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A HETEROGENEOUS INTERPOLANT

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In this note we exhibit an interpolant for a certain valid implication $\models \varphi \to \psi$, where φ and ψ come from the infinitary language $L_{\omega_1\omega_1}$. The existence of this interpolant follows from Takeuti's heterogeneous interpolation theorem [5], but unfortunately the proof in [5] is not explicit enough to allow one to find the interpolant explicitly. Takeuti's theorem asserts the existence of an interpolant in the class $\bar{L}_{\omega_1\omega_1}$ of *heterogeneous formulas*, which admits the rules of formation of $L_{\omega_1\omega_1}$ plus the following additional rule: if $\varphi \in \bar{L}_{\omega_1\omega_1}$ and $\langle Q_\alpha \rangle_{\alpha<\beta}$ is a sequence of quantifiers (i.e. $Q_\alpha = \exists$ or $Q_\alpha = \forall$) then $Q_0 x_{i_0} \cdots Q_\alpha x_{i_\alpha} \cdots {}_{(\alpha<\beta)}\varphi \in \bar{L}_{\omega_1\omega_1}$. (The semantic interpretation is the obvious one; consult [2, § C] or [4].)

The present interpolation example (whose investigation was suggested by J. Malitz) will be presented *as a definability theorem*. Namely we give a formula explicitly defining an isomorphism between two isomorphic well-founded extensional relations.

We take unary predicates A_1, A_2 and binary predicates E_1, E_2, F (written medially). Let σ be the conjunction of the following sentences.

$$\forall x[(A_1x \vee A_2x) \wedge \neg(A_1x \wedge A_2x)]$$

$$\forall x \forall y[xE_iy \to A_ix \wedge A_iy] \qquad (i = 1, 2)$$

$$\forall x \forall y[A_ix \wedge A_iy \to [x \simeq y \leftrightarrow \forall z(zE_ix \leftrightarrow zE_iy)]] \qquad (i = 1, 2)$$

$$\forall x_0 \forall x_1 \cdots [\neg \bigwedge_{j\in\omega} x_{j+1}E_ix_j] \qquad (i = 1, 2)$$

$$\forall x \forall y[xFy \to A_1x \wedge A_2y]$$

$$\forall x[A_1x \to \exists\,!\, y(xFy)]$$

$$\forall y[A_2y \to \exists\,!\, x(xFy)]$$

$$\forall u \forall v \forall x \forall y[xFy \wedge uFv \to (uE_1x \leftrightarrow vE_2y)] .$$

One may easily check that if

$$\langle U; A_1, A_2, E_1, E_2, F \rangle \models \sigma$$

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and

$$\langle U; A_1, A_2, E_1, E_2, F'\rangle \models \sigma ,$$

then $F = F'$. Thus F is implicitly defined by σ, and hence by Takeuti's theorem together with the usual argument for Beth's theorem, there is a (heterogeneous) formula $\Phi(x, y)$ such that $\sigma \models xFy \leftrightarrow \Phi(x, y)$, with F not appearing in Φ. (Such Φ cannot be in any $L_{\kappa\lambda}$, as follows from the proof of Malitz [3, Theorem 4.2].) The aim of this note is to explicitly exhibit Φ.

Let C be the set of finite sequences of 0's and 1's (including the empty sequence $\Box$). For $\Box \neq \sigma = a_0 a_1 \cdots a_{n-1} a_n \in C$, we let $\hat{\sigma} = a_0 \cdots a_{n-1}$. For all $\sigma \in C$ we take variables x_σ and y_σ. For all $\sigma \in C$, let Q_σ stand for[1]

$$(\forall x_{\sigma 0} \in A_1)(\exists y_{\sigma 0} \in A_2)(\forall y_{\sigma 1} \in A_2)(\exists x_{\sigma 1} \in A_1) .$$

Now let $\Phi(x, y)$ be

$$(A_1 x) \wedge (A_2 y) \wedge Q_\Box \cdots \underset{\substack{\sigma,\tau \in C \\ |\sigma| \leq |\tau|}}{Q_\sigma \cdots Q_\tau} \cdots \bigwedge_{\substack{\sigma \in C \\ \sigma \neq \Box}} (x_\sigma E_1 x_{\hat{\sigma}} \leftrightarrow y_\sigma E_2 y_{\hat{\sigma}}) ,$$

where $|\sigma|$ denotes the length of σ. We claim that $\sigma \models (\Phi(x, y) \leftrightarrow xFy)$. To see this, let $\mathfrak{A} = \langle U; A_1, A_2, E_1, E_2, F\rangle$ be any model of σ. We then need to see that for $a, b \in U, \mathfrak{A} \models \Phi\ [a, b]$ if and only if $(a, b) \in F$. Certainly if $(a, b) \in F$, then we may obviously use the isomorphism F to continue to establish the satisfaction of Φ in $\mathfrak{A}$. Conversely suppose that for some a and b, $a \in A_1$ and $b \in A_2$,

$$(*) \qquad \mathfrak{A} \models \Phi[a, b] \quad \text{but } (a, b) \notin F .$$

We let a be E_1-minimal among those a for which such b exists, and let b be E_2-minimal such that $(*)$ holds for b and this value of a. Notice that $\Phi(x, y)$ is logically equivalent to

$$\begin{aligned} &\forall x_0 \exists y_0[(x_0 E_1 x \leftrightarrow y_0 E_2 y) \wedge \Phi(x_0, y_0)] \wedge \\ &\forall y_1 \exists x_1[(x_1 E_1 x \leftrightarrow y_1 E_2 y) \wedge \Phi(x_1, y_1)] . \end{aligned}$$

Thus we know that

$$\mathfrak{A} \models \begin{bmatrix} \forall x_0 \exists y_0[(x_0 E_1 a \leftrightarrow y_0 E_2 b) \wedge \Phi(x_0, y_0)] \wedge \\ \forall y_1 \exists x_1[(x_1 E_1 a \leftrightarrow y_1 E_2 b) \wedge \Phi(x_1, y_1)] \end{bmatrix}$$

[1] We express the quantifiers Q_σ with "$\in$" for cognitive purposes. Clearly it is possible to write Φ in strict accordance with the formation rule mentioned above.

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By the minimality of a and b, we know that therefore

$$\mathfrak{A} \models \begin{bmatrix} \forall x_0 \exists y_0 [(x_0 E_1 a \leftrightarrow y_0 E_2 b) \wedge x_0 F y_0] \wedge \\ \forall y_1 \exists x_1 [(x_1 E_1 a \leftrightarrow y_1 E_2 b) \wedge x_1 F y_1] \end{bmatrix}.$$

But since F is an isomorphism and since E_1 and E_2 are each extensional, we see that $(a, b) \in F$.

References

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Added July 18, 1973 The following articles give further information on Takeuti's (and other) interpolation theorems:

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