\sigma \in [T]$ with $\sigma^{-1}\tau \in V$.

By Lemma 32 in [**34**], for each $1 \leqslant i \leqslant n$, there exists $\sigma_i \in [T]$ such that

(1) $\sigma_i(P_i) = \tau(P_i)$ and if $\tau(P_i) = P_i$, then $\sigma_i = \mathrm{Id}$;

(2) $\displaystyle\left|\frac{\mathrm{d}\mu \circ \sigma_i}{\mathrm{d}\mu}(x) - \frac{\mu(\tau(P_i))}{\mu(P_i)}\right| < \varepsilon$ for μ-a.e. $x \in P_i$.

Let $\sigma \in [T]$ be given by $\sigma|_{P_i} = \sigma_i$. As a consequence of (2), one has for μ-a.e. $x \in X$,

$$\left|\frac{\mathrm{d}\mu \circ \sigma}{\mathrm{d}\mu}(x) - \frac{\mathrm{d}\mu \circ \tau}{\mathrm{d}\mu}(x)\right| < \varepsilon.$$

Moreover, as $\mathcal{P} \prec \mathcal{Q}$, one has $\sigma^{-1}\tau(A) = A$ for each $A \in \mathcal{Q}$, and so $\sigma^{-1}\tau \in V$, as required. □

Remark 6.5. We do not know if the group $\mathrm{Aut}^*(X,\mu)_{\mathrm{u}}$ with the uniform topology is (extremely) amenable.

6.2. Groups of isometries of L^p

In this subsection we will prove the following result.

Theorem 6.6. *Let (X,μ) be a standard Borel space with a non-atomic measure and let $1 \leqslant p < \infty$. The group of isometries $\mathrm{Iso}(L^p(X,\mu))$ with the strong operator topology is a Lévy group (and therefore extremely amenable).*

In case $p=2$, the result by Gromov and Milman [**30**] that the unitary group $U(\ell^2)$ with the strong operator topology is a Lévy group has started the current problematics. For $p \neq 2$, however, the proof is different.

A description of the isometries of $L^p(X,\mu)$ was obtained by Banach [**1**, Theorem 11.5.I, p. 178], but the proof was apparently never published. The first available proof (of a more general result concerning not necessarily surjective isometries) belongs to Lamperti [**41**], whose statement of the main result (Theorem 3.1) was not quite correct, as noticed in [**5**] (see also [**31**]).

Theorem 6.7 (Banach). *Let $1 \leqslant p < \infty$, $p \neq 2$, let (X,μ) be a finite measure space, and let T be a surjective isometry of $L^p(X,\mu)$. Then there is an invertible measure class preserving transformation σ of X and a measurable function h with $|h|=1$ such that*

$$Tf = h\,{}^{\sigma}\!f. \tag{6.1}$$

□

Recall that $\mathrm{Aut}^*(X,\mu)_{\mathrm{w}}$ can be identified with a (closed) topological subgroup of the group $\mathrm{Iso}(L^p(X,\mu))$, equipped with the strong operator topology, through the left quasi-regular representation:

$${}^{\sigma}\!f(x) = f(\sigma^{-1}x)\biggl(\frac{\mathrm{d}(\mu\circ\sigma^{-1})}{\mathrm{d}\mu}(x)\biggr)^{1/p}, \quad \sigma \in \mathrm{Aut}^*(X,\mu).$$

Invertible (the same as onto) isometries of $L^p(X,\mu)$ correspond to the invertible regular set isomorphisms, that is, invertible measure class preserving transformations of (X,μ), in which case $h\phi(f) = {}^{\phi^{-1}}\!f$. We obtain the following corollary.

Corollary 6.8. *Let $1 \leqslant p < \infty$, $p \neq 2$, let (X,μ) be a finite measure space. The group $\mathrm{Iso}(L^p(X,\mu))$ is the semidirect product of the subgroup $\mathrm{Aut}^*(X,\mu)$ and the normal subgroup $L^0(X,\mu;U(1))$ of all measurable maps from X to the circle rotation group. Moreover, the group $\mathrm{Iso}(L^p(X,\mu))$ with the strong operator topology is the semidirect product of $\mathrm{Aut}^*(X,\mu)_{\mathrm{w}}$ and the group $L^0(X,\mu;U(1))$ equipped with the topology of convergence in measure.*

Proof. According to Banach's theorem (Theorem 6.7), every element $T \in \mathrm{Iso}(L^p(X,\mu))$ admits a (clearly unique) decomposition of the form $Tf = h\cdot{}^{\sigma}\!f$, for $f \in L^p(X,\mu)$, where $h \in L^0(X,\mu;U(1))$ and $\sigma \in \mathrm{Aut}^*(X,\mu)$. To establish the first claim, it is therefore enough to prove that $L^0(X,\mu;U(1))$ is normal in $\mathrm{Iso}(L^p(X,\mu))$ or, which is the same,

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invariant under inner automorphisms generated by the elements of $\mathrm{Aut}^*(X,\mu)$. Let $h \in L^0(X,\mu;U(1))$ and $\sigma \in \mathrm{Aut}^*(X,\mu)$. One has

$$\begin{aligned}(\sigma^{-1}h\sigma)(f)(x) &\equiv {}^{\sigma^{-1}}(h\cdot {}^{\sigma}f)(x)\\ &= (h\cdot {}^{\sigma}f)(\sigma x)\biggl(\frac{\mathrm{d}(\mu\circ\sigma)}{\mathrm{d}\mu}(x)\biggr)^{1/p}\\ &= h(\sigma x)f(\sigma^{-1}\sigma x)\biggl(\frac{\mathrm{d}(\mu\circ\sigma^{-1})}{\mathrm{d}\mu}(\sigma x)\biggr)^{1/p}\biggl(\frac{\mathrm{d}(\mu\circ\sigma)}{\mathrm{d}\mu}(x)\biggr)^{1/p}\\ &= \sigma^{-1}(h)(x)f(x),\end{aligned}$$

and therefore $\sigma^{-1}h\sigma = \sigma^{-1}(h) \in L^0(X,\mu;U(1))$.

We have previously noted that $\mathrm{Aut}^*(X,\mu)_{\mathrm{w}}$ is a closed topological subgroup of $\mathrm{Iso}(L^p(X,\mu))$. The same is true of $L^0(X,\mu;U(1))$. Indeed, every L^p-metric, $1 \leqslant p < \infty$, induces the topology of convergence in measure on $L^0(X,\mu;U(1))$ (see, for example, Example 13.33 in [**36**]). Using this observation, it is easy to verify that, first, for every $f \in L^p(X,\mu)$ the orbit map of the multiplication action

$$L^0(X,\mu;U(1)) \ni h \mapsto h\cdot f \in L^p(X,\mu)$$

is continuous, while for $f(x)\equiv 1$ it is a homeomorphic embedding.

As a corollary (which can be also checked directly), the action of $\mathrm{Aut}^*(X,\mu)_{\mathrm{w}}$ on $L^0(X,\mu;U(1))$, defined by $(\sigma,h)\mapsto\sigma^{-1}(h)$, is continuous as a map

$$\mathrm{Aut}^*(X,\mu)_{\mathrm{w}} \times L^0(X,\mu;U(1)) \to L^0(X,\mu;U(1)).$$

Therefore, the semi-direct product $\mathrm{Aut}^*(X,\mu)_{\mathrm{w}} \ltimes L^0(X,\mu;U(1))$ is a (Polish) topological group.

The algebraic automorphism given by the multiplication map

$$\mathrm{Aut}^*(X,\mu)_{\mathrm{w}} \ltimes L^0(X,\mu;U(1)) \to \mathrm{Iso}(L^p(X,\mu))$$

is continuous, and since all groups involved are Polish, it is a topological isomorphism by the standard Open Mapping Theorem for Polish groups (cf. Corollary 3 (p. 98) and Theorem 3 (p. 90) in [**37**]). □

Proof of Theorem 6.6. As in § 5.2, we will identify the measure space (X,μ) with the compact abelian group $X=\prod\{0,1\}$ equipped with an ergodic non-atomic quasi-invariant measure μ. Let L_n denote the subgroup of $L^0(X,\mu;U(1))$ consisting of functions constant on cylinder sets determined by the first n coordinates. It is clear that the group L_n is invariant under all inner automorphisms generated by elements of the permutation group $\mathrm{Bij}(n)$ of the finite set S_n.

The normalized Haar measures on $\mathrm{Bij}(n)$ concentrate in $\mathrm{Aut}^*(X,\mu)_{\mathrm{w}}$ by Corollary 2.13. Since concentration depends on the uniform structure only, one can assume without loss in generality that the groups L_n are equipped with the $L^1(\mu)$-metric induced from $L^0(X,\mu;U(1))$ (because it induces the topology of convergence in measure on the former

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group and is translation invariant, therefore induces the additive uniformity). By Corollary 2.11, the normalized Haar measures on L_n, $n \in \mathbb{N}$, concentrate in $L^0(X, \mu; U(1))$.

The semidirect product groups $\mathrm{Bij}(n) \ltimes L_n$ are compact, and the normalized Haar measures on them are products of Haar measures, respectively, on $\mathrm{Bij}(n)$ and on L_n. Therefore, those measures concentrate in $\mathrm{Iso}(L^p(X, \mu))$ by Lemma 2.1.

Finally, the union of subgroups $\mathrm{Bij}(n) \ltimes L_n$ is everywhere dense in $\mathrm{Iso}(L^p(X, \mu))$ by Corollary 6.8, because the union of $\mathrm{Bij}(n)$ is uniformly dense in its own full group by Lemma 5.2, and therefore weakly dense in $\mathrm{Aut}^*(X, \mu)$ by Proposition 6.2, while the groups L_n of simple functions are dense in $L^0(X, \mu; U(1))$. □

Theorem 6.6 was conjectured by one of the present authors in [**52**].

7. Concentration to a non-trivial space

Gromov [**29**, Chapter $3\frac{1}{2}$, § 45, p. 200] has defined a metric on the space of equivalence classes of Polish *mm*-spaces in such a way that a sequence $X_n = (X_n, d_n, \mu_n)$ of *mm*-spaces forms a Lévy family if and only if it converges to the trivial *mm*-space $\{*\}$.

This approach allows one to talk of *concentration to a non-trivial mm-space*. According to Gromov, this type of concentration commonly occurs in statistical physics. At the same time, there are very few known non-trivial examples of this kind in the context of transformation groups. The aim of this section is to give some natural examples of concentration of this type.

We will recall Gromov's construction. Let $\mu^{(1)}$ denote the Lebesgue measure on the unit interval $\mathbb{I} = [0, 1]$. On the space $L^0(0, 1)$ of all (equivalence classes) of measurable real-valued functions define the metric me_1, generating the topology of convergence in measure, as follows: $\mathrm{me}_1(h_1, h_2)$ is the infimum of all $\lambda > 0$ with the property

$$\mu^{(1)}\{x \in \mathbb{I} : |h_1(x) - h_2(x)| > \lambda\} < \lambda.$$

Let $X = (X, d_X, \mu_X)$ and $Y = (Y, d_Y, \mu_Y)$ be two Polish *mm*-spaces. There exist measurable maps $f : \mathbb{I} \to X$, $g : \mathbb{I} \to Y$ such that $\mu_X = f_* \mu^{(1)}$ and $\mu_Y = g_* \mu^{(1)}$. Denote by L_f the set of all functions of the form $h = h_1 \circ f$, where $h_1 : X \to \mathbb{R}$ is 1-Lipschitz. Similarly, define the set L_g.

Let $\underline{H}_1\mathcal{L}\iota(X, Y)$ be the infimum of Hausdorff distances between L_f and L_g (with regard to the metric me_1), taken over all parametrizations f and g as above. One can prove that $\underline{H}_1\mathcal{L}\iota$ is a metric on the space of (isomorphism classes of) all Polish *mm*-spaces. (The difficult part of the proof is verifying the first axiom of a metric.)

A sequence of *mm*-spaces $X_n = (X_n, d_n, \mu_n)$ forms a Lévy family if and only if it converges to the trivial (one-point) *mm*-space in the metric $\underline{H}_1\mathcal{L}\iota$:

$$X_n \xrightarrow{\underline{H}_1\mathcal{L}\iota} \{*\}.$$

(This is a reformulation of Lemma 2.4.)

Theorem 7.1 (Gromov). *Let $Y = (Y, d_Y, \nu)$ and $Z_n = (Z_n, d_n, \mu_n)$, $n \in \mathbb{N}$, be mm-spaces, where (Z_n) forms a Lévy family. Let $d_Y \oplus^{(2)} d_n$ denote the ℓ_2-type sum*

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of the metrics d_Y and d_n on the product space $Y \times Z_n$. Then the family of mm-spaces $(Y \times Z_n,\ d_Y \oplus^{(2)} d_n,\ \nu \otimes \mu_n)$ concentrates to the mm-space (Y, d_Y, ν):

$$(Y \times Z_n, d_Y \oplus^{(2)} d_n, \nu \otimes \mu_n) \xrightarrow{H_1\mathcal{L}\iota} (Y, d_Y, \nu). \qquad \square$$

We are going to show a situation where Gromov's theorem applies naturally.

Lemma 7.2. *Let G be a metrizable topological group which is, as a topological group, a semidirect product of a compact subgroup H and a closed normal subgroup N:*

$$G = H \ltimes N.$$

Assume that N is a Lévy group, that is, there is a directed family of compact subgroups (K_α) of N, having an everywhere dense union and such that (K_α, μ_α) form a Lévy family with regard to the right uniform structure on G. Let ν denote the normalized Haar measure on H. Then there exist a right-invariant compatible metric ρ on G such that the family of mm-spaces $H \cdot K_\alpha$, equipped with the metric ρ and the convolution of normalized Haar measures, concentrates to the mm-space $(H, \rho|_H, \nu)$:

$$(H \cdot K_\alpha, \rho, \nu * \mu_\alpha) \xrightarrow{H_1\mathcal{L}\iota} (H, \rho|_H, \nu).$$

If in addition each of the subgroups (K_α) is invariant under the action of H by conjugations, then every $H \cdot K_\alpha$ is a compact subgroup of G.

Proof. Denote by τ the continuous action of H on N by topological group isomorphisms, so that $G = H \ltimes_\tau N$. Fix any compatible right-invariant metric, d, on N, and define a new metric, ς, as follows: for $x, y \in N$, set

$$\varsigma_N(x, y) = \int_H d(\tau_h x, \tau_h y)\, \mathrm{d}\nu(h).$$

This ς_N is also a right-invariant compatible metric on N, which in addition is invariant under the action τ.

Now choose a bi-invariant compatible metric, ς_H, on the compact group H. Define a metric ρ on G as the ℓ_2-type sum of ς_H and ς_N:

$$\rho((h, x), (h', x')) = \sqrt{\varsigma_H(h, h')^2 + \varsigma_N(x, x')^2}.$$

It is easily verified that ρ is a right-invariant compatible metric on G. Besides, ρ is invariant under left multiplication by elements of H:

$$\begin{aligned}
\rho((h, e) \cdot (g, x), (h, e) \cdot (g', x')) &= \rho((hg, \tau_h x), (hg', \tau_h x')) \\
&= (\varsigma_H(hg, hg')^2 + \varsigma_N(\tau_h x, \tau_h x')^2)^{1/2} \\
&= (\varsigma_H(g, g')^2 + \varsigma_N(x, x')^2)^{1/2} \\
&= \rho((g, x), (g', x')).
\end{aligned}$$

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The restriction of ρ to each $H \cdot K_\alpha$ is just the ℓ_2-type sum of ς_H with the restriction of ς_N to K_α.

Since the multiplication mapping $H \times K_\alpha \to H \cdot K_\alpha$ is one-to-one (by assumption) and also of course continuous, it is a homeomorphic embedding of $H \times K_\alpha$ into our topological group G. In particular, the convolution of measures $\nu * \mu_\alpha$ is in fact the product measure, $\nu \otimes \mu_\alpha$. Now Gromov's theorem (Theorem 7.1) applies.

The last statement of the lemma is obvious. □

Remark 7.3. As a direct consequence of [**52**, Lemma 2.3], under the assumptions of Lemma 7.2, H forms the universal minimal flow of the topological group G. It would be interesting to find more general situations where concentration of subobjects of G to a non-trivial compact space X would mean that X is a universal minimal flow of G.

Using the description of the automorphism groups of injective factors of type III, we now present concrete examples of concentration to a non-trivial space. In particular, in Example 7.6 below we get a sequence of compact subgroups concentrating to a non-trivial space with regard to one group topology, and to a trivial space with regard to another group topology.

Example 7.4. For $0 < \lambda < 1$, let us consider $G = \oplus(\mathbb{Z}/2)$ acting freely and ergodically on

$$(X, \mu_\lambda) = \prod\left(\{0,1\}, \left\{\frac{1}{1+\lambda}, \frac{\lambda}{1+\lambda}\right\}\right),$$

and let R_λ denote the von Neumann factor of type III_λ, obtained by the group measure space construction. Recall that R_λ is approximately finite dimensional and by [**9**] is up to isomorphism the unique injective factor of type III_λ. To any automorphism α of R_λ is associated its modulus, $\mathrm{mod}\,\alpha$, which belongs to $\mathbb{R}/(\log\lambda)\mathbb{Z}$ (see, for example, [**56**, Chapter XII]). If $\mathrm{Aut}(R_\lambda)$ is endowed with the u-topology, then the map $\mathrm{mod} : \mathrm{Aut}(R_\lambda) \to \mathbb{R}/(\log\lambda)\mathbb{Z}$ is a continuous homomorphism. Moreover (see [**56**, Theorem XVIII.1.11]), there is a natural short exact sequence

$$1 \to \overline{\mathrm{Inn}(R_\lambda)} \to \mathrm{Aut}(R_\lambda) \xrightarrow{\mathrm{mod}} \mathbb{R}/(\log\lambda)\mathbb{Z} \to 1,$$

and a continuous action $\tilde{\theta} : \mathbb{R}/(\log\lambda)\mathbb{Z} \to \mathrm{Aut}(R_\lambda)$ with

$$\mathrm{mod}(\tilde{\theta}(s)) = s, \quad s \in \mathbb{R}/(\log\lambda)\mathbb{Z}.$$

By Proposition 3.9, $\mathrm{Inn}(R_\lambda)$ and consequently $\overline{\mathrm{Inn}(R_\lambda)}$ are Lévy groups. Hence by Lemma 7.2 the group $\mathrm{Aut}(R_\lambda)$ contains a chain of compact mm-spaces with everywhere dense union which concentrates to $\mathbb{R}/(\log\lambda)\mathbb{Z}$.

Example 7.5. Let G be a countable discrete group acting freely and ergodically on the Lebesgue space (X,μ). By Remark 5.8, if (X,μ) is an amenable G-space, then the full group $[G]$, endowed with the u-topology, is extremely amenable.

If the dynamical system (X,μ,G) is of type III_λ, $0 < \lambda < 1$, then there is a split exact sequence

$$1 \to \overline{[G]}^{u\text{-top}} \to N[G] \xrightarrow{\mathrm{mod}} \mathbb{R}/(\log\lambda)\mathbb{Z} \to 1$$

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(see [**11**, 3.3]). As $\overline{[G]}^{u\text{-top}}$ is extremely amenable and moreover a Lévy group, $N[G]$ contains a chain of compact mm-spaces with everywhere dense union which concentrates to $\mathbb{R}/(\log\lambda)\mathbb{Z}$.

Example 7.6. Let R_λ be the injective factor of type III_λ (see Example 7.4), realized as an infinite tensor product of factors of type I_2, i.e. $R_\lambda = \otimes(M_2(\mathbb{C}), \varphi_\lambda)$, where $\varphi_\lambda = \mathrm{Tr}(h_\lambda, \cdot)$ and h_λ is the density matrix

$$\begin{pmatrix} \dfrac{1}{1+\lambda} & \\ & \dfrac{\lambda}{1+\lambda} \end{pmatrix}.$$

If φ denotes the state $\otimes\varphi_\lambda$, then the modular group of automorphisms σ^φ is given by

$$\sigma_t^\varphi = \otimes \operatorname{Ad}\begin{pmatrix} 1 & \\ & \lambda^{\mathrm{i}t} \end{pmatrix}, \quad t \in \mathbb{R}.$$

For $m \geqslant 1$, let N_m be the type I_{2^m} subfactor of M generated by

$$\{x_1 \otimes x_2 \otimes \cdots \otimes x_m \otimes 1 \otimes 1 \otimes \cdots : x_k \in M_2(\mathbb{C})\}.$$

Let K_m denote the compact subgroup of $\mathrm{Inn}(R_\alpha)$ given by $\{\mathrm{Ad}(u) : u \in U(N_m)\}$. Let $\mathrm{Cnt}(R_\lambda) \lhd \mathrm{Aut}(R_\lambda)$ be the normal subgroup of centrally trivial automorphisms of R_λ. Let $T = -2\pi/\log\lambda$. By [**56**, Theorem XVIII.1.11],

$$\mathrm{Cnt}(R_\lambda) = \mathrm{Inn}(R_\lambda) \rtimes_{\sigma^\varphi} \mathbb{R}/T\mathbb{Z}. \tag{7.1}$$

By Propositions 3.5 and 3.9, $\{K_m\}_{m\geqslant 1}$ forms a Lévy family of subgroups of $\mathrm{Inn}(R_\lambda)$. As for $t \in \mathbb{R}$, σ_t^φ leaves N_m globally fixed,

$$K_m \cdot \mathbb{R}/T\mathbb{Z} = \{\operatorname{Ad} u \circ \sigma_s^\varphi : u \in U(N_m),\ s \in \mathbb{R}\}$$

is a subgroup of $\mathrm{Cnt}(R_\lambda)$.

We will now consider two group topologies on $\mathrm{Cnt}(R_\lambda)$, with regard to which the family of compact subgroups $(K_m \cdot \mathbb{R}/T\mathbb{Z})_{m=1}^\infty$ exhibit different kinds of concentration behaviour.

The first one is the topology of semidirect product using the decomposition in equation (7.1). Consider $\mathrm{Inn}(R_\lambda)$ as the topological factor-group of the group $U(R_\lambda)$, equipped with the strong operator topology (or the $s(R_\lambda, (R_\lambda)_*)$-topology. As a factor-group of a Polish group by a compact subgroup (the central subgroup isomorphic to the circle rotation group), $\mathrm{Inn}(R_\lambda)$ is a Polish group itself. It is easy to check that the action σ^φ of $\mathbb{R}/T\mathbb{Z}$ on $\mathrm{Inn}(R_\lambda)$ is continuous. Therefore, $\mathrm{Cnt}(R_\lambda)$ is a Polish group, and it contains a chain of compact subgroups $K_m \cdot \mathbb{R}/T\mathbb{Z}$, $m \in \mathbb{N}$, with everywhere dense union, which family of subgroups concentrate in Gromov's sense to the circle rotation group $\mathbb{R}/T\mathbb{Z}$ by force of Lemma 7.2.

Notice also that the compact space $\mathbb{R}/T\mathbb{Z}$ forms the universal minimal flow of the topological group $\mathrm{Cnt}(R_\lambda)$ (cf. Remark 7.3).

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The second topology on the group $\mathrm{Cnt}(R_\lambda)$ is the familiar u-topology. In this case, $\mathrm{Inn}(R_\lambda)$ is everywhere dense in $\mathrm{Cnt}(R_\lambda)$, and in particular the group $\mathrm{Cnt}(R_\lambda)$ is extremely amenable. Let d be a compatible metric on $\mathrm{Cnt}(R_\lambda)$, invariant on one side (e.g. on the left). For every $\varepsilon > 0$, if m is sufficiently large, the ε-neighbourhood of K_m contains the subgroup $\mathbb{R}/T\mathbb{Z}$ and therefore all of $\mathrm{Cnt}(R_\lambda)$. Now one can easily see that the compact subgroups $K_m \cdot \mathbb{R}/T\mathbb{Z}$, $m \in \mathbb{N}$, form a Lévy family and thus, in contrast to the previous situation, concentrate to a one-point space.

Acknowledgements. The investigation has been partially supported by NSERC operating grants and by the Swiss Fund for Scientific Research (both authors), as well as the Marsden Fund of the Royal Society of New Zealand and a University of Ottawa start-up grant (V.P. only). Both authors are grateful to Alexander Kechris for useful comments, and to Pierre de la Harpe for his hospitality at the University of Geneva where the present research collaboration started in April 1999. The authors are also grateful to the referee whose suggestions have led to a considerable reworking of the article: several proofs have been simplified and made clearer.

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