Dynamics of non-classical interval exchanges

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Abstract. A natural generalization of interval exchange maps are linear involutions, first
introduced by Danthony and Nogueira [Measured foliations on non-orientable surfaces.
Ann. Sci. Éc. Norm. Supér. (4) **26**(6) (1993), 645–664]. Recurrent train tracks with
a single switch provide a subclass of linear involutions. We call such linear involutions
non-classical interval exchanges. They are related to measured foliations on orientable
flat surfaces. Non-classical interval exchanges can be studied as a dynamical system
by considering Rauzy induction in this context. This gives a refinement process on the
parameter space similar to Kerckhoff's simplicial systems. We show that the refinement
process gives an expansion that has a key dynamical property called *uniform distortion*.
We use uniform distortion to prove normality of the expansion. Consequently, we prove
an analog of Keane's conjecture: almost every non-classical interval exchange is uniquely
ergodic. Uniform distortion has been independently shown in [A. Avila and M. Resende.
Exponential mixing for the Teichmüller flow in the space of quadratic differentials,
http://arxiv.org/abs/0908.1102].

1. *Introduction*
Here, we are interested in the dynamical properties of non-classical interval exchanges.
In a classical interval exchange, an interval I is partitioned into d subintervals; these
subintervals are permuted and glued back, preserving orientation to get I. This gives a
Lebesgue-measure-preserving map from I to itself. The parameter data that completely
determines the map is as follows: first, the lengths of the subintervals, and second, the
permutation for reshuffling the subintervals. There is a way to draw these maps pictorially.

We draw the original interval I horizontally and then thicken it vertically to get two
copies I_+ and I_-, the top and the bottom intervals. Divide I_+ into d subintervals with the
prescribed lengths. Divide I_- also into d subintervals but incorporate the permutation to
decide the lengths. Each subinterval of I_+ pairs off, by the permutation, with a subinterval
of I_- with the same length. Join these subintervals by a band of uniform width equal to
their common length. For example, Figure 1.1 shows a classical interval exchange with
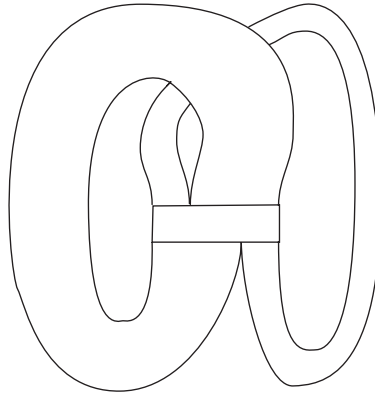
FIGURE 1.1. A classical interval exchange.

two bands. The interval exchange is exhibited as a map from I_+ to I_+ given by the vertical flow up the bands from I_+ to I_- followed by switching to I_+ by identity. The inverse of the interval exchange is then a map from I_- to I_- by flowing down the bands from I_- followed by switching to I_- by identity.

Train tracks enter the discussion as follows: to each band we associate a central edge that goes from I_+ to I_-. If we collapse each band to its central edge, then collapse the thickened interval horizontally to a vertical edge and finally collapse the vertical edge, we get a train track with a single switch. The train track has the property that every branch outgoing from one side of the switch is incoming from the other side. A generalization is obtained by relaxing this last property. The associated picture is then a thickened interval with bands in which bands from I_+ to I_+, or I_- to I_-, are allowed. In the terminology of [4], the resulting dynamical system is a linear involution with no flips. The bands from I_+ to I_+ or I_- to I_- will be called orientation-reversing bands. As before, the space of parameters is the widths of the bands and the exchange combinatorics.

Here, we restrict to recurrent tracks. This imposes the condition that there are orientation-reversing bands on both I_+ and I_-. We call such linear involutions non-classical interval exchanges. They define a Lebesgue-measure-preserving dynamical system on $I_+ \sqcup I_-$. See §2 for precise definitions and details. Non-classical interval exchanges are first-return maps induced on a transverse interval by vertical measured foliations on flat surfaces defined by quadratic differentials that are not squares of abelian differentials.

In contrast to classical interval exchanges, the widths of bands satisfy an additional constraint: the sum of the widths of the orientation-reversing bands on I_+ is equal to the sum of the widths of the orientation-reversing bands on I_-.

1.1. *Keane's conjecture.* For classical interval exchanges, Keane conjectured that if the reshuffle permutation is irreducible, then for almost all widths of the bands, the associated exchange is uniquely ergodic with the Lebesgue measure the only invariant probability measure. This conjecture was first proved, independently, by Masur [11] and Veech [17]. See also Rees [16] and Zorich [19]. A key tool was Rauzy induction, alternatively

called splitting: this is the process of inducing a classical interval exchange on a suitable subinterval by considering the first-return map. Veech [**17**] used this to develop a continued fraction expansion for classical interval exchanges. In fact, when $d = 2$, the expansion is equivalent to the continued fraction expansion of the ratio of the widths of the two bands. Veech showed that the expansion has most of the nice properties of the classical Gauss map. He derived unique ergodicity as a consequence of this.

A similar conjecture was also made for transverse measured foliations on a Riemann surface. Masur proved this conjecture along with the original Keane conjecture in [**11**]; his approach relied on techniques from Teichmüller theory and applied to both contexts.

1.1.1. *Kerckhoff's approach.*

Later, Kerckhoff [**8**] developed a different and more elementary approach, which does not rely on Teichmüller theory, to include both conjectures under a unified setup. In his formulation, Rauzy induction on one hand and train track splittings on the other are both encoded as a refinement process on the associated parameter spaces.

For a classical interval exchange with d bands, the parameter space at each stage can be identified with the standard simplex of dimension $(d - 1)$. Rauzy induction splits this simplex into two halves, each a $(d - 1)$-simplex itself. A point in the original simplex picks out the half in which it lies. Additionally, there is a map from the new parameter space to the initial parameter space identifying the standard simplex with the half that is picked out. After this identification, the induction is applied to the simplex given by the new stage and the process continues. Associated to a finite sequence of inductions, there is a map from the current simplex back into the starting simplex. This map is defined iteratively as a composition of maps in the individual steps of the sequence.

It turns out that, almost surely, iterations give an infinite expansion consisting of nested simplices. There is a canonical identification of the infinite intersection of these nested simplices with the set of invariant probability measures. The refinement process on a simplex can be defined purely abstractly; Kerckhoff called this a simplicial system and the resulting expansion, a simplicial expansion. He showed that if a simplicial system satisfies a combinatorial condition called the 'absence of isolated blocks', then, for almost every initial point in the starting simplex, the resulting simplicial expansion is normal: almost surely, every finite sequence of inductions that can occur in the expansion does occur infinitely often. Finally, normality and the existence of finite sequences which shrink diameter by a definite amount ensure that, almost surely, the nested sequence of simplices actually nests down to a point. To finish the proof for classical interval exchanges, Kerckhoff showed that if the reshuffle permutation is irreducible, then the associated simplicial system has no isolated blocks. Consequently, almost surely, the expansion nests down to a point and so there is a unique invariant probability measure.

For general train tracks, the parameter space is the set of possible weights carried by the train track, normalized so that their sum is one. This is a convex subset of the ambient simplex, cut out by the switch conditions of the train track. In this context, some aspects of the refinement process on the associated parameter space are understood. See [**13**]. However, we do not know how to carry out Kerckhoff's approach in this general setup.

Here, we restrict to non-classical interval exchanges instead. In terms of encoding measured foliations that arise in a given stratum of quadratic differentials, there is no loss

of generality incurred by this restriction. The advantages are as follows: first, the parameter space has codimension one in the standard $(d-1)$-dimensional simplex and is easier to analyze, second, the combinatorics of the Rauzy diagram is better understood by the work of Boissy and Lanneau [2]. The main theorem we prove is the following.

THEOREM 1.1. (Normality) *For non-classical interval exchanges, almost surely, the expansion is normal, i.e., every finite non-transient sequence that can occur in the expansion does occur, infinitely often.*

The adjective 'non-transient' will become clear in §5.

As we shall see in §6, combinatorially, there are finitely many subsets inside the standard $(d-1)$-simplex which can be the parameter space for a non-classical interval exchange. We call each such subset a configuration space and denote them by W_r. For all subsequent discussions, we fix a combinatorial type π_0 for the initial non-classical interval exchange and call it the starting stage. The starting stage is assumed to be strongly irreducible and in an attractor of the Rauzy diagram (see §6 for the precise details). Let W_0 be the configuration space at π_0. As we shall see in §6, for almost every point in W_0, iterated Rauzy induction defines an infinite expansion. Let π' be a subsequent stage obtained from π_0 by a finite sequence of inductions. Let W' be the configuration space at π'. The associated map f' relates the new widths of the bands to the starting widths in π_0, by giving a diffeomorphism from Δ onto a subset of itself, such that $f'(W') = W_0 \cap f'(\Delta)$. Fix a constant $C > 1$. The stage π is C-uniformly distorted if, for any pair of points $\mathbf{y}$, $\bar{\mathbf{y}}$ in W, the Jacobian of f', thought of as a map from W to W_0, satisfies

$$\frac{1}{C} < \frac{\mathcal{J}(f')(\mathbf{y})}{\mathcal{J}(f')(\bar{\mathbf{y}})} < C.$$

Let π be any stage obtained from π_0 by a finite sequence of inductions. Let W be the configuration space at π and f the associated map. The main technical theorem required to prove Theorem 1.1 is the uniform distortion theorem, stated below.

THEOREM 1.2. (Uniform distortion) *There exists a constant $C > 1$, independent of the stage π, such that, for almost every $\mathbf{x} \in f(W)$, the associated expansion of $\mathbf{x}$ after π has some stage π', depending on $\mathbf{x}$, such that π' is C-uniformly distorted. Moreover, the combinatorics of π' can be assumed to be the same as π_0.*

Theorem 1.2 is proved independently in [1] as a key step in studying Teichmüller flow on strata of quadratic differentials. See [1, Theorem 4.2]. In [6], we apply Theorem 1.2 to a question about random walks on mapping class groups. In a different direction, Theorem 1.2 also fixes a small gap in the proof in [9] that the handlebody limit set has measure zero [7].

Theorem 1.2 implies that the relative probability that a permissible finite sequence occurs right after a uniformly distorted stage is roughly the same as the probability that an expansion begins with that sequence. Normality is a straightforward consequence of this phenomenon.

This approach was outlined by Kerckhoff in [8] and carried out for classical interval exchanges. For general train tracks, there are two issues. First, as we shall see in an example in §7, the probability of a split which is the proportion of the volume of the part

of the configuration space that is inside the smaller simplex picked out by the split can be very different from the ratio of the volumes of the ambient simplices. Second, splitting sequences of general train tracks can have isolated blocks. Because of these issues, the proof of unique ergodicity for measured foliations in [**8**] is incomplete.

We do not know how to fix this for general train tracks. Nevertheless, we resolve these problems for non-classical interval exchanges by proving Theorems 1.2 and 1.1. As a consequence, we also show in Theorem 13.1 that the map on the parameter space given by Rauzy induction is ergodic.

1.2. *Outline of the paper.* In §2, we see how the definition of a classical interval exchange generalizes naturally to a non-classical interval exchange. In §3, we define Rauzy induction and explain how it is encoded by matrices. In §4, we define the associated directed graph called the Rauzy diagram. In §5, we discuss irreducibility for non-classical interval exchanges. We explain the connection, established in [**2**], of the concept of irreducibility to the dynamics of the non-classical interval exchanges and attractors of the Rauzy diagram. We also explain the relevance of these issues to the main theorems, Theorems 1.2 and 1.1. In §6, we provide the details for the refinement process on the parameter space. We show that for almost all parameters, we get an infinite expansion. We then formulate the main theorem, Theorem 6.3, a more precise version of Theorem 1.2. In §7, we illustrate by an example the main ideas of Kerckhoff's original proof for classical interval exchanges. We also illustrate by way of an example the key issues that make the task harder for non-classical interval exchanges. One of the issues is that the parameter space for non-classical interval exchanges is codimension one in the ambient simplex. Resolving it requires understanding the Jacobian of the restriction of the projective linear map to this subset, instead of considering the full simplex. This is done in §8. In §9, we recall the main results in the measure theory of projective linear maps with non-negative entries from a standard simplex into itself, and use it to compute the measures of some standard subsets that we encounter later. Section 10 gives the detailed proof of Theorem 6.3. In §11, we prove normality from Theorem 6.3. Finally, in §12, we use normality to show unique ergodicity. As a consequence, we show in §13 that the Rauzy map on the parameter space is ergodic.

2. *Non-classical interval exchanges*

For a precise definition of a classical interval exchange, see [**18**]. Here, we focus on representing it pictorially.

Let $\mathcal{A}$ denote an alphabet over d letters. In the definition that follows, the set $\mathcal{A}$ labels the bands. A classical interval exchange is determined by the lengths (λ_α), $\alpha \in \mathcal{A}$ of the subintervals and bijections p_0 and p_1 from $\mathcal{A}$ to the set $\{1, \ldots, d\}$ as follows: in the plane, draw the interval $I = [0, \sum_\alpha \lambda_\alpha)$ along the horizontal axis and then thicken it slightly in the vertical direction to get two copies, I_+ and I_-. Call them top interval and bottom interval, respectively. Let $\epsilon : I_+ \sqcup I_- \to I_+ \sqcup I_-$ be the map that switches the intervals, i.e., $\epsilon(I_+) = I_-$ and $\epsilon(I_-) = I_+$. Subdivide I_+ into d subintervals with widths $\lambda_{p_0^{-1}(1)}, \ldots, \lambda_{p_0^{-1}(d)}$ from left to right. Subdivide I_- into d subintervals with widths $\lambda_{p_1^{-1}(1)}, \ldots, \lambda_{p_1^{-1}(d)}$ from left to right. For each $\alpha \in \mathcal{A}$, join the $p_0(\alpha)$ subinterval of I_+

to the $p_1(\alpha)$ subinterval of I_- by a band of uniform width λ_α. The vertical flow along the bands from I_+ to I_-, followed by ϵ, exhibits the classical interval exchange as a map from I_+ to I_+. Similarly, the inverse of the interval exchange is realized as a map from I_- to itself by flowing in reverse along the bands, followed by ϵ. The ambiguity in the definition at the endpoints of the subintervals is removed by requiring the endpoint flow along the band that lies to the left.

One can construct a train track from this picture. A *train track* is a one-dimensional CW complex with some additional structure. The edges are called *branches* and the vertices are called *switches*. There is a common point of tangency to all branches meeting at a switch. This splits the set of branches incident at a switch into two disjoint subsets, arbitrarily assigned as incoming and outgoing branches at that switch. Additionally, one assigns non-negative weights to the branches so that the *switch conditions* are satisfied: at each switch the sum of the weights of the outgoing branches is equal to the sum of the weights of the incoming branches.

In our picture, each band has a central edge which joins the midpoint of the subinterval of I_+ to the midpoint of the corresponding subinterval of I_-. Retract each band to its central edge and the thickened interval to a vertical edge. It is clear that this can be done in a way such that the edges associated to the bands share a vertical line of tangency on each side of the vertical edge. Finally, retract the central vertical edge to a point while preserving the vertical tangency. The result is a train track with a single switch. The branches are in bijection with the bands and every outgoing branch on one side of the switch is incoming from the other. If we assign the width of the band as the weight on the corresponding branch, then the weights satisfy the single-switch condition.

The first step towards defining non-classical interval exchanges is to relax the constraint that every outgoing branch from one side of the switch is incoming from the other, i.e., to allow bands from I_+ to I_+ and I_- to I_-. We call such bands *orientation reversing* because the flow along such a band reverses the orientation of a subinterval of I_+ or I_-.

A train track is *recurrent* if there is an assignment of weights satisfying the switch conditions, such that all weights are positive. Here, this implies that if there are orientation-reversing bands on one side then there has to be at least one orientation-reversing band on the other side.

Definition 2.1. A *non-classical interval exchange* is the pair of intervals I_+ and I_- with bands such that there are orientation-reversing bands on both I_+ and I_-, i.e., the underlying train track is recurrent. The transformation $T : I_+ \sqcup I_- \to I_+ \sqcup I_-$ defined by it is the following composition: except for the endpoints of the subintervals, every $x \in I_+ \sqcup I_-$ lies in exactly one band. Flow x along this band to its other end to get a point x'. Set $T(x)$ to be $\epsilon(x')$.

We shall no longer distinguish between the picture and the transformation, i.e., by the dynamics of a non-classical interval exchange, we shall mean the dynamics of T. It is clear from the definition, that the Lebesgue measure on $I_+ \sqcup I_-$ is invariant under T.

2.1. *Linear involutions.* To relate this to [**2**, Definition 2.1], the map $\tilde{T}$ in their notation is exactly the map given by the flow along the bands. Our requirement that there are orientation-reversing bands on I_+ and I_- is equivalent in their definition to imposing that there are subintervals of I_+ and I_- that $\tilde{T}$ maps to I_+ and I_-, respectively.
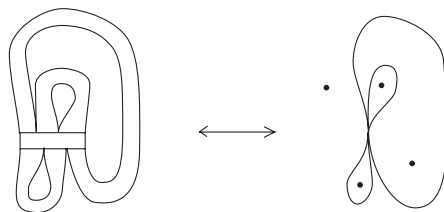
FIGURE 2.1. A non-classical interval exchange on S^2 minus 4 points.

The labeling of the bands by $\mathcal{A}$ can be thought of as given by *a generalized permutation*, as defined in [**2**]. A generalized permutation π is a 2-1 map from $\{1, \ldots, 2d\}$ to $\mathcal{A}$. Thus, $\pi^{-1}\alpha$ denotes the two ends of the band labeled α. The permutation is of type (l, m), where $l + m = 2d$ if the set $\{1, \ldots, l\}$ enumerates the subintervals of I_+ from left to right and the set $(l + 1, \ldots, l + m = 2d\}$ enumerates the subintervals of I_- from left to right. A generalized permutation defines a fixed point free involution σ of $\{1, \ldots, 2d\}$ by

$$\pi^{-1}(\pi(i)) = \{i, \sigma(i)\}.$$

Our definition implies that the generalized permutation we get does not arise from a true permutation $p = p_1 p_0^{-1}$, i.e., there is a positive integer i with $i, \sigma(i) \le l$ and a positive integer j with $l + 1 \le j, \sigma(j)$. The equivalence classes under σ are indexed by the elements of $\mathcal{A}$ and correspond to the bands. Following Kerckhoff, we shall call the positions that are rightmost on the intervals I_+ and I_-, the *critical* positions.

2.2. *Non-classical interval exchanges and strata of quadratic differentials.*   A train track on an oriented surface S with non-negative Euler characteristic is *large* if every region complementary to it in S is a polygon or a once-punctured polygon. As a result of the tangency condition at the switches, the complementary regions are *ideal* in the sense that the internal angles at all their vertices are zero, and hence they are *cusps*. For example, in Figure 2.1, there are four complementary regions all of which are once-punctured monogons: each contains one puncture and has one cusp. See [**15**] or [**12**] for background on train tracks.

A large train track on S is said to belong to a particular stratum of quadratic differentials if each ideal polygon or once-punctured ideal polygon contains, respectively, exactly a single zero or pole, and the number of cusps of each region is the order of the zero or the pole. A non-classical interval exchange belongs to a particular stratum if the underlying train track embeds into S such that the embedding belongs to the stratum. As an example, [**5**, Figure 17] shows a non-classical interval exchange in the principal stratum on a 5-punctured sphere, and [**5**, Figure 19] shows a non-classical interval exchange in the principal stratum of a genus-2 surface.

3. *Rauzy induction*
We now describe Rauzy induction. Since the underlying picture of intervals with bands is similar to classical interval exchanges, Rauzy induction for a non-classical interval exchange is similarly defined. The precise definition is given in [**2**, §2.2]. Here, we concentrate on encoding iterations by products of elementary matrices.
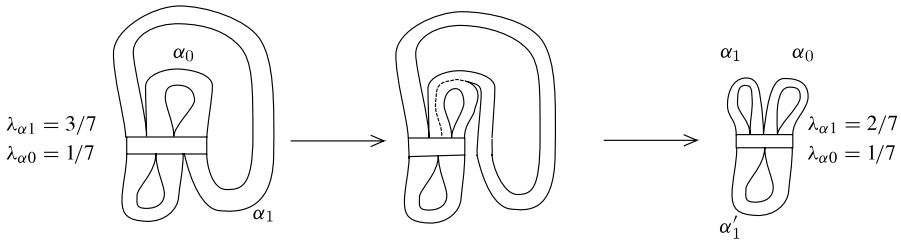
FIGURE 3.1. Rauzy induction.

Iterations of Rauzy induction of a classical interval exchange give an expansion analogous to the continued fraction expansion. In fact, when $d = 2$, the expansion is exactly the continued fraction expansion of the ratio of the widths of the two bands.

Suppose T is a non-classical interval exchange. Let α_0 and α_1 be the bands in the critical positions with α_0 on I_+. First, suppose that $\lambda_{\alpha_0} > \lambda_{\alpha_1}$. Then we slice as shown in Figure 3.1 until we hit $I_+ \sqcup I_-$ for the first time. The α_0 band remains in its critical position, but typically a band with a different label α_1' moves into the other critical position. Furthermore, the new width of α_0 is $\lambda_{\alpha_0} - \lambda_{\alpha_1}$. All other widths remain unchanged. If instead $\lambda_{\alpha_1} < \lambda_{\alpha_0}$, then we slice in the opposite direction, which in Figure 3.1 would be the analogous operation after flipping the picture about the horizontal axis. In either case, we get new non-classical interval exchanges with combinatorics and widths, as described above. The operation we just described is called *Rauzy induction*. Since Rauzy induction is represented pictorially by one band being split by another, it is also called a *split*. This is consistent with the notion of a split in the context of train-tracks; if the interval exchange is thought of as a train track, then Rauzy induction is the same as splitting the train track. Iterations of Rauzy inductions are called *splitting sequences*.

For a classical interval exchange, Rauzy induction is the first-return map to the interval $I' = [0, \sum_{\alpha \neq \alpha_0} \lambda_\alpha)$ in the first instance and $I' = [0, \sum_{\alpha \neq \alpha_1} \lambda_\alpha)$ in the second. Similarly, for non-classical interval exchanges, if I_+' and I_-' denote the copies of I' in I_+ and I_-, respectively, then Rauzy induction is the first-return map to $I_+' \sqcup I_-'$, in either instance. In this context, not all instances of Rauzy induction are defined. For example:

(1) when $\alpha_0 = \alpha_1$, i.e., both ends of a single band are in the critical positions, neither of the splits are defined;

(2) when $\lambda_{\alpha_0} = \lambda_{\alpha_1}$, then neither of the splits is defined;

(3) when α_0 is an orientation-reversing band on I_+ and α_1 is the only orientation-reversing band on I_-, then α_0 can split α_1 but not the other way round, i.e., only one of the splits is defined.

As we shall see later, case (1) is ruled out by the assumption that the non-classical interval exchange is irreducible and case (2) represents a set of measure zero. Eventually, we show in §6, that, almost surely, all iterations of Rauzy induction are well defined.

3.1. *Encoding Rauzy expansions by matrices.*

3.1.1. *Description of the parameter space.* All non-classical interval exchanges sharing the same generalized permutation π are parameterized by the widths of the bands.

Consider the vector space $\mathbb{R}^{\mathcal{A}}$ and let $\mathbb{R}^{\mathcal{A}}_{\geq 0}$ be the set of points with non-negative coordinates. Let Δ denote the standard $(d-1)$-simplex in $\mathbb{R}^{\mathcal{A}}$ given by the constraint that the sum of the coordinates is 1. An assignment of widths to the bands is a point in $\mathbb{R}^{\mathcal{A}}$. Normalizing the widths so that their sum is 1 restricts us to Δ.

To be consistent with π, any assignment of normalized widths must satisfy the switch condition defined by π. We denote the set of such widths by W. Let $\mathcal{A}_+$ and $\mathcal{A}_-$ be the set of orientation-reversing bands in $\mathcal{A}$ that are incident on I_+ and I_-, respectively. Then the points in W satisfy the additional constraint

$$\sum_{\alpha \in \mathcal{A}_+} \lambda_\alpha = \sum_{\alpha \in \mathcal{A}_-} \lambda_\alpha.$$

Thus, W is the intersection with Δ of a codimension-one subspace of $\mathbb{R}^{\mathcal{A}}$. For $\alpha \in \mathcal{A}_+$ and $\beta \in \mathcal{A}_-$, let $e_{\alpha\beta}$ be the midpoint of the edge $[e_\alpha, e_\beta]$ of Δ joining the vertices e_α and e_β. The subset W is the convex hull of the points $e_{\alpha\beta}$ and e_ρ for $\rho \notin \mathcal{A}_+ \cup \mathcal{A}_-$.

There are finitely many generalized permutations of an alphabet $\mathcal{A}$ over d letters, and hence finitely many convex codimension-one subsets of Δ that could be W. We call the subsets of Δ that the generalized permutations define *configuration spaces*. Whenever it is necessary to keep track, we index, in a fixed manner, the configuration spaces as W_r. The full parameter space for non-classical interval exchanges with d bands is a disjoint union of the configuration spaces W_r.

3.1.2. *Matrices.* Let I denote the $d \times d$ identity matrix on $\mathbb{R}^{\mathcal{A}}$. For $\alpha, \beta \in \mathcal{A}$, let $M_{\alpha\beta}$ be the $d \times d$-matrix with the (α, β) entry 1 and all other entries 0. After Rauzy induction, the relationship between the old and new width data is expressed by

$$\lambda = E\lambda',$$

where the matrix E has the form $E = I + M$. In the first instance of the split, when $\lambda_{\alpha_0} > \lambda_{\alpha_1}$, the matrix $M = M_{\alpha_0\alpha_1}$; in the second instance of the split, when $\lambda_{\alpha_1} > \lambda_{\alpha_0}$, the matrix $M = M_{\alpha_1\alpha_0}$. Thus, in either case the matrix E is an elementary matrix; in particular $E \in \mathrm{SL}(d; \mathbb{Z})$. If B is any $d \times d$ matrix, then in the instance when $\lambda_{\alpha_0} > \lambda_{\alpha_1}$, the action on B by right multiplication by E has the effect that the α_1th column of B is replaced by the sum of the α_0th column and α_1th column of B. We phrase this as follows: in the split, the α_0th column *moves* the α_1th column. A similar statement holds when $\lambda_{\alpha_0} > \lambda_{\alpha_1}$.

4. *Rauzy diagram*

For non-classical interval exchanges, one constructs an oriented graph similar to the Rauzy diagram for a classical interval exchange. However, as we shall see, there are some key differences in the non-classical context.

Construct an oriented graph G as follows: the nodes of the graph are generalized permutations π of an alphabet $\mathcal{A}$ over d letters, satisfying the conditions imposed by Definition 2.1 and summarized in §2.1. We draw an arrow from π to π' if π' results from splitting π. For each node π, there are at most two arrows coming out of it. A splitting sequence gives us a directed path in G.

In the context of classical interval exchanges, irreducibility of the permutation is sufficient to ensure that each connected component of the Rauzy diagram is an *attractor*,

i.e., any node can be joined to any other node by a directed path. Each component is called a Rauzy class. Veech showed a bijective correspondence between extended Rauzy classes and connected components of the corresponding strata of abelian differentials [**17**], and Kontsevich and Zorich gave a classification scheme for them [**10**].

The Rauzy diagram for non-classical interval exchanges is more complicated and need not have such strong recurrence properties. See the examples in the Appendix of [**2**] or see [**5**, §10]. The notion of strong irreducibility of generalized permutations defined by Boissy–Lanneau [**2**] is needed to characterize attractors. We shall explain these issues and their relevance to Theorem 6.3 in the next section.

5. Irreducibility

Let $|A|$ denote the Lebesgue measure of a measurable subset $A \subset I_+ \sqcup I_-$. We shall first consider the simplest notion of reducibility.

Definition 5.1. A generalized permutation π is *combinatorially reducible* if $\mathcal{A}$ can be written as a disjoint union $\mathcal{A}_1 \sqcup \mathcal{A}_2$ of non-empty subsets such that for all $\alpha \in \mathcal{A}_1$ and $\beta \in \mathcal{A}_2$, the ends $\pi^{-1}(\alpha)$ of band α occur to the left of the ends $\pi^{-1}(\beta)$ of the band β.

This means that for *some* choice of widths, the intervals $I_\pm$ can be cut into two intervals $I_\pm(1) \sqcup I_\pm(2)$ such that $|I_+(1)| = |I_-(1)|$ and $|I_+(2)| = |I_-(2)|$ and all the $\mathcal{A}_1$ bands are incident on $I_\pm(1)$ and all the $\mathcal{A}_2$ bands are incident on $I_\pm(2)$. Thus, $I_\pm(1)$ and $I_\pm(2)$ define interval exchanges by themselves with $\#\mathcal{A}_1$ and $\#\mathcal{A}_2$ bands, respectively, i.e., for some choice of widths, the original exchange is obtained by concatenating two disjoint exchanges with fewer bands. In particular, the dynamics of the original exchange is non-minimal.

Definition 5.2. A generalized permutation π is said to be *measure reduced* if, for all widths of the bands, the non-classical interval exchange decomposes into disjoint exchanges given by the $\mathcal{A}_1$ and $\mathcal{A}_2$ bands.

A generalized permutation π is *irreducible* if it has no measure reduction.

LEMMA 5.3. *A generalized permutation π, combinatorially reducible as $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2$, is measure reduced if and only if all the orientation-reversing bands in π are entirely in $\mathcal{A}_1$ or entirely in $\mathcal{A}_2$.*

Proof. Let $\mathcal{A}_{1,-}$ and $\mathcal{A}_{1,+}$ denote the set of orientation-reversing bands in $\mathcal{A}_1$ that are incident on the bottom and top, respectively. The generalized permutation π is measure reduced if and only if the additional constraint

$$\sum_{\alpha \in \mathcal{A}_{1,-}} \lambda_\alpha = \sum_{\beta \in \mathcal{A}_{1,+}} \lambda_\beta \tag{5.1}$$

gives the defining equation of W or is vacuous. If the former, then all the orientation-reversing bands of π have to be in $\mathcal{A}_1$. If the latter, then all the orientation-reversing bands of π have to be in $\mathcal{A}_2$. $\square$

A train track on a surface S with non-negative Euler characteristic is *complete* if all its complementary regions are ideal triangles or once-punctured monogons. This means that

a generic measured foliation carried by it is the vertical foliation of a quadratic differential in the principal stratum, i.e., have simple zeros and poles. We shall call a non-classical interval exchange *complete* if the underlying train track embeds as a complete train track on some surface S. Technically, the train track is required to be transversely recurrent in addition to being recurrent, but transverse recurrence of non-classical interval exchanges follows by a direct application of [**15**, Corollary 1.3.5]. So, we skip the definition of transverse recurrence and refer the reader to [**15**]. In the proposition below, we show that in the principal stratum, combinatorial reducibility is never measure reducibility.

PROPOSITION 5.4. *A generalized permutation π of a complete non-classical interval exchange is irreducible.*

Proof. Suppose that π admits a measure reduction $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2$. By the previous lemma, all the orientation-reversing bands in π must belong entirely to, say, $\mathcal{A}_2$. Mark the points $p_\pm$ on $I_\pm$ that are the common endpoints of the intervals $I_\pm(1)$ and $I_\pm(2)$. Suppose α and β are the bands incident on $I_+(1)$ and $I_+(2)$ that are adjacent at the common endpoint p_+. Let $\mathcal{R}$ be the complementary region with a cusp at p_+ and α and β as sides. For $\mathcal{R}$ to be a once-punctured monogon, α has to be the same as β, which is not possible since π is measure reduced. So, $\mathcal{R}$ has to be an ideal triangle. Then the remaining two cusps of $\mathcal{R}$ have to be incident on $I_-(1)$ and $I_\pm(2)$, respectively. But this implies that there is a band whose one end is incident on $I_-(1)$ and its other end is incident on $I_\pm(2)$. This contradicts the fact that π is measure reduced. $\square$

5.1. *Strong irreducibility and attractors.* In contrast to Proposition 5.4, in some of the other strata, there are combinatorially reducible non-classical interval exchanges that are also measure reduced. This can give rise to the following phenomena: the Rauzy diagram may contain generalized permutations that are not combinatorially reducible but which split with positive probability to measure-reduced ones. In fact, in some cases, the dynamics of the exchange is minimal for a set of widths of intermediate measure, and non-minimal because of a measure reduction for the complementary widths. See [**2**, Figure 15] for an example. This makes the attractors of the Rauzy diagram harder to characterize.

In [**2**], Boissy and Lanneau define a stronger notion of reducibility and prove the following.

THEOREM 5.5. [**2**, Theorem C] *Let G_{irr} be the subset of nodes of the Rauzy diagram G corresponding to the strongly irreducible generalized permutations. Then G_{irr} is closed under forward iterations of Rauzy induction. Moreover, each connected component of G_{irr} is strongly connected, i.e., any node π in a connected component of G_{irr} can be connected to any other node π' in the same component of G_{irr} by a sequence of splits.*

The theorem above implies that a strongly irreducible non-classical interval exchange can never split to a measure-reduced one. Thus, by restricting to strongly irreducible non-classical interval exchanges, one can avoid the issues mentioned.

From the point of view of uniform distortion, i.e., Theorem 6.3, these issues become relevant to Propositions 6.1 and 10.1. For Proposition 6.1, minimality of the non-classical interval exchange is necessary. For Proposition 10.1, it is necessary to know that the set of widths for which a combinatorial reduction decomposes a non-classical interval exchange

into two exchanges with fewer bands has measure zero, i.e, it is necessary to know that the expansion never gives a measure-reduced generalized permutation. As shown in [**2**], strong irreducibility implies both these properties.

For us, it is enough to assume that the initial generalized permutation π_0 satisfies the following possibly weaker properties: first, the set of widths that define a minimal non-classical interval exchange has full measure, and second, the generalized permutation π_0 never splits to a measure-reduced one. We do not use the actual definition of strong irreducibility at any point in the argument. Hence, we shall skip the details of the definition, and refer the reader to [**2**]. It should be pointed out that in the principal stratum, all generalized permutations split in finite time to strongly irreducible ones. So, in the principal stratum, strong irreducibility is not required to prove Theorem 6.3.

Similarly, for Theorem 1.1, it is enough to assume that the initial generalized permutation π_0 is such that all splitting sequences starting from π_0 end up in some attractor.

The assumption that π_0 is strongly irreducible implies all the required hypotheses. So, for the rest of the paper, we shall assume that the initial generalized permutation π_0 is strongly irreducible, in which case there are no transient nodes at all.

6. *Dynamics*

In this section, we analyze the expansion by splitting sequences on the space of widths of the bands. It turns out that the space of invariant probability measures embeds into the space of widths and the iterative refinement process defined by the splitting sequences, in the limit, nests down to the set of invariant probability measures. So, the basic idea, due to Kerckhoff, is to show that, for almost every non-classical interval exchange, one nests down to a single point.

6.0.1. *Preliminary notation.*
Given a matrix A with non-negative entries, we define the projectivization ΓA as a map from Δ to itself by

$$\Gamma A(\mathbf{y}) = \frac{A\mathbf{y}}{|A\mathbf{y}|},$$

where, if $\mathbf{y} = (y_1, y_2, \ldots, y_d)$ in coordinates, then $|\mathbf{y}| = \sum |y_i|$. This will be the norm used throughout. The norm is additive on $\mathbb{R}^d_{\geq 0}$, i.e., for $\mathbf{y}$, $\mathbf{y}'$ in $\mathbb{R}^d_{\geq 0}$, $|\mathbf{y} + \mathbf{y}'| = |\mathbf{y}| + |\mathbf{y}'|$.

6.1. *Iterations of Rauzy induction.*
We fix a generalized permutation π_0 that is strongly irreducible, and hence belongs to an attractor of G. Non-classical interval exchanges with generalized permutation π_0 are points in the configuration space W_0 defined by π_0. Let $\mathbf{x} = (\lambda_\alpha)$ be a point in W_0. First, we show that almost every $\mathbf{x} \in W_0$ has an infinite expansion.

Recall from §3 that Rauzy induction stops in cases (1) and (2). If case (1) is true, then the underlying generalized permutation is measure reduced, which rules out case (1). In case (2), except when the critical bands α_0 and α_1 are the only orientation-reversing bands on I_+ and I_-, the set of widths satisfying $\lambda_{\alpha_0} = \lambda_{\alpha_1}$ forms a codimension-one subset of the associated configuration space. If α_0 and α_1 are the only orientation-reversing bands on I_+ and I_-, then we can simply amalgamate together the ends of α_0 and α_1 in the critical

positions into a single band and cut those parts out of the intervals $I_{\pm}$. This makes it equivalent to a classical interval exchange with $(d - 1)$ bands for all widths, contradicting the assumption that the generic vertical foliation carried is non-orientable. The conclusion is that when the expansion stops in finite time, the widths belong to a codimension-one subset of the associated configuration space.

Finally, the subset of points whose Rauzy expansion stops in a finite number of steps is a countable union of codimension-one sets, and hence measure zero. So, for almost every $\mathbf{x}$, iterated splitting gives an infinite expansion. An infinite expansion determines an infinite directed path in the attractor.

A finite directed path $\pi_0 \to \pi_1 \to \cdots \to \pi_n$ in the attractor will be called a *stage*. Let E_i denote the elementary matrix associated to the split $\pi_{i-1} \to \pi_i$. Let W_i denote the configuration spaces corresponding to the π_i. The projective linear maps ΓE_i have the property that the inverse image of the configuration space W_{i-1} is the configuration space W_i. The matrix Q_n associated to the stage is given by the product

$$Q_n = E_1 E_2 \cdots E_n.$$

The image of ΓQ_n in W_0, i.e., the set $\Gamma Q_n(W_n)$, is the set of all $\mathbf{x} \in W_0$ whose expansion begins with this finite sequence $\pi_0 \to \pi_1 \to \cdots \to \pi_n$.

For a point $\mathbf{x}$ with an infinite expansion, whenever it is necessary to emphasize the dependence of the directed path on the initial point $\mathbf{x}$, we shall denote the nodes in the directed path by $\pi_{\mathbf{x},i}$, the configuration spaces defined by $\pi_{\mathbf{x},i}$ by $W_{\mathbf{x},i}$, and the elementary matrices associated to the splits $\pi_{\mathbf{x},i-1} \to \pi_{\mathbf{x},i}$ by $E_{\mathbf{x},i}$. Thus, given a stage $\pi_0 \to \pi_1 \to \cdots \to \pi_n$, the set $\Gamma Q_n(W_n)$ is precisely the set of all $\mathbf{x} \in W_0$ for which $\pi_{\mathbf{x},i} = \pi_i$ for all $i \leq n$.

In the expansion for $\mathbf{x}$, the actual (or un-normalized) widths $\lambda^{(n)}$ at any stage are related by the equation

$$\mathbf{x} = Q_n \lambda^{(n)}.$$

The projectivization $\mathbf{x}^{(n)} = \lambda^{(n)}/|\lambda^{(n)}|$ lies in the configuration space W_n. Thus, we get a sequence of points $\mathbf{x}^{(n)} \in W_n$ such that $\mathbf{x} = \Gamma Q_{\mathbf{x},n} \mathbf{x}^{(n)}$. The sets $\Gamma Q_{\mathbf{x},n}(W_n)$ form a nested sequence in W_0, all containing $\mathbf{x}$. Let

$$C(\mathbf{x}) = \bigcap_n \Gamma Q_{\mathbf{x},n}(W_n).$$

Let μ be a probability measure on the disjoint union $I_+ \sqcup I_-$, invariant under the non-classical interval exchange T, defined by $\mathbf{x} = (\lambda_\alpha)$. Let (λ_α^μ) be the widths assigned by μ to the bands. If T is minimal, then μ is absolutely continuous with respect to l, where l is the standard Lebesgue measure on $I_+ \sqcup I_-$. Let $t \in I_+ \sqcup I_-$ and consider the subinterval $[0, t)$ in the same component I_+ or I_- as t. Set $H_\mu(t) = \mu([0, t))$ to be in the same component as t. The map H_μ is a homeomorphism of $I_+ \sqcup I_-$. Define $T_\mu = H_\mu \circ T \circ H_\mu^{-1}$. It is easy to see that the transformation T_μ is a non-classical interval exchange with generalized permutation π_0. Following the exact argument as in the first proposition of [**18**, §4.4], we get the following proposition.

PROPOSITION 6.1. *The map $\mu \to (\lambda_\alpha^\mu)$ is a linear homeomorphism from the set of T-invariant probability measures onto the set $C(\mathbf{x})$. In particular, the non-classical interval exchange T is uniquely ergodic if and only if $C(\mathbf{x}) = \mathbf{x}$.*

Subsequently, we shall be interested in estimating the Lebesgue measure of subsets of $\Gamma Q_n(W_n)$. Here, Lebesgue measure means the probability measure on W_0 given by the $(d-2)$-volume form induced on it as a submanifold of Δ, normalized so that the total volume of W_0 is 1. We shall denote it by ℓ.

For example, to get an estimate of $\ell(\Gamma Q_n(W_n))$, we first push forward, by ΓQ_n, the volume form on W_n. Since there are finitely many configuration spaces, the volumes of any two configuration spaces W_r and W_s differ up to some factor that depends only on d. So, now compare the actual measure on W_0 to the push-forward. The Radon–Nikodym derivative of the actual measure with respect to the push-forward is just the Jacobian of ΓQ_n, restricted as a map from W_n to W_0. So, integrating the Jacobian over W_n gives us $\ell(\Gamma Q_n(W_n))$ up to the factor that relates the volumes of the two configuration spaces. This shows that to give quantitative estimates, one needs to understand the Jacobian of ΓQ_n restricted as a map from W_n to W_0. We denote this Jacobian by $\mathcal{J}(\Gamma Q_n)$.

Suppose π_n is the same as π_0 at some stage in the expansion and suppose $\jmath$ is a finite splitting sequence starting from π_0. If the Jacobian $\mathcal{J}(\Gamma Q_n)$ is roughly the same at all points, then the relative probability that $\jmath$ follows π_n is also roughly the same as the probability that an expansion starts with $\jmath$. We make the notion of the Jacobian being roughly the same at all points precise below.

Definition 6.2. Suppose $\pi_0 \to \pi_1 \to \cdots \to \pi_n$ is a finite directed path in the attractor and Q_n the associated matrix. For $C > 1$, we say that the stage π_n is *C-uniformly distorted* if, for all $\mathbf{y}, \mathbf{y}' \in W_n$,

$$\frac{1}{C} \leq \frac{\mathcal{J}(\Gamma Q_n)(\mathbf{y})}{\mathcal{J}(\Gamma Q_n)(\mathbf{y}')} \leq C.$$

At this point, we adopt some conventions: at any stage, points in W_0 or, more generally, in its ambient simplex will be denoted by $\mathbf{x}$s, points in W_n or, more generally, in its ambient simplex by $\mathbf{y}$s, points in $Q_n W_n$ or, more generally, in the image under Q_n of the ambient simplex by $\mathbf{w}$s. In all cases, we use suitable subscripts whenever necessary. From the previous discussion $\mathbf{y} = \mathbf{x}^{(n)}$, where $\mathbf{x}^{(n)}$ is the nth point constructed iteratively in the expansion of $\mathbf{x}$. The main technical theorem is as follows.

THEOREM 6.3. *Suppose $\pi_0 \to \pi_1 \to \cdots \to \pi_n$ is a stage in the expansion. There exists a constant $C > 1$, independent of the stage, such that, for almost every $\mathbf{x} \in \Gamma Q_n(W_n)$, there is some $m > n$, depending on $\mathbf{x}$, such that the stage $\pi_{\mathbf{x},m}$ is C-uniformly distorted.*

7. *Kerckhoff's approach: some examples*

We fix the following notation: let Q be the matrix associated to a stage in the expansion. We denote the αth column of Q by $Q(\alpha)$.

Before we present examples, we note that for classical interval exchanges, equation (8.1) implies that C-uniform distortion is equivalent to the $C^{1/d}$-distribution of the columns of the associated matrix, i.e., the ratio of the norms of any two columns of Q must be in the interval $(1/C^{1/d}, C^{1/d})$.

As the first example, consider an irreducible classical interval exchange with two bands. There is just one possible combinatorial type and hence the Rauzy diagram has just a single vertex. At the starting stage, we shall label the band in the critical position on the bottom

as 1 and the other band as 2. At every stage, we normalize so that the sum of the widths of the bands is one. If, at any stage, band 1 splits band 2, then we denote the split by the symbol L and if it happens the other way round, then we denote it by R. The matrices corresponding to L and R, which we also denote by the same letters, are

$$L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Suppose Q is the matrix at some stage in the expansion and suppose that $|Q(2)| > |Q(1)|$. Subsequent to this stage, as long as the split L keeps occurring, the column $Q(1)$ keeps moving column $Q(2)$. Hence, the new matrix Q' has columns

$$Q'(2) = Q(2) + k Q(1), \quad Q'(1) = Q(1),$$

where k is the number of times L has occurred. In this case, the norm of the second column keeps increasing while the first column stays the same, making the inequality $|Q'(2)| > |Q'(1)|$ more and more pronounced. However, as soon as R occurs, the second column moves the first, and, by additivity of the norm, we get

$$\frac{1}{2} < \frac{|Q'(1)|}{|Q'(2)|} < 2,$$

i.e., the columns become 2-distributed. To summarize, as long as a sequence of Ls occur, the columns get farther and farther from being nicely distributed and the resulting stage is farther and farther from being uniformly distorted. But as soon as a R occurs after that, the columns get 2-distributed, resulting in a stage that is uniformly distorted.

Now, using the measure theory of projective linear maps from a 1-simplex to itself, specifically equation (9.3), it can be shown that the probability of the second column increasing in norm by a factor of K as a result of a sequence of Ls, is bounded away from 1 by a quantity that depends only on K and is independent of the stage. This implies that, with a definite probability, the split R must occur, giving us 2-distribution and an instance of the theorem.

To handle classical interval exchanges with d bands, Kerckhoff [8] first proves a similar proposition about *increase in norm*: at any stage, the probability that a band is never split, before the norm of its column increases by a large enough factor K, is bounded away from 1. The bound depends only on K and d and is independent of the stage. In fact, the bound monotonically goes to zero as K goes to infinity. As in the example above, the proof of this proposition uses equation (9.3).

After this, Kerckhoff proves the following inductive step: suppose our stage has a collection of C'-distributed columns that also includes the column with the largest norm. Then there is a definite probability that one gets a larger collection of C''-distributed columns that also includes the largest column at the new stage. The constant C'' depends only on C' and d and is independent of the stage. Iterating the inductive step shows that, with a definite probability, one must get C-distributed.

The basic idea behind the inductive step is as follows. Before the norms of the columns in the collection increase by a factor of K, suppose one of the following happens: either an outside column becomes the column with the largest norm or a column in the collection moves a column from outside. At this point, if we add the outside band to our collection,

then, similar to the example above, the ratio of the norms of any two columns in it is within suitable bounds, even if, initially, the ratio is way out. So, it remains to show that, with a definite probability, one of the two events happens.

The key idea is that this happens provided there are no isolated blocks. An *isolated block* is a splitting sequence in which there is a collection of bands that satisfy the following properties: first, every band in the collection is moved, at least once, by some other band in the collection, and second, every band in the collection moves some other band in the collection but never moves a band outside the collection. Kerckhoff shows that for irreducible classical interval exchanges, isolated blocks are absent.

The proposition about increase in norm implies that, with a definite probability, every band in the collection has to move some other band before its norm increases by K. But it could so happen that it moves a band in the collection itself. If this repeats enough times, then Kerckhoff shows that there is a subcollection that forms an isolated block. The number of times it needs to repeat is independent of the stage. Finally, since isolated blocks are absent, there is a definite probability that a band in the collection has to move a band outside, which is exactly the kind of split we want to finish the proof of the inductive step.

With non-classical interval exchanges, there are two issues. First, the probabilities of splits are different from the ratios of the measures of the ambient simplices. Because of this, for non-classical interval exchanges, the proposition about increase in norm is incorrect as it stands. Second, isolated blocks are possible.

To illustrate the first issue, consider the stage given by the splitting sequence in Figure 7.1. Let Δ denote the standard simplex in $\mathbb{R}^4$, and let Q denote the matrix associated to the stage. Now, suppose that following this stage, band 1 splits band 2. Let E denote the elementary matrix associated to this split. In the subsequent computation, we show that the probability that this split happens is $(n + 3)/2(n + 2)$, which approaches $1/2$ as n becomes large. On the other hand, when band 1 splits band 2, the column $Q(2)$ moves the column $Q(1)$. The computation shows that this increases $|Q(1)|$ by a factor of $2(n + 2)/3$, which is unbounded as n becomes large. It is also interesting to compare the probability of the split to the ratio $\ell(T(QE)(\Delta))/\ell(\Gamma Q(\Delta))$ of the volumes of the ambient simplices. By Lemma 9.1, the ratio $\ell(T(QE)(\Delta))/\ell(\Gamma Q(\Delta))$ is equal to the reciprocal of the factor by which $|Q(1)|$ increases, i.e., equal to $3/2(n + 2)$. This is completely different from the probability of the split. The example thus illustrates that for non-classical interval exchanges Proposition 1.1 from Kerckhoff [**8**] cannot hold as it stands.

However, a closer look at the proof of the inductive step reveals that it suffices that the proposition about increase in norm be true for the *largest* columns around. In Proposition 10.5, we show that this is indeed the case for non-classical interval exchanges.

The computations in the example now follow. The matrix Q is given by

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & n+1 & n+2 & 0 \\ 2 & n & n+1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Denote the simplex with the columns of Q as vertices by $\Delta(Q)$. Let the image under Q of the configuration space for the stage be $W(Q)$.
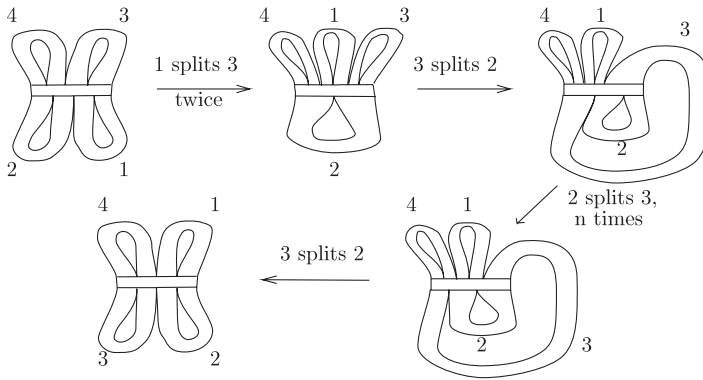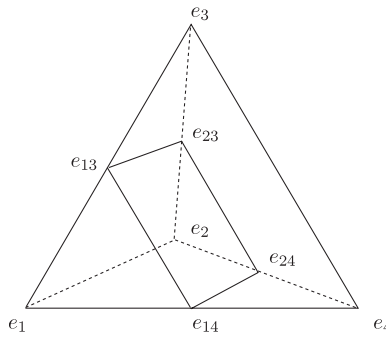
FIGURE 7.1. Splitting sequence.

FIGURE 7.2. W in Δ.

As in §6, let e_{ij} be the midpoint of the edge $[e_i, e_j]$. As shown in Figure 7.2, in the standard simplex, the initial configuration space W is the quadrilateral with vertices e_{13}, e_{14}, e_{24} and e_{23}.

Let $Q(ij)$ denote the midpoint of the edge $[Q(i), Q(j)]$ of $\Delta(Q)$. As shown in Figure 7.3, $W(Q)$ is the quadrilateral in $\Delta(Q)$ with vertices $Q(12)$, $Q(13)$, $Q(34)$ and $Q(24)$.

The columns representing the vertices of $W(Q)$ are

$$
Q(12) = \begin{bmatrix} 1/2 \\ (n+1)/2 \\ (n+2)/2 \\ 0 \end{bmatrix}, \quad
Q(13) = \begin{bmatrix} 1/2 \\ (n+2)/2 \\ (n+3)/2 \\ 0 \end{bmatrix},
$$

$$
Q(34) = \begin{bmatrix} 0 \\ (n+2)/2 \\ (n+1)/2 \\ 1/2 \end{bmatrix}, \quad
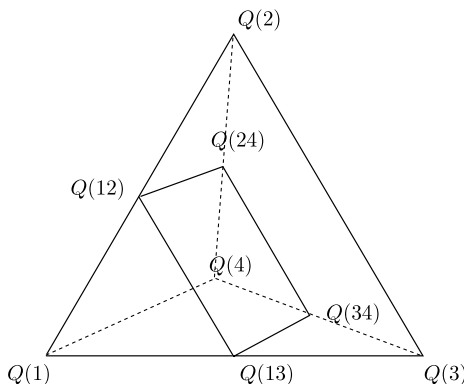Q(24) = \begin{bmatrix} 0 \\ (n+1)/2 \\ n/2 \\ 1/2 \end{bmatrix}.
$$

FIGURE 7.3. $W(Q)$ in $\Delta(Q)$.

Projectivize to find the images of these vertices in the configuration space W of the starting stage

$$\Gamma Q(12) = \begin{bmatrix} 1/2(n+2) \\ (n+1)/2(n+2) \\ 1/2 \\ 0 \end{bmatrix}, \quad \Gamma Q(13) = \begin{bmatrix} 1/2(n+3) \\ (n+2)/2(n+3) \\ 1/2 \\ 0 \end{bmatrix},$$

$$\Gamma Q(34) = \begin{bmatrix} 0 \\ 1/2 \\ (n+1)/2(n+2) \\ 1/2(n+2) \end{bmatrix}, \quad \Gamma Q(24) = \begin{bmatrix} 0 \\ 1/2 \\ n/2(n+1) \\ 1/2(n+1) \end{bmatrix}.$$

In terms of the vertices of the quadrilateral W, we get the linear combinations

$$\Gamma Q(12) = \left(\frac{1}{n+2}\right)e_{13} + \left(\frac{n+1}{n+2}\right)e_{23},$$

$$\Gamma Q(13) = \left(\frac{1}{n+3}\right)e_{13} + \left(\frac{n+2}{n+3}\right)e_{23},$$

$$\Gamma Q(34) = \left(\frac{n+1}{n+2}\right)e_{23} + \left(\frac{1}{n+2}\right)e_{24},$$

$$\Gamma Q(24) = \left(\frac{n}{n+1}\right)e_{23} + \left(\frac{1}{n+1}\right)e_{24}.$$

By symmetry, the quadrilateral W is a square with side length $1/\sqrt{2}$. $W(Q)$ projects down, as shown in Figure 7.4. From this we calculate the area of the quadrilateral $\Gamma Q(12)\Gamma Q(24)\Gamma Q(34)\Gamma Q(13)$ to be $1/2(n+1)(n+2)(n+3)$.

Now, suppose that in the next split band 1 splits band 2. As shown in Figure 7.5, the split is represented by the smaller simplex with vertices $Q(2)$, $Q(3)$, $Q(4)$ and $Q(12)$. The part of $W(Q)$ that lies inside this smaller simplex is the part of $W(Q)$ that lies above the shaded plane in Figure 7.5. This is the triangle $Q(12)Q(24)Q(34)$. Projecting the triangle down to W gives the triangle $\Gamma Q(12)\Gamma Q(24)\Gamma Q(34)$ in Figure 7.4. The area of this triangle is $1/4(n+1)(n+2)^2$.
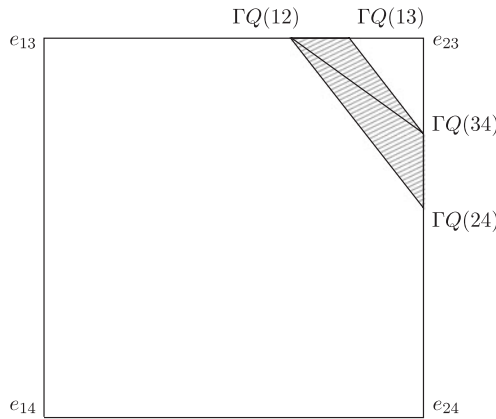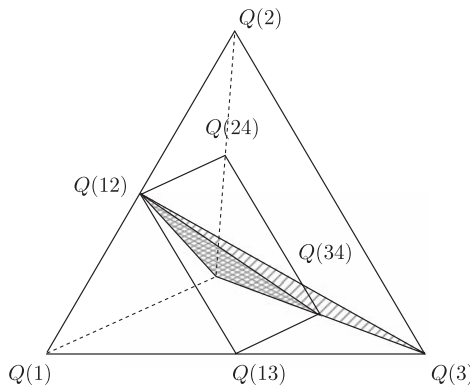
FIGURE 7.4. Projection of $W(Q)$.

FIGURE 7.5. Band 1 splits band 2.

Hence, the probability of the split is the ratio of the area of the triangle $\Gamma Q(12)\Gamma Q(24)\Gamma Q(34)$ to the area of quadrilateral $\Gamma Q(12)\Gamma Q(24)\Gamma Q(34)\Gamma Q(13)$, i.e., $(n + 3)/2(n + 2)$. We also see that $|Q(1)|$ increases by a factor of $2(n + 2)/3$ in the split, finishing the computations.

The second issue is that for non-classical interval exchanges, isolated blocks are possible when there is a combinatorial reduction. They arise as follows. Write the combinatorial reduction as a concatenation of two non-classical interval exchanges with fewer bands, and consider the non-classical interval exchange on the right. If there is a splitting sequence of this smaller exchange in which every column moves every other column, then this splitting sequence is an isolated block for the original non-classical interval exchange. This point is overlooked in Kerckhoff [8] and also in [14, Proposition 3.6] and [4, Lemma 1.7]. However, irreducibility of a non-classical interval exchange implies that one splits out of these blocks with a definite probability. See Step 1 of Proposition 10.10.

From the next section on, we begin the technical details starting with the analysis of the Jacobian of the restriction of the projective linear map to the configuration spaces.

8. *Jacobian of the restriction*

Let $\pi_0 \to \pi_1 \to \cdots \to \pi_n$ be a stage. For the rest of the paper, we simplify the notation by dropping the subscripts, i.e., henceforth we shall denote π_n by π, the configuration space W_n defined by π_n as W and the associated matrix Q_n as Q. In this section, we analyze the Jacobian of ΓQ restricted as a map from W to W_0. The expression for the Jacobian of the full map ΓQ from Δ to itself is well known; see [3]. At a point in the configuration space W, we write a matrix for the derivative of the full map ΓQ, with respect to suitable decompositions, as direct sums of the tangent spaces to Δ at the point itself and its image under ΓQ. Then using the expression for the full Jacobian and the particular form of this matrix, we get an expression for the Jacobian of ΓQ as a map from W to W_0.

We fix some terminology. Given an affine subspace L of $\mathbb{R}^{\mathcal{A}}$ and some $\mathbf{y} \in \mathbb{R}^{\mathcal{A}}$, which need not be in L, we let $T_{\mathbf{y}}L$ be the subspace of the tangent space at $\mathbf{y} \in \mathbb{R}^{\mathcal{A}}$ parallel to L. We will denote a vector in $\mathbb{R}^{\mathcal{A}}$ and also in the tangent space at any point $\mathbf{y}$ by the same letter $\mathbf{u}$, whenever the context is clear. Associated to a codimension-one subspace L transverse to a vector $\mathbf{u}$, there is a projection map $\phi_{\mathbf{u}} : T_{\mathbf{y}}\mathbb{R}^{\mathcal{A}} \to T_{\mathbf{y}}L$ by projecting along lines parallel to $\mathbf{u}$ until one hits L. A dilation of $\mathbb{R}^{\mathcal{A}}$ by $t > 0$ will be denoted by ρ_t. The derivative of a linear map A acting on tangent spaces is denoted by A itself.

Let $\Delta(Q)$ be the simplex with vertices the columns of Q. The map $\Gamma Q : \Delta \to \Delta$ is a diffeomorphism onto its image. This means that, for any point $\mathbf{y} \in \Delta$, the vector $\mathbf{w} = Q\mathbf{y}$ is transverse to both $\Delta(Q)$ and Δ. Hence, at the level of tangent spaces, $\mathbf{w}$ is transverse to $T_{\mathbf{w}}\Delta(Q)$ and $T_{\mathbf{w}}\Delta$. So, the projection map $\phi_{\mathbf{w}} : T_{\mathbf{w}}\mathbb{R}^{\mathcal{A}} \to T_{\mathbf{w}}\Delta$ by projecting along lines parallel to $\mathbf{w}$ is an isomorphism from $T_{\mathbf{w}}\Delta(Q)$ to $T_{\mathbf{w}}\Delta$. The full derivative $\mathbf{D}\Gamma Q(\mathbf{y}) : T_{\mathbf{y}}\Delta \to T_{\Gamma Q(\mathbf{y})}\Delta$ is the composition

$$T_{\mathbf{y}}\Delta \xrightarrow{Q} T_{\mathbf{w}}\Delta(Q) \xrightarrow{\phi_{\mathbf{w}}} T_{\mathbf{w}}\Delta \xrightarrow{\rho_{|\mathbf{w}|^{-1}}} T_{\Gamma Q(\mathbf{y})}\Delta.$$

It is known that (see Bufetov [3]) the Jacobian $\mathcal{J}_{\Delta}(\Gamma Q)$ of the above composition is given by

$$\mathcal{J}_{\Delta}(\Gamma Q)(\mathbf{y}) = \frac{1}{|\mathbf{w}|^d} = \frac{1}{|Q\mathbf{y}|^d}. \tag{8.1}$$

Fix unit vectors $\mathbf{m}_0$ and $\mathbf{m}$ normal to W_0 and W, respectively, such that the tangent bundle $T\Delta$ has the orthogonal decompositions

$$T\Delta = (\mathbb{R}\mathbf{m}_0) \oplus TW_0, \quad T\Delta = (\mathbb{R}\mathbf{m}) \oplus TW,$$

where TW_0 and TW are the tangent bundles of the configuration spaces.

Since ΓQ restricted to W maps it into W_0 and is a diffeomorphism onto its image, at any point $\mathbf{y} \in W$, the full derivative $\mathbf{D}\Gamma Q$ restricts to an isomorphism from $T_{\mathbf{y}}W$ to $T_{\Gamma Q(\mathbf{y})}W_0$. So, with respect to the orthogonal decompositions fixed above, the matrix at $\mathbf{y}$ for $\mathbf{D}\Gamma Q$ has the form

$$\begin{pmatrix} & & & & * \\ & \mathbf{D}\Gamma Q|_{T_{\mathbf{y}}W \to T_{\Gamma Q(\mathbf{y})}W_0} & & & * \\ & & & & * \\ 0 & \cdots & & 0 & c(\mathbf{y}) \end{pmatrix}.$$
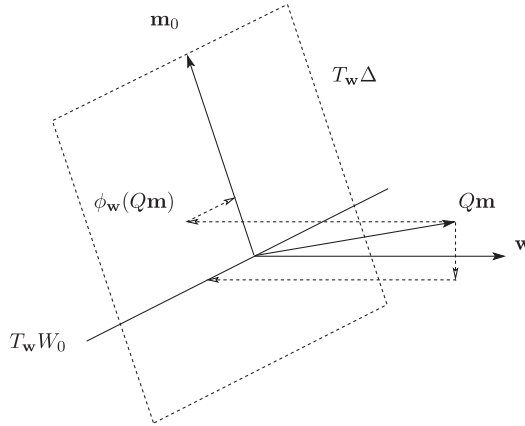
FIGURE 8.1. Schematic picture.

From the matrix form we get

$$\mathcal{J}_\Delta(\Gamma Q)(\mathbf{y}) = c(\mathbf{y})\mathcal{J}(\Gamma Q)(\mathbf{y}). \tag{8.2}$$

To compute $c(\mathbf{y})$, we use the fact that

$$\mathbf{D}\Gamma Q(\mathbf{y}) = \rho_{|\mathbf{w}|^{-1}} \circ \phi_{\mathbf{w}} \circ Q.$$

In the composition, we project the vector $Q\mathbf{m} \in T_{\mathbf{w}}\Delta(Q)$ by lines parallel to $\mathbf{w}$ to get $\phi_{\mathbf{w}}(Q\mathbf{m}) \in T_{\mathbf{w}}\Delta$. Next, we write $\phi_{\mathbf{w}}(Q\mathbf{m})$ as the unique linear combination

$$\phi_{\mathbf{w}}(Q\mathbf{m}) = a(\mathbf{y})\mathbf{m}_0 + \mathbf{p}$$

in $T_{\mathbf{w}}\Delta$, where $\mathbf{p} \in T_{\mathbf{w}}W_0$. Last, we apply the dilation $\rho_{|\mathbf{w}|^{-1}}$ to get $c(\mathbf{y}) = a(\mathbf{y})/|\mathbf{w}|$. Hence, from equation (8.2), we get

$$\mathcal{J}(\Gamma Q)(\mathbf{y}) = \frac{1}{a(\mathbf{y})|\mathbf{w}|^{d-1}} = \frac{1}{a(\mathbf{y})|Q\mathbf{y}|^{d-1}}. \tag{8.3}$$

LEMMA 8.1. *As $\mathbf{y}$ varies over W, the quantity $a(\mathbf{y})$ remains constant.*

Proof. To prove the lemma, we need to understand the projection $\phi_{\mathbf{w}} : T_{\mathbf{w}}\Delta(Q) \to T_{\mathbf{w}}\Delta$. We refer to Figure 8.1, which is a schematic picture of $T_{\mathbf{w}}\mathbb{R}^{\mathcal{A}}$. In the figure, the projection $\phi_{\mathbf{w}}$ by lines parallel to $\mathbf{w}$ is represented by horizontal dashed arrows. Let L be the codimension-one subspace in $T_{\mathbf{w}}\mathbb{R}^{\mathcal{A}}$ spanned by $\mathbf{w}$ and $T_{\mathbf{w}}W_0$. If A_0 denotes the subspace of $\mathbb{R}^{\mathcal{A}}$ spanned by W_0, i.e., the subspace of $\mathbb{R}^{\mathcal{A}}$ satisfying the equation that defines W_0, then $L = T_{\mathbf{w}}A_0$. So, L does not depend on $\mathbf{w}$.

Since $\phi_{\mathbf{w}}$ gives an isomorphism from $T_{\mathbf{w}}W(Q)$ to $T_{\mathbf{w}}W_0$, $T_{\mathbf{w}}W(Q)$ is in L. In the figure, we represent L by the horizontal (x, y)-plane. The vector $\mathbf{m}_0$ is orthogonal to $T_{\mathbf{w}}W_0$ and transverse to $\mathbf{w}$, so in fact transverse to L.

For the remainder of the proof, we shall denote the $(d - 1)$-dimensional subspace of $\mathbb{R}^{\mathcal{A}}$ that is parallel to Δ by Δ itself. The essential point from the discussion above is

as follows: thinking of each tangent space $T_{\mathbf{w}}\mathbb{R}^{\mathcal{A}}$ as $\mathbb{R}^{\mathcal{A}}$ itself, there are two $(d-1)$-dimensional transverse subspaces L and Δ with a vector $\mathbf{m}_0 \in \Delta$ not in L. Moreover, as $\mathbf{y}$ varies over W, which is equivalent to saying as $\mathbf{w}$ varies over $W(Q)$, the vector $\mathbf{w}$ is constrained to lie in L. Any vector $\mathbf{v} \in T_{\mathbf{w}}\mathbb{R}^{\mathcal{A}}$ can be uniquely written as

$$\mathbf{v} = a\mathbf{m}_0 + \mathbf{q},$$

where $\mathbf{q} \in L$. Applying the projection $\phi_{\mathbf{w}} : T_{\mathbf{w}}\mathbb{R}^{\mathcal{A}} \to T_{\mathbf{w}}\Delta$,

$$\phi_{\mathbf{w}}(\mathbf{v}) = a\phi_{\mathbf{w}}(\mathbf{m}_0) + \phi_{\mathbf{w}}(\mathbf{q}) = a\mathbf{m}_0 + \phi_{\mathbf{w}}(\mathbf{q}).$$

The projection $\phi_{\mathbf{w}}$ has the form $\phi_{\mathbf{w}}(\mathbf{v}) = \mathbf{v} - b\mathbf{w}$. This implies that the vector $\phi_{\mathbf{w}}(\mathbf{q})$ belongs to L. So, the component of $\phi_{\mathbf{w}}(\mathbf{v})$ along $\mathbf{m}_0$ remains $a\mathbf{m}_0$, even as $\mathbf{w}$ varies. Choosing $\mathbf{v} = Q\mathbf{m}$, we are done. $\qquad\square$

By the above lemma, we can drop the dependence of $a(\mathbf{y})$ on $\mathbf{y}$ in the expression for the Jacobian and write it just as a. However, the number a does depend on the stage of the expansion. Observe that, up to the number a, the expression for the Jacobian of the restriction looks similar to the Jacobian of a determinant-1 projective linear map with non-negative entries in dimension $(d-1)$. We shall make this observation precise and use it to compute measures in §10.

Next, we give the definition for a set of C-distributed vectors.

Definition 8.2. Let $C > 1$. A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is said to be *C-distributed* if

$$\frac{1}{C} < \frac{|\mathbf{u}_i|}{|\mathbf{u}_j|} < C,$$

for all i, j such that $1 \le i$, $j \le k$.

We conclude from the previous lemma that a stage π is C-uniformly distorted if and only if the vertices of $W(Q)$ are $C^{1/(d-1)}$-distributed. Moreover if the vertices of $\Delta(Q)$ are C-distributed, then the vertices of $W(Q)$ are also C-distributed. So, to show that, for almost every $\mathbf{x}$, there exists a stage $\pi_{\mathbf{x},n}$ that is C-uniformly distorted, it is enough to show that, for almost every $\mathbf{x}$, there exists a stage $\pi_{\mathbf{x},n}$ such that the columns of $Q_{\mathbf{x},n}$ are C-distributed. The proof of this roughly follows Kerckhoff's original proof in the case of classical interval exchanges while weakening various hypotheses.

In the next section, we recall the measure theory for determinant-1 projective linear maps with non-negative entries, from the standard simplex into itself. Additionally, we evaluate measures of certain subsets of the standard simplex, which will be useful later.

9. *Measure theory*

9.1. *Measure theory of projective linear maps.* Let $Q : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}}$ be a linear map with non-negative entries and determinant 1. Projectivize Q to get the map $\Gamma Q : \Delta \to \Delta$. We recall §4.4, Lemma 1 from Yoccoz [18].

LEMMA 9.1.
$$\frac{\ell(\Gamma Q(\Delta))}{\ell(\Delta)} = \frac{1}{\prod_{\alpha=1}^{d} |Q(\alpha)|}.$$
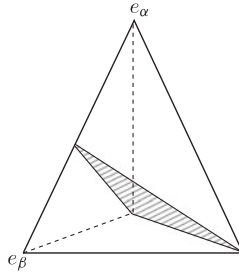
FIGURE 9.1. $\Delta_R(\alpha, \beta)$.

Remark 9.2. The formula for $\ell(\Gamma Q(\Delta))$ can be derived by integrating expression (8.1) for $\mathcal{J}_\Delta(\Gamma Q)$ on Δ.

The next corollary does not need the assumption that the matrix Q has determinant 1. This will be relevant in Step 5 of Proposition 10.5.

COROLLARY 9.3. *Let E be an elementary matrix with the off-diagonal (β, α) entry equal to $R > 0$, and let Q be a linear map with non-negative entries. Then*

$$\frac{\ell(\Gamma(QE)(\Delta))}{\ell(\Gamma Q(\Delta))} = \frac{|Q(\alpha)|}{|Q(\alpha)| + R|Q(\beta)|}.$$

Proof. As both sides of the claimed equality are unchanged if we replace Q by tQ, we can reduce to the case $\det(Q) = 1$. By applying Lemma 9.1, we get

$$\frac{\ell(\Gamma(QE)(\Delta))}{\ell(\Gamma Q(\Delta))} = \frac{\prod_{\gamma=1}^d |Q(\gamma)|}{\prod_{\gamma=1}^d |QE(\gamma)|} = \frac{|Q(\alpha)|}{|QE(\alpha)|} = \frac{|Q(\alpha)|}{|Q(\alpha)| + R|Q(\beta)|}. \qquad \square$$

9.2. *Evaluating measures.* We apply Corollary 9.3 to evaluate measures of certain subsets of Δ that we shall encounter in §9.

Wedges. For a pair of distinct labels $\alpha, \beta \in \mathcal{A}$ and a non-negative constant R, let $\Delta_R(\alpha, \beta)$ be the set of points in Δ whose convex combination $\mathbf{x} = \sum \lambda_\gamma e_\gamma$ satisfies

$$\frac{\lambda_\beta}{\lambda_\alpha} \geq R. \tag{9.1}$$

Figure 9.1 shows a schematic picture of $\Delta_R(\alpha, \beta)$. The shaded plane $F_R(\alpha, \beta)$ represents the set of points for which $\lambda_\beta/\lambda_\alpha = R$. The set $\Delta_R(\alpha, \beta)$ is the region in Figure 9.1 that lies below the shaded plane. Thus, it looks like a *wedge* in Δ.

Let $Q : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}}$ be a linear map with non-negative entries. Write R in the form

$$R = K \frac{|Q(\alpha)|}{|Q(\beta)|} \tag{9.2}$$

for some $K > 0$. Then

$$\Gamma Q(\Delta_R(\alpha, \beta)) = \Gamma(QE)(\Delta),$$

where E is the elementary matrix with the (β, α) entry equal to R. Applying Corollary 9.3, we get

$$\frac{\ell(\Gamma Q(\Delta_R(\alpha, \beta)))}{\ell(\Gamma Q(\Delta))} = \frac{1}{1 + K}. \tag{9.3}$$

Half-space intersections. Having established equation (9.3) for the relative measure of wedges, we move on to more general subsets of Δ, which we call *thin half-space intersections*. The wedges $\Delta_R(\alpha, \beta)$ are an instance of these. We encounter thin half-space intersections in the proof of Proposition 10.5.

A half-space H in $\mathbb{R}^{\mathcal{A}}$ that intersects Δ and does not contain the vertex e_α will be called a half-space *opposite* α. Let S_H denote the set of those $\gamma \neq \alpha$ for which the intersection $\partial H \cap [e_\alpha, e_\gamma]$ is non-empty, where $[e_\alpha, e_\gamma]$ denotes the edge of Δ joining e_α with e_γ. For $R_0 > 0$, a half-space H opposite α is said to be R_0-th*in* if, for every $\gamma \in S_H$, the intersection point $a_{\alpha\gamma} = \partial H \cap [e_\alpha, e_\gamma]$ has a convex combination that satisfies

$$\frac{\lambda_\gamma}{\lambda_\alpha} \geq R_0.$$

We have the following inclusion.

LEMMA 9.4. *For $R = R_0/d$, we have*

$$H \cap \Delta \subset \bigcup_{\gamma \in S_H} \Delta_R(\alpha, \gamma).$$

Proof. Let U be the subsimplex of Δ whose vertices are indexed by $S_H \cup \{\alpha\}$, and let $U(R_0)$ denote the set of points in U whose convex combination in U satisfies $\lambda_\alpha \leq 1/(1 + R_0)$. Let $\Delta(R_0)$ be the convex hull of $U(R_0)$ with the remaining vertices of Δ.

CLAIM 9.5.
$$H \cap \Delta \subset \Delta(R_0).$$

Proof. Since both $H \cap \Delta$ and $\Delta(R_0)$ are convex sets, it is enough to show that every vertex of $H \cap \Delta$ is in $\Delta(R_0)$. By virtue of being R_0-thin, the vertices $a_{\alpha\gamma}$ are in $\Delta(R_0)$. All the other vertices of $H \cap \Delta$ lie in $F(\alpha)$, the face of Δ opposite e_α, which itself lies in $\Delta(R_0)$. $\square$

The lemma then follows from the following claim.

CLAIM 9.6.
$$\Delta(R_0) \subset \bigcup_{\gamma \in S_H} \Delta_R(\alpha, \gamma).$$

Proof. Notice that the face $F(\alpha)$ of Δ sits in the union above. The vertices of $\Delta(R_0)$ that are not in U lie in $F(\alpha)$. So, it is enough to show that $U(R_0)$ lies in the union. The convex combination of a point in $U(R_0)$ satisfies $\lambda_\alpha \leq 1/(1 + R_0)$. This implies that for at least one $\gamma \in S_H$,

$$\lambda_\gamma \geq \left(\frac{1}{\#S_H}\right)\left(1 - \frac{1}{1 + R_0}\right) > \frac{R_0}{d(1 + R_0)}.$$

It follows that

$$\frac{\lambda_\gamma}{\lambda_\alpha} > \frac{R_0}{d}$$

and so the point belongs to $\Delta_R(\alpha, \gamma)$. $\square$

In Proposition 10.5, we need an upper bound on the measure of half-space intersections. This upper bound is derived there by using the inclusion of Lemma 9.4 and then bounding the measures of the individual wedges in the union.

We now have all the preliminaries to carry out the proof of Theorem 6.3.

10. *Proof of Theorem 6.3*

Let π be a stage in the expansion. Let W be the configuration space at π and let Q be the matrix associated to π.

PROPOSITION 10.1. *For almost every* $\mathbf{x} \in \Gamma Q(W)$*, every band* $\alpha \in \mathcal{A}$ *is split infinitely often and splits other bands infinitely often.*

Proof. The first claim is the following.

CLAIM 10.2. *Suppose in the expansion for some* $\mathbf{x}$*, a band* $\alpha \in \mathcal{A}$ *occurs in the critical positions infinitely often. Then* α *is split infinitely often and splits other bands infinitely often.*

Proof. Suppose α is split finitely many times. Then there exists a future stage $\pi_{\mathbf{x},m}$ after which α never gets split. So, the actual width of α remains unchanged from $\pi_{\mathbf{x},m}$ onwards. On the other hand, after $\pi_{\mathbf{x},m}$, the band α splits some other band γ infinitely often; this is impossible if the actual width of α is to remain fixed.

Alternatively, suppose α splits other bands finitely many times. A band leaves a critical position only if it splits some other band. Since α gets split infinitely often, there exists a future stage $\pi_{\mathbf{x},m}$ after which α remains fixed in one of the critical positions and is the only band split thereafter. But then from $\pi_{\mathbf{x},m}$ onwards the actual widths of the rest of the bands remain unchanged because they never get split. This makes it impossible to split α *ad infinitum*. □

Thus, to prove Proposition 10.1, it is enough to show that, almost surely, every band α occurs in the critical positions infinitely often. Let Z be the set of those $\mathbf{x}$ in $\Gamma Q(W)$ for which there exists a band α that occurs in the critical positions finitely often in the expansion of $\mathbf{x}$. Fix such a point $\mathbf{x}$ and let $\mathcal{A}_1$ be the subset of those bands that occur in the critical positions finitely many times in the expansion of $\mathbf{x}$. Then there is a future stage $\pi_{\mathbf{x},n}$ such that in the expansion of $\mathbf{x}$ after $\pi_{\mathbf{x},n}$:
(1) the bands in $\mathcal{A}_1$ never occur in the critical positions; and
(2) every band in $\mathcal{A}_2 = \mathcal{A} \backslash \mathcal{A}_1$ occurs in the critical positions infinitely often.
Moreover, we assume that $\pi_{\mathbf{x},n}$ is the first instance in the expansion of $\mathbf{x}$ in which the above properties hold.

CLAIM 10.3. $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2$ *is a combinatorial reduction of* $\pi_{\mathbf{x},n}$ *and all future stages.*

Proof. Without loss of generality, suppose some $\beta \in \mathcal{A}_2$ occurs to the left of some $\alpha \in \mathcal{A}_1$ on the bottom. Since α never enters a critical position, whenever β is split, the split has to begin at the other end of β. This increases the number of bands $k(\alpha)$ to the left of α on the bottom by 1. Moreover, since β is split infinitely often, the number $k(\alpha)$ has to become large enough to force α into the critical position on the bottom, which is a contradiction. This proves the claim. □

Let H be the set of finite splitting sequences $\jmath$, starting from π, such that the final stage of $\jmath$ is combinatorially reducible. The set H is countable because it is a subset of the countable set of finite splitting sequences.

By the above claim, we can define a map $\psi : Z \to H$. Let $Z_\jmath = \psi^{-1}(\jmath)$. If the set $Z_\jmath$ is non-empty, then the expansion after π of each point in $Z_\jmath$ begins with the sequence $\jmath$. Since H is a countable set, $\ell(Z) = \sum_{\jmath \in H} \ell(Z_\jmath)$. So, to prove Proposition 10.1, it is enough to show that $\ell(Z_\jmath) = 0$ for all $\jmath$ in H.

CLAIM 10.4. *For every $\jmath$ in H, $\ell(Z_\jmath) = 0$.*

Proof. Let $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2$ be the combinatorial reduction for the final stage π' of $\jmath$. Let (λ'_α) denote the normalized widths at stage π'. We shall show that a point in $Z_\jmath$ must satisfy constraint (5.1) in the widths (λ'_α). Strong irreducibility implies that the set given by this constraint has measure zero, which then proves the claim.

For a point $\mathbf{x}$ that does not satisfy the constraint, set

$$D = \sum_{\alpha \in \mathcal{A}_{2,-}} \lambda'_\alpha - \sum_{\beta \in \mathcal{A}_{2,+}} \lambda'_\beta$$

and, without loss of generality, assume that $D > 0$. We claim that for a point with $D > 0$, there is a stage in the expansion after π' in which twice the sum of the actual widths of the bands in $\mathcal{A}_2$ is equal to D. When this happens, a band in $\mathcal{A}_1$ is forced into the critical position on the bottom. This would show that such a point cannot be in $Z_\jmath$.

Let λ^∞_α denote the limit of the actual widths of α in the expansion of $\mathbf{x}$. To show that there is a stage in the expansion of $\mathbf{x}$ after π' in which the sum of the actual widths of the bands in $\mathcal{A}_2$ is equal to D, we show that $\lambda^\infty_\alpha = 0$ for all $\alpha \in \mathcal{A}_2$. We prove this by showing that for any $\epsilon > 0$, there exists a future stage at which the actual widths of all bands in $\mathcal{A}_2$ are at most ϵ.

For $\epsilon > 0$, there exists a stage $\pi_{\mathbf{x},N}$ after π' such that the actual widths $\lambda^{(N)}_\alpha$ at $\pi_{\mathbf{x},N}$ satisfy $\lambda^\infty_\alpha < \lambda^{(N)}_\alpha < \lambda^\infty_\alpha + \epsilon$ for all α in $\mathcal{A}_2$. Then for every subsequent split after $\pi_{\mathbf{x},N}$, the actual width of the band split can be reduced by at most ϵ. This means that the actual width of a band doing the splitting is at most ϵ. For the bands in $\mathcal{A}_2$, there is a future stage $\pi_{\mathbf{x},N'}$ such that every band in $\mathcal{A}_2$ has split other bands at least once after $\pi_{\mathbf{x},N}$ but before $\pi_{\mathbf{x},N'}$. By the previous observation, the actual widths at $\pi_{\mathbf{x},N'}$ of all bands in $\mathcal{A}_2$ are at most ϵ and we are done. $\square$

This concludes the proof of Proposition 10.1. $\square$

For $\alpha \in \mathcal{A}$ and $\mathbf{x} \in \Gamma Q(W)$, let $n(\mathbf{x}, \alpha)$ be the index of the first stage after π such that, in the split $\pi_{n(\mathbf{x},\alpha)} \to \pi_{n(\mathbf{x},\alpha)+1}$, the band split is α; if there is no such integer, set $n(\mathbf{x}, \alpha) = \infty$. From Proposition 10.1, we see that $n(\mathbf{x}, \alpha)$ is finite, almost surely.

For $M > 1$, let $X_{M,\alpha} = \{\mathbf{x} \in \Gamma Q(W) : |Q_{n(\mathbf{x},\alpha)}(\alpha)| > M|Q(\alpha)|\}$. The set $X_{M,\alpha}$ is thus the set of those $\mathbf{x}$ in whose expansion the α-column increases in norm by a factor greater than M before α is split.

PROPOSITION 10.5. *Let α be a band and $C > 1$ be a constant such that*

$$|Q(\alpha)| > \frac{1}{C} \max_{\gamma \in \mathcal{A}} |Q(\gamma)|.$$

Then there exists a constant $M > 1$, depending only on C and d, such that

$$\ell(X_{M,\alpha}) < \frac{1}{2d}\ell(\Gamma Q(W)).$$

In fact, for all $M' > M$, the proportion of $X_{M',\alpha}$ in $\Gamma Q(W)$ has an upper bound that depends only on M'; moreover, the bound tends 0 as $M' \to \infty$.

Proof. We shall prove Proposition 10.5 in a number of steps.

Step 1. As a first step, we shall include the set $X_{M,\alpha}$ in a finite union of sets whose measures are easier to estimate.

Let $Y_{M,\alpha} = (\Gamma Q)^{-1}(X_{M,\alpha})$. For a $\beta \in \mathcal{A}$ and a positive constant R, recall from §9 that $\Delta_R(\alpha, \beta)$ is the set of points in Δ whose convex combination satisfies

$$\frac{\lambda_\beta}{\lambda_\alpha} \geq R.$$

Write R as

$$R = K\frac{|Q(\alpha)|}{|Q(\beta)|}$$

for some $K > 0$, let $Y_R(\alpha, \beta) = \Delta_R(\alpha, \beta) \cap W$ and let $X_R(\alpha, \beta) = \Gamma Q(Y_R(\alpha, \beta))$. We shall show that

LEMMA 10.6. *For $K = (M - 1)/(d - 1)$ we have the inclusion*

$$Y_{M,\alpha} \subseteq \bigcup_{\beta \neq \alpha} Y_R(\alpha, \beta).$$

or, equivalently,

$$X_{M,\alpha} \subseteq \bigcup_{\beta \neq \alpha} X_R(\alpha, \beta).$$

Proof. Since α does not get split until $\pi_{n(\mathbf{x},\alpha)}$, the column $Q_{n(\mathbf{x},\alpha)}(\alpha)$ has the form

$$Q_{n(\mathbf{x},\alpha)}(\alpha) = Q(\alpha) + \sum_{\beta \neq \alpha} c_\beta Q(\beta)$$

for non-negative integers c_β.

CLAIM 10.7. *Let $\mathbf{y} \in W$ be the pre-image of $\mathbf{x}$ under ΓQ. The convex combination $\mathbf{y} = \sum \lambda_\gamma e_\gamma$ must satisfy*

$$\frac{\lambda_\beta}{\lambda_\alpha} \geq c_\beta$$

for all β.

Proof. To simplify notation, let $m = n(\mathbf{x}, \alpha)$. Let T denote the matrix associated to the splitting sequence $\pi \to \cdots \to \pi_m$. Starting with the normalized widths (λ_γ), $\gamma \in \mathcal{A}$ at π, let $(\lambda_\gamma^{(m)})$ denote the actual widths at π_m resulting from the splitting sequence. Since α is never split until π_m, the width of α remains unchanged, i.e., $\lambda_\alpha^{(m)} = \lambda_\alpha$. Additionally, the

entries of the αth column of T are given by $T_{\beta\alpha} = c_\beta$ for $\beta \neq \alpha$ and $T_{\alpha\alpha} = 1$. From §6, the relationship between the old and the new widths is

$$\lambda_\beta = \sum_{\gamma \in A} T_{\beta\gamma} \lambda_\gamma^{(m)}.$$

Since all terms on the right-hand side above are non-negative, we get the inequality

$$\lambda_\beta \geq T_{\beta\alpha} \lambda_\gamma^{(m)} = c_\beta \lambda_\alpha,$$

finishing the proof of the claim. □

CLAIM 10.8. *If $|Q_{n(\mathbf{x},\alpha)}(\alpha)| \geq M|Q(\alpha)|$ for some constant $M > 1$, then, for at least one β,*

$$c_\beta \geq \frac{M-1}{d-1} \frac{|Q(\alpha)|}{|Q(\beta)|}.$$

Proof. By the additive property of the norm on $(\mathbb{R}_{\geq 0})^d$,

$$|Q_{n(\mathbf{x},\alpha)}(\alpha)| = |Q(\alpha)| + \sum_{\beta \neq \alpha} c_\beta |Q(\beta)|.$$

If $|Q_{n(\mathbf{x},\alpha)}(\alpha)| \geq M|Q(\alpha)|$, then

$$\sum_{\beta \neq \alpha} c_\beta |Q(\beta)| \geq (M-1)|Q(\alpha)|,$$

which implies the claim. □

By Claims 10.7 and 10.8, for each $\mathbf{y} \in Y_{M,\alpha}$, there is some β such that the convex combination for $\mathbf{y}$ satisfies

$$\frac{\lambda_\beta}{\lambda_\alpha} \geq K \frac{|Q(\alpha)|}{|Q(\beta)|}$$

for $K = (M-1)/(d-1)$. This proves the lemma. □

To show that there is a choice of M large enough such that $\ell(X_{M,\alpha})$ is bounded above by $(1/2d)\ell(\Gamma Q(W))$, it is enough to show that there is an M large enough such that for every β, the quantity $\ell(X_R(\alpha, \beta)) = \ell(\Gamma Q(Y_R(\alpha, \beta)))$ is bounded above by $(1/2d^2)\ell(\Gamma Q(W))$. So, for the remainder of this proof, we focus on one such $Y_R(\alpha, \beta)$.

Step 2. Figure 10.1 shows, for some β, a schematic picture of the sets $\Delta_R(\alpha, \beta)$ and $Y_R(\alpha, \beta)$ inside Δ. The simplex is drawn such that the vertex e_α is on the top and the opposite face $F(\alpha)$ is in the horizontal plane forming the floor of the simplex.

The picture on the left is an instance when α is orientation preserving and the picture on the right is an instance when α is orientation reversing. Let $e_R(\alpha, \beta)$ be the point on the edge $[e_\alpha, e_\beta]$ whose convex combination satisfies $\lambda_\beta/\lambda_\alpha = R$. Let $F_R(\alpha, \beta)$ denote the convex hull of $e_R(\alpha, \beta)$ and the vertices of the simplex other than e_α and e_β. In the picture, this is represented by the plane that is shaded. Then $\Delta_R(\alpha, \beta)$ is the *wedge* in the simplex bounded by $F_R(\alpha, \beta)$ on the top and $F(\alpha)$ on the bottom, i.e., it is the region in the simplex below the shaded plane. As a result, $Y_R(\alpha, \beta)$ is the part of W that lies below
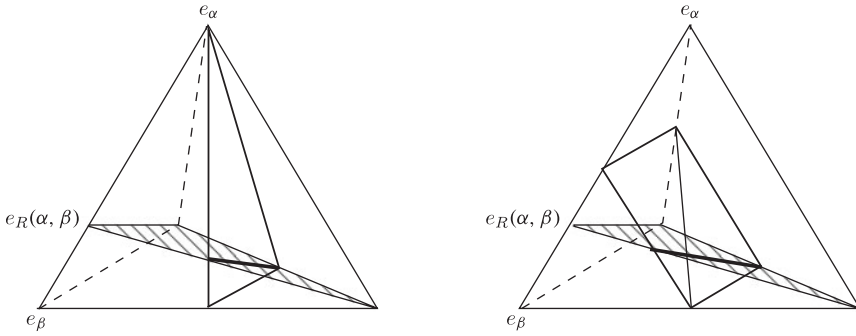
FIGURE 10.1. Schematic picture.

the intersection of the shaded plane and W, which is shown by the bold line segment in each picture.

The position of $e_R(\alpha, \beta)$ on the edge joining e_α and e_β is determined by the value of K and the ratio $|Q(\alpha)|/|Q(\beta)|$. The assumption $|Q(\alpha)| > (1/C) \max_{\gamma \in \mathcal{A}} |Q(\gamma)|$ implies that the ratio $|Q(\alpha)|/|Q(\beta)|$ is bounded below by $1/C$. Using this lower bound, we see that the points in $\Delta_R(\alpha, \beta)$, and hence in $Y_R(\alpha, \beta)$, satisfy

$$\frac{\lambda_\beta}{\lambda_\alpha} \geq \frac{K}{C}. \tag{10.1}$$

This means that, for every β, the distance along the edge of the point $e_R(\alpha, \beta)$ from e_β is bounded above by a quantity that depends only on M and C. Moreover, this quantity goes to 0 as M goes to infinity. In other words, each wedge is uniformly thin.

Step 3. In this step, we triangulate W by $(d-2)$-dimensional simplices $\Delta(i)$ in a specific way. Then, to get the upper bound for measure relative to $\Gamma Q(W)$, it is enough to get the same bound for measure of the intersection with each simplex in the triangulation.

Recall from §6 that each configuration space W is a convex hull of a finite number of vertices, either of type $e_{\alpha\beta}$ or e_γ. Fix a triangulation of W by $(d-2)$-dimensional simplices as follows.

(1)   Triangulate ∂W without introducing new vertices. This can be done in any manner by adding an appropriate number of new diagonals.

(2)   Fix the vertex p of W where, if α is orientation preserving, then $p = e_\alpha$, and if α is orientation reversing, then $p = e_{\alpha\gamma}$ for some γ. Cone off the triangulation of ∂W to p to get a triangulation of W.

For example, in the picture on the right in Figure 10.1, we have triangulated by adding in the diagonal of the quadrilateral. The number of simplices $\Delta(i)$ in a triangulation is bounded above by some number $n(d)$ that depends only on d.

Let the individual simplices in the triangulation of W be $\Delta(i)$. By construction, each $\Delta(i)$ contains the vertex p of W. To show the required bound for $\ell(\Gamma Q(Y_R(\alpha, \beta)))$, it is enough to show that for M large enough, the ratio $\ell(\Gamma Q(Y_R(\alpha, \beta) \cap \Delta(i)))/\ell(\Gamma Q(\Delta(i)))$ is bounded above by $1/2d^2$ for each i. Taking union over all i implies the required bound for $\Gamma Q(Y_R(\alpha, \beta))$. So, for the remainder of this proof, we shall focus on one such $\Delta(i)$.

Step 4. The intersection $Y_R(\alpha, \beta) \cap \Delta(i)$ is a half-space intersection $H \cap \Delta(i)$ with $\partial H = F_R(\alpha, \beta)$ (see §9). In inequality (10.1), if $K/C > 1/2$, then the half-space intersection is opposite p. In this step, we shall show that $H \cap \Delta(i)$ opposite p is $R/2$-thin. Hence, we can use Lemma 9.4 to include the half-space in a union of wedges in $\Delta(i)$. This reduces the task to showing a suitable upper bound for the ratio in each of these wedges.

The edges of $\Delta(i)$ with endpoint p that have non-empty intersection with $F_R(\alpha, \beta)$ are the ones joining p to a vertex q of $\Delta(i)$ in $F(\alpha)$. Denote the set of vertices of $\Delta(i)$ in $F(\alpha)$ by S_H and, for each vertex $q \in S_H$, let a_q denote the point of intersection $F_R(\alpha, \beta) \cap [p, q]$. Since $a_q \in F_R(\alpha, \beta)$, its convex combination in the full simplex Δ satisfies

$$\frac{\lambda_\beta}{\lambda_\alpha} = R. \tag{10.2}$$

Let a_q have the convex combination $a_q = \lambda_p p + \lambda_q q$ in $\Delta(i)$. Write

$$p = \frac{e_\alpha + e_\gamma}{2}, \quad q = \frac{e_\beta + e_\eta}{2}, \tag{10.3}$$

where $\gamma = \alpha$ or $\gamma \neq \alpha$, depending on whether α is orientation preserving or orientation reversing, and similarly $\eta = \beta$ or $\eta \neq \beta$. This means that $\lambda_\alpha = \lambda_p$ or $\lambda_\alpha = \lambda_p/2$, and similarly, $\lambda_\beta = \lambda_q$ or $\lambda_\beta = \lambda_q/2$. In any case, combining this with equation (10.2) implies that convex combinations in $\Delta(i)$ of the points on the line segment joining a_q to q satisfy

$$\frac{\lambda_q}{\lambda_p} \geq \frac{R}{2}.$$

Thus, our half-space intersection with $\Delta(i)$ opposite p is $R/2$-thin. By applying Lemma 9.4 to the intersection $Y_R(\alpha, \beta) \cap \Delta(i)$, we get for $R_0 = R/2(d-1)$, the inclusion

$$Y_R(\alpha, \beta) \cap \Delta(i) \subset \bigcup_{q \in S_H} \Delta(i)_{R_0}(p, q). \tag{10.4}$$

Consider the vectors Qp, $Qq \in \mathbb{R}^\mathcal{A}$. Using equation (10.3), the ratio $|Qp|/|Qq|$ satisfies

$$\frac{|Qp|}{|Qq|} = \frac{|Q(\alpha)| + |Q(\gamma)|}{|Q(\beta)| + |Q(\eta)|} < (C+1)\frac{|Q(\alpha)|}{|Q(\beta)|}.$$

So, if we set

$$R_1 = \left(\frac{K}{4dC}\right)\frac{|Qp|}{|Qq|},$$

then $R_0 > R_1$. This implies the inclusion

$$\Delta(i)_{R_0}(p, q) \subset \Delta(i)_{R_1}(p, q)$$

for each $q \in S_H$. Combining this with inclusion (10.4), we get

$$Y_R(\alpha, \beta) \cap \Delta(i) \subset \bigcup_{q \in S_H} \Delta(i)_{R_1}(p, q).$$

The number of simplices $\Delta(i)$ is at most $n(d)$. So, to establish the required upper bound on $\ell(\Gamma Q(Y_R(\alpha, \beta) \cap \Delta(i)))$, it is enough to show that there is a choice of M and consequently of R_1 such that

$$\ell(\Gamma Q(\Delta(i)_{R_1}(p, q))) \leq \frac{1}{2d^2 n(d)}\ell(\Gamma Q(\Delta(i))). \tag{10.5}$$

Step 5. Equation (9.3) applies to projective linear maps with non-negative entries of the standard $(d-1)$-dimensional simplex into itself. So, to apply equation (9.3) to get estimate (10.5), we need to extract such a map $\Gamma\widehat{Q}: \Delta_{d-2} \to \Delta_{d-2}$ from the restriction of ΓQ to $\Delta(i)$. This creates two issues. First, to extract the map $\Gamma\widehat{Q}$, we have to identify $\Delta(i)$ with Δ_{d-2}; this introduces a distortion in the standard measure on Δ_{d-2}. So, we need to show that this distortion is universally bounded. Second, we need to show that the Jacobian $\mathcal{J}(\Gamma Q)$ restricted to $\Delta(i)$ is, up to a universal constant, the same as the full Jacobian $\mathcal{J}_{\Delta_{d-2}}(\Gamma\widehat{Q})$ of the extracted map. In the final step, we show how to do this.

Recall from §6 that the volume forms on configuration spaces are normalized so that the volume of each configuration space is 1. Hence, up to the ratios of the configuration space volumes, for a measurable subset X of $\Gamma Q(W)$,

$$\ell(X) \approx \int_W \mathcal{J}(\Gamma Q)(\mathbf{y})\chi_Y(\mathbf{y}) \, d\mathbf{y}, \tag{10.6}$$

where χ_Y is the characteristic function of the set $Y = (\Gamma Q)^{-1}(X)$. Using (10.6), we get

$$\frac{\ell(\Gamma Q(\Delta(i)_{R_1}(p, q)))}{\ell(\Gamma Q(\Delta(i)))} = \frac{\int_{\Delta(i)_{R_1}(p,q)} \mathcal{J}(\Gamma Q)(\mathbf{y}) \, d\mathbf{y}}{\int_{\Delta(i)} \mathcal{J}(\Gamma Q)(\mathbf{y}) \, d\mathbf{y}}.$$

Recall from §8 the expression for the Jacobian of the restriction to W:

$$\mathcal{J}(\Gamma Q)(\mathbf{y}) = \frac{1}{a|Q\mathbf{y}|^{d-1}}.$$

So,

$$\frac{\ell(\Gamma Q(\Delta(i)_{R_1}(p, q)))}{\ell(\Gamma Q(\Delta(i)))} = \frac{\int_{\Delta(i)_{R_1}(p,q)} (1/|Q\mathbf{y}|^{d-1}) \, d\mathbf{y}}{\int_{\Delta(i)} (1/|Q\mathbf{y}|^{d-1}) \, d\mathbf{y}}. \tag{10.7}$$

In equation (10.7), the integrand looks similar to the Jacobian of a determinant-1 projective linear map, with non-negative entries, of a standard $(d-2)$-dimensional simplex Δ_{d-2}. We indicate how to make this observation precise and then use it to bound the ratio from above.

Let A_0 be the subspace in $\mathbb{R}^{\mathcal{A}}$ spanned by W_0 and let A be the subspace in $\mathbb{R}^{\mathcal{A}}$ spanned by W. Thus, A_0 and A are subspaces of $\mathbb{R}^{\mathcal{A}}$ satisfying the equations that define W_0 and W, respectively. Fix some linear isomorphism $F: \mathbb{R}^{d-1} \to A_0$ such that $F(\Delta_{d-2})$ contains W_0. Also, up to a permutation of the standard basis in $\mathbb{R}^{d-1}$, there is a unique linear isomorphism $G(i): \mathbb{R}^{d-1} \to A$ such that $G(i)(\Delta_{d-2}) = \Delta(i)$. Use the identification $G(i)$ to label the vertices of Δ_{d-2} by the corresponding labels of $\Delta(i)$.

Using the map F and the identifications $G(i)$, we define a linear map $\widehat{Q} = F^{-1} \circ Q \circ G(i)$. Since the projectivization $\Gamma\widehat{Q}$ maps Δ_{d-2} into itself, the map $\widehat{Q}$ has non-negative entries.

Since there are finitely many configuration spaces and finitely many simplices in the triangulation of each configuration space, the distortion by $G(i)$ of the standard Lebesgue measure on Δ_{d-2} is bounded above by a constant that depends only on d. This means that, for a point $\mathbf{y} \in \Delta_{d-2}$, the quantity $1/|QG(i)\mathbf{y}|^{d-1}$ (the integrand on the right in (10.7)) is the same as the full Jacobian $\mathcal{J}_{\Delta_{d-2}}(\Gamma\widehat{Q})(\mathbf{y}) = 1/|\widehat{Q}\mathbf{y}|^{d-1}$ up to a constant that depends

only on d. This also means that up to a universal constant, the norm of the rth column $|\widehat{Q}(r)|$ is the same as $|Qr|$.

The restriction $\Delta(i)_{R_1}(p, q) \rightarrow \Gamma Q(\Delta(i)_{R_1}(p, q))$ corresponds to $\Gamma(\widehat{Q}E) : \Delta_{d-2} \rightarrow \Delta_{d-2}$, where E is the elementary matrix whose off-diagonal entry in the (q, p) place is R_1.

Applying Corollary 9.3, we get

$$
\begin{aligned}
\frac{\ell(\Gamma Q(\Delta(i)_{R_1}(p, q)))}{\ell(\Gamma Q(\Delta(i)))} &\approx \frac{|\widehat{Q}(p)|}{|\widehat{Q}(p)| + R_1|\widehat{Q}(q)|} \\
&= |\widehat{Q}(p)| \left[|\widehat{Q}(p)| + \left(\frac{K}{4dC} \right) \frac{|Qp||\widehat{Q}(q)|}{|Qq|} \right]^{-1} \\
&\approx \frac{4dC}{4dC + K} < \frac{4dC}{K}.
\end{aligned}
$$

Finally, if M is large enough such that $K > 8d^3 n(d)C$, then we get estimate (10.5); moreover, it is easily checked that as $M \rightarrow \infty$, $K \rightarrow \infty$ and so the second part of the Proposition 10.5 also follows. □

In the next lemma, we use the technique in Step 5 of Proposition 10.5 to estimate from below the probability of certain splits. This will be useful later in Proposition 10.10.

LEMMA 10.9. *Let γ and σ be the bands in critical positions at some stage π. Let $C > 1$ be a constant such that*

$$
|Q(\gamma)| > \frac{1}{C} \max_{\alpha \in \mathcal{A}} |Q(\alpha)|.
$$

Then, up to a universal constant depending only on d, the probability that γ splits σ is bounded below by the quantity $1/n(d)(2C)^d$.

Proof. Let π' be the generalized permutation that results from γ splitting σ. Let W and W' denote the configuration spaces defined by π and π'.

A schematic picture is shown in Figure 10.2 with the shaded part X' in W representing the split. In other words, the shaded part X' is the image of W' under the projectivization of the elementary matrix associated to the split. Similarly, let X be $W \setminus X'$; this corresponds to σ splitting γ instead. Except when γ is the only orientation-reversing band on its side, X is non-empty. When X is empty, the split occurs with probability 1; so, we focus on the split in which X is non-empty. Let Z denote $\partial X' \cap \partial X$, i.e., the part of the boundary that separates the two sets.

The vertices of W fall into the following categories.
(1)   If a vertex has the linear combination $(1/2)e_\sigma + (1/2)e_\tau$ with $\tau \neq \gamma$, then it lies in $X' \setminus Z$.
(2)   If a vertex has the linear combination $(1/2)e_\gamma + (1/2)e_\tau$ with $\tau \neq \sigma$, then it lies in $X \setminus Z$.
(3)   All other vertices of W lie in Z.

Just as in Proposition 10.5, let $\Delta(i)$ be the triangulation of X. Denote the set of vertices of $\Delta(i)$ by $V(i)$. Let k be the index among all i, for which the measure $\ell(\Gamma Q(\Delta(k)))$ is the maximum. The triangulation has at most $n(d)$ simplices. Hence, we get the estimate
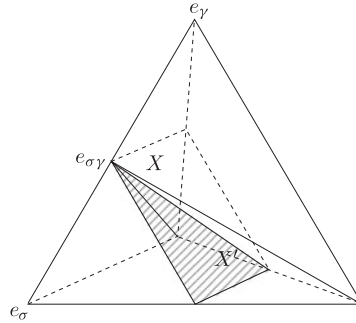
$$
\ell(\Gamma Q(X)) \leq n(d)\ell(\Gamma Q(\Delta(k))).
$$

FIGURE 10.2. γ splits σ.

Let V_Z be the subset of vertices of $\Delta(k)$ that lie in Z. Construct any triangulation of X' such that there is a simplex $\Delta'(j)$ for which the points in V_Z are vertices. Then

$$\frac{\ell(\Gamma Q(X'))}{\ell(\Gamma Q(X))} > \left(\frac{1}{n(d)}\right)\frac{\ell(\Gamma Q(\Delta'(j)))}{\ell(\Gamma Q(\Delta(k)))}.$$

So, it is enough to show a lower bound for the right side in the equation above. Let $V'(j)$ denote the set of vertices of $\Delta'(j)$. Associated to the simplices $\Delta(k)$ and $\Delta'(j)$, we define maps $\widehat{Q}_k = F^{-1} \circ Q \circ G(k)$ and $\widehat{Q}_j = F^{-1} \circ Q \circ G'(j)$ from the standard $(d-2)$-dimensional simplex into itself. See Step 5 of Proposition 10.5 for the definition of these maps. The gist of the discussion in that step is that, up to a universal constant introduced in the identifications,

$$\frac{\ell(\Gamma Q(\Delta'(j)))}{\ell(\Gamma Q(\Delta(k)))} \approx \frac{\ell(\Gamma \widehat{Q}_j(\Delta_{d-2}))}{\ell(\Gamma \widehat{Q}_k(\Delta_{d-2}))}. \tag{10.8}$$

There is a unique linear isomorphism $G(jk)$ of $\mathbb{R}^{d-1}$ such that $\widehat{Q}_k = \widehat{Q}_j \circ G(jk)$. Since the number of identification maps $G(*)$ is finite, the number of linear isomorphisms that relate them is also finite. Hence, the ratio of the determinants of $\widehat{Q}_*$ is universally bounded. This bound can be incorporated in applying Lemma 9.1 to evaluate the right-hand side in (10.8). Recall that the norm of the rth column $\widehat{Q}_*(r)$ is, up to a universal constant, the same as the norm of the vector Qr in $\mathbb{R}^{\mathcal{A}}$. In conclusion,

$$\frac{\ell(\Gamma \widehat{Q}_j(\Delta_{d-2}))}{\ell(\Gamma \widehat{Q}_k(\Delta_{d-2}))} \approx \frac{\prod_{p \in V(k)} |Qp|}{\prod_{q \in V'(j)} |Qq|} = \frac{\prod_{p \in V(k)\backslash V_Z} |Qp|}{\prod_{q \in V'(j)\backslash V_Z} |Qq|}. \tag{10.9}$$

In (10.9), the number of vertices in the numerator is exactly equal to the number of vertices in the denominator. So, the terms in the products in the numerator and the denominator pair off. In addition, any vertex p in $V(k)\backslash V_Z$ is a vertex in $X\backslash Z$, i.e., in our categorization, the vertex belongs to category (2). Hence its linear combination is $p = (1/2)e_\gamma + (1/2)e_\tau$, and so it satisfies $|Qp| > |Q(\gamma)|/2$. Hence, for a vertex q in $V'(j)\backslash V_Z$, we have the estimate

$$\frac{|Qp|}{|Qq|} > \frac{|Q(\gamma)|/2}{\max_{\alpha \in \mathcal{A}} |Q(\alpha)|} > \frac{1}{2C}.$$

Using the estimate in (10.9) gives the lower bound

$$\frac{\ell(\Gamma \widehat{Q}_j(\Delta_{d-2}))}{\ell(\Gamma \widehat{Q}_k(\Delta_{d-2}))} > \left(\frac{1}{2C}\right)^{d-1}.$$

Thus, from (10.8), we get the lower bound for the probability of the split

$$\frac{\ell(\Gamma Q(X))}{\ell(\Gamma Q(W))} \succ \frac{1}{1 + n(d)(2C)^{d-1}} > \frac{1}{n(d)(2C)^d}. \qquad \square$$

The next proposition, which goes back to Kerckhoff [8], has the following idea: one starts off by arranging the columns $Q(\alpha)$ in the order of decreasing norm and then picks out a C_0-distributed subset $\mathcal{A}_0$ starting with the column with the largest norm and following the order. Then the proposition shows that there is a definite probability, independent of the stage, that some future stage has a larger subset $\mathcal{A}_1 \supsetneq \mathcal{A}_0$ that is C_1-distributed and contains the column with the largest norm, with the constant C_1 independent of the stage. Thus, the proposition can be used as an iterative step, which we iterate over a finite number of times, to get, with a definite probability, a future stage that is C-distributed. The precise statement of the proposition here is identical to §4.4, Lemma 5 in Yoccoz [18], except that it is formulated for non-classical interval exchanges.

PROPOSITION 10.10. *Let $C_0 > 1$ be a constant and let $\mathcal{A}_0 \subsetneq \mathcal{A}$ be a set of C_0-distributed columns which contains the largest column in norm. Then there exist constants $c_1 \in (0, 1)$ and $C_1 > 1$, depending only on C_0 and d, and a finite set of future stages π_θ, after π, that satisfy the following two conditions.*

(1) *Denote the matrix corresponding to each π_θ by Q_θ and the configuration space by W_θ. The sets $\Gamma Q_\theta(W_\theta)$ have disjoint interiors and*

$$\sum_\theta \ell(\Gamma Q_\theta(W_\theta)) \geq c_1 \ell(\Gamma Q(W)).$$

(2) *For every θ, there exists a set of C_1-distributed columns $\mathcal{A}_\theta \supsetneq \mathcal{A}_0$ that contains the largest column at the stage π_θ.*

Proof. We recall the basic idea of the proof from §7. Before the norms of columns in $\mathcal{A}_0$ increase by some factor, suppose one of the following happens: either a column outside $\mathcal{A}_0$ becomes the column with the largest norm or a column in $\mathcal{A}_0$ moves a column outside. At this point, if we let $\mathcal{A}_1$ be $\mathcal{A}_0$ union this outside column, then, as we shall see, the new collection is C_1-distributed, where C_1 depends only on the initial constants. Then it remains to show that, with a definite probability, one of the two events happens.

The proof here follows the proof of Lemma 5 of §4.4 in Yoccoz [18] closely, except that the individual steps require justification for non-classical interval exchanges. The steps are as follows.

Step 1. To make it possible for a column in $\mathcal{A}_0$ to move a column outside, a column outside $\mathcal{A}_0$ needs to land in one of the critical positions. In this step, we show that there is a definite probability that this happens before the maximum of the norms of the columns increases by a definite factor. Specifically, we claim that π can be split to a future stage π_t such that, for constants $c_1' \in (0, 1)$ and $C_1' > 1$ depending only on C_0 and d, we have

$$\max_{\alpha \in \mathcal{A}} |Q_t(\alpha)| < C_1' \max_{\alpha \in \mathcal{A}} |Q(\alpha)|, \qquad (10.10)$$

$$\ell(\Gamma Q_t(W_t)) > c_1' \ell(\Gamma Q(W)) \qquad (10.11)$$

and at least one of the bands in the critical positions in π_t does not belong to $\mathcal{A}_0$.

Proof. By Proposition 10.1, for each node π in the attractor and for each band β in $\mathcal{A}$, there are splitting sequences which bring an end of β to a critical position. For each pair (π, β), choose one such splitting sequence. The number of chosen sequences is equal to d times the number of nodes in the attractor. This means that there is an upper bound h, depending only on d, on the lengths of all chosen sequences.

For some band β in $S \setminus \mathcal{A}_0$, let $\iota(\beta)$ be the chosen splitting sequence starting from π that brings β into one of the critical positions. Let ι be the shortest prefix of $\iota(\beta)$, for which a band not in $\mathcal{A}_0$ is in a critical position.

By the above discussion, the length of ι is at most h. This implies that the norm of a column in $\mathcal{A}_0$ participating in ι can increase by a factor of at most 2^h. So, we get the estimate

$$\max_{\alpha \in \mathcal{A}_0} |Q_\iota(\alpha)| < 2^h \max_{\alpha \in \mathcal{A}} |Q(\alpha)|.$$

Thus, (10.10) holds with $C_1' = 2^h$.

To show the bound (10.11) for the measures, it is enough to show that every split in ι has relative probability bounded below by a constant that depends only on C_0 and d.

Suppose we are at some intermediate stage π_1 of ι. Following π_1, suppose a band γ in $\mathcal{A}_0$ splits a band σ in $\mathcal{A}_0$ to give us the next stage π_2. The column $Q_1(\gamma)$ satisfies the estimate

$$|Q_1(\gamma)| > |Q(\gamma)| > \frac{1}{C_0} \max_{\alpha \in \mathcal{A}} |Q(\alpha)| > \frac{1}{C_0 C_1'} \max_{\alpha \in \mathcal{A}} |Q_1(\alpha)|.$$

By Lemma 10.9, the probability of this split is bounded below by $1/n(d)(2C_0 C_1')^d$, up to a universal constant that depends only on d. Finally, using the fact that the length of ι is bounded above by h, we get

$$\ell(\Gamma Q_\iota(W_\iota)) \succ \frac{1}{n(d)^h (2C_0 C_1')^{dh}} \ell(\Gamma Q(W)).$$

So, setting

$$c_1' \approx \frac{1}{n(d)^h (2C_0 C_1')^{dh}}$$

proves Step 1.   □

Step 2. Notice that for every $\alpha \in \mathcal{A}_0$, we have $|Q_\iota(\alpha)| \geq (1/C_1') \max_{\gamma \in \mathcal{A}} |Q_\iota(\gamma)|$. So, we can apply Proposition 10.5 to the stage π_ι. Let

$$X_{M,\alpha} = \{\mathbf{x} \in \Gamma Q_\iota(W_\iota) : |Q_{n(\mathbf{x},\alpha)}(\alpha)| > M |Q_\iota(\alpha)|\}.$$

By Proposition 10.5, there exists $M > 1$, depending on C_1' and d, such that $\ell(X_{M,\alpha}) < (1/2d)\ell(\Gamma Q_\iota(W_\iota))$. If we let

$$X = \Gamma Q_\iota(W_\iota) \setminus \bigcup_{\alpha \in \mathcal{A}_0} X_{M,\alpha},$$

then we have the estimate

$$\frac{\ell(X)}{\ell(\Gamma Q_\iota(W_\iota))} \geq 1 - \sum_{\alpha \in \mathcal{A}_0} \frac{\ell(X_{M,\alpha})}{\ell(\Gamma Q_\iota(W_\iota))} > 1 - \sum_{\alpha \in \mathcal{A}_0} \frac{1}{2d} > \frac{1}{2}. \tag{10.12}$$

From this point on, the rest of the proof follows Yoccoz [18] almost verbatim. We include it here for completeness.

Step 3. Since at π_ι a band outside $\mathcal{A}_0$ is in a critical position, it is not possible to get a stage in which both bands in the critical positions are in $\mathcal{A}_0$ unless a column in $\mathcal{A}_0$ moves a column outside. This observation can be exploited to show that, with a definite probability, either a band outside $\mathcal{A}_0$ becomes large enough in norm or a band in $\mathcal{A}_0$ moves a band outside.

For each $\mathbf{x} \in X$, let $\pi_{\mathbf{x},m}$ denote the stage after π_ι such that $\pi_{\mathbf{x},m-1} \to \pi_{\mathbf{x},m}$ is the first instance when a band in $\mathcal{A}_0$ is split. To each $\mathbf{x} \in X$, we associate a stage $\pi(\mathbf{x})$ in the path $\pi_\iota \to \cdots \to \pi_{\mathbf{x},m}$ as follows.

Case 1. If at some intermediate stage π',

$$|Q'(\beta)| \geq \max_{\alpha \in \mathcal{A}} |Q_\iota(\alpha)| \tag{10.13}$$

for some $\beta \in \mathcal{A} \backslash \mathcal{A}_0$, then we let $\pi(\mathbf{x})$ be the first instance when inequality (10.13) is true for some band in $S \backslash \mathcal{A}_0$. This also means that in $\pi(\mathbf{x})$ there is a unique band $\beta \in \mathcal{A} \backslash \mathcal{A}_0$ that satisfies inequality (10.13).

Case 2. If no band in $S \backslash \mathcal{A}_0$ satisfies inequality (10.13) at any intermediate stage, we set $\pi(\mathbf{x}) = \pi_{\mathbf{x},m}$. Here, a band $\beta \in \mathcal{A} \backslash \mathcal{A}_0$ splits a band $\alpha \in \mathcal{A}_0$ in the final split before $\pi(\mathbf{x})$. So,

$$|Q_\theta(\beta)| = |Q_{\theta-1}(\beta)| + |Q_{\theta-1}(\alpha)|. \tag{10.14}$$

Step 4. Given the estimate (10.12), it is possible to select a finite number of stages π_θ from the collection of stages $\pi(\mathbf{x})$ constructed in Step 3 such that $\Gamma Q_\theta(W_\theta)$ have disjoint interiors and

$$\ell(\cup_\theta \Gamma Q_\theta(W_\theta)) > \tfrac{1}{2}\ell(X) > \tfrac{1}{4}\ell(\Gamma Q_\iota(W_\iota)).$$

For each π_θ as above, let $\mathcal{A}_\theta$ be the set of $\gamma \in \mathcal{A}$ that satisfies

$$|Q_\theta(\gamma)| > C_0^{-1} \max_{\alpha \in \mathcal{A}_0} |Q(\alpha)|. \tag{10.15}$$

CLAIM 10.11. *For each θ, the collection $\mathcal{A}_\theta$ is strictly larger than $\mathcal{A}_0$ and is $C_0 C_1'(1 + M)$-distributed.*

Proof. The stage π_θ belongs to one of the two cases in Step 3. In both cases, the distinguished band β is in $\mathcal{A}_\theta \backslash \mathcal{A}_0$.

Given the lower bound (10.15) in the definition of $\mathcal{A}_\theta$, it is enough to show that the largest column in norm has an appropriate upper bound. Because we are dealing with subsets of X, every band $\alpha \in \mathcal{A}_0$ satisfies

$$|Q_\theta(\alpha)| < M|Q_\iota(\alpha)|. \tag{10.16}$$

This implies

$$\max_{\alpha \in \mathcal{A}_0} |Q_\theta(\alpha)| < M \max_{\alpha \in \mathcal{A}_0} |Q_\iota(\alpha)|. \tag{10.17}$$

If the largest column at π_θ is in $\mathcal{A}_0$, then estimates (10.10), (10.15) and (10.17) imply that $\mathcal{A}_\theta$ is $C_0 C_1' M$-distributed. So, it is also $C_0 C_1'(1 + M)$-distributed and we are done. Hence, to finish the proof, we shall assume that the largest column is not in $\mathcal{A}_0$. A stage π_θ can belong to Case 1 or Case 2 of Step 3. We argue the two cases separately.

Case 1. If π_θ belongs to Case 1, then there is a unique band $\beta \in \mathcal{A}_\theta \backslash \mathcal{A}_0$ that satisfies inequality (10.13). By assumption, the largest column is not in $\mathcal{A}_0$; hence, the column $Q_\theta(\beta)$ has to be the largest. At every stage before π_θ, no column in $\mathcal{A} \backslash \mathcal{A}_0$ satisfies inequality (10.13) and no column in $\mathcal{A}_0$ moves a column in $\mathcal{A} \backslash \mathcal{A}_0$. So, we get the bound

$$|Q_\theta(\beta)| < 2 \max_{\alpha \in \mathcal{A}_0} |Q_\iota(\alpha)| < (1 + M) \max_{\alpha \in \mathcal{A}_0} |Q_\iota(\alpha)|.$$

Combining with inequality (10.10) results in the upper bound

$$|Q_\theta(\beta)| < (1 + M) \max_{\alpha \in \mathcal{A}_0} |Q_\iota(\alpha)| < C_1'(1 + M) \max_{\alpha \in \mathcal{A}_0} |Q(\alpha)|$$

and so the collection $\mathcal{A}_\theta$ is $C_0 C_1'(1 + M)$-distributed.

Case 2. If π_θ belongs to Case 2, then, in the final split before π_θ, some band α in $\mathcal{A}_0$ is split by some band β in $\mathcal{A} \backslash \mathcal{A}_0$. So, it satisfies (10.14). Along the sequence $\pi_\iota \to \cdots \to \pi_\theta$, no band in $\mathcal{A} \backslash \mathcal{A}_0$ satisfies inequality (10.13) at any intermediate stage. This implies that the column $Q_\theta(\beta)$ has to be the largest in norm. It also implies that

$$|Q_{\theta-1}(\beta)| < \max_{\alpha \in \mathcal{A}_0} |Q_\iota(\alpha)|. \tag{10.18}$$

Inequalities (10.10) and (10.18) give the upper bound

$$|Q_\theta(\beta)| < (1 + M) \max_{\alpha \in \mathcal{A}_0} |Q_\iota(\alpha)| < C_1'(1 + M) \max_{\alpha \in \mathcal{A}_0} |Q(\alpha)| \tag{10.19}$$

and so the collection $\mathcal{A}_\theta$ is $C_0 C_1'(1 + M)$-distributed.   □

Setting $c_1 = (1/4)c_1'$ and $C_1 = C_0 C_1'(1 + M)$, we conclude the proof of Proposition 10.10.   □

Iterating Proposition 10.10, we get the following proposition.

PROPOSITION 10.12. *There exist constants $c \in (0, 1)$ and $C > 1$ that depend only on d, and a finite number of future stages π_τ after π, such that:*
(1)   *the sets $\Gamma Q_\tau(W_\tau)$ have disjoint interiors and*

$$\sum_\tau \ell(\Gamma Q_\tau(W_\tau)) \geq c \cdot \ell(\Gamma Q(W));$$

(2)   *each stage π_τ is C-distributed.*

To prove Theorem 6.3, apply Proposition 10.12 to π to get a set of stages, which we now denote by $\pi_\tau^{(1)}$, that are C-distributed. Let $X^{(1)}$ be the union of the sets $\Gamma Q_\tau(W_\tau)$. Consider the complement $\Gamma Q(W) \backslash X^{(1)}$. Write it as a union of stages with disjoint interiors. Apply Proposition 10.12 to each of them to get a new set of stages, which we denote by $\pi_\tau^{(2)}$, that are C-distributed. Let $X^{(2)}$ denote the union of the sets that correspond to the new C-distributed stages. Now, consider the complement $\Gamma Q(W) \backslash (X^{(1)} \cup X^{(2)})$ and iterate the process.

As a result, we construct an infinite sequence of sets $X^{(n)}$ with pairwise intersections of measure zero, such that they are a union of C-distributed stages. The set of $\mathbf{x} \in \Gamma Q(W)$ whose expansion gets C-distributed at some stage after π is equal to the infinite union of the sets $X^{(n)}$. Let

$$Y^{(n)} = \Gamma Q(W) \Big\backslash \bigcup_{i=1}^{n-1} X^{(i)}.$$

By Proposition 10.12, we know that $\ell(Y^{(n)}) \leq (1 - c)\ell(Y^{(n-1)})$. Hence, by induction,

$$\lim_{n \to \infty} \ell(Y^{(n)}) = 0.$$

So, the infinite union of the sets $X^{(n)}$ has full measure, which concludes the proof of Theorem 6.3. An immediate consequence of Theorem 6.3 is the following theorem.

THEOREM 10.13. *For almost every* $\mathbf{x} \in W_0$, *the expansion of* $\mathbf{x}$ *becomes C-distributed infinitely often.*

Proof. Theorem 6.3 says that the set whose expansion never gets C-distributed is measure zero. The set of $\mathbf{x}$ whose expansion gets C-distributed finitely many times is a countable union of such measure-zero sets. Hence, it is measure zero, proving the theorem. □

11. *Normality*

In this section, we prove normality, i.e., Theorem 1.1. For every π' in the attractor, choose a *shortest* splitting sequence from π' to π_0. This gives a finite number of chosen splitting sequences. This means that, for any C-distributed π':

(1)   with probability bounded below by a constant $c' > 0$ that depends only on d, it is possible to split from the C-distributed π' to a stage that ends at π_0 using the sequence chosen above;

(2)   splitting from π' to a stage as above, introduces a bounded amount of distortion in the measure, where the bound depends only on d.

The statements above imply that in Proposition 10.12, for different constants $c \in (0, 1)$ and $C > 1$ which depend only on d, one gets C-distributed with a relative probability greater than c, such that the C-distributed stages have generalized permutation π_0. In other words, the uniformly distorted stages in Theorem 6.3 can be assumed to have generalized permutation π_0. Moreover, because of this additional conclusion in Theorem 6.3 and the fact that we are in the attractor, it is enough to prove normality for finite sequences that start from π_0.

Let $J = \pi_0 \to \cdots \to \pi$ be a finite sequence of splits. We prove the stronger form of normality stated below, which is necessary to prove unique ergodicity.

THEOREM 11.1. (Strong normality) *In almost every expansion, for any finite sequence J starting from π_0, there are infinitely many instances in which J immediately follows a C-distributed stage.*

Proof. By Theorem 10.13, for every $k \in \mathbb{N}$, there is a kth instance of C-distribution in the expansion of almost every $\mathbf{x} \in W_0$. Let $\pi_{\mathbf{x}, n_k}$ be the C-distributed stage that is the kth instance. For a finite sequence J in the attractor, let X_k denote the set of all $\mathbf{x}$ for which the expansion after $\pi_{\mathbf{x}, n_k}$ begins with J. Recall from §8 that a C-distributed stage is C^{d-1}-uniformly distorted. So, the probability that J follows any C-distributed stage is, up to a constant that depends only on C and d, the same as the probability that an expansion begins with J. Let this probability be $\mu > 0$. So, we have $\ell(X_k) \approx \mu$. Since

$$\sum_{k=1}^{\infty} \ell(X_k) = \infty$$

by the standard Borel–Cantelli lemma, the set of $\mathbf{x}$ that belong to infinitely many X_k has full measure.   □

12.   *Unique ergodicity*

By Proposition 6.1, to show unique ergodicity, it is enough to show the following theorem.

THEOREM 12.1. *For almost every* $\mathbf{x}$ *in* W_0, *the nested sequence of sets* $\Gamma Q_n(W_n)$ *given by the expansion of* $\mathbf{x}$ *satisfies*

$$\bigcap \Gamma Q_n(W_n) = \mathbf{x}.$$

In other words, almost surely, the infinite intersection nests down to a point.

Proof. It is enough to prove that, almost surely, the sets $\Gamma Q_n(\Delta)$ nest down to a point. To do this, suppose there exists a finite sequence $\jmath : \pi_0 \to \cdots \to \pi_j$ with the *diameter-shrinking* property: there exists a universal positive constant $R < 1$ such that, when any C-distributed stage π is followed by $\jmath$, the diameter becomes less than the diameter at π by a factor less than R. By strong normality, in almost every expansion, the sequence $\jmath$ follows a C-distributed stage infinitely often. This implies that, for almost every $\mathbf{x}$, the diameter shrinking occurs infinitely often and so the infinite intersection of the sets $\Gamma Q_n(\Delta)$ has diameter zero, i.e., it is a point.

It is not immediate from the definition of C-distribution that a sequence $\jmath$ that shrinks the diameter of the initial stage by $R < 1$ also shrinks the diameter at every C-distributed stage by R (up to a constant that depends only on C and d), i.e., that $\jmath$ has the diameter-shrinking property. The subtle issue is that a C-distributed stage is defined using the Jacobian. So, it is not immediate that the distortion is uniform along line segments in Δ. The following lemma verifies that, for a C-distributed stage, the distortion along line segments in Δ is C^2-uniform.

Let π be a C-distributed stage and let Q be the associated matrix. Let L be a line segment in Δ with endpoints $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\partial \Delta$. Denote the unit tangent vector along L from $\mathbf{u}_1$ to $\mathbf{u}_2$ by v. Let $\mathbf{w}_1 = Q\mathbf{u}_1$ and $\mathbf{w}_2 = Q\mathbf{u}_2$. As in §8, denote the full derivative of ΓQ as a map from Δ to itself by $\mathbf{D}\Gamma Q$.

LEMMA 12.2. *For any pair of points* $\mathbf{y}_1$ *and* $\mathbf{y}_2$ *in* L,

$$\frac{1}{C^2} |\mathbf{D}\Gamma Q(\mathbf{y}_1)(v)| < |\mathbf{D}\Gamma Q(\mathbf{y}_2)(v)| < C^2 |\mathbf{D}\Gamma Q(\mathbf{y}_1)(v)|.$$

Proof. Parameterize $Q(L)$ by $f(t) = (1-t)\mathbf{w}_1 + t\mathbf{w}_2$. Then $g(t) = f(t)/|f(t)|$ is a parameterization of $\Gamma Q(L)$. By additivity of the norm, $|f(t)| = (1-t)|\mathbf{w}_1| + t|\mathbf{w}_2|$. Computing the derivative with respect to t, we get $|f(t)|' = |\mathbf{w}_2| - |\mathbf{w}_1|$. This gives

$$
\begin{aligned}
g'(t) &= \frac{|f(t)|f'(t) - f(t)|f(t)|'}{|f(t)|^2} \\
&= \frac{((1-t)|\mathbf{w}_1| + t|\mathbf{w}_2|)(\mathbf{w}_2 - \mathbf{w}_1) - ((1-t)\mathbf{w}_1 + t\mathbf{w}_2)(|\mathbf{w}_2| - |\mathbf{w}_1|)}{|f(t)|^2} \\
&= \frac{|\mathbf{w}_1|\mathbf{w}_2 - |\mathbf{w}_2|\mathbf{w}_1}{|f(t)|^2}.
\end{aligned}
$$

This means that, for any pair of values t_1 and t_2 in the interval $[0, 1]$,

$$\frac{1}{C^2} < \frac{|g'(t_1)|}{|g'(t_2)|} < C^2. \tag{12.1}$$

Let $h(t) = (1 - t)\mathbf{u}_1 + t\mathbf{u}_2$ parameterize L. The map ΓQ restricted to L is given by $\Gamma Q(\mathbf{y}) = g \circ h^{-1}(\mathbf{y})$. Since h is a parameterization of L with constant speed, the estimate (12.1) concludes the proof. □

Similar to Corollary 4 from §4.3 of Yoccoz [**18**], there exists a splitting sequence $\jmath$ from π_0 whose associated matrix has every entry positive. For non-classical interval exchanges, the corollary is a consequence of Proposition 10.1 using an argument identical to the original proof in Yoccoz. The corollary implies that the image of Δ under the associated map is in the interior of Δ. We fix this sequence $\jmath$ for the rest of this section.

Let t_α be the vertices $\Gamma Q_\jmath(\alpha)$ of $\Gamma Q_\jmath(\Delta)$. Let s denote the minimum distance between t_α and $\partial \Delta$. The number s is related to the largest entry of $Q_\jmath$.

Extend the edge $[t_\alpha, t_\beta]$ of $\Gamma Q_\jmath(\alpha)$ in either direction to a line segment $L[\alpha, \beta]$ with corresponding endpoints u_α and u_β in $\partial \Delta$. Then the distances $d(t_\alpha, u_\alpha)$ and $d(t_\beta, u_\beta)$ are greater than s. So, the length of $[t_\alpha, t_\beta]$ is at most $(1 - 2s)$ times the length of $L[\alpha, \beta]$.

PROPOSITION 12.3. *Suppose π is a C-distributed stage and let π' be the stage obtained by $\jmath$ following π. Then there exists a positive constant $R < 1$ that depends on s, C and d such that*

$$\operatorname{diam}(\Gamma Q'(\Delta)) < R \operatorname{diam}(\Gamma Q(\Delta)).$$

Proof. Since $\Gamma Q'(\Delta)$ is a simplex, the longest line segment in it is a side joining a pair of vertices. Since $Q' = Q \cdot Q_\jmath$, we assume that this side is $\Gamma Q([t_\alpha, t_\beta])$. Let v be the unit tangent vector along the segment $L[\alpha, \beta]$. By Lemma 12.2, the distortion of v is C^2-uniform on $L[\alpha, \beta]$. Combining this with the fact that the length of $[t_\alpha, t_\beta]$ is at most $(1 - 2s)$ times the length of $L[\alpha, \beta]$, we get the distance estimate

$$\frac{d(\Gamma Q(t_\alpha), \Gamma Q(t_\beta))}{d(\Gamma Q(u_\alpha), \Gamma Q(u_\beta))} < \frac{C^2(1 - 2s)}{C^2(1 - 2s) + 2s}.$$

Taking R to be the right-hand side in the above inequality, we finish the proof of Proposition 12.3. □

Strong normality applied to the diameter-shrinking sequence $\jmath$ implies that diameter shrinking happens infinitely often in almost every expansion, and so, almost surely, the infinite intersection is a point. □

Finally, Theorem 12.1 implies that, for almost every $\mathbf{x}$, there is a unique transverse probability measure. Consequently, almost every non-classical interval exchange is uniquely ergodic.

13. *Rauzy map on the parameter space*

Consider the full parameter space for a strongly irreducible non-classical interval exchange. It is a disjoint union of configuration spaces, each defined by a generalized permutation in the attractor. Rauzy induction induces a map from this parameter space to

itself, as described in §6. Each configuration space in the full parameter space carries a natural Lebesgue measure. A consequence of Theorems 10.13 and 12.1 is the following theorem.

THEOREM 13.1. *The Lebesgue measure on the full parameter space is ergodic with respect to the Rauzy map.*

Proof. Let A be a Borel set in the full parameter space invariant under the Rauzy map and suppose that A is not full measure. Then A is not full measure in some configuration space W defined by some generalized permutation π. Let $\mathbf{x}$ be a point of Lebesgue density for the complement $W \setminus A$. Given $1/2 > \epsilon > 0$, there is a ball $B(\mathbf{x})$ centered at $\mathbf{x}$ such that the proportion of A in this ball is less than ϵ.

Starting with π as the initial generalized permutation, Theorems 10.13 and 12.1 imply that the ball $B(\mathbf{x})$ can be closely approximated by a union of C-distributed stages contained in $B(\mathbf{x})$. Furthermore, the argument at the beginning of §11 implies that, given a generalized permutation π' in the attractor, the C-distributed stages approximating $B(\mathbf{x})$ can be all arranged to have generalized permutation π'. Let us suppose the approximation is chosen so that the union of these C-distributed stages has measure greater than $(1 - \epsilon)\ell(B(\mathbf{x}))$. If the proportion of A in each of these C-distributed stages were greater than 2ϵ, then that would imply that the proportion of A in $B(\mathbf{x})$ is greater than $2\epsilon(1 - \epsilon) > \epsilon$, a contradiction. This means that there exists at least one C-distributed stage such that the proportion of A in it is at most 2ϵ.

Let Q be the matrix associated to this particular stage and let W' be the configuration space defined by π'. By the invariance of A under the Rauzy map,

$$\frac{\ell(A \cap W')}{\ell(W')} = \frac{\ell(A \cap \Gamma Q(W'))}{\ell(\Gamma Q(W'))} \leq 2\epsilon,$$

i.e., the proportion of A in W' is at most 2ϵ. Since ϵ can be made arbitrarily small and π' was chosen to be any generalized permutation in the attractor, the conclusion is that A must have measure zero. Thus, the invariant sets for the Rauzy map on the parameter space have zero or full measure, i.e., the Rauzy map is ergodic. □

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