

COMPOSITIO MATHEMATICA

On Lipschitz normally embedded complex surface germs

André Belotto da Silva, Lorenzo Fantini and Anne Pichon

Compositio Math. **158** (2022), 623–653.

[doi:10.1112/S0010437X22007357](https://doi.org/10.1112/S0010437X22007357)



FOUNDATION
COMPOSITIO
MATHEMATICA



LONDON
MATHEMATICAL
SOCIETY
EST. 1865





On Lipschitz normally embedded complex surface germs

André Belotto da Silva, Lorenzo Fantini and Anne Pichon

This work is dedicated to Norbert A'Campo

ABSTRACT

We undertake a systematic study of Lipschitz normally embedded normal complex surface germs. We prove, in particular, that the topological type of such a germ determines the combinatorics of its minimal resolution which factors through the blowup of its maximal ideal and through its Nash transform, as well as the polar curve and the discriminant curve of a generic plane projection, thus generalizing results of Spivakovsky and Bondil that were known for minimal surface singularities. In an appendix, we give a new example of a Lipschitz normally embedded surface singularity.

1. Introduction

A germ of a real or complex analytic space $(X, 0)$ embedded in $(\mathbb{R}^n, 0)$ or in $(\mathbb{C}^n, 0)$ is equipped with two natural metrics: its *outer metric* d_{out} , induced by the standard metric of the ambient space, and its *inner metric* d_{inn} , which is the associated arc-length metric on the germ. These two metrics are usually studied up to bi-Lipschitz local homeomorphisms, because they then give rise to tame classifications of singular sets, as was proven in various geometric contexts by Pham and Teissier [PT69], Mostowski [Mos85, Mos88], Parusiński [Par88, Par94], and Valette [Val05].

The germ $(X, 0)$ is said to be *Lipschitz normally embedded* (LNE) if the identity map of $(X, 0)$ is a bi-Lipschitz homeomorphism between the inner and the outer metric, that is, if there exist a neighborhood U of 0 in X and a constant $K \geq 1$ such that

$$d_{\text{inn}}(x, y) \leq K d_{\text{out}}(x, y)$$

for all x and y in U . As the inner and the outer geometries of $(X, 0)$ are invariant under bi-Lipschitz homeomorphisms (see [Pic20, Proposition 7.2.13]), this property only depends on the analytic type of $(X, 0)$, and not on the choice of an embedding in some smooth ambient space $(\mathbb{R}^n, 0)$ or $(\mathbb{C}^n, 0)$.

The study of LNE singularities is a very active research area with many recent results, for example by Birbrair, Bobadilla, Fernandes, Heinze, Kerner, Mendes, Misev, Neumann, Pedersen, Pereira, Pichon, Ruas, and Sampaio (see [BM18, FHPS21, FS19, KPR18, NPP20a, NPP20b]), but despite the current progress it is still in its infancy.

Although an irreducible complex curve germ $(X, 0)$ is LNE if and only if it is smooth (see [PT69, Fer03, NP07]), the situation is far richer already for complex surface germs. LNE germs

Received 8 January 2021, accepted in final form 12 January 2022, published online 27 May 2022.

2020 Mathematics Subject Classification 32S25 (primary), 13A18, 14B05 (secondary).

Keywords: complex surface singularities, Lipschitz geometry, Lipschitz normal embeddings, polar varieties, discriminant varieties, valuation spaces.

© 2022 The Author(s). The publishing rights in this article are licensed to Foundation Compositio Mathematica under an exclusive licence.

are fairly common in this context, including, in particular, all minimal surface singularities (as proven in [NPP20b] exploiting a characterization obtained in [NPP20a]), and the superisolated surface singularities with LNE tangent cone (see [MP21]). In this paper we prove several properties of a general complex LNE normal surface, describing in particular its generic polar curves and the discriminant curves of its generic plane projections. We also give a new example of a LNE normal singularity which is neither minimal nor superisolated, showing that the class of LNE normal surface singularities contains more elements than those already discovered (see Appendix B).

Among LNE surface singularities, the most widely studied are minimal singularities, which have been introduced in greater generality in [Kol85]. In dimension two, they are the rational surface singularities with reduced fundamental cycle, and they have the remarkable property that the topological type of $(X, 0)$ determines the following data, which is *a priori* of analytic nature.

- (1) The dual graph of the minimal good resolution of $(X, 0)$ which factors through the blowup of the maximal ideal and through the Nash transform, decorated by two families of arrows corresponding to the strict transform of a generic hyperplane section and to the strict transform of the polar curve of a generic plane projection.
- (2) The topological type of the discriminant curve of a generic projection. Moreover, this data can be computed explicitly from the dual graph of the minimal good resolution of $(X, 0)$.

The first property is a deep result of Spivakovsky [Spi90, III, Theorem 5.4], the second was later proven by Bondil (see [Bon03, Theorem 4.1] and [Bon16, Proposition 5.4]).

Observe that by good resolution of $(X, 0)$ we mean a proper bimeromorphic morphism $\pi: X_\pi \rightarrow X$ from a smooth surface X_π to X which is an isomorphism outside of a simple normal crossing divisor $E = \pi^{-1}(0)$, and the vertices of the dual graph Γ_π of E carry as weights the genera and self-intersections of the corresponding irreducible components of E . The fact that the topological type of a surface germ determines the dual graph of its minimal resolution is a classical result of Neumann [Neu81].

The two main results of the present paper extend the theorems of Spivakovsky and Bondil to all LNE surface singularities. Furthermore, we strengthen Spivakovsky’s result by showing that another important datum is an invariant of the topological type of $(X, 0)$, namely the inner rates of $(X, 0)$, an infinite family of rational numbers which measures the local metric structure of the germ $(X, 0)$ with respect to its inner metric. If E_v is a component of the exceptional divisor of a good resolution of $(X, 0)$, then its inner rate q_v , introduced in [BNP14] and further studied in [BFP22], measures the shrinking rate of the piece of the link of $(X, 0)$ that corresponds to E_v (see [BFP22, §§ 1 and 3]). These results show the crucial role played by generic projections and polar varieties, notions introduced and studied by Teissier [Tei82], in the understanding of LNE singularities.

To give a precise statement of our results we need to introduce some additional notation. Let $\pi: X_\pi \rightarrow X$ be a good resolution of $(X, 0)$ and denote by $V(\Gamma_\pi)$ the set of vertices of the dual graph Γ_π of π , so that every element v of $V(\Gamma_\pi)$ corresponds to a component E_v of the exceptional divisor $E = \pi^{-1}(0)$ of π . We denote by $Z_{\max}(X, 0) = \sum_{v \in V(\Gamma_\pi)} m_v E_v$ the maximal ideal divisor of $(X, 0)$, that is, the divisor of X_π supported on E and whose coefficient m_v , called *multiplicity of v* , is the multiplicity along the component E_v of the pullback via π of a generic linear form $h: (X, 0) \rightarrow (\mathbb{C}, 0)$ on $(X, 0)$. Although, in general, the divisor $Z_{\max}(X, 0)$ depends on the analytic type of $(X, 0)$, there is another divisor supported on E , namely the

fundamental cycle Z_{\min} of Γ_π , defined as the unique minimal nonzero element of the Lipman cone of Γ_π (see §2 for the relevant definitions), which only depends on the graph Γ_π . Finally, we denote by Z_{Γ_π} the canonical cycle of Γ_π , that is, the divisor supported on E determined by $Z_{\Gamma_\pi} \cdot E_v = -E_v^2 + 2g(E_v) - 2$ for every vertex v of Γ_π .

For each vertex v of Γ_π , set $l_v = -Z_{\max}(X, 0) \cdot E_v$, that is, l_v is the intersection multiplicity of E_v with the strict transform of a generic hyperplane section $h^{-1}(0)$ of $(X, 0)$ via π . We call \mathcal{L} -vector of $(X, 0)$ the vector $L_\pi = (l_v)_{v \in V(\Gamma_\pi)} \in \mathbb{Z}_{\geq 0}^{V(\Gamma_\pi)}$. Recall that the blowup $\text{Bl}_0 X$ of the maximal ideal of $(X, 0)$ is the minimal resolution of the base points of the family of generic hyperplane sections of $(X, 0)$. Whenever $\pi: X_\pi \rightarrow X$ factors through $\text{Bl}_0 X$, the strict transform of such a generic hyperplane section via π consists of a disjoint union of smooth curves that intersect transversely E at smooth points of E (see Appendix A), and l_v is the number of such curves passing through the component E_v ; we then call \mathcal{L} -node of Γ_π (or simply of $(X, 0)$) any vertex v such that $l_v > 0$. Similarly, we denote by p_v the intersection multiplicity of the strict transform of the polar curve of a generic plane projection $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ with E_v and we call the \mathcal{P} -vector of $(X, 0)$ the vector $P_\pi = (p_v)_{v \in V(\Gamma_\pi)} \in \mathbb{Z}_{\geq 0}^{V(\Gamma_\pi)}$. The Nash transform ν of $(X, 0)$ is the minimal resolution of the base points of the family of generic polar curves of $(X, 0)$ (see [Spi90, Section III, Theorem 1.2]). Whenever $\pi: X_\pi \rightarrow X$ factors through ν , then such a strict transform consists of smooth curves intersecting E transversely at smooth points, and p_v equals the number of such curves through E_v (see again Appendix A).

We then call the \mathcal{P} -node of Γ_π (or simply of $(X, 0)$) any vertex v such that $p_v > 0$. Finally, whenever $\pi: X_\pi \rightarrow X$ factors through the blowup of the maximal ideal, we define a natural distance d on Γ_π by declaring the length of an edge e between two vertices v and v' of Γ_π to be $1/\text{lcm}(m_v, m_{v'})$.

We can now state our first main theorem, which generalizes Spivakovsky’s result [Spi90, III, Theorem 5.4] to all LNE normal surface germs.

THEOREM 1.1. *Let $(X, 0)$ be a LNE normal surface germ, let $\pi: X_\pi \rightarrow X$ be the minimal good resolution of $(X, 0)$, and let Γ_π be the dual graph of π . Then the following properties hold.*

- (i) *The resolution π factors through the blowup of the maximal ideal of $(X, 0)$ and all \mathcal{L} -nodes have multiplicity one.*
- (ii) *The maximal ideal divisor $Z_{\max}(X, 0)$ of $(X, 0)$ coincides with the fundamental cycle Z_{\min} of Γ_π . In particular, Γ_π determines the multiplicity m_v associated with every vertex v of Γ_π , and therefore also the set $V_{\mathcal{L}}$ of \mathcal{L} -nodes of Γ_π , the \mathcal{L} -vector L_π of $(X, 0)$, and the distance d on Γ_π .*
- (iii) *The inner rate q_v of each vertex (or, more generally, of each divisorial point) of Γ_π is given by*

$$q_v = d(v, V_{\mathcal{L}}) + 1.$$

- (iv) *The \mathcal{P} -vector P_π of $(X, 0)$ is determined by*

$$p_v = -E_v \cdot \left(\sum_{v'} (m_{v'} q_{v'} - 1) E_{v'} - (Z_{\Gamma_\pi} - Z_{\min}) \right)$$

for every vertex v of Γ_π . Moreover, if v is an \mathcal{L} -node, this formula simplifies to $p_v = 2(g(E_v) + l_v - 1)$.

- (v) *Let π' be the minimal good resolution of $(X, 0)$ that factors through its Nash transform. A vertex v of $\Gamma_{\pi'}$ is a \mathcal{P} -node of $(X, 0)$ if and only if either $l_v > 1$ or there exist two distinct vertices v' and v'' of $\Gamma_{\pi'}$ adjacent to v and such that $q_{v'}, q_{v''} < q_v$.*

- (vi) *The resolution π' is obtained by composing π with a finite sequence of blowups of double points of the respective exceptional divisor: at each step, a double point in $E_v \cap E_{v'}$ has to be blown up if and only if $|q_v - q_{v'}| < d(v, v')$. In particular, an edge $e = [v, v']$ of Γ_π contains a \mathcal{P} -node of $(X, 0)$ in its interior (that is, such a \mathcal{P} -node appears as a vertex after blowing up finitely many double points of the exceptional divisor, starting with the blowup of the double point associated with e) if and only if $|q_v - q_{v'}| < d(v, v')$; when this is the case, e contains exactly one \mathcal{P} -node w , and the inner rate of w is $q_w = (d(v, v') + q_v + q_{v'})/2$.*

In particular, we can build from Γ_π the resolution graph $\Gamma_{\pi'}$ of π' , decorated by arrows corresponding to the components of the polar curve of a generic plane projection $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ and by the inner rate of each vertex. Observe also that parts (i) and (ii) imply that the multiplicity of a LNE normal germ is determined by its topological type.

Although the first two parts of the theorem are quite elementary, the remaining parts rely heavily on a careful study of generic projections of LNE surfaces (see Lemma 4.1), building on results from [NPP20a]. Parts (iii) and (iv) also depend on the study of inner rates of [BFP22], and, in particular, on the so-called Laplacian formula of [BFP22].

We then move our attention to the study of the discriminant curve Δ of a generic plane projection $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ of $(X, 0)$. Our second main result, which generalizes Bondil's results (see [Bon03, Theorem 4.1] and [Bon16, Proposition 5.4]), can be stated as follows.

THEOREM 1.2. *Let $(X, 0)$ be a LNE normal surface germ and let $\pi: X_\pi \rightarrow X$ be the minimal good resolution of $(X, 0)$. Then the dual graph Γ_π of π determines the embedded topological type of the discriminant curve of a generic plane projection $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ of $(X, 0)$.*

To be more precise, the embedded topological type of a plane curve can be conveniently encoded in a combinatorial object, its *Eggers–Wall tree*, whose construction is recalled in § 7 (see also [GGP19, Definition 3.9]). We give a more precise statement of Theorem 1.2 in Theorem 7.5, showing explicitly how to obtain the Eggers–Wall tree of the discriminant curve Δ of a generic plane projection $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ of $(X, 0)$ as the quotient of the graph $\Gamma_{\pi'}$ by a suitable equivalence relation.

Part (iv) of Theorem 1.1 can be thought of as the uniqueness of a solution, within the class of LNE surface singularities, to what we refer to as the problem of *polar exploration* of surface singularities, which asks to determine the possible configurations of arrows of a finite graph that can be realized as polar curves of a complex surface germ $(X, 0)$. Recall that surface singularities can be resolved either by a sequence of normalized point blowups, following the seminal work of Zariski [Zar39] from the late 1930s, or by a sequence of normalized Nash transforms, as was done half a century later by Spivakovsky [Spi90]. The relationship between these two resolution algorithms, and therefore between hyperplane sections and polar curves of a surface singularity, is still quite mysterious, and they seem to be in some sense dual, as was observed by Lê [Lê00, § 4.3].

More precisely, recall that the incidence matrix of the dual graph Γ_π associated with a good resolution $\pi: X_\pi \rightarrow X$ of $(X, 0)$ is negative definite by a classical result of Mumford [Mum61, § 1]. Moreover, Grauert [Gra62] proved that every weighted graph Γ without loops and with negative-definite incidence matrix can be realized as dual graph Γ_π associated with a good resolution of some normal complex surface germ $(X, 0)$. It is well known that the weighted graph Γ_π determines the topology of $(X, 0)$, because Γ_π is a plumbing graph of the link of $(X, 0)$ and, conversely, as we have already mentioned, Neumann [Neu81] proved that the plumbing graph Γ_π is determined up to a natural equivalence relation by the topology of the surface germ. It is, thus, natural to

consider the plumbing graph Γ_π endowed with an \mathcal{L} - and a \mathcal{P} -vector. From this point of view, our result implies the following statement.

COROLLARY 1.3. *Let Γ be a finite connected graph without loops weighted by attaching to each vertex v a genus $g(v) \geq 0$ and a self-intersection $e(v) < 0$. Then there exists at most one pair (L, P) of vectors $L = (l_v)$ and $P = (p_v) \in (\mathbb{Z}_{\geq 0})^{V(\Gamma)}$ such that there exist a LNE normal surface singularity $(X, 0)$ and a good resolution π of $(X, 0)$ satisfying*

$$(\Gamma, L, P) = (\Gamma_\pi, L_\pi, P_\pi).$$

Observe that not all weighted graphs can be realized as resolution graph of a LNE surface germ. For instance, one topological restriction comes from parts (i) and (ii) of Theorem 1.1: if Γ is the resolution graph of a LNE normal surface germ, then for every vertex v of Γ such that $E_v \cdot Z_{\min} < 0$ the component E_v has to have multiplicity one in Z_{\min} .

2. Surface germs with unique \mathcal{L} -vector

In this section we prove parts (i) and (ii) of Theorem 1.1. More generally, we are interested in finding a suitable geometric condition yielding a class of complex surfaces $(X, 0)$ whose \mathcal{L} -vector is completely determined by the topology of a resolution. To achieve this, we recall the precise definitions of the divisors $Z_{\max}(X, 0)$ and Z_{\min} that have been mentioned in the introduction, and determine a condition that guarantees their equality.

We begin by recalling the notion of Lipman cone. A more thorough discussion of the objects described in this section can be found in [Ném99]. Let Γ be a finite connected graph without loops and such that each vertex $v \in V(\Gamma)$ is weighted by two integers $g(v) \geq 0$, called genus, and $e(v)$, called self-intersection. We assume that the incidence matrix induced by the self-intersections of the vertices of Γ , that is the matrix $I_\Gamma \in \mathbb{Z}^{V(\Gamma)}$ whose (v, v') th entry is $e(v)$ if $v = v'$, and the number of edges of Γ connecting v to v' otherwise, is negative definite. Let $E = \bigcup_{v \in V(\Gamma)} E_v$ be a configuration of curves whose dual graph is Γ , so that $I_\Gamma = (E_v \cdot E_{v'})$, and consider the free additive group \mathcal{G} generated by the irreducible components of E , that is,

$$\mathcal{G} = \left\{ D = \sum_{v \in V(\Gamma)} m_v E_v \mid m_v \in \mathbb{Z} \right\}.$$

By a slight abuse of notation, we refer to the elements of \mathcal{G} as *divisors on Γ* . On \mathcal{G} there is a natural intersection pairing $D \cdot D'$, described by the incidence matrix I_Γ , and a natural partial ordering given by setting $\sum m_v E_v \leq \sum m'_v E_v$ if and only if $m_v \leq m'_v$ for every $v \in V(\Gamma)$.

The *Lipman cone* of Γ is the semi-group \mathcal{E}^+ of \mathcal{G} defined as

$$\mathcal{E}^+ = \{ D \in \mathcal{G} \mid D \cdot E_v \leq 0 \text{ for all } v \in V(\Gamma) \}.$$

Remark 2.1. By looking at the coefficients of a divisor we can identify \mathcal{G} with the additive group $\mathbb{Z}^{V(\Gamma)}$. Then the Lipman cone \mathcal{E}^+ of Γ is naturally identified with the cone $\mathbb{Z}_{\geq 0}^{V(\Gamma)} \cap -I_\Gamma^{-1}(\mathbb{Q}_{\geq 0}^{V(\Gamma)})$, because, by definition, a divisor $\sum m_v E_v$ belongs to \mathcal{E}^+ if and only if the vector $I_\Gamma \cdot (m_v)_{v \in V(\Gamma)}$ belongs to $\mathbb{Z}_{\leq 0}^{V(\Gamma)}$.

A cardinal property of the Lipman cone \mathcal{E}^+ , proven in [Art66, Proposition 2], is that it has a unique nonzero minimal element Z_{\min} , called the *fundamental cycle* of Γ , and that moreover $Z_{\min} > 0$, that is, the coefficients of Z_{\min} are all positive. Observe that the existence of the fundamental cycle and the fact that $Z_{\min} > 0$ are equivalent to the fact that $D > 0$ for every nonzero divisor D in \mathcal{E}^+ .

Assume from now on that Γ is the dual graph of a good resolution of a normal surface singularity $(X, 0)$. Note that the Lipman cone, and therefore its fundamental cycle, only depend on the graph Γ , that is, on the topology of $(X, 0)$, and not on the complex geometry of $(X, 0)$; the fundamental cycle Z_{\min} can be explicitly computed from Γ by using Laufer’s algorithm from [Lau72, Proposition 4.1].

Consider now a germ of analytic function $f: (X, 0) \rightarrow (\mathbb{C}, 0)$. The *total transform* of f by π is the divisor $(f) = (f)_{\Gamma} + f^*$ on X_{π} , where f^* is the strict transform of f and $(f)_{\Gamma} = \sum_{v \in V(\Gamma)} m_v(f)E_v$ is the divisor supported on E such that $m_v(f)$ is the multiplicity of $f \circ \pi$ along E_v . By [Lau71, Theorem 2.6], we have

$$(f) \cdot E_v = 0 \quad \text{for all } v \in V(\Gamma). \tag{1}$$

In particular, $(f)_{\Gamma}$ belongs to the Lipman cone \mathcal{E}^+ of Γ , and therefore the semi-group $\mathcal{A}_X^+ = \{(f)_{\Gamma} \mid f \in \mathcal{O}_{(X,0)}\}$ of \mathcal{G} is contained in \mathcal{E}^+ ; it has a unique nonzero minimal element $Z_{\max}(X, 0)$, which is called the *maximal ideal divisor* of $(X, 0)$. Observe that the divisor $Z_{\max}(X, 0)$ coincides with the cycle $(h)_{\Gamma}$ of a generic linear form $h: (X, 0) \rightarrow (\mathbb{C}, 0)$, and that by the definition of the fundamental cycle we have $Z_{\min} \leq Z_{\max}(X, 0)$.

The following proposition is the main result of this section.

PROPOSITION 2.2. *Let $(X, 0)$ be a normal surface singularity and let $\pi: (X_{\pi}, E) \rightarrow (X, 0)$ be the minimal good resolution of $(X, 0)$. If a generic hyperplane section of $(X, 0)$ is a union of smooth curves, then:*

- (i) π factors through the blowup of the maximal ideal of $(X, 0)$ and all \mathcal{L} -nodes have multiplicity one;
- (ii) the maximal ideal divisor $Z_{\max}(X, 0)$ of $(X, 0)$ coincides with the fundamental cycle Z_{\min} of Γ_{π} .

Proof. Let $\pi': X_{\pi'} \rightarrow X$ be the minimal good resolution of $(X, 0)$ which factors through the blowup of its maximal ideal and let E_v be a component of $(\pi')^{-1}(0)$. Let γ^* be a curvette of E_v , that is, a smooth complex curve germ intersecting transversely E_v at a smooth point of $(\pi')^{-1}(0)$, and let $h': (X, 0) \rightarrow (\mathbb{C}, 0)$ be a generic linear form of $(X, 0)$ such that the strict transform of $(h')^{-1}(0)$ via π' does not pass through the point $p = \gamma^* \cap E_v$. Then the multiplicity $\text{mult}(\gamma, 0)$ of $\gamma = \pi'(\gamma^*)$ at 0 can be computed as the intersection multiplicity of γ with a Milnor fiber $\{h = t\}$ of h in a small neighborhood of 0. Let us choose local coordinates (u, v) centered at p such that $u = 0$ is a local equation for E_v and $v = 0$ a local equation for γ^* . Then by the definition of m_v we have $(h' \circ \pi)(u, v) = u^{m_v} \alpha(u, v)$ where $\alpha(u, v)$ is a unity in $\mathbb{C}\{u, v\}$ and, therefore, $m_v = \text{mult}(\gamma, 0)$.

If v is an \mathcal{L} -node of $(X, 0)$ and $h: (X, 0) \rightarrow (\mathbb{C}, 0)$ is a generic linear form of $(X, 0)$, so that $h^{-1}(0)$ is a generic hyperplane section of $(X, 0)$, then there exists an irreducible component γ of $h^{-1}(0)$ whose strict transform γ^* by π' intersects E_v . By hypothesis, the curve γ is smooth, therefore it has multiplicity 1 and γ^* is a curvette of E_v . This proves that $m_v = 1$.

Assume now that π does not factor through the blowup of the maximal ideal, so that $\pi' = \pi \circ \alpha$, where α is a finite composition of point blowups. By minimality of π' there exists an \mathcal{L} -node v_0 of $(X, 0)$ which is associated with the exceptional component of one of the point blowups in α . Let α_1 be the first blowup in the sequence α , that is, α_1 is the blowup of X_{π} at a point p of $E_v \cap h^*$, where E_v is a component of $\pi^{-1}(0)$, and let E_w be the exceptional curve of α_1 . As h^* passes through p , we have $m_w = m_w(h) > m_v(h) \geq 1$. As this argument can be repeated for every blowup forming α , we deduce that $m_{v_0}(h) > 1$ as well, contradicting the first part of the proof. This implies that α must be an isomorphism, proving part (i).

To prove part (ii), write $Z_{\max} = Z_{\max}(X, 0) = \sum_{v \in V(\Gamma)} m_v E_v$ and $Z_{\min} = \sum_{v \in V(\Gamma)} \tilde{m}_v E_v$, and for every v in $V(\Gamma)$ consider the non-negative integers

$$l_v = -Z_{\max} \cdot E_v \quad \text{and} \quad \tilde{l}_v = -Z_{\min} \cdot E_v.$$

As $Z_{\min} \leq Z_{\max}$ by the definition of Z_{\min} , it is enough to prove that $Z_{\min} \geq Z_{\max}$. As I_Γ is negative definite, it is therefore sufficient to show that the integer $(Z_{\min} - Z_{\max}) \cdot E_v = l_v - \tilde{l}_v$ is at most zero for every vertex v of Γ . Whenever $l_v = 0$, this follows immediately from the definition, so let us fix a vertex v such that $l_v > 0$. From part (i), we know that $m_v = 1$. It follows from the inequality $0 < \tilde{m}_v \leq m_v = 1$ that $\tilde{m}_v = 1$ as well. We therefore obtain

$$l_v - \tilde{l}_v = (Z_{\min} - Z_{\max}) \cdot E_v = \sum_{w \in V(\Gamma)} (\tilde{m}_w - m_w) E_w \cdot E_v = \sum_{w \neq v} (\tilde{m}_w - m_w) E_w \cdot E_v \leq 0,$$

because $E_w \cdot E_v \geq 0$ whenever $w \neq v$ and $\tilde{m}_w \leq m_w$ at all vertices. □

The hypothesis of Proposition 2.2 is quite weak, as it is satisfied by every normal surface germ with *reduced tangent cone* (in which case the components of a generic hyperplane section are not only smooth but also transverse, see, for example, [GL97, §1]), for example, by every minimal surface singularity. More generally, the hypothesis holds for all LNE surface germs, as was proven in [FS19, Theorem 3.10]. In particular, the proposition implies parts (i) and (ii) of Theorem 1.1.

Observe that it follows by (1) that the vector $-I_{\Gamma_\pi} \cdot Z_{\max}(X, 0)$ of $\mathbb{Z}_{\geq 0}^{V(\Gamma_\pi)}$ coincides with the \mathcal{L} -vector L_π of $(X, 0)$ considered in the introduction. Therefore, whenever $Z_{\max}(X, 0)$ is determined by the topological type of $(X, 0)$, the same holds true for L_π . We collect this result, which is the first step towards the proof of Corollary 1.3, in the following corollary.

COROLLARY 2.3. *Let Γ be a weighted graph. Then there exists at most one vector $L \in \mathbb{Z}^{V(\Gamma)}$ such that there exist a normal surface germ $(X, 0)$ whose generic hyperplane section is a union of smooth curves and a good resolution $\pi: X_\pi \rightarrow (X, 0)$ of $(X, 0)$ satisfying $(\Gamma, L) = (\Gamma_\pi, L_\pi)$.*

3. A lemma on generic projections

In this section, we introduce three notions that prove fundamental in the remaining part of the paper, namely generic projections, non-archimedean links, and local degrees. We also prove an important result, Lemma 3.1, that shows the compatibility of generic projections with minimal resolutions.

We begin by discussing the notion of generic projection, which is based on seminal work of Teissier. Fix an embedding of $(X, 0)$ in a smooth germ $(\mathbb{C}^n, 0)$, and consider the morphism $\ell_{\mathcal{D}}: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ obtained as the restriction to X of the projection along an $(n - 2)$ -dimensional linear subspace \mathcal{D} of \mathbb{C}^n . Recall that whenever $\ell_{\mathcal{D}}$ is finite, the associated *polar curve* $\Pi_{\mathcal{D}}$ is the closure in $(X, 0)$ of the ramification locus of the restriction of $\ell_{\mathcal{D}}$ to $X \setminus \{0\}$, and the associated *discriminant curve* is the plane curve $\Delta_{\mathcal{D}} = \ell_{\mathcal{D}}(\Pi_{\mathcal{D}})$. The Grassmannian variety $\text{Gr}(n - 2, \mathbb{C}^n)$ of $(n - 2)$ -planes in \mathbb{C}^n contains an analytic dense open subset Ω such that, for every \mathcal{D} in Ω , the projection $\ell_{\mathcal{D}}$ is finite and the families $\{\Pi_{\mathcal{D}}\}_{\mathcal{D} \in \Omega}$ and $\{\Delta_{\mathcal{D}}\}_{\mathcal{D} \in \Omega}$ are both well behaved (for example, they are equisingular in a strong sense). We say that a morphism $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is a *generic projection* of $(X, 0)$ if $\ell = \ell_{\mathcal{D}}$ for some \mathcal{D} in Ω . A discussion of the properties satisfied by a generic projection, leading to a precise definition of Ω , can be found in [NPP20a, §2], building on work of Teissier

(see, in particular, [Tei82, Lemme-clé V 1.2.2]); we will come back to this matter later in this section.

We now recall the definition of the non-archimedean link $NL(X, 0)$ of the germ $(X, 0)$. Indeed, our goal for this section is to study the map induced by a generic projection $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ on the dual graph of a good resolution of $(X, 0)$. In principle, for this to make sense, it is necessary to choose a suitable good resolution $\pi: X_\pi \rightarrow X$ of $(X, 0)$ and a compatible sequence of point blowups $\sigma: Y_\sigma \rightarrow \mathbb{C}^2$ of $(\mathbb{C}^2, 0)$ in order for ℓ to induce a map $|\Gamma_\pi| \rightarrow |\Gamma_\sigma|$ between the topological spaces underlying Γ_π and Γ_σ . In this paper, we use $NL(X, 0)$ as a convenient way of encoding intrinsically all the dual graphs of good resolutions of $(X, 0)$; for this purpose, we can adopt the following *ad hoc* definition. Recall that, if $\pi: X_\pi \rightarrow X$ and $\pi': X_{\pi'} \rightarrow X$ are two good resolutions of $(X, 0)$ such that π' dominates π (that is, π' factors through π), then we have a natural inclusion $|\Gamma_\pi| \hookrightarrow |\Gamma_{\pi'}|$ between the topological spaces underlying the dual graphs Γ_π and $\Gamma_{\pi'}$, and a retraction $|\Gamma_{\pi'}| \rightarrow |\Gamma_\pi|$ obtained by contracting the trees in $|\Gamma_{\pi'}| \setminus |\Gamma_\pi|$. The non-archimedean link can then be seen as the inverse limit $NL(X, 0) = \varprojlim_\pi |\Gamma_\pi|$ in the category of topological spaces and with respect to the various retraction morphisms, where the limit runs over the poset of good resolutions of $(X, 0)$, ordered by domination. In particular, $NL(X, 0)$ contains a copy the dual graph of each good resolution of $(X, 0)$, and it can be seen as a compactification of the infinite union $\bigcup_\pi |\Gamma_\pi|$ of all the dual graphs of the good resolutions of $(X, 0)$. As such, it can be thought of as a universal dual graph of the singularity $(X, 0)$. To unburden the notation, in the remaining part of the paper we usually identify a dual graph Γ_π with its image $|\Gamma_\pi|$ in $NL(X, 0)$. This point of view makes it convenient to think of \mathcal{L} - and \mathcal{P} -nodes abstractly as points of $NL(X, 0)$.

Traditionally, the non-archimedean link $NL(X, 0)$ is built as a space of normalized semivaluations on the complete local ring $\widehat{\mathcal{O}_{X,0}}$ of X at 0. In particular, if $\pi: X_\pi \rightarrow X$ is a good resolution of $(X, 0)$ and E_v is a component of its exceptional divisor $\pi^{-1}(0)$, the corresponding vertex of Γ_π is identified with the corresponding *divisorial valuation* $v: \widehat{\mathcal{O}_{X,0}} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ defined by $v(f) = \text{ord}_{E_v}(\pi^*f)/m_v$, where $\text{ord}_{E_v}(\pi^*f)$ denotes the order of vanishing along E_v of the pull-back of f via π . Throughout the paper, we freely make use of this terminology, calling *divisorial point* of $NL(X, 0)$ (or of a given dual graph Γ_π) any point that can arise in this way, and denoting by E_v any exceptional curve corresponding to a divisorial point v . Observe that the subset of $NL(X, 0)$ consisting of its divisorial points is dense in the non-archimedean link; this corresponds to the fact that any given dual graph Γ_π can be refined *ad infinitum* by passing to resolutions dominating π , subdividing each edge $e = [v, v']$ into smaller edges by successively blowing up double points starting with the blowup of X_π at the point of $E_v \cap E_{v'}$ that corresponds to e . In particular, a divisorial point of $NL(X, 0)$ is contained in the interior of e if and only if it is associated with an exceptional component that appears after blowing up only double points as above. We refer the reader to [BFP22, § 2.1] and [Fan18] for further details on this point of view.

The morphism $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ induces a natural map $\tilde{\ell}: NL(X, 0) \rightarrow NL(\mathbb{C}^2, 0)$. From the point of view of semivaluations, this is simply defined functorially by pre-composing a semivaluation on $\widehat{\mathcal{O}_{X,0}}$ with the morphism of complete local rings $\widehat{\mathcal{O}_{\mathbb{C}^2,0}} \rightarrow \widehat{\mathcal{O}_{X,0}}$ induced by ℓ .

Concretely, $\tilde{\ell}(v)$ can also be computed explicitly on a divisorial point v of $NL(X, 0)$ as follows: we can find a sequence of point blowups $\sigma_{\ell,v}: Y_{\sigma_{\ell,v}} \rightarrow \mathbb{C}^2$ of $(\mathbb{C}^2, 0)$ and a good resolution $\pi_{\ell,v}: X_{\pi_{\ell,v}} \rightarrow X$ of $(X, 0)$ such that v corresponds to a component E_v of the exceptional divisor of $\pi_{\ell,v}$, the composition $\ell \circ \pi_{\ell,v}: X_{\pi_{\ell,v}} \rightarrow \mathbb{C}^2$ factors through a map $\widehat{\ell}: X_{\pi_{\ell,v}} \rightarrow Y_{\sigma_{\ell,v}}$ making the

following diagram commute

$$\begin{array}{ccc}
 X_{\pi_{\ell,v}} & \xrightarrow{\pi_{\ell,v}} & X \\
 \widehat{\ell} \downarrow & & \downarrow \ell \\
 Y_{\sigma_{\ell,v}} & \xrightarrow{\sigma_{\ell,v}} & \mathbb{C}^2
 \end{array} \tag{2}$$

and such that E_v is mapped by $\widehat{\ell}$ surjectively onto a component E_w of the exceptional divisor of $\sigma_{\ell,v}$; we then have $\widehat{\ell}(v) = w$.

Let $\pi: X_\pi \rightarrow X$ be a good resolution of $(X, 0)$. As ℓ is a finite map ramified precisely over the associated polar curve, the induced map $\ell|_{\Gamma_\pi}: \Gamma_\pi \rightarrow \widetilde{\ell}(\Gamma_\pi)$ is itself a finite cover, which on a set contained in the set of \mathcal{P} -nodes of $(X, 0)$ that are contained in Γ_π (not necessarily as vertices but possibly in the interior of some edges). In particular, $\widetilde{\ell}$ cannot contract an edge of Γ_π , but it may fold one if it contains a \mathcal{P} -node in its interior.

Observe that the map $\widetilde{\ell}$ clearly depends on the choice of ℓ . Indeed, if $\ell': (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is another generic projection obtained by composing ℓ with an automorphism ϕ of $(\mathbb{C}^2, 0)$, then ϕ induces a nontrivial automorphism $\widetilde{\varphi}$ of $\text{NL}(\mathbb{C}^2, 0)$, and we have $\widetilde{\ell} = \widetilde{\varphi} \circ \widetilde{\ell}'$. Although, in general, two generic projections of $(X, 0)$ do not differ by an automorphism of $(\mathbb{C}^2, 0)$, it is possible to control this phenomenon if we restrict $\widetilde{\ell}$ to the dual graph Γ_π of the minimal good resolution $\pi: X_\pi \rightarrow X$ of $(X, 0)$ that factors through its Nash transform, as we explain in Lemma 3.1.

To do this, we need to dive deeper into the definition of generic projections, to be able to study the polar curves and the discriminant curves of $(X, 0)$ in families. Let us begin by recalling the precise notion of strong equiresolution of singularities given in [Tei76, 3.1.1 and 3.1.5]. Given a morphism $\beta: M \rightarrow \Lambda$ with reduced fibers between smooth connected complex manifolds and a simple normal crossing divisor E of M , we say that β is *simple* (with respect to E) if β is smooth and its restriction $\beta|_E: E \rightarrow \Lambda$ to E is proper and locally a trivial deformation along its fibers. If we have another morphism $\sigma: M' \rightarrow M$, we say that σ is β -compatible if the composition $\beta' = \beta \circ \sigma$ is simple (with respect to $E' = \sigma^{-1}(E)$). Finally, given a (singular) subvariety X of M , we say that an embedded resolution of singularities $\pi: \widetilde{M} \rightarrow M$ of X is a *strong equiresolution* (along Λ) of X if π is β -compatible and all of its restrictions π_λ over $\lambda \in \Lambda$ are good embedded resolutions of X_λ .

According to [Tei82, Lemme-clé V 1.2.2] (see [NPP20a, Proposition 2.3] for an English presentation), there exists an analytic open dense subset Ω of the Grassmannian $\text{Gr}(n-2, \mathbb{C}^n)$ where the family $\{(\Delta_{\mathcal{D}}, \mathcal{D})\}_{\mathcal{D} \in \Omega}$ of discriminant curves, which can be seen as a surface in $(\mathbb{C}^2, 0) \times \Omega$ fibered over Ω via the projection $\beta: (\mathbb{C}^2, 0) \rightarrow \Omega$ on the second factor, admits a strong equiresolution

$$\begin{array}{ccc}
 (\mathcal{Y}, \mathcal{F}) & \xrightarrow{\sigma} & (\mathbb{C}^2, 0) \times \Omega \\
 & \searrow \beta_{\mathcal{Y}} & \downarrow \beta \\
 & & \Omega
 \end{array}$$

with $F = \sigma^{-1}(\{0\} \times \Omega)$ a simple normal crossing divisor of \mathcal{Y} .

For each \mathcal{D} in Ω , denote by $\sigma_{\mathcal{D}}: (\mathcal{Y}_{\mathcal{D}}, F_{\mathcal{D}}) \rightarrow (\mathbb{C}^2, 0)$ the restriction of σ to the fiber $\beta_{\mathcal{Y}}^{-1}(\mathcal{D})$, which is a sequence of point blowups of $(\mathbb{C}^2, 0)$. Given two elements \mathcal{D} and \mathcal{D}' of Ω , this allows us to define an isomorphism of graphs $\eta_{\mathcal{D}, \mathcal{D}'}: \Gamma_{\sigma_{\mathcal{D}}} \xrightarrow{\sim} \Gamma_{\sigma_{\mathcal{D}'}}$ as follows. For each $v \in V(\Gamma_{\sigma_{\mathcal{D}}})$, if we denote by $F_v^{\mathcal{D}}$ the corresponding irreducible component of $\mathcal{F} = \sigma_{\mathcal{D}}^{-1}(0)$, there

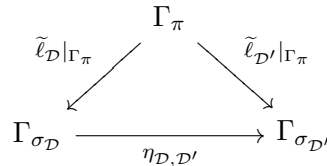
is a unique irreducible component $\mathcal{F}_v^{\mathcal{D}}$ of $\sigma^{-1}(\{0\} \times \Omega)$ such that $F_v^{\mathcal{D}} = \mathcal{F}_v^{\mathcal{D}} \cap \sigma_{\mathcal{D}}^{-1}(0)$. We then set $\eta_{\mathcal{D},\mathcal{D}'}(v) = v'$, where v' is the vertex of $\Gamma_{\mathcal{D}'}$ such that $\mathcal{F}_{v'}^{\mathcal{D}'} = \mathcal{F}_v^{\mathcal{D}}$ (that is, equivalently, such that $\mathcal{F}_v^{\mathcal{D}} \cap \sigma_{\mathcal{D}'}^{-1}(0) = F_{v'}^{\mathcal{D}'}$). This yields a bijection $V(\Gamma_{\sigma_{\mathcal{D}}}) \rightarrow V(\Gamma_{\sigma_{\mathcal{D}'}})$ which extends to a natural homeomorphism

$$\eta_{\mathcal{D},\mathcal{D}'} : \Gamma_{\sigma_{\mathcal{D}}} \rightarrow \Gamma_{\sigma_{\mathcal{D}'}}$$

defined on the divisorial points of $\Gamma_{\sigma_{\mathcal{D}}}$ as follows. Fix $\mathcal{D} \in \Omega$ and consider a divisorial point v on an edge $[v_1, v_2]$ of $\Gamma_{\sigma_{\mathcal{D}}}$. Then $E_v^{\mathcal{D}}$ is created by a finite sequence of blowups of double points of the previous exceptional divisor, starting with the blowup of the point $F_{v_1}^{\mathcal{D}} \cap F_{v_2}^{\mathcal{D}}$. We can perform this blowups in family by blowing up along successive intersections of the form $\mathcal{F}_{w_1}^{\mathcal{D}} \cap \mathcal{F}_{w_2}^{\mathcal{D}}$, starting with the blowup along $\mathcal{F}_{v_1}^{\mathcal{D}} \cap \mathcal{F}_{v_2}^{\mathcal{D}}$. By composing this sequence of blowups with σ , we obtain a (β -compatible) morphism $\sigma_v : \mathcal{Y}_{\sigma_v} \rightarrow (\mathbb{C}^2, 0) \times \Omega$. The last blowup creates an irreducible new component $\mathcal{F}_v^{\mathcal{D}}$ in the exceptional divisor, and as before we define $v' = \eta_{\mathcal{D},\mathcal{D}'}(v)$ by declaring that the corresponding irreducible component $F_{v'}^{\mathcal{D}'}$ is the intersection $\mathcal{F}_v^{\mathcal{D}} \cap (\sigma')^{-1}(0, \mathcal{D}')$. Observe that, because multiplicities are constant along a smooth family, we have $m_v = m_{\eta_{\mathcal{D},\mathcal{D}'}(v)}$ for every divisorial point v of $\Gamma_{\sigma_{\mathcal{D}}}$.

The following lemma relating the graph $\Gamma_{\sigma_{\mathcal{D}}}$ to Γ_{π} plays a crucial role in several arguments in the rest of the paper.

LEMMA 3.1. *Let $(X, 0)$ be a normal surface singularity, let $\pi : X_{\pi} \rightarrow X$ be the minimal good resolution of $(X, 0)$ that factors through the blowup of its maximal ideal and its Nash transform, and let $\Gamma_{\pi} \subset \text{NL}(X, 0)$ be the dual graph of π . Then for all \mathcal{D} and \mathcal{D}' in Ω the diagram*



obtained by restricting to the graph Γ_{π} the two induced morphisms of non-archimedean links $\tilde{\ell}_{\mathcal{D}}, \tilde{\ell}_{\mathcal{D}'} : \text{NL}(X, 0) \rightarrow \text{NL}(\mathbb{C}^2, 0)$, is commutative.

Before moving to the proof of the lemma, which is rather technical, we observe that the homeomorphism $\eta_{\mathcal{D},\mathcal{D}'} : \Gamma_{\sigma_{\mathcal{D}}} \rightarrow \Gamma_{\sigma_{\mathcal{D}'}}$ lifts naturally to an automorphism $\eta_{\mathcal{D},\mathcal{D}'}$ of the dual graph $\Gamma_{\pi'}$ of any good resolution $\pi' : X_{\pi'} \rightarrow X$ of $(X, 0)$. However, the commutativity $\eta_{\mathcal{D},\mathcal{D}'} \circ \tilde{\ell}_{\mathcal{D}}$ does not necessarily hold on the whole of $\Gamma_{\pi'}$. We defer an illustration of this phenomenon to Example 4.4, because showing this now would require a lengthy local computation, while after proving Lemma 4.1 we can give a more conceptual explanation.

As our needs go slightly beyond what was done by Teissier, let us explain how to adapt his constructions accordingly. We start by proving a technical lemma about resolution in families of surfaces, much in the spirit of [Tei76, 4.1 and 4.2].

LEMMA 3.2. *Let M and Ω be connected complex manifold such that $\dim(M) = \dim(\Omega) + 2$, let E be a simple normal crossing divisor of M , and let $\beta : M \rightarrow \Omega$ be a simple morphism (with respect to E). Consider a finite sequence of (adapted) smooth blowups $\sigma : (M', E') \rightarrow (M, E)$ whose centers have codimension at least 2. Then, up to shrinking the size of the dense open Ω (and, therefore, of M and M'), the composition $\beta' = \beta \circ \sigma$ is simple (with respect to E').*

Proof. It is enough to prove the claim in the case that σ is a single blowup with center \mathcal{C} . By Remmert’s proper mapping theorem applied to $\beta|_E$, the image $\beta(\mathcal{C})$ is a closed analytic subset

of Ω . If $\dim(\beta(\mathcal{C})) < \dim(\Omega)$, set $Z = \beta(\mathcal{C})$ and note that, once we replace Ω by $\Omega \setminus Z$, the result easily follows from the fact that $\sigma: M' \rightarrow M$ is an isomorphism. We can therefore assume that $\dim(\beta(\mathcal{C})) = \dim(\Omega)$, so that $\beta(\mathcal{C}) = \Omega$. Since $\dim(\mathcal{C}) \leq \dim(\Omega)$ by hypothesis, we conclude that $\dim(\mathcal{C}) = \dim(\Omega)$, and, in particular, the restriction $\beta|_{\mathcal{C}}: \mathcal{C} \rightarrow \Omega$ is generically a local isomorphism. Let $Y \subset \mathcal{C}$ be the set of critical points of $\beta|_{\mathcal{C}}$, which is a proper closed analytic subset of \mathcal{C} . Again by Remmert's proper mapping theorem, the image $Z' = \beta(Y)$ is a closed analytic subset of $\text{Gr}(n - 2, \mathbb{C}^n)$, properly contained in $\text{Gr}(n - 2, \mathbb{C}^n)$ because $\dim(Y) < \dim(\text{Gr}(n - 2, \mathbb{C}^n))$. Now, after replacing Ω by $\Omega \setminus Z'$, we can assume that $\beta: \mathcal{C} \rightarrow \Omega$ is everywhere a local isomorphism. We now claim that β' is simple via direct computation. Indeed, because smoothness can be verified locally, let us fix a point $p \in \mathcal{C}$, and denote by f_1 and $f_2 \in \mathcal{O}_p$ local generators of \mathcal{C} . As β is simple at p , there exists an (analytic) local coordinate system (λ, x_1, x_2) at p such that $\beta(\lambda, x_1, x_2) = \lambda$ and E is locally contained in $(x_1 x_2 = 0)$. As $\pi: \mathcal{C} \rightarrow \Omega$ is a local isomorphism around p and \mathcal{C} is smooth and adapted to E , it follows that the map $(\lambda, x_1, x_2) \rightarrow (\lambda, f_1, f_2)$ is a local isomorphism and $E \subset (f_1 f_2 = 0)$. Therefore, up to a local change of variables, we can assume that $f_1 = x_1$ and $f_2 = x_2$, and we easily conclude that $\beta': M' \rightarrow \Omega$ is simple. \square

Now, recall that we have an embedding of $(X, 0)$ in $(\mathbb{C}^n, 0)$ and let $\Phi: (X, 0) \times \Omega \rightarrow (\mathbb{C}^2, 0) \times \Omega$ be the morphism defined by $\Phi(x, \mathcal{D}) = (\ell_{\mathcal{D}}(x), \mathcal{D})$, which is generically of maximal rank. Let $\pi: (X_{\pi}, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$ which factors through the blowup of its maximal ideal and through its Nash transform. We note that, by using [Lau71, Lemma 5.2] (a special case of the direct image theorem of Grauert), resolution of singularities, and the universal property of blowups, there exists a sequence of blowups $\alpha: (\mathcal{Z}, \mathcal{G}) \rightarrow (X_{\pi}, E) \times \Omega$ and an analytic morphism $\Psi: (\mathcal{Z}, \mathcal{G}) \rightarrow (\mathcal{Y}, \mathcal{F})$ such that $\Psi^{-1}(\mathcal{F})_{\text{red}} = \mathcal{G}_{\text{red}}$ and the following diagram is commutative

$$\begin{array}{ccccc}
 (\mathcal{Z}, \mathcal{G}) & \xrightarrow{\alpha} & (X_{\pi}, E) \times \Omega & \xrightarrow{\pi \times \text{Id}} & (X, 0) \times \Omega \\
 \downarrow \Psi & & & & \downarrow \Phi \\
 (\mathcal{Y}, \mathcal{F}) & \xrightarrow{\sigma} & (\mathbb{C}^2, 0) \times \Omega & \xrightarrow{\beta} & \Omega
 \end{array} \tag{3}$$

with $\beta_{\mathcal{Y}} = \beta \circ \sigma$ simple. Thanks to Lemma 3.2, up to shrinking the size of the open Ω if necessary, the morphism $\beta_{\mathcal{Z}} = \beta_{\mathcal{Y}} \circ \Psi$ is simple as well. We are now ready to complete the proof of Lemma 3.1.

Proof of Lemma 3.1. The map $\tilde{\ell}|_{\Gamma_{\pi}}$ is determined by its restriction to the set of divisorial points of Γ_{π} , as those form a dense subset of Γ_{π} . As $\beta_{\mathcal{Z}}$ and $\beta_{\mathcal{Y}}$ are simple, for every pair of elements $\mathcal{D}, \mathcal{D}'$ of Ω , the following diagram commutes

$$\begin{array}{ccc}
 & V(\Gamma_{\pi}) & \\
 \tilde{\ell}_{\mathcal{D}} \swarrow & & \searrow \tilde{\ell}_{\mathcal{D}'} \\
 V(\Gamma_{\sigma_{\mathcal{D}}}) & \xrightarrow{\tilde{\rho}_{\mathcal{D}, \mathcal{D}'}} & V(\Gamma_{\sigma_{\mathcal{D}'}})
 \end{array}$$

We now need to prove the result on the divisorial points of Γ_{π} which are not vertices of Γ_{π} . It is sufficient to consider the case where v is the divisorial point associated with the exceptional curve of the blowup $\pi': (X'_{\pi}, E') \rightarrow (X_{\pi}, E)$ of center $E_{v_1} \cap E_{v_2}$, because the same argument can then be repeated verbatim for general sequence of point blowups. Observe that if $E_v \times \Omega$ is already a component of \mathcal{G} , then $\ell_{\mathcal{D}}(v) \in \Gamma_{\sigma_{\mathcal{D}}}$ for every $\mathcal{D} \in \Omega$, and we conclude easily. If $\mathcal{G}_v = E_v \times \Omega$

is not a component of \mathcal{G} , we note that $\mathcal{G}_{v_1} \cap \mathcal{G}_{v_2} = (E_{v_1} \cap E_{v_2}) \times \Omega$ is an admissible center in $(\mathcal{Z}, \mathcal{G})$, because all blowups in α are admissible. We therefore may perform this extra blowup $\alpha': (\mathcal{Z}', \mathcal{G}') \rightarrow (\mathcal{Z}, \mathcal{G})$, whose exceptional divisor $\mathcal{G}_v = E_v \times \Omega$ is trivial with respect to the family structure. Fix $\mathcal{D} \in \Omega$, set $w_1 = \ell_{\mathcal{D}}(v_1)$ and $w_2 = \ell_{\mathcal{D}}(v_2)$, and consider the associated components $\mathcal{F}_{w_1}^{\mathcal{D}}$ and $\mathcal{F}_{w_2}^{\mathcal{D}}$ of \mathcal{F} . Then, after performing a sequence of combinatorial blowups $\rho: (Y', \mathcal{F}') \rightarrow (Y, \mathcal{F})$, starting with blowing up the center $\mathcal{F}_{w_1}^{\mathcal{D}} \cap \mathcal{F}_{w_2}^{\mathcal{D}}$, the projection $\tilde{\ell}_{\mathcal{D}'}(v)$ belongs to the graph of $\Gamma_{\rho_{\mathcal{D}'} \circ \sigma_{\mathcal{D}'}}$ for every \mathcal{D}' in Ω . We have obtained, without the need to shrink the size of Ω , the following commutative diagram:

$$\begin{array}{ccccccc}
 (\mathcal{Z}', \mathcal{G}') & \xrightarrow{\alpha'} & (\mathcal{Z}, \mathcal{G}) & \xrightarrow{\alpha} & (X_{\pi}, E) \times \Omega & \xrightarrow{\pi \times \text{Id}} & (X, 0) \times \Omega \\
 \Psi' \downarrow & & \downarrow \Psi & & & & \downarrow \Phi \\
 (\mathcal{Y}', \mathcal{F}') & \xrightarrow{\rho} & (\mathcal{Y}, \mathcal{F}) & \xrightarrow{\sigma} & (\mathbb{C}^2, 0) \times \Omega & \xrightarrow{\beta} & \Omega
 \end{array}$$

where $\beta_{\mathcal{Y}'} = \beta \circ \sigma \circ \rho$ and $\beta_{\mathcal{Z}'} = \beta_{\mathcal{Y}'} \circ \Psi'$ are simple morphisms. We conclude easily. □

Remark 3.3. If $(X, 0)$ is a hypersurface in $(\mathbb{C}^3, 0)$, shrinking the open set Ω is not necessary when applying Lemma 3.2, because a resolution of the family can be constructed everywhere by performing a Hirzebruch–Jung process in family, exploiting the fact that, thanks to [Pop02, Corollary 3.4] (or, more generally, to [Pop04, Theorem 5.1]), the combinatorial data of the quasi-ordinary singularities that appear during the process are constant in the family.

We conclude the section by recalling the definition of the local degree of a divisorial point v of $\text{NL}(X, 0)$, as it will be very important in the remaining part of the paper. Let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection of $(X, 0)$ and consider the diagram (2). For each component E_{ν} of $\pi_{\ell, v}^{-1}(0)$ (respectively $E_{\nu'}$ of $\sigma_{\ell, v}^{-1}(0)$), let us choose a tubular neighborhood disc bundle $N(E_{\nu})$ (resp. $N(E_{\nu'})$), and consider the two sets

$$\mathcal{N}(E_v) = N(E_v) \setminus \bigcup_{E_{\nu} \neq E_v} N(E_{\nu}) \quad \text{and} \quad \mathcal{N}(E_{\tilde{\ell}(v)}) = N(E_{\tilde{\ell}(v)}) \setminus \bigcup_{E_{\nu'} \neq E_{\tilde{\ell}(v)}} N(E_{\nu'})$$

in $X_{\pi_{\ell, v}}$ and $Y_{\sigma_{\ell, v}}$, respectively. We can then adjust the disc bundles $N(E_{\nu})$ and $N(E_{\nu'})$ in such a way that the cover ℓ restricts to a cover

$$\ell_v: \pi_{\ell, v}(\mathcal{N}(E_v)) \longrightarrow \sigma_{\ell, v}(\mathcal{N}(E_{\tilde{\ell}(v)})) \tag{4}$$

branched precisely on the polar curve of ℓ (if v is not a \mathcal{P} -node, the branching locus is just the origin). Using a resolution in family over Ω as in the proof of the Lemma 3.1, it is easy to deduce the following result.

LEMMA 3.4. *For every divisorial point v of $\text{NL}(X, 0)$, the degree $\text{deg}(\ell_v)$ of the cover ℓ_v does not depend on the choice of a generic projection $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$.*

Therefore, we can set $\text{deg}(v) = \text{deg}(\ell_v)$. We call this integer the *local degree of a generic projection of $(X, 0)$ at v* , or simply the *local degree of $(X, 0)$ at v* . Note that if $\text{deg}(v) = 1$, then the map (4) is an isomorphism.

4. Generic projections of LNE surfaces

In this section, we study LNE surface germs by establishing some properties related to their generic projections.

We begin by proving the invariance of multiplicities under generic projections, and showing a characterization of the \mathcal{P} -nodes of a LNE normal surface in terms of their local degrees. More precisely, we prove the following result.

LEMMA 4.1. *Let $(X, 0)$ be a LNE normal surface germ, let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection, let $\pi: X_\pi \rightarrow X$ be the minimal good resolution of $(X, 0)$ which factors through its Nash transform, and let v be a divisorial point of $\Gamma_\pi \subset \text{NL}(X, 0)$. Then:*

- (i) $m_v = m_{\tilde{\ell}(v)}$;
- (ii) v is a \mathcal{P} -node of $(X, 0)$ if and only if $\deg v > 1$.

Before delving into the proof of the lemma, let us recall the notions of inner and outer contact and that of inner rate. Let $(\gamma, 0)$ and $(\gamma', 0)$ be two distinct real or complex curve germs on the surface germ $(X, 0) \subset (\mathbb{C}^n, 0)$ and denote S_ϵ the sphere in \mathbb{C}^n having center 0 and radius $\epsilon > 0$. The *inner contact* between γ and γ' is the rational number $q_{\text{inn}} = q_{\text{inn}}(\gamma, \gamma')$ defined by

$$d_{\text{inn}}(\gamma \cap S_\epsilon, \gamma' \cap S_\epsilon) = \Theta(\epsilon^{q_{\text{inn}}}),$$

where Θ stands for the big-Theta asymptotic notation of Bachmann–Landau, which is defined as follows: given two function germs $f, g: ([0, \infty), 0) \rightarrow ([0, \infty), 0)$ we say that f is *big-Theta* of g , and we write $f(t) = \Theta(g(t))$, if there exist real numbers $\eta > 0$ and $K > 0$ such that $K^{-1}g(t) \leq f(t) \leq Kg(t)$ for all $t \geq 0$ satisfying $f(t) \leq \eta$. The *outer contact* $q_{\text{out}}(\gamma, \gamma')$ is defined in an analogous way, by using the outer metric d_{out} instead of the inner metric d_{inn} . Observe that if $(X, 0)$ is LNE then $q_{\text{inn}}(\gamma, \gamma') = q_{\text{out}}(\gamma, \gamma')$. Recall that the *inner rate* q_v of a divisorial point v of $\text{NL}(X, 0)$ is defined as the inner contact $q_{\text{inn}}(\gamma, \gamma')$, where $\gamma, \gamma' \subset (X, 0)$ are two curve germs that pullback to two curvettes through distinct points of the divisor E_v associated with v via any good resolution $\pi: X_\pi \rightarrow X$ of $(X, 0)$ that makes the divisor E_v appear. This definition only depends on the divisorial point v (see [BFP22, Lemma 3.2]).

Proof. We begin by proving part (i). Write $\ell = \ell_{\mathcal{D}}$ and set $w = \tilde{\ell}_{\mathcal{D}}(v)$. Consider maps

$$\begin{array}{ccc} X_{\pi_{\ell, \mathcal{D}}} & \xrightarrow{\pi_{\ell, \mathcal{D}}} & X \\ \widehat{\ell}_{\mathcal{D}} \downarrow & & \downarrow \ell_{\mathcal{D}} \\ Y_{\sigma_{\ell, v\mathcal{D}}} & \xrightarrow{\sigma_{\ell, \mathcal{D}}} & \mathbb{C}^2 \end{array}$$

as in diagram (2) such that $\pi_{v, \mathcal{D}}$ is a good resolution of $(X, 0)$ factoring through its Nash transform (and, therefore, through π), $\sigma_{v, \mathcal{D}}$ is a sequence of point blowups of $(\mathbb{C}^2, 0)$ such that w is associated with a component E_w of $(\sigma_{v, \mathcal{D}})^{-1}(0)$, and the component E_v of $(\pi_{v, \mathcal{D}})^{-1}(0)$ associated with v is sent by $\widehat{\ell}$ surjectively onto E_w .

Take a curvette γ^* of E_w which does not intersect a component of the strict transform of the discriminant curve $\Delta_{\mathcal{D}}$ of $\ell_{\mathcal{D}}$ and let $(\gamma, 0) \subset (\mathbb{C}^2, 0)$ be the irreducible curve germ defined by $\gamma = \sigma_{v, \mathcal{D}}(\gamma^*)$, so that we have $m_w = \text{mult}(\gamma)$. Up to replacing γ^* by a nearby curvette, among the components of $(\ell_{\mathcal{D}})^{-1}(\gamma)$ we can find an irreducible curve germ $\widehat{\gamma}$ on $(X, 0)$ whose strict transform by $\pi_{v, \mathcal{D}}$ is a curvette of E_v , so that we have $m_v = \text{mult}(\widehat{\gamma})$. We then have $\text{mult}(\widehat{\gamma}) = k \text{mult}(\gamma)$, where k is the degree of the covering $\widehat{\gamma} \rightarrow \gamma$ induced by ℓ .

We argue by contradiction. Assume that $\text{mult}(\widehat{\gamma}) \neq \text{mult}(\gamma)$, that is, that $k > 1$. Our goal will be to construct two real arcs $\widehat{\delta}_1$ and $\widehat{\delta}_2$ inside $\widehat{\gamma}$ whose inner and outer contacts do not coincide; this will then imply that $(X, 0)$ is not LNE, contradicting our hypothesis. To do so, we consider another generic projection $\ell_{\mathcal{D}'}: (X, 0) \rightarrow (\mathbb{C}^2, 0)$, chosen to be generic with respect to

the curve $\widehat{\gamma}$ as well, and set $\gamma' = \ell_{\mathcal{D}'}(\widehat{\gamma})$. Then the cover $\widehat{\gamma} \rightarrow \gamma'$ induced by $\ell_{\mathcal{D}'}$ has degree 1, and thus $\widehat{\gamma}$ and γ' have the same multiplicity because $\text{mult}(\widehat{\gamma}) = \text{degree}(\ell_{\mathcal{D}'}|_{\widehat{\gamma}}) \text{mult}(\gamma') = \text{mult}(\gamma')$. Set $w' = \widetilde{\ell}_{\mathcal{D}'}(v)$. By Lemma 3.1, we have $w' = \eta_{\mathcal{D},\mathcal{D}'}(w)$ and, thus, $m_{w'} = m_w$ and $q_{w'} = q_v = q_w$. Moreover, by the definition of $\eta_{\mathcal{D},\mathcal{D}'}$, the strict transform of γ' by $\sigma_{v,\mathcal{D}'}$ intersects $(\sigma_{v,\mathcal{D}'})^{-1}(0)$ in a smooth point p of $E_{w'}$.

Observe that, because the plane curve germ γ is the image through $\sigma_{v,\mathcal{D}}$ of a curvette of E_w , it has no characteristic Puiseux exponent strictly greater than the inner rate q_w of w . On the other hand, the strict transform of γ' by $\sigma_{v,\mathcal{D}'}$ cannot be a curvette of $E_{w'}$ because $\text{mult}(\gamma') = \text{mult}(\widehat{\gamma}) = km_{w'} > m_{w'}$. Therefore, the minimal good embedded resolution of γ' is obtained by composing $\sigma_{v,\mathcal{D}'}$ with a nontrivial sequence of point blowups, starting with the blowup of $Y_{\sigma_{v,\mathcal{D}'}}$ at p . Let $E_{w''}$ be the last irreducible curve created by this sequence, so that the strict transform of γ' is a curvette of $E_{w''}$. Then the inner rate $q_{w''}$ of $E_{w''}$, which is strictly greater than $q_{w'} = q_w$, is a characteristic Puiseux exponent of γ' .

Let us choose an embedding $(X, 0) \subset (\mathbb{C}^n, 0)$ and coordinates (x_1, \dots, x_n) of \mathbb{C}^n such that $\ell_{\mathcal{D}'}(x) = (x_1, x_2)$ and γ' is not tangent to the line $x_1 = 0$. Then, since $q_{w''}$ is a characteristic Puiseux exponent of γ' , we can find a pair of real arcs δ'_1 and δ'_2 among the components of the intersection $\gamma' \cap \{x_1 = t \mid t \in \mathbb{R}\}$ such that their contact $q(\delta'_1, \delta'_2)$ is equal to $q_{w''}$ (we refer to [NP14, § 3] for details on this classical result about Puiseux expansions). Let $\widehat{\delta}_1$ and $\widehat{\delta}_2$ be two liftings of δ'_1 and δ'_2 via ℓ' . As the projection $\ell_{\mathcal{D}'}$ is generic with respect to $\widehat{\gamma}$, it induces by [Tei82, pp. 352–354] a bi-Lipschitz homeomorphism for the outer metric from $\widehat{\gamma}$ onto γ' and, therefore, the outer contacts $q_{\text{out}}(\widehat{\delta}_1, \widehat{\delta}_2)$ and $q(\delta'_1, \delta'_2)$ coincide, so that, in particular, we have

$$q_{\text{out}}(\widehat{\delta}_1, \widehat{\delta}_2) = q_{w''} > q_v. \tag{5}$$

We now show that the inner contact $q_{\text{inn}}(\widehat{\delta}_1, \widehat{\delta}_2)$ between $\widehat{\delta}_1$ and $\widehat{\delta}_2$ is at most q_v , which will yield the contradiction we were after. Observe that the inner contact $q = q_{\text{inn}}^X(\widehat{\delta}_1, \widehat{\delta}_2)$ between $\widehat{\delta}_1$ and $\widehat{\delta}_2$ can also be computed as $d_{\text{inn}}^{F_t}(\widehat{\delta}_1(t), \widehat{\delta}_2(t)) = \Theta(t^q)$, where $d_{\text{inn}}^{F_t}(\widehat{\delta}_1(t), \widehat{\delta}_2(t))$ denotes the inner distance between $\widehat{\delta}_1(t)$ and $\widehat{\delta}_2(t)$ inside the Milnor fiber $F_t = X \cap \{x_1 = t\}$, that is, the distance measured by taking the infimum of the inner lengths of the paths joining $\widehat{\delta}_1(t)$ to $\widehat{\delta}_2(t)$ inside F_t . This is a consequence of the fact that, by [BNP14] and in the language therein, the subset $\pi(\mathcal{N}(E_v))$ of $(X, 0)$ is a $B(q_v)$ -piece fibered by the restriction of the generic linear form x_1 whenever $q_v > 1$, whereas it is a conical piece if $q_v = 1$.

To conclude, consider a small disc D contained in the divisor E_v and centered at the point $\widehat{\gamma}^* \cap E_v$ and let $N \cong D \times D'$ be a trivialization of the normal disc-bundle to E_v over D such that $\widehat{\gamma}^* = \{0\} \times D'$. The intersection $F_t \cap \pi(N)$ consists of m_v disjoint discs each centered at one of the m_v distinct points of $\widehat{\gamma} \cap F_t$. As $\delta_1(t)$ and $\delta_2(t)$ are two of these points, then they are the centers of two of these discs, D_1 and D_2 , respectively. As these two discs have diameters $\Theta(t^{q_v})$, any path from $\widehat{\delta}_1(t)$ to $\widehat{\delta}_2(t)$ inside F_t will have intersections with D_1 and D_2 of length at least $\Theta(t^{q_v})$. Therefore, $q_{\text{inn}}(\widehat{\delta}_1, \widehat{\delta}_2) \leq q_v$, and so $q_{\text{inn}}(\widehat{\delta}_1, \widehat{\delta}_2) < q_{\text{out}}(\widehat{\delta}_1, \widehat{\delta}_2)$, which contradicts the fact that $(X, 0)$ is LNE. This completes the proof that $m_v = m_{\widetilde{\ell}_{\mathcal{D}}(v)}$.

Let us now prove part (ii). If v is a \mathcal{P} -node, then it immediately follows from the definition of degree that $\text{deg}(v) > 1$, because the cover ℓ is ramified in a neighborhood of the polar curve. Assume that v is not a \mathcal{P} -node and that $\text{deg } v > 1$. We use again the plane curve $\gamma = \sigma_{v,\mathcal{D}}(\gamma^*)$ introduced in the proof of part (i).

By the definition of $\text{deg}(v)$, the curve $\ell^{-1}(\gamma)$ has k_v irreducible components whose strict transforms by $\pi_{v,\mathcal{D}}$ are curvettes of E_v , where k_v divides $\text{deg}(v)$, and we have $m_v = m_{\widetilde{\ell}(v)} \text{deg}(v) / k_v$. As $m_v = m_{\widetilde{\ell}(v)}$ by part (i), then $\text{deg}(v) = k_v$, so $k_v > 1$. Let $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ be two components of

$\ell^{-1}(\gamma)$ whose strict transforms by $\pi_{v,\mathcal{D}}$ are curvettes of E_v (as was the case in part (i), two such components can always be found after replacing γ^* by a nearby curvette if necessary), and let us consider two real arcs $\widehat{\delta}_1 \subset \widehat{\gamma}_1$ and $\widehat{\delta}_2 \subset \widehat{\gamma}_2$ such that $\ell_{\mathcal{D}}(\widehat{\delta}_1) = \ell_{\mathcal{D}}(\widehat{\delta}_2)$. By the definition of q_v , we have $q_{\text{inn}}(\widehat{\gamma}_1, \widehat{\gamma}_2) = q_v$ and then $q_{\text{inn}}(\widehat{\delta}_1, \widehat{\delta}_2) = q_v$. As v is not a \mathcal{P} -node, the lifted Gauss map λ on X_π (see [NPP20a, Definition 6.11]) is constant along E_v and we then have $\lambda(p_1) = \lambda(p_2)$, see [NPP20a, p. 19]. By [NPP20a, Lemma 9.1], this implies that $q_{\text{inn}}(\delta_1, \delta_2) < q_{\text{out}}(\delta_1, \delta_2)$, therefore $(X, 0)$ is not LNE, contradicting the hypothesis. \square

Remark 4.2. Whenever v is not a \mathcal{P} -node, then part (i) of the lemma is an immediate consequence of part (ii) (whose proof is more elementary and independent on the proof of part (i)). Indeed, consider the degree $\deg(v)$ cover $\ell_v: \pi_{\ell,v}(\mathcal{N}(E_v)) \rightarrow \sigma_{\ell,v}(\mathcal{N}(E_{\widetilde{\ell}(v)}))$ of equation (4), and choose coordinates of \mathbb{C}^2 so that $\ell = (z_1, z_2)$, with $h = z_1$ a generic linear form on $(X, 0)$. Then ℓ_v restricts to a degree $\deg(v)$ cover from the intersection $F_v = \mathcal{N}(E_v) \cap \{h = t\}$ to its image $\ell(F_v)$, implying that $m_v = km_{\widetilde{\ell}(v)}$, where the integer k divides $\deg(v)$.

As a simple consequence of Lemma 4.1(ii) we deduce the following result.

COROLLARY 4.3. *Let $(X, 0)$ be a LNE normal surface germ, let v be a divisorial point of $\text{NL}(X, 0)$, and assume that v is associated with a genus $g > 0$ component E_v of the exceptional divisor of some good resolution of $(X, 0)$. Then v is a \mathcal{P} -node of $(X, 0)$.*

Proof. Consider again the finite cover $\ell_v: \pi_{\ell,v}(\mathcal{N}(E_v)) \rightarrow \sigma_{\ell,v}(\mathcal{N}(E_{\widetilde{\ell}(v)}))$ of equation (4), and assume that v is not a \mathcal{P} -node. Then ℓ_v is a homeomorphism by Lemma 4.1(ii), and so is its restriction $\ell_v|_{\mathcal{N}(E_v) \cap E_v}: \mathcal{N}(E_v) \cap E_v \rightarrow \mathcal{N}(E_{\widetilde{\ell}(v)}) \cap E_{\widetilde{\ell}(v)}$. Observe that $\mathcal{N}(E_v) \cap E_v$ (respectively $\mathcal{N}(E_{\widetilde{\ell}(v)}) \cap E_{\widetilde{\ell}(v)}$) is the complex curve E_v (respectively, $E_{\widetilde{\ell}(v)}$) with a finite union of discs removed. As $E_{\widetilde{\ell}(v)}$ has genus zero, this implies that E_v also has genus zero. \square

Example 4.4. Let us show with an example that the minimality of π is a necessary hypothesis in Lemma 4.1. Let $(X, 0)$ be the standard singularity A_2 , which is the hypersurface singularity in $(\mathbb{C}^3, 0)$ defined by the equation $x^2 + y^2 + z^3 = 0$. A good resolution $\pi: X_\pi \rightarrow X$ of $(X, 0)$ can be obtained by the method described in [Lau71, Chapter II]. It considers the generic projection $\ell = \ell_{\mathcal{D}} = (y, z): (X, 0) \rightarrow (\mathbb{C}^2, 0)$ and, given a suitable embedded resolution $\sigma_\Delta: Y_{\sigma_\Delta} \rightarrow \mathbb{C}^2$ of the associated discriminant curve $\Delta: y^2 + z^3 = 0$, gives a simple algorithm to compute a resolution of $(X, 0)$ as a cover of Y . In this example, Δ is a cusp and the dual graph Γ_{σ_Δ} of its minimal embedded resolution σ_Δ is depicted on the left of Figure 1. Its vertices are labeled as w_0, w_1 , and w_2 in their order of appearance as exceptional divisors of point blowups in the resolution process, the negative number attached to each vertex denotes the self-intersection of the corresponding exceptional curve, the positive numbers in parentheses denote the multiplicities, and the arrow denotes the strict transform of Δ . In this case, Laufer’s method gives us the dual graph of a good resolution π_ℓ of $(X, 0)$ such that $\ell \circ \pi_\ell$ factors through σ_Δ , appearing as the graph in the middle of Figure 1. Again, all exceptional components are rational, each vertex is decorated by the self-intersection of the corresponding exceptional curve and with its multiplicity, the arrow denotes the strict transform of Δ , and the vertices are labeled in a way that $\widetilde{\ell}(v_0) = \ell(v'_0) = w_0$, $\widetilde{\ell}(v_1) = w_1$, and $\widetilde{\ell}(v_2) = w_2$. Observe that the vertex v_1 has multiplicity 2, but it is sent by $\widetilde{\ell}$ to the vertex w_1 , which has multiplicity 1. However, the rational curve E_{v_1} associated with the vertex v_1 has self-intersection -1 and can, thus, be contracted. The resulting map $\pi: X_\pi \rightarrow X$, which no longer factors through σ_Δ , is the minimal resolution of $(X, 0)$ factoring through its Nash transform. Observe that its \mathcal{P} -node v_2 can also be contracted, yielding the minimal good resolution of $(X, 0)$, which in this case does not factor through the Nash transform of $(X, 0)$.

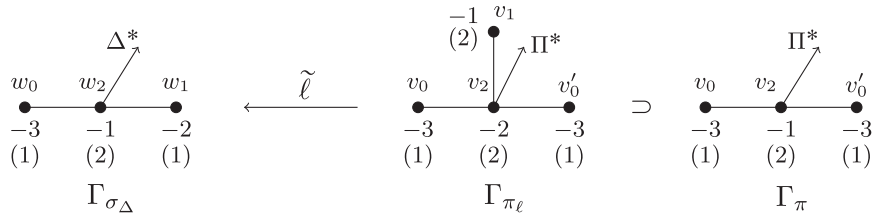


FIGURE 1. Dual resolution graphs for the plane curve Δ (left) and for the surface singularity $X = A_2$ (middle and right).

In the proof of Lemma 4.1, the minimality of π is only required to apply Lemma 3.1. Therefore, this example also shows how the commutativity of the diagram of Lemma 3.1 may fail to hold on a larger dual graph such as Γ_{π_ℓ} .

We can now move our focus to the morphism $\tilde{\ell}$ induced by a generic projection $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$, and more precisely to its restriction to the dual graph Γ_π of some good resolution of $(X, 0)$. Recall that, given a graph Γ , we denote by $V(\Gamma)$ the set of its vertices. In general, even whenever π factors through the Nash transform of $(X, 0)$, it is not possible to find a suitable sequence of point blowups $\sigma: Y \rightarrow \mathbb{C}^2$ of $(\mathbb{C}^2, 0)$ such that $\tilde{\ell}$ induces a morphism of graphs $\tilde{\ell}|_{\Gamma_\pi} \rightarrow \Gamma_\sigma$, because to make the elements of $\tilde{\ell}(V(\Gamma_\pi))$ appear among the vertices of Γ_σ , one usually introduces too many additional vertices, so that the image $\ell(e)$ of some edge e of Γ_π is not an edge of Γ_σ , but only a string of several edges. Remarkably, thanks to Lemma 4.1(ii), in the case of LNE surfaces we can control this phenomenon completely. Indeed, the following proposition explains that in this case we do obtain a morphism of graphs, provided that we restrict our attention to a subgraph of Γ_π that does not contain a \mathcal{P} -node of $(X, 0)$ in its interior.

PROPOSITION 4.5. *Let $(X, 0)$ be a LNE normal surface germ, let $\pi: X_\pi \rightarrow X$ be the minimal good resolution of $(X, 0)$ that factors through its Nash transform, let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection, and let $\tilde{\ell}: \text{NL}(X, 0) \rightarrow \text{NL}(\mathbb{C}^2, 0)$ be the map induced by ℓ . Let S be a subset of $V(\Gamma_\pi)$ which contains all \mathcal{P} -nodes. Let W be one of the connected components of $\Gamma_\pi \setminus S$, and let Γ_0 be the subgraph of Γ_π whose underlying topological space is the closure of W in Γ_π . Let $\sigma_{\Gamma_0}: Y_{\sigma_{\Gamma_0}} \rightarrow \mathbb{C}^2$ be the minimal sequence of point blowups such that $\tilde{\ell}(V(\Gamma_0)) \subset V(\Gamma_{\sigma_{\Gamma_0}})$. Then:*

- (i) $\sigma_{\Gamma_0}: Y_{\sigma_{\Gamma_0}} \rightarrow \mathbb{C}^2$ coincides with the minimal sequence of point blowups of $(\mathbb{C}^2, 0)$ such that $\tilde{\ell}(V(\partial W)) \subset V(\Gamma_{\sigma_{\Gamma_0}})$;
- (ii) the restriction $\tilde{\ell}|_{\Gamma_0}: \Gamma_0 \rightarrow \text{NL}(\mathbb{C}^2, 0)$ induces an isomorphism of graphs from Γ_0 onto its image, which is the subgraph of $\Gamma_{\sigma_{\Gamma_0}}$ whose underlying topological space is the closure of the connected component of $\Gamma_{\sigma_{\Gamma_0}} \setminus \tilde{\ell}(\partial W)$ which contains $\tilde{\ell}(W)$;
- (iii) the isomorphism of part (ii) respects the weights of all vertices of Γ_0 that are contained in W .

Let v be a vertex of Γ_π which is not a \mathcal{P} -node, let W be the connected component of $\Gamma_\pi \setminus \{\mathcal{P}\text{-nodes}\}$ containing v , and let Γ_0 be any subgraph of Γ_π contained in the closure of W and such that $v \in V(\Gamma_0)$ and all edges of Γ_π at v are edges of Γ_0 . Then the last part of the statement of Proposition 4.5 tells us that $g(E_v) = g(E_{\tilde{\ell}(v)}) = 0$ and $E_v^2 = E_{\tilde{\ell}(v)}^2$, where the self-intersection of E_v is computed in X_π and that of $E_{\tilde{\ell}(v)}$ is computed in $Y_{\sigma_{\Gamma_0}}$.

Proof. Write $\partial W = \{z_1, \dots, z_n\} \subset V(\Gamma_\pi)$. Observe that, because Γ_0 contains no \mathcal{P} -node in its interior, $\tilde{\ell}$ does not fold it, and, thus, $\partial(\tilde{\ell}(W)) = \tilde{\ell}(\partial W)$ as subsets of $V(\Gamma_{\sigma_{\Gamma_0}})$. If W contains at least one vertex of Γ_0 , that is, if Γ_0 has at least a vertex which is not a point of S , denote by Γ_0° the maximal subgraph of Γ_0 contained in W and set $V(\Gamma_0^\circ) = \{v_1, \dots, v_r\}$, so that $V(\Gamma_0) = \{v_1, \dots, v_r, z_1, \dots, z_n\}$. Let U be a tubular neighborhood of the curve $C = E_{v_1} \cup \dots \cup E_{v_r}$ in X_π . As the incidence matrix of Γ_0° is negative definite, the analytic contraction $\eta: (U, C) \rightarrow (S, p)$ of the curve C onto a point p defines a normal surface singularity (S, p) . Observe that, because π is the minimal resolution of $(X, 0)$ which factors through its Nash transform, the only components of $\pi^{-1}(0)$ that could be contracted while retaining smoothness of the ambient surface are associated with \mathcal{P} -nodes of $(X, 0)$. As $\Gamma_0^\circ \subset W \subset \Gamma_\pi \setminus S$ contains no \mathcal{P} -node, this implies that $\eta: (U, C) \rightarrow (S, p)$ is the minimal good resolution of the surface germ (S, p) . On the other hand, if W contains no vertex of Γ_0 , then Γ_0 consists of two vertices v and v' in S and a single edge corresponding to an intersection point $p = E_v \cap E_{v'}$, in which case we set $(U, C) = (S, p) = (X_\pi, p)$ and $\eta = \text{Id}_U$.

Let $\hat{\pi}: X_{\hat{\pi}} \rightarrow X$ be the minimal resolution of $(X, 0)$ that factors through its Nash transform and such that $\ell \circ \hat{\pi}$ factors through σ_{Γ_0} via a map $\hat{\ell}: X_{\hat{\pi}} \rightarrow Y_{\Gamma_0}$. Then $\hat{\pi}$ factors through π by minimality of the latter, so that we obtain a commutative diagram as follows.

$$\begin{array}{ccccc}
 & & \hat{\pi} & & \\
 & \curvearrowright & & \curvearrowleft & \\
 X_{\hat{\pi}} & \xrightarrow{\beta} & X_\pi & \xrightarrow{\pi} & X \\
 \hat{\ell} \downarrow & & & & \downarrow \ell \\
 Y_{\Gamma_0} & \xrightarrow{\sigma_{\Gamma_0}} & & & \mathbb{C}^2
 \end{array}$$

Set $\hat{U} = \beta^{-1}(U)$ and $\hat{C} = \beta^{-1}(C)$, so that (\hat{U}, \hat{C}) contracts to (S, p) via $\hat{\pi} = \pi \circ \beta$. Set $U' = \hat{\ell}(\hat{U})$ and consider the curve $C' = \hat{\ell}(\hat{C}) = E_{w_1} \cup \dots \cup E_{w_s} \subset \sigma_{\Gamma_0}^{-1}(0)$ in Y_{Γ_0} . Observe that, because $\partial(\tilde{\ell}(W)) = \tilde{\ell}(\partial W) \subset V(\Gamma_{\sigma_{\Gamma_0}})$, we have $\{w_1, \dots, w_s\} = \tilde{\ell}(W) \cap V(\Gamma_{\sigma_{\Gamma_0}})$. Moreover, because $\hat{U} \cap \hat{\pi}^{-1}(0) \subset \hat{C} \cup E_{z_1} \cup \dots \cup E_{z_n}$ and no curve among the E_{z_i} is contracted by $\hat{\ell}$, we deduce that U' is open in Y_{Γ_0} . It follows that U' is a tubular neighborhood of C' . This also shows that the set $\tilde{\ell}(\Gamma_0)$ is the closure of the connected component of $\Gamma_{\sigma_{\Gamma_0}} \setminus \tilde{\ell}(\partial W)$ which contains $\tilde{\ell}(W)$. To establish the proposition, it is then sufficient to prove that ℓ induces an isomorphism between the pairs (U, C) and (U', C') .

Similarly as above, the contraction of the curve C' in U' defines a normal surface singularity (S', p') and an analytic map $\eta': (U', C') \rightarrow (S', p')$ which is a good resolution of (S', p') . Moreover, the restriction $\hat{\ell}|_U$ induces a finite analytic map $\check{\ell}: (S, p) \rightarrow (S', p')$. As W contains no \mathcal{P} -node, by Lemma 4.1 we have $\deg(w) = 1$ for every divisorial point w of $W \cap \Gamma_0$. This implies that $\hat{\ell}$ is a one-sheeted analytic covering between normal complex analytic spaces, thus an isomorphism by [Rem94, Proposition 14.7]. It follows that $\check{\ell}^{-1} \circ \eta'$ is a good resolution of (S, p) . Therefore, by minimality of the resolution η , the map $\check{\ell}^{-1} \circ \eta'$ factors through η via a finite sequence of point blowups $\alpha: (U', C') \rightarrow (U, C)$, so that we obtain the following commutative diagram.

$$\begin{array}{ccccc}
 (\hat{U}, \hat{C}) & \xrightarrow{\beta} & (U, C) & \xrightarrow{\eta} & (S, p) \\
 \hat{\ell} \downarrow & \nearrow \alpha & & & \uparrow \check{\ell}^{-1} \\
 (U', C') & & & \xrightarrow{\eta'} & (S', p')
 \end{array}$$

It remains to show that α is an isomorphism. If this is not the case, then the exceptional component of the last point blowup forming α , which is contractible by definition, is the image through $\tilde{\ell}$ of the exceptional component of one of the point blowups forming β . As this contradicts the minimality condition in the definition of σ_{Γ_0} , this proves both parts (ii) and (iii) of the proposition. Part (i) is then a consequence of the minimality of π . \square

What might prevent Proposition 4.5 from holding globally on Γ_π is that, for example, there might exist an edge e of Γ_π such that $\tilde{\ell}(e)$ contains in its interior one (and, for the sake of the example, exactly one) point of the form $\tilde{\ell}(v)$ for some vertex v elsewhere in Γ_π . However, whenever this happens it is always possible to *refine* the graph Γ_π , performing a blowup of the double point of the exceptional divisor of X_π that corresponds to e and, thus, subdividing the edge e by adding a new vertex w , and this vertex satisfies $\tilde{\ell}(w) = \tilde{\ell}(v)$. Observe that, if $(X, 0)$ were arbitrary, this could still fail to give a morphism of graphs because the vertex associated with the blowup would not necessarily be sent to $\tilde{\ell}(v)$ by $\tilde{\ell}$. The fact that this does not occur in the case of LNE surfaces and that, therefore, we can refine Γ_π to obtain a morphism of graphs, is the content of the following corollary.

COROLLARY 4.6. *Let $(X, 0)$ be a LNE normal surface germ, let $\pi: X_\pi \rightarrow X$ be the minimal good resolution of $(X, 0)$ that factors through its Nash transform, let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection, let $\tilde{\ell}: \text{NL}(X, 0) \rightarrow \text{NL}(\mathbb{C}^2, 0)$ be the map induced by ℓ , and let $\sigma_\ell: Y_{\sigma_\ell} \rightarrow \mathbb{C}^2$ be the minimal sequence of point blowups of $(\mathbb{C}^2, 0)$ such that $\tilde{\ell}(V(\Gamma_\pi)) \subset V(\Gamma_{\sigma_\ell})$. Then there exists a good resolution $\pi': X_{\pi'} \rightarrow X$ of $(X, 0)$, obtained by composing π with a finite sequence of blowups of double points of the successive exceptional divisors, such that $\tilde{\ell}$ induces a morphism of graphs $\tilde{\ell}|_{\Gamma_{\pi'}}: \Gamma_{\pi'} \rightarrow \Gamma_{\sigma_\ell}$.*

Proof. If Γ_π consists of a single vertex, then there is nothing to prove, so that we can assume without loss of generality that Γ_π has at least an edge. Let e be an edge of Γ_π and let Γ_0 be the subgraph of Γ_π that consists of e and of the two vertices v and v' to which the latter is adjacent. Let σ_{Γ_0} be the minimal sequence of point blowups of $(\mathbb{C}^2, 0)$ such that $\tilde{\ell}(v), \tilde{\ell}(v') \in V(\Gamma_{\sigma_{\Gamma_0}})$. By Proposition 4.5, $\tilde{\ell}(e)$ is an edge of $\Gamma_{\sigma_{\Gamma_0}}$, and in particular $\tilde{\ell}$ induces an isomorphism of smooth germs $\alpha: (X_\pi, E_v \cap E_{v'}) \xrightarrow{\sim} (Y_{\sigma_{\Gamma_0}}, E_{\tilde{\ell}(v)} \cap E_{\tilde{\ell}(v')})$. Now, σ_ℓ factors through σ_{Γ_0} by minimality of the latter. In particular, a finite sequence of point blowups above $E_{\tilde{\ell}(v)} \cap E_{\tilde{\ell}(v')}$ occur in this factorization. By performing the same sequence of blowups on $(X_\pi, E_v \cap E_{v'})$ via the isomorphism α , we subdivide the edge e in a chain of edges that is sent isomorphically to a subgraph of Γ_{σ_ℓ} via $\tilde{\ell}$. Repeating this procedure for every edge e of Γ_π , we obtain the resolution π' that we were after. \square

Observe that the resulting morphism of graphs $\tilde{\ell}|_{\Gamma_{\pi'}}: \Gamma_{\pi'} \rightarrow \Gamma_{\sigma_\ell}$ is not surjective, as is clear from Example 4.4. This issue is discussed further in § 7.

We conclude the section by discussing a remarkable property of the Nash transforms of LNE normal surface germs. The singularity (S, p) appearing in the course of the proof of Proposition 4.5, being isomorphic to the singularity (S', p') appearing in a modification of $(\mathbb{C}^2, 0)$, is *sandwiched*, which means that it admits a proper bimeromorphic morphism to a smooth surface germ. Sandwiched singularities play an important role in Spivakovsky’s proof of resolution of singularities of surfaces via normalized Nash transforms [Spi90], because Hironaka [Hir83] proved that it is possible to reduce any singularity to a sandwiched singularity by a finite sequence of normalized Nash transforms. As Proposition 4.5 applies, in particular, to any connected component of the complement in $\text{NL}(X, 0)$ of its \mathcal{P} -nodes, and the \mathcal{P} -nodes are precisely the divisorial

valuations corresponding to the exceptional components of the Nash transform of $(X, 0)$, we deduce the following result.

COROLLARY 4.7. *Let $(X, 0)$ be a LNE normal surface germ and let $\nu: \mathcal{N}(X) \rightarrow X$ be the Nash transform of $(X, 0)$. Then all the singularities of $\mathcal{N}(X)$ are sandwiched.*

5. Inner rates on LNE surface germs

In this section, we move to the study of the inner rates of LNE surface germs, whose definition was recalled immediately before the proof of Lemma 4.1, and prove parts (iii) and (iv) of Theorem 1.1.

We begin by endowing the dual graph of a good resolution of $(X, 0)$ with a natural metric. Let $\pi: X_\pi \rightarrow X$ be a good resolution of $(X, 0)$ factoring through the blowup of its maximal ideal, and denote by $|\Gamma_\pi|$ the topological space underlying the graph Γ_π . We endow $|\Gamma_\pi|$ with the metric defined by declaring that the length of an edge connecting two vertices v and w is equal to $1/\text{lcm}(m_v, m_w)$, and denote by d the associated distance function. Observe that, because the exceptional component of the blowup of an intersection point between the two components associated with v and w has multiplicity $m_v + m_w$, and $1/\text{lcm}(m_v, m_w) = 1/\text{lcm}(m_v, m_v + m_w) + 1/\text{lcm}(m_v + m_w, m_w)$, the metric on $|\Gamma_\pi|$ is compatible with subdividing the edges of the graph Γ_π by blowing up X_π at double points of $\pi^{-1}(0)$, and thus induces a metric on $\text{NL}(X, 0)$. The reader should be warned that this metric on Γ_π is not the same as that defined in [BFP22, § 2.1], albeit it is strictly related to the latter and was already briefly used in Lemma 5.5 of [BFP22].

The following proposition is strictly stronger than part (iii) of Theorem 1.1, as it computes inner rates on the whole $\text{NL}(X, 0)$ rather than on a specific resolution graph.

PROPOSITION 5.1. *Let $(X, 0)$ be a LNE normal surface germ. Then, for every divisorial point v of $\text{NL}(X, 0)$, the inner rate q_v of v equals $d(v, V_{\mathcal{L}}) + 1$, where $d(v, V_{\mathcal{L}})$ denotes the distance of v from the set $V_{\mathcal{L}}$ of all \mathcal{L} -nodes of $(X, 0)$.*

Proof. Let $\pi: X_\pi \rightarrow X$ be the minimal good resolution of $(X, 0)$ which factors through its Nash transform and let $\Gamma_{\pi'}$ be a refinement of Γ_π as in Corollary 4.6. We begin by proving the wanted equality for divisorial points contained in $\Gamma_{\pi'}$. Denote by w_0 the unique \mathcal{L} -node of $(\mathbb{C}^2, 0)$, that is, the divisorial point associated with the blowup of \mathbb{C}^2 at 0. For every divisorial point w of $\text{NL}(\mathbb{C}^2, 0)$ the inner rate of w is $d(w, w_0) + 1$ by [BFP22, Lemma 5.5] (or, in a more elementary way, by a simple computation using Lemma 3.6 of [BFP22]). As the inner rates on $(X, 0)$ and $(\mathbb{C}^2, 0)$ commute with the map $\tilde{\ell}$ (see [BFP22, Lemma 3.2]), we need to show that $d(v, V_{\mathcal{L}}) = d(\tilde{\ell}(v), w_0)$. We claim that, if γ is an injective path in $\Gamma_{\pi'}$ connecting two divisorial points v_1 and v_2 , then the length of γ is greater or equal to the length of its image $\tilde{\ell}(\gamma)$ in $\text{NL}(\mathbb{C}^2, 0)$, with equality holding as long as $\tilde{\ell}$ maps γ injectively onto its image. Indeed, any edge e in γ is sent via $\tilde{\ell}$ to an edge $\tilde{\ell}(e)$ of the dual graph of some sequence of point blowups of $(\mathbb{C}^2, 0)$ thanks to Corollary 4.6. It then follows from Lemma 4.1(i) that the edges e and $\tilde{\ell}(e)$ have the same length, which implies our claim. In particular, because $V_{\mathcal{L}} = \tilde{\ell}^{-1}(w_0)$, we deduce that $d(v, V_{\mathcal{L}}) \geq d(\tilde{\ell}(v), w_0)$. To obtain the converse inequality it is sufficient to prove that there exists a path γ from v to an element of $V_{\mathcal{L}}$ where $\tilde{\ell}$ is injective. This follows from the fact that there exists such a path along which the inner rate function is strictly decreasing (and, hence, injective), which was proven in [BFP22, Proposition 3.9]. The fact that the equality holds on the whole of $\text{NL}(X, 0)$ is a consequence of [BFP22, Lemma 5.5] (which is itself based on the same computations using Lemma 3.6 of [BFP22] that appears above). □

Remark 5.2. Proposition 5.1 shows that the inner rate function generalizes the function s used by Spivakovsky in [Spi90, Definition 5.1] to study minimal and sandwiched surface singularities.

To prove part (iv) of Theorem 1.1, we need to rely on a deeper result, the so-called *Laplacian formula* for the inner rate function that we obtained in [BFP22] and that we now briefly recall. To state this formula we introduce two additional vectors indexed by the vertices of the dual graph Γ_π of a good resolution $\pi: X_\pi \rightarrow X$ of $(X, 0)$. Let L_π and P_π be the \mathcal{L} - and the \mathcal{P} -vector of $(X, 0)$, respectively, as before. For every vertex v of Γ_π , set $k_v = \text{val}_{\Gamma_\pi}(v) + 2g(v) - 2$ and $a_v = m_v q_v$, and consider the vectors $K_\pi = (k_v)_{v \in V(\Gamma_\pi)}$ and $A_\pi = (a_v)_{v \in V(\Gamma_\pi)}$. Denote by I_{Γ_π} the incidence matrix of the exceptional divisor of π . Then the following equality holds:

$$I_{\Gamma_\pi} \cdot A_\pi = K_\pi + L_\pi - P_\pi. \tag{6}$$

This equality is an effective version (see [BFP22, Proposition 5.3]) of the main result of [BFP22].

Proof of part (iv) of Theorem 1.1. For every vertex v of Γ_π , equation (6) yields

$$m_v q_v E_v^2 + \sum_{v'} m_{v'} q_{v'} = \text{val}_{\Gamma_\pi}(v) + 2g(E_v) - 2 + l_v - p_v, \tag{7}$$

where the sum runs over the vertices v' of Γ_π adjacent to v . Then the equality we are after follows from the fact that $E_v \cdot \sum_{v' \in V(\Gamma_\pi)} E_{v'} = E_v^2 + \text{val}_{\Gamma_\pi}(v)$, that $l_v = -E_v \cdot Z_{\max}(X, 0)$ by the definition of l_v , that $Z_{\max}(X, 0) = Z_{\min}$ by part (ii) of the theorem, and that $E_v \cdot Z_{\Gamma_\pi} = -E_v^2 + 2g(E_v) - 2$ by the definition of Z_{Γ_π} . Whenever v is an \mathcal{L} -node, we have $q_v = 1$ and $q_{v'} = 1 + 1/m_{v'}$ by Proposition 5.1. Therefore, the left-hand side of equation (7) becomes equal to $E_v \cdot Z_{\max}(X, 0) + \sum_{v'} 1 = -l_v + \text{val}_{\Gamma_\pi}(v)$, and we deduce that $p_v = 2(g(E_v) + l_v - 1)$. \square

6. End of the proof of Theorem 1.1

In this section, we conclude the proof of Theorem 1.1, showing parts (v) and (vi), which means that we are interested in determining the \mathcal{P} -nodes of the LNE surface germ $(X, 0)$.

We begin with two definitions. Let π denote a good resolution of $(X, 0)$ that factors through the blowup of its maximal ideal and through its Nash transform, let v be a vertex of Γ_π , and let $e = [v, v']$ be an edge of Γ_π adjacent to v . We say that e is *incoming* at v if we have $q_v > q_{v'}$. Following [Spi90, Definition 5.3], we say that it is a *central node* of Γ_π if v has at least two incoming edges.

Observe that the \mathcal{L} -nodes of Γ_π have no incoming edges, and that the number of incoming edges at a vertex v does not depend on the choice of a resolution such that v is a vertex of the associated dual graph, because the inner rates increase along any new edge introduced by blowing up a smooth point. In the LNE case, we can prove the following more precise result, building on the local degree formula [BFP22, Lemma 4.18].

LEMMA 6.1. *Let $(X, 0)$ be a LNE normal surface germ, let π be a good resolution of $(X, 0)$ that factors through its Nash transform, and let v be a vertex of Γ_π . Then the local degree $\text{deg}(v)$ at v equals l_v if v is an \mathcal{L} -node of Γ_π , or the number of incoming edges of Γ_π at v otherwise.*

Proof. Denote by $\ell : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ a generic projection. Assume first that v is an \mathcal{L} -node. In this case, we can compute the degree directly via the definition, using a generic linear form $h : (X, 0) \rightarrow (\mathbb{C}, 0)$ that factors through ℓ (that is, such that there exists a linear projection $\ell_h : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ satisfying $h = \ell_h \circ \ell$). More precisely, let γ be the curve $h^{-1}(0) \cap X$. As h factors through ℓ , we know that $\ell(\gamma)$ is a line in $(\mathbb{C}^2, 0)$. Now, l_v corresponds to the number of distinct irreducible components of the strict transform of γ by π that intersect E_v . As each of

those components is smooth, we conclude that $\deg(v) = l_v$ by the definition of degree. Assume now that v is not an \mathcal{L} -node. By Corollary 4.6, there exist a refinement $\Gamma_{\pi'}$ of Γ_{π} and a sequence of point blowups $\sigma : Y \rightarrow \mathbb{C}^2$ such that $\tilde{\ell}$ induces a morphism of graphs $\Gamma_{\pi'} \rightarrow \Gamma_{\sigma}$. In particular, because $\tilde{\ell}$ respects inner rates (see again [BFP22, Lemma 3.2]), all edges of $\Gamma_{\pi'}$ that are incoming at v are sent to the unique edge of Γ_{σ} that is incoming at $\tilde{\ell}(v)$ (its uniqueness can for example be seen as a consequence of Proposition 5.1). The lemma follows now by applying the formula of [BFP22, Lemma 4.18], observing that the local degree $\deg(e)$ along every edge adjacent to v equals 1 by Lemma 4.1(ii) and that, even if further point blowups may be needed to pass from π' to a resolution adapted to ℓ , no new edge can be incoming at v . \square

We can now complete the proof of our main theorem.

End of proof of Theorem 1.1. Let $(X, 0)$ be a LNE normal surface germ, denote by $\pi : X_{\pi} \rightarrow X$ be the minimal good resolution of $(X, 0)$, let $\pi' : X_{\pi'} \rightarrow X$ the minimal one that factors through the Nash transform, and let v be a vertex of $\Gamma_{\pi'}$. By combining the lemmas 4.1(ii) and 6.1, we obtain the following:

$$v \text{ is a } \mathcal{P} \text{ - node of } \Gamma_{\pi'} \text{ if and only if either } l_v > 1, \text{ or } v \text{ is a central node of } \Gamma_{\pi'} \quad (*)$$

which establishes part (v) of the theorem. We claim that this also implies that the \mathcal{P} -nodes of $\Gamma_{\pi'}$ are already on the graph $\Gamma_{\pi} \subset \Gamma_{\pi'}$ (possibly in the interior of some edge). Indeed, if v is a vertex of $\Gamma_{\pi'} \setminus \Gamma_{\pi}$, because π' is obtained from π by a sequence of point blowups, we deduce from Proposition 5.1 that there is only one incoming edge at v (observe that all \mathcal{L} -nodes of $(X, 0)$ are contained in Γ_{π} thanks to Proposition 2.2), so that the claim follows from (*). Therefore, we obtain π' by successive blowups of double points on the exceptional divisor of π . Now, let e be an edge of Γ_{π} . If $e = [v, v']$ contains no \mathcal{P} -node, then it is also an edge of $\Gamma_{\pi'}$, and by applying Proposition 4.5 to its closure we deduce that its image through the map induced by a generic projection $\ell : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is an edge $\tilde{\ell}(e) = [\tilde{\ell}(v), \tilde{\ell}(v')]$ of $\Gamma_{\sigma_{\ell}}$. Therefore we have $|q_{\tilde{\ell}(v)} - q_{\tilde{\ell}(v')}| = d(\tilde{\ell}(v), \tilde{\ell}(v'))$, as it can, for example, be seen by Proposition 5.1, and because the inner rate map commutes with $\tilde{\ell}$, and $d(v, v') = d(\tilde{\ell}(v), \tilde{\ell}(v'))$ by Lemma 4.1(i), we deduce that $|q_v - q_{v'}| = d(v, v')$. This shows that if $|q_v - q_{v'}| < d(v, v')$, then e must contain a \mathcal{P} -node. Conversely, if e contains a \mathcal{P} -node w , then, as it can only contain one \mathcal{P} -node, e is folded in two by the projection $\tilde{\ell}$. It follows that, with respect to the distance d , the inner rate grows linearly with slope 1 from v to w , and then decreases linearly with slope 1 from w to v' , so that $|q_v - q_{v'}| < d(v, v')$. We also deduce that $d(v, v') = d(v, w) + d(w, v') = (q_w - q_v) + (q_w - q_{v'})$, and therefore $q_w = (d(v, v') + q_v + q_{v'})/2$. This reasoning can be repeated after blowing up the double point of $\pi^{-1}(0)$ corresponding to e and is, therefore, sufficient to establish part (vi) of Theorem 1.1 and, thus, conclude its proof. \square

7. Discriminant curves

In this section, we focus our attention on the discriminant curve of a generic plane projection of a LNE normal surface germ. We describe those curves completely, proving in Theorem 7.5 a more precise version of Theorem 1.2 from the introduction. To do so, we need to pursue in greater depth the study of the properties of the map $\tilde{\ell}$ already undertaken in § 4.

Let $(X, 0)$ be a LNE normal surface singularity, let $\pi : X_{\pi} \rightarrow X$ be the minimal good resolution of $(X, 0)$ which factors through its Nash transform, let $\ell : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection, let $\tilde{\ell} : \text{NL}(X, 0) \rightarrow \text{NL}(\mathbb{C}^2, 0)$ be the induced morphism, and, as in § 4, let $\sigma_{\ell} : Y_{\sigma_{\ell}} \rightarrow \mathbb{C}^2$ be the minimal sequence of point blowups of $(\mathbb{C}^2, 0)$ such that $\tilde{\ell}(V(\Gamma_{\pi})) \subset V(\Gamma_{\sigma_{\ell}})$.

We call a Δ -node of Γ_{σ_ℓ} any vertex v which is the image by $\tilde{\ell}$ of a \mathcal{P} -node of Γ_π , and we call a *root vertex* of Γ_{σ_ℓ} the image by $\tilde{\ell}$ of the \mathcal{L} -nodes of Γ_π . Observe that the root vertex of Γ_{σ_ℓ} is the divisorial point associated with the exceptional divisor of blowup of \mathbb{C}^2 at 0, which can also be seen as the unique \mathcal{L} -node of $(\mathbb{C}^2, 0)$. Moreover, a vertex v of Γ_{σ_ℓ} is a Δ -node if and only if the associated exceptional component $E_v \subset Y_{\sigma_\ell}$ intersects the strict transform Δ^* of the discriminant curve Δ of ℓ via σ_ℓ .

The following proposition explains that for LNE surfaces the morphism σ_ℓ coincides with the minimal good embedded resolution of the discriminant curve Δ .

PROPOSITION 7.1. *Let $(X, 0)$ be a LNE normal surface germ, let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection of $(X, 0)$, and let $\pi: X_\pi \rightarrow X$ be the minimal good resolution of $(X, 0)$ that factors through its Nash transform. Consider the three finite sequences of point blowups of $(\mathbb{C}^2, 0)$ defined as follows:*

- (i) $\sigma_\Delta: Y_{\sigma_\Delta} \rightarrow \mathbb{C}^2$ is the minimal good embedded resolution of the discriminant curve Δ associated with ℓ ;
- (ii) $\sigma_\Omega: Y_\sigma \rightarrow \mathbb{C}^2$ is the minimal sequence which resolves the base points of the family of projected generic polar curves $\{\ell(\Pi_{\mathcal{D}})\}_{\mathcal{D} \in \Omega}$;
- (iii) $\sigma_\ell: Y_{\sigma_\ell} \rightarrow \mathbb{C}^2$ is the minimal sequence such that $V(\Gamma_{\sigma_\ell})$ contains $\tilde{\ell}(V(\Gamma_\pi))$.

Then σ_Δ , σ_Ω , and σ_ℓ coincide.

Proof. Let us assume for now that the graph Γ_π does not consist of a single vertex. We begin by showing that σ_ℓ and σ_Ω coincide. Denote by S the set of \mathcal{P} -nodes of Γ_π and by W_1, \dots, W_r the connected components of $\Gamma_\pi \setminus S$. For each $i = 1, \dots, r$, following the notation of Proposition 4.5, let $\Gamma_{0,i}$ be the subgraph of Γ_π induced on the topological closure of W_i in Γ_π , and let $\sigma_{\Gamma_{0,i}}: Y_{\sigma_{\Gamma_{0,i}}} \rightarrow \mathbb{C}^2$ be the minimal sequence of point blowups $(\mathbb{C}^2, 0)$ such that $\tilde{\ell}(V(\Gamma_{0,i})) \subset V(\Gamma_{\sigma_{\Gamma_{0,i}}})$. As $V(\Gamma_\pi) = \bigcup_i V(\Gamma_{0,i})$, it follows that σ_ℓ is the minimal sequence of point blowups of $(\mathbb{C}^2, 0)$ that factors through all the maps $\sigma_{\Gamma_{0,i}}$. On the other hand, by Proposition 4.5(i), $\sigma_{\Gamma_{0,i}}$ coincides with the minimal sequence $\sigma_{\Omega,i}$ of point blowups of $(\mathbb{C}^2, 0)$ such that $\tilde{\ell}(V(\partial W_i)) \subset V(\Gamma_{\sigma_{\Gamma_{0,i}}})$. As $\bigcup_i \partial W_i = S$, this implies that σ_ℓ is the minimal sequence of blowups of \mathbb{C}^2 over 0 such that $V(\Gamma_{\sigma_\ell})$ contains the set $\tilde{\ell}(S)$ of the Δ -nodes, which is, by definition, σ_Ω .

Let us now prove that σ_Ω factors through σ_Δ by showing that it is a good embedded resolution of the curve Δ . Assume, by contradiction, that this is not the case, so that there exist a Δ -node w and a component Δ_0 of Δ whose strict transform by σ_Ω , while intersecting E_w at a smooth point p of an exceptional component E_w , is not a curvette of E_w . This implies that the multiplicity of Δ_0 is strictly greater than m_w . Let Π_0 be the component of the polar curve Π of ℓ such that $\Delta_0 = \ell(\Pi_0)$. As π is minimal, the strict transform of Π_0 by π is a curvette on an exceptional component E_v such that $\tilde{\ell}(v) = w$, so that the multiplicity of Π_0 equals m_v . As $\tilde{\ell}(v) = w$, by Lemma 4.1(ii) we have $m_v = m_w$ and, therefore, $\text{mult}(\Delta_0) > \text{mult}(\Pi_0)$. However, the restriction $\ell|_{\Pi_0}: \Pi_0 \rightarrow \Delta_0$ is a bi-Lipschitz homeomorphism with respect to the outer metric by [Tei82, pp. 352–354], so that, in particular, we have $\text{mult}(\Delta_0) = \text{mult}(\Pi_0)$, yielding a contradiction. This proves that σ_Ω is a good embedded resolution of Δ .

It is now sufficient to show that σ_Δ also resolves the base points of the family $\{\ell(\Pi_{\mathcal{D}})\}_{\mathcal{D} \in \Omega}$, so that it factors through σ_Ω . Assume, by contradiction, that this is not the case, so that there exists a component Δ_0 of Δ whose strict transform by σ_Δ meets the exceptional divisor $\sigma_\Delta^{-1}(0)$ at a (smooth) point p which is a base point of the family $\{\ell(\Pi_{\mathcal{D}})\}_{\mathcal{D} \in \Omega}$. Let w be the vertex of

Γ_{σ_Ω} such that E_w is the irreducible component of $\sigma_\Omega^{-1}(0)$ that contains p . The base point p is resolved by a sequence of point blowups δ which creates a bamboo (that is, a chain of two-valent vertices ending with a univalent vertex) B living inside $\Gamma_{\sigma_\Omega} \setminus \Gamma_{\sigma_\Delta}$, stemming from the vertex w and having the corresponding Δ -node w' at its extremity. As σ_Ω is a resolution of Δ , we can construct a resolution of $(X, 0)$ by performing the Hirzebruch–Jung resolution process starting from the morphisms ℓ and σ_Ω . For this, we begin by taking the strict transform of $(X, 0)$ by the fiber product of ℓ and σ_Ω and normalize it. As ℓ is a cover which ramifies over the discriminant curve Δ , we obtain a normal surface Z and a finite cover $\ell': Z \rightarrow Y_{\sigma_\Omega}$ which ramifies over the total transform $\sigma_\Omega^{-1}(\Delta)$ of the discriminant curve Δ . Resolving the singularities of Z , we obtain a resolution $\pi': X_{\pi'} \rightarrow X$ of $(X, 0)$ which we can describe as follows. As Γ_{σ_Ω} contains all Δ -nodes, then $\Gamma_{\pi'}$ contains all \mathcal{P} -nodes of $(X, 0)$ and, therefore, π' factors through π . As by the previous part σ_Ω factors through σ_Δ , the total transform $\sigma_\Omega^{-1}(\Delta)$ has normal crossings in Y_{σ_Ω} . Moreover, each singularity of Z is branched over a double point of $\sigma_\Omega^{-1}(\Delta)$ and has a resolution whose exceptional divisor is a string of rational curves, and the strict transform of the branching locus consists of two curvettes, one at each extremity of the string. This implies that the bamboo B lifts via $\tilde{\ell}$ to a bamboo B' in the resolution graph $\Gamma_{\pi'}$ with a \mathcal{P} -node at its extremity. This gives a \mathcal{P} -node with a unique inward edge in $\Gamma_{\pi'}$ and, therefore, a unique inward edge in Γ_π , contradicting the statement (*).

In the special case where Γ_π consists of a single vertex, then the morphisms σ_Δ , σ_Ω , and σ_ℓ all coincide with a single blowup of \mathbb{C}^2 along its origin. To see this, the only part which is not immediate is the factorization of σ_Ω through σ_Δ , but the argument given above remains valid in this special case and, thus, the proof of the proposition is complete. \square

Recall that, as shown by Example 4.4, the map $\tilde{\ell}: \Gamma_\pi \rightarrow \Gamma_{\sigma_\ell}$ may fail to be surjective. As a first step to better understand the situation, the following proposition, which refines the techniques we employed in the course of the proof of Lemma 4.1, allows us to describe $\tilde{\ell}$ more explicitly.

LEMMA 7.2. *Let $(X, 0)$ be a LNE normal surface germ, let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection, let $\pi: X_\pi \rightarrow X$ be the minimal good resolution of $(X, 0)$ which factors through its Nash transform, and let v and v' be two divisorial points of $\Gamma_\pi \subset \text{NL}(X, 0)$. Then $\tilde{\ell}(v) = \tilde{\ell}(v')$ if and only if the two following conditions are satisfied:*

- (i) $q_v = q_{v'}$;
- (ii) *there exists a path τ in Γ_π between v and v' such that the inner rate of any point in τ is greater than or equal to q_v .*

Proof. Let us begin by proving the ‘only if’ part of the statement. Assume that $\tilde{\ell}(v) = \tilde{\ell}(v')$. Then $q_v = q_{v'}$ because both inner rates are equal to $q_{\tilde{\ell}(v)}$ by [BFP22, Lemma 3.2]. Assume, by contradiction, that the condition (ii) is not satisfied, and let γ be a curve in $(\mathbb{C}^2, 0)$ which is the image through σ_ℓ of a curvette of $E_{\tilde{\ell}(v)}$. Let $\hat{\gamma}$ (respectively, $\hat{\gamma}'$) be a component of $\ell^{-1}(\gamma)$ which is the image of a curvette of E_v (respectively, $E_{v'}$) via a suitable resolution factoring through π . As Γ_π is path connected but part (ii) is not satisfied, then the inner contact $q_{\text{inn}}(\hat{\gamma}, \hat{\gamma}')$ between $\hat{\gamma}$ and $\hat{\gamma}'$ is strictly smaller than q_v by [NPP20a, Proposition 15.3]. On the other hand, by taking a different generic projection $\ell_{\mathcal{D}'}$ which is also generic with respect to $\hat{\gamma} \cup \hat{\gamma}'$, and observing (in a similar way as in the proof of Lemma 4.1) that $\ell_{\mathcal{D}'}$ induces a bi-Lipschitz homeomorphism for the outer metric from $\hat{\gamma} \cup \hat{\gamma}'$ onto its image by [Tei82, pp. 352–354], we deduce that the outer contact $q_{\text{out}}(\hat{\gamma}, \hat{\gamma}')$ between $\hat{\gamma}$ and $\hat{\gamma}'$ equals $q_{\text{out}}(\ell_{\mathcal{D}'}(\hat{\gamma}), \ell_{\mathcal{D}'}(\hat{\gamma}')) = q_{\text{inn}}(\ell_{\mathcal{D}'}(\hat{\gamma}), \ell_{\mathcal{D}'}(\hat{\gamma}'))$. By Lemma 3.1 we have $\ell_{\mathcal{D}'}(v) = \ell_{\mathcal{D}'}(v')$ and, thus, the curves $\ell_{\mathcal{D}'}(\hat{\gamma})$ and $\ell_{\mathcal{D}'}(\hat{\gamma}')$ lift to the same divisor $E_{\tilde{\ell}_{\mathcal{D}'}(v)}$.

Therefore, their inner contact is bigger than or equal to $q_{\tilde{\ell}_{\mathcal{D}'}(v)} = q_v$, which contradicts the fact that $(X, 0)$ is LNE.

To prove the converse implication, observe that if w and w' are two points of $NL(\mathbb{C}^2, 0)$, then there exists a unique injective path $\tau_{w,w'}$ between w and w' in $NL(\mathbb{C}^2, 0)$, because the latter is a connected infinite tree. Moreover, for each point w'' in the interior of $\tau_{w,w'}$ we have $q_{w''} < \max\{q_w, q_{w'}\}$ (for example, this can be derived from Proposition 5.1). Now assume that v and v' are two points of Γ_π that satisfy the conditions (i) and (ii) and let τ be any path in Γ_π between v and v' . By continuity of the projection, $\tilde{\ell}(\tau)$ must contain $\tau_{\tilde{\ell}(v), \tilde{\ell}(v')}$, and the latter has nonempty interior as soon as $\tilde{\ell}(v) \neq \tilde{\ell}(v')$, therefore we deduce that when this is the case then τ contains a point v'' mapping to the interior of $\tau_{\tilde{\ell}(v), \tilde{\ell}(v')}$, so that $q_{v''} = q_{\tilde{\ell}(v'')} < q_{\tilde{\ell}(v)} = q_v$. As this would contradict condition (ii), we must have $\tilde{\ell}(v) = \tilde{\ell}(v')$. \square

To make good use of this result we need to introduce some additional notation. We denote by $V_N(\Gamma_\pi)$ the set of *nodes* of Γ_π , that is, the subset of $V(\Gamma_\pi)$ consisting of the \mathcal{P} -nodes, the \mathcal{L} -nodes, and of all the vertices that have valency at least three in Γ_π (that is, those with at least three adjacent edges). Similarly, we call *node* of Γ_{σ_ℓ} a vertex which is either the root vertex, a Δ -node, or a vertex of valency at least three in Γ_{σ_ℓ} , and we denote by $V_N(\Gamma_{\sigma_\ell})$ the set of nodes of Γ_{σ_ℓ} .

Let Γ be either of the two graphs Γ_π or Γ_{σ_ℓ} . We call *principal part* of Γ the subgraph Γ' of Γ generated by the set $V_N(\Gamma)$ of nodes of Γ , that is, the subgraph defined as the union of all injective paths connecting pairs of points of $V_N(\Gamma)$. The closure of each component of $\Gamma \setminus \Gamma'$ is a *bamboo* (that is, a chain of valency 2 vertices ending with a valency 1 vertex) stemming from a node of Γ .

Lemma 7.2 prompts us to consider an equivalence relation \sim on the graph Γ'_π defined by declaring that two vertices v and v' of Γ'_π are equivalent if the two conditions (i) and (ii) of the lemma hold, and two edges $e = [v_1, v_2]$ and $e' = [v'_1, v'_2]$ are equivalent if and only if $v_1 \sim v'_1$ and $v_2 \sim v'_2$. The following proposition relates the nodes of Γ_π to those of Γ_{σ_ℓ} and explains how the equivalence relation \sim allows to retrieve the principal part Γ'_{σ_ℓ} of Γ_{σ_ℓ} from that of Γ_π .

PROPOSITION 7.3. *Let $(X, 0)$, π , ℓ , and σ_ℓ be as above. Then we have:*

- (i) $\tilde{\ell}(V_N(\Gamma_\pi)) = V_N(\Gamma_{\sigma_\ell})$;
- (ii) *the map $\tilde{\ell}|_{\Gamma'_\pi} : \Gamma'_\pi \rightarrow \Gamma'_{\sigma_\ell}$ identifies the graph Γ'_{σ_ℓ} with the quotient graph Γ'_π / \sim .*

Proof. Observe that Γ_{σ_ℓ} and Γ_{σ_Δ} coincide thanks to Proposition 7.1, and the latter, being the minimal embedded resolution graph of the plane curve Δ , has a very simple shape (see, for example, [BK86]). In particular, we deduce that the principal part Γ'_{σ_ℓ} of Γ_{σ_ℓ} coincides with the union of the injective paths connecting the root vertex of Γ_{σ_ℓ} to one of its Δ -nodes. It follows that the image $\tilde{\ell}(\Gamma_\pi)$ of Γ_π via $\tilde{\ell}$ contains Γ'_{σ_ℓ} . Indeed, any injective path from an \mathcal{L} -node to a \mathcal{P} -node v in Γ_π is sent by $\tilde{\ell}$ to the unique injective path in Γ_{σ_ℓ} from the root vertex to the Δ -node $\tilde{\ell}(v)$. Moreover, any vertex of Γ_{σ_ℓ} contained in $\tilde{\ell}(\Gamma_\pi)$ is the image of a vertex of Γ_π , as follows readily from Proposition 4.5. From the particular shape of Γ_{σ_ℓ} we also deduce that, if w is a vertex of Γ_{σ_ℓ} , then at most one of the edges of Γ_{σ_ℓ} that are outgoing (that is, not incoming in the sense of § 6) at w may fail to be contained in the principal part Γ'_{σ_ℓ} .

By definition, a vertex w of Γ_{σ_ℓ} is the root vertex (respectively, a Δ -node) of Γ_{σ_ℓ} if and only if $\tilde{\ell}^{-1}(w)$ contains an \mathcal{L} -node (respectively a \mathcal{P} -node) of $(X, 0)$. To prove part (i) of the proposition it is then sufficient to establish the following claim: a vertex w of Γ_{σ_ℓ} that is not

the root vertex nor a Δ -node has valency at least three if and only if $\tilde{\ell}^{-1}(w)$ contains at least a vertex of valency at least three in Γ_π .

The ‘if’ part of the claim can be easily obtained by taking a vertex v in $\tilde{\ell}^{-1}(w)$ having valency at least three and applying Proposition 4.5 to the subgraph of Γ_π consisting of the topological closure of v and its adjacent edges.

Let us prove the ‘only if’ part of the claim. We begin by showing that all edges of Γ_{σ_ℓ} that are outgoing at w belong to $\tilde{\ell}(\Gamma_\pi)$. Assume, by contradiction, that this is not the case, and let e be such an edge that is not contained in $\tilde{\ell}(\Gamma_\pi)$. In particular, e is not contained in Γ'_{σ_ℓ} either, and recalling that σ_ℓ coincides with the minimal resolution of the plane curve Δ , we deduce that the connected component of $\sigma_\ell \setminus \{w\}$ containing e is a bamboo which cannot be contracted. Therefore, in the notation of Proposition 4.5, if Γ_0 is the graph induced on the closure of any connected component of $\Gamma_\pi \setminus \{\mathcal{P}\text{-nodes}\}$ which intersects $\tilde{\ell}^{-1}(w)$ nontrivially, then e is contained in the dual graph $\Gamma_{\sigma_{\Gamma_0}}$. As w is not a Δ -node of Γ_{σ_ℓ} , we deduce from Proposition 4.5 that e is contained in $\tilde{\ell}(\Gamma_\pi)$. To conclude the proof of the claim we make use of Lemma 7.2. Let e and e' be two distinct outgoing edges of Γ_{σ_ℓ} at w . If there exists a vertex v of Γ_π such that e and e' are both images of edges of Γ_π adjacent to v , then v has valency at least three in Γ_π , because not being an \mathcal{L} -node it must also have an incoming edge and, therefore, there is nothing else to prove. We can, thus, assume without loss of generality that there exist two distinct vertices v and v' of Γ_π such that $\tilde{\ell}(v) = \tilde{\ell}(v') = w$ and two edges, \tilde{e} adjacent to v and \tilde{e}' adjacent to v' , whose images contain e and e' , respectively. As $\tilde{\ell}(v) = \tilde{\ell}(v')$, it follows from Lemma 7.2 that there exists an injective path τ from v to v' passing through outgoing edges only. However, τ cannot leave v from \tilde{e} and reach v' through \tilde{e}' , as if that were the case then τ would become a loop in Γ_{σ_ℓ} , which is a tree. This implies that at least one among the vertices v and v' must have a second outgoing edge, and must therefore be trivalent. This concludes the proof of the claim, and therefore of part (i) of the proposition.

To conclude the proof, observe that part (i) implies that $\tilde{\ell}$ restricts to a surjective map $\tilde{\ell}|_{\Gamma'_\pi} : \Gamma'_\pi \rightarrow \Gamma'_{\sigma_\ell}$ between the principal parts of Γ_π and Γ_{σ_ℓ} . Lemma 7.2 then shows that setwise the map $\tilde{\ell}$ identifies Γ'_{σ_ℓ} with the quotient Γ'_π / \sim . The fact that it gives an isomorphism of graphs follows then from the fact that, as we already observed, any vertex of Γ_{σ_ℓ} contained in $\tilde{\ell}(\Gamma_\pi)$ is the image of a vertex of Γ_π , and so $\tilde{\ell}(V(\Gamma'_\pi)) = V(\Gamma'_{\sigma_\ell})$. \square

Remarks 7.4. (i) Observe that Lemma 7.2 fails outside of the dual graph of the minimal resolution π . Indeed, it is sufficient to compose π with two point blowups, choosing as center two distinct points of the same component of the exceptional divisor of π which are identified by a suitable lifting of ℓ , to obtain two divisorial points v and v' such that $\tilde{\ell}(v) = \tilde{\ell}(v')$ and for which condition (ii) does not hold.

(ii) If w is the root vertex of Γ_{σ_ℓ} , then $\tilde{\ell}^{-1}(w)$ is exactly the set of \mathcal{L} -nodes of $(X, 0)$, so that $\tilde{\ell}^{-1}(w) \subset V_N(\Gamma_\pi)$. However, if w is a Δ -node of Γ_{σ_ℓ} , not all vertices in $\tilde{\ell}^{-1}(w)$ need to be \mathcal{P} -nodes of $(X, 0)$ (nor, more generally, nodes of Γ_π), as [NPP20b, Example 3.13] shows. If w is a node of Γ_{σ_ℓ} which is not a Δ -node and which has valency at least three in Γ_{σ_ℓ} , we do not know whether $\tilde{\ell}^{-1}(w)$ may contain vertices having valency less than three in Γ_π .

(iii) In the course of the proof of Proposition 7.3, we have shown that $\Gamma_{\sigma_\ell} \setminus \tilde{\ell}(\Gamma_\pi)$ consists of bamboos stemming from Δ -nodes. Moreover, over a bamboo of Γ_{σ_ℓ} stemming from a vertex w which is not a Δ -node, there is a copy of same bamboo stemming from any vertex v of Γ_π such that $\tilde{\ell}(v) = w$. It then follows from Lemma 7.2 that there can only be one vertex v of Γ_π which is sent to such a vertex w by $\tilde{\ell}$.

We have now collected all the results we need to move to the study of the embedded topological type of the discriminant curve $(\Delta, 0) \subset (\mathbb{C}^2, 0)$. Fix once and for all a set of coordinates (x_1, x_2) on $(\mathbb{C}^2, 0)$ such that $x_1 = 0$ is transverse to Δ . The topological type we are interested in is then completely determined by the characteristic exponents of the Newton–Puiseux expansion with respect to x_1 of each branch of Δ and by the coincident exponents between each pair of branches. This data is encoded by another combinatorial object, the so-called Eggers–Wall tree $\Theta(\Delta) = \Theta_{x_1}(\Delta)$ of Δ . We refer the reader to [GGP19, §3] for a thorough introduction to this object, and in particular to Definition 3.8 and Remark 3.14 of [GGP19] for a formal definition starting from Newton–Puiseux expansions and an interesting historical remark. From our point of view, it is more convenient to describe the Eggers–Wall tree $\Theta(C)$ of a plane curve germ $(C, 0) \subset (\mathbb{C}^2, 0)$ starting from the dual graph of a good embedded resolution of Δ and from the invariants we already consider, namely multiplicities and inner rates. This follows the philosophy of §8 of [GGP19], where an embedding of $\Theta(C)$ in a valuation space homeomorphic to $\text{NL}(\mathbb{C}^2, 0)$ is described (see, in particular, Theorem 8.19 there). The procedure is as follows.

ALGORITHM A. Denote by $\sigma_C: Y_{\sigma_C} \rightarrow C^2$ the minimal good embedded resolution of the curve C . The set of nodes $V_N(\Gamma_{\sigma_C})$ of the dual graph Γ_{σ_C} of σ_C is, by definition, the set consisting of its root, its C -nodes, which are the vertices corresponding to the components of $\sigma_C^{-1}(0)$ intersecting the strict transform of C , and its vertices of valency at least three. The Eggers–Wall tree $\Theta(C)$ is obtained from the set of nodes $V_N(\Gamma_{\sigma_C})$ of the tree Γ_{σ_C} , from its principal part Γ'_{σ_C} , and from the multiplicities and the inner rates of the vertices of Γ'_{σ_C} , as follows.

- (i) From Γ'_{σ_C} , attach one extra edge to the root and one to each C -node w for every branch of C passing through E_w .
- (ii) Decorate each node $w \in V_N(\Gamma_{\sigma_C})$ (this includes vertices that have valency larger than three in Γ_{σ_C} but less than three in Γ'_{σ_C}) with the rational number $e_C(w) = q_w$.
- (iii) If $e = [w, w']$ is an edge of Γ'_{σ_C} , decorate it with the integer $i(e) = \text{lcm}(m_w, m_{w'})$.
- (iv) If e is one of the new edges of $\Theta(C)$ adjacent to a vertex w , decorate it with the integer $i(e) = m_w$.

The rational numbers $e_C(w)$ on the nodes w on the path connecting the root to a C -node w' are then precisely the characteristic exponents of any branch of C passing through E_w , whereas the coincident exponent between two branches can be computed from the functions e_C and i_C , as explained in [GGP19, Theorem 3.25].

To describe the embedded topological type of the discriminant curve Δ , it remains to show how to combine results we proved in §§4, 6, and 7 to determine the input of Algorithm A from the minimal good resolution of $(X, 0)$. This can be done as follows.

ALGORITHM B. Denote by $\pi_0: X_{\pi_0} \rightarrow X$ the minimal good resolution of $(X, 0)$.

- (i) The multiplicities and the inner rates of the vertices of Γ_{π_0} are uniquely determined by parts (ii) and (iii) of Theorem 1.1.
- (ii) The minimal resolution $\pi: X_{\pi} \rightarrow X$ of $(X, 0)$ factoring through its Nash transform, decorated with its multiplicities and inner rates, is obtained from π_0 applying the algorithm of part (vi) of Theorem 1.1. This also determines the set of nodes $V_N(\Gamma_{\pi})$ of Γ_{π} and its principal part Γ'_{π} .
- (iii) Recall that we have $\sigma_{\ell} = \sigma_{\Delta}$ by Proposition 7.1. Therefore, by Proposition 7.3 we obtain the principal part $\Gamma'_{\sigma_{\Delta}}$ of $\Gamma_{\sigma_{\Delta}}$ and the subset $V_N(\Gamma_{\sigma_C})$ consisting of the nodes of $\Gamma_{\sigma_{\Delta}}$.
- (iv) The multiplicities of the vertices of $\Gamma'_{\sigma_{\Delta}}$ are determined by those of the vertices of Γ_{π} thanks to Lemma 4.1(i).

- (v) The inner rates of the vertices of Γ'_{σ_Δ} are determined by those of the vertices of Γ_π because inner rates commute with $\tilde{\ell}$ thanks to [BFP22, Lemma 3.2].

We have proven the following result, which is a more precise version of Theorem 1.2 from the introduction.

THEOREM 7.5. *Let $(X, 0)$ be a LNE normal surface germ and let $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic projection. Then the embedded topology of the discriminant curve Δ of ℓ is completely determined by the topology of $(X, 0)$. More precisely, the Eggers–Wall tree of Δ can be obtained by applying Algorithm B followed by Algorithm A to the dual graph of the minimal resolution of $(X, 0)$.*

Appendix A. Generic polar curves and Nash transform

In this appendix we give a comprehensive proof of a result stated in the introduction.

PROPOSITION A.1. *Let $(X, 0)$ be a normal surface singularity, let $\pi: X_\pi \rightarrow X$ be a good resolution of $(X, 0)$, and let $h: (X, 0) \rightarrow (\mathbb{C}, 0)$ and $\ell: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a generic linear form and a generic plane projection of $(X, 0)$, respectively.*

- (i) *If π factors through the blowup of the maximal ideal of $(X, 0)$, then the strict transform via π of the hyperplane section $h^{-1}(0)$ associated with h consists of smooth curves intersecting the exceptional divisor $\pi^{-1}(0)$ transversely at smooth points.*
- (ii) *If π factors through the Nash transform of $(X, 0)$, then the strict transform via π of the polar curve of ℓ consists of smooth curves intersecting $\pi^{-1}(0)$ transversely at smooth points.*

Observe that, in the statement, by the word *generic* we also mean generic with respect to π , which means that the strict transforms by π of the associated linear form and polar curve intersect $\pi^{-1}(0)$ only at \mathcal{L} - and \mathcal{P} -nodes, respectively, rather than moving to other components created by additional blowups (this is a standard condition; see, for example, [BFP22, 2.2]). We note that this result seems to be accepted by the experts working in the field (see, for instance, the discussion of [Bon05, Section 2] regarding the first part of the proposition), but we have not been able to locate a proof of it in the literature and therefore we provide one here.

Proof. Let us prove a more general version of part (i). Denote by \mathfrak{M} the maximal ideal of $(X, 0)$ and let I be a \mathfrak{M} -primary ideal (that is, I contains a power of \mathfrak{M}). Choose a system of generators (f_1, \dots, f_k) of I , consider the blowup $\text{Bl}_I(X): X_I \rightarrow X$ of I , which is defined as the closure in $X \times \mathbb{P}^{k-1}$ of the set $\{(x, [f_1(x) : \dots : f_k(x)]) \mid x \in X \setminus V(I)\}$, and let $n_I: (\overline{X}_I, \overline{E}_I) \rightarrow (X_I, E_I)$ be its normalization. Denote by $p_2: E_I \rightarrow \mathbb{P}^{k-1}$ the projection on the second factor. By Bertini’s theorem, a generic hyperplane $H: a_1 z_1 + \dots + a_k z_k = 0$ of \mathbb{P}^{k-1} intersects the complex curve $p_2(E_I)$ transversely at smooth points. Therefore, the multigerms $(p_2 \circ n_I)^{-1}(H)$ consists of disjoint smooth curves intersecting transversely \overline{E}_I at a finite numbers of smooth points. On the other hand, $(p_2 \circ n_I)^{-1}(H)$ is exactly the strict transform of $V(h)$ by $\text{Bl}_I(X) \circ n_I$, where h is the element of I defined by $h = a_1 f_1 + \dots + a_k f_k$. Applying this to $I = \mathfrak{M}$, because π factors through the normalized blowup of the maximal ideal, we obtain part (i) of the proposition.

Let us now prove part (ii). Recall that the Nash transform $\nu: \mathcal{N}(X) \rightarrow X$ of $(X, 0)$ is the projection on the first factor of the closure of $\{(x, T_x X) \mid x \in X \setminus \{0\}\}$ in $X \times \text{Gr}(2, \mathbb{C}^k)$. Set $\bar{\nu} = \nu \circ n$ where $n: \overline{\mathcal{N}(X)} \rightarrow \mathcal{N}(X)$ is the normalization of $\mathcal{N}(X)$. The Gauss map $X \setminus \{0\} \rightarrow \text{Gr}(2, \mathbb{C}^k)$ which sends x to $T_x X$ lifts to a well-defined map $\bar{\lambda}: \overline{\mathcal{N}(X)} \rightarrow \text{Gr}(2, \mathbb{C}^k)$. To complete the proof of the proposition it is now sufficient to show that the strict transform by $\bar{\nu}$ of a generic polar curve of $(X, 0)$ intersects transversely the exceptional divisor $E = \bar{\nu}^{-1}(0)$ at smooth points.

The proof is similar in spirit to the proof of (i), but relies on a construction from [BNP14, p. 210]. Let Ω be a dense open subset of $\text{Gr}(k - 2, \mathbb{C}^k)$ parametrizing generic plane projections of $(X, 0)$, let \mathcal{D} be an element of Ω , and let $\ell_{\mathcal{D}}: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be the generic plane projection of $(X, 0)$ associated with \mathcal{D} . Assume first that $k = 3$ and denote by $L_{\mathcal{D}}$ the projective line in $\text{Gr}(2, \mathbb{C}^3) \cong \mathbb{P}^2$ consisting of the 2-planes of \mathbb{C}^3 which contain the line \mathcal{D} . By Bertini's theorem, $L_{\mathcal{D}}$ intersects the complex curve $\bar{\lambda}(E)$ transversely at smooth points. Therefore, the multigerms $(\bar{\lambda})^{-1}(L_{\mathcal{D}})$ consists of disjoint smooth curves intersecting transversely E at a finite numbers of smooth points. As $\Pi_{\mathcal{D}} \setminus \{0\}$ is the set of critical points of the restriction $\ell_{\mathcal{D}}: (X, 0) \rightarrow (\mathbb{C}^2, 0)$ of the linear projection $\mathbb{C}^k \rightarrow \mathbb{C}^2$ with kernel \mathcal{D} to $(X, 0)$, then the strict transform $\Pi_{\mathcal{D}}^*$ by $\bar{\nu}$ is the multigerms defined by $\Pi_{\mathcal{D}}^* = \bar{\lambda}^{-1}(L_{\mathcal{D}})$. This proves part (ii) when $k = 3$.

Assume now that $k \geq 3$ and let us choose a $(k - 3)$ -dimensional subspace $W \subset \mathbb{C}^k$ transverse to the 2-planes which belong to the finite set $B = \bar{\lambda}(A)$ where $A = \Pi_{\mathcal{D}}^* \cap E$. Let $\text{Gr}(2, \mathbb{C}^k; W)$ denote the set of 2-planes in \mathbb{C}^k transverse to W , so that the projection $p: \mathbb{C}^k \rightarrow \mathbb{C}^k/W$ induces a map $p': \text{Gr}(2, \mathbb{C}^k; W) \rightarrow \text{Gr}(2, \mathbb{C}^k/W) \cong P^2\mathbb{C}$. Observe that $\text{Gr}(2, \mathbb{C}^k; W)$ is a Zariski open subset of $\text{Gr}(2, \mathbb{C}^k)$ containing B , therefore on a small neighborhood of A in $\mathcal{N}(X)$ we can define the map $\lambda' = p' \circ \bar{\lambda}$. We now follow the lines of the argument in the case $k = 3$ but using λ' instead of $\bar{\lambda}$. Let $L_{\mathcal{D}}$ be the projective line in $\text{Gr}(2, \mathbb{C}^k/W) \cong P^2\mathbb{C}$ image by p' of the set of 2-planes that intersect \mathcal{D} non-trivially. By genericity of \mathcal{D} and by Bertini's theorem, $L_{\mathcal{D}}$ intersects the complex curve $\bar{\lambda}(E)$ transversely at smooth points. Therefore, the multigerms $\Pi_{\mathcal{D}}^* = (\lambda')^{-1}(L_{\mathcal{D}})$ consists of disjoint smooth curves intersecting transversely E at smooth points. \square

Appendix B. A new example of a LNE normal surface singularity

The aim of this appendix is to prove the following result.

PROPOSITION B.1. *The hypersurface singularity $(X, 0) \subset (\mathbb{C}^3, 0)$ defined by the equation $x^5 + y^5 + z^5 + xyz = 0$ is LNE.*

This is a cusp singularity (see [Lau77]). The blowup of $(X, 0)$ at 0 has as exceptional divisor a loop formed by three rational curves, and three singular points (all being A_1 singularities) where two of those intersect. The minimal resolution $\pi: X_{\pi} \rightarrow X$ is then obtained by composing this blowup with the blowups of those three singular points. The dual graph Γ_{π} of π is shown in Figure B.1, where the negative numbers denote the self-intersections of the corresponding components. The vertices v_1, v_2 , and v_3 are the \mathcal{L} -nodes of Γ_{π} and carry one arrow each, representing a component of the strict transform to X_{π} of a generic linear form $h: (X, 0) \rightarrow (\mathbb{C}^2, 0)$.

Therefore, $(X, 0)$ gives an example of a LNE normal surface singularity which was not previously known. Indeed, the examples of LNE surface germs already known are either minimal surface singularities [NPP20b] or superisolated surface singularities with LNE tangent

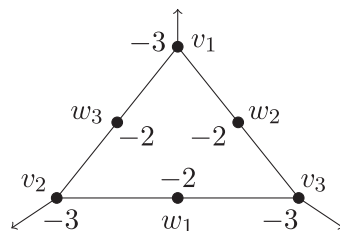


FIGURE B.1. The dual graph Γ_{π} of π .

cone [MP21]. The cusp singularity $(X, 0): x^5 + y^5 + z^5 + xyz = 0$ is not minimal, because it is not rational (its resolution graph is not a tree), nor superisolated, because it is not resolved by the blowup of its maximal ideal.

Proof. A generic polar curve Π of $(X, 0)$ has equation $g(x, y, z) = 0$, where g is a generic linear combination $\alpha f_x + \beta f_y + \gamma f_z$ of the three partial derivatives $f_x = 5x^4 + yz$, $f_y = 5y^4 + xz$ and $f_z = 5z^4 + xy$ of f . A direct computation in the charts of the point blowups forming π shows that the family of generic polar curves has no base points on X_π , that is, π factors through the Nash transform of $(X, 0)$, and that the \mathcal{P} -nodes of $(X, 0)$ are the three vertices w_1, w_2 , and w_3 , with $p_{w_i} = 2$. Observe that it is easy to see from the defining equation that the projectivized tangent cone of $(X, 0)$ consists of three distinct projective lines C_{v_1}, C_{v_2} , and C_{v_3} , each one corresponding to one of the three \mathcal{L} -nodes of $(X, 0)$. The Gauss map $\lambda: X_\pi \rightarrow \text{Gr}(2, \mathbb{C}^3)$ induces a natural map $\tilde{\lambda}: \Gamma_\pi \rightarrow \text{Gr}(2, \mathbb{C}^3)$ which is constant on each connected component of $\Gamma_\pi \setminus \{\mathcal{P} - \text{nodes}\}$, sending any point of the connecting component containing the \mathcal{L} -node v_i to the projective line C_{v_i} . Then the fact that $(X, 0)$ is LNE is a direct consequence of [MP21, Lemma 5.2] and the test curve criterion [NPP20a, Theorem 3.8], repeating the arguments of [MP21, § 5]. \square

ACKNOWLEDGEMENTS

We thank Patrick Popescu-Pampu and Bernard Teissier for interesting discussions about quasi-ordinary singularities and generic plane projections, and Javier Fernández de Bobadilla for giving us an argument that simplified the proof of Proposition A.1. We would also like to thank Camille Le Van and Delphine Ménard for fruitful conversations. This work has been partially supported by the project Lipschitz geometry of singularities (LISA) of the Agence Nationale de la Recherche (project ANR-17-CE40-0023) and by the PEPS–JCJC Métriques singulières, valuations et géométrie Lipschitz des variétés of the Institut National des Sciences Mathématiques et de leurs Interactions of the Centre National de la Recherche Scientifique. The second author has also been supported by a Research Fellowship of the Alexander von Humboldt Foundation.

REFERENCES

- Art66 M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. **88** (1966), 129–136.
- BFP22 A. Belotto da Silva, L. Fantini and A. Pichon, *Inner geometry of complex surfaces: a valuative approach*, Geom. Topol. **26** (2022), 163–219.
- BM18 L. Birbrair and R. Mendes, *Arc criterion of normal embedding*, in *Singularities and foliations. Geometry, topology and applications*, Springer Proceedings in Mathematics and Statistics, vol. 222 (Springer, Cham, 2018), 549–553.
- BNP14 L. Birbrair, W. D. Neumann and A. Pichon, *The thick-thin decomposition and the Bilipschitz classification of normal surface singularities*, Acta Math. **212** (2014), 199–256.
- Bon03 R. Bondil, *Discriminant of a generic projection of a minimal normal surface singularity*, C. R. Math. Acad. Sci. Paris **337** (2003), 195–200.
- Bon05 R. Bondil, *General elements of an m -primary ideal on a normal surface singularity*, in *Singularités Franco-Japonaises*, Séminaires & Congrès, vol. 10 (Société Mathématique de France, Paris, 2005), 11–20.
- Bon16 R. Bondil, *Fine polar invariants of minimal singularities of surfaces*, J. Singul. **14** (2016), 91–112.
- BK86 E. Brieskorn and H. Knörrer, *Plane algebraic curves* (Birkhäuser, Basel, 1986); translated from the German by John Stillwell.

- Fan18 L. Fantini, *Normalized Berkovich spaces and surface singularities*, *Trans. Amer. Math. Soc.* **370** (2018), 7815–7859.
- Fer03 A. Fernandes, *Topological equivalence of complex curves and bi-Lipschitz homeomorphisms*, *Michigan Math. J.* **51** (2003), 593–606.
- FS19 A. Fernandes and J. E. Sampaio, *Tangent cones of Lipschitz normally embedded sets are Lipschitz normally embedded. Appendix by Anne Pichon and Walter D. Neumann*, *Int. Math. Res. Not. IMRN* **2019** (2019), 4880–4897.
- FHPS21 J. Fernández de Bobadilla, S. Heinze, M. P. Pereira and J. E. Sampaio, *Moderately discontinuous homology*, *Commun. Pure Appl. Math.* (2021), [doi:10.1002/cpa.22013](https://doi.org/10.1002/cpa.22013).
- GGP19 E. R. García Barroso, P. D. González Pérez and P. Popescu-Pampu, *The valuative tree is the projective limit of Eggers–Wall trees*, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **113** (2019), 4051–4105.
- Gra62 H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, *Math. Ann.* **146** (1962), 331–368.
- GL97 G. Gonzalez-Sprinberg and M. Lejeune-Jalabert, *Families of smooth curves on surface singularities and wedges*, *Ann. Polon. Math.* **67** (1997), 179–190.
- Hir83 H. Hironaka, *On Nash blowing-up*, in *Arithmetic and geometry, Vol. II*, *Progress in Mathematics*, vol. 36 (Birkhäuser, Boston, Mass., 1983), 103–111.
- KPR18 D. Kerner, H. M. Pedersen and M. A. S. Ruas, *Lipschitz normal embeddings in the space of matrices*, *Math. Z.* **290** (2018), 485–507.
- Kol85 J. Kollár, *Toward moduli of singular varieties*, *Compos. Math.* **56** (1985), 369–398.
- Lau71 H. B. Laufer, *Normal two-dimensional singularities*, *Annals of Mathematics Studies*, vol. 71 (Princeton University Press; University of Tokyo Press, Princeton, NJ; Tokyo, 1971).
- Lau72 H. B. Laufer, *On rational singularities*, *Amer. J. Math.* **94** (1972), 597–608.
- Lau77 H. B. Laufer, *On minimally elliptic singularities*, *Amer. J. Math.* **99** (1977), 1257–1295.
- Lê00 D. T. Lê, *Geometry of complex surface singularities*, in *Singularities—Sapporo 1998*, *Advanced Studies in Pure Mathematics*, vol. 29 (Kinokuniya, Tokyo, 2000), 163–180.
- MP21 F. Misev and A. Pichon, *Lipschitz normal embedding among superisolated singularities*, *Int. Math. Res. Not. IMRN* **2021** (2021), 13546–13569.
- Mos85 T. Mostowski, *Lipschitz equisingularity*, *Dissertationes Math. (Rozprawy Mat.)* **243** (1985), 46.
- Mos88 T. Mostowski, *Tangent cones and Lipschitz stratifications*, in *Singularities (Warsaw, 1985)*, *Banach Center Publications*, vol. 20 (PWN, Warsaw, 1988), 303–322.
- Mum61 D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, *Publ. Math. Inst. Hautes Études Sci.* **9** (1961), 5–22.
- Ném99 A. Némethi, *Five lectures on normal surface singularities*, in *Low dimensional topology (Eger, 1996/Budapest, 1998)*, *Bolyai Society Mathematical Studies*, vol. 8 (János Bolyai Mathematical Society, Budapest, 1999), 269–351, with the assistance of Ágnes Szilárd and Sándor Kovács.
- Neu81 W. D. Neumann, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, *Trans. Amer. Math. Soc.* **268** (1981), 299–344.
- NP07 W. D. Neumann and A. Pichon, *Complex analytic realization of links*, in *Intelligence of low dimensional topology 2006*, *Knots Everything*, vol. 40 (World Scientific, Hackensack, NJ, 2007), 231–238.
- NP14 W. D. Neumann and A. Pichon, *Lipschitz geometry of complex curves*, *J. Singul.* **10** (2014), 225–234.
- NPP20a W. D. Neumann, H. M. Pedersen and A. Pichon, *A characterization of Lipschitz normally embedded surface singularities*, *J. Lond. Math. Soc. (2)* **101** (2020), 612–640.

- NPP20b W. D. Neumann, H. M. Pedersen and A. Pichon, *Minimal surface singularities are Lipschitz normally embedded*, J. Lond. Math. Soc. (2) **101** (2020), 641–658.
- Par88 A. Parusiński, *Lipschitz properties of semi-analytic sets*, Ann. Inst. Fourier (Grenoble) **38** (1988), 189–213.
- Par94 A. Parusiński, *Lipschitz stratification of subanalytic sets*, Ann. Sci. Éc. Norm. Sup. (4) **27** (1994), 661–696.
- PT69 F. Pham and B. Teissier, *Fractions lipschitziennes d’une algèbre analytique complexe et saturation de Zariski*. *Prépublications Ecole Polytechnique* (1969), No. M17.0669.
- Pic20 A. Pichon, *An introduction to Lipschitz geometry of complex singularities*, in *Introduction to Lipschitz geometry of singularities*, Lecture Notes in Mathematics, vol. 2280 (Springer, Cham, 2020), 167–216.
- Pop02 P. Popescu-Pampu, *On the invariance of the semigroup of a quasi-ordinary surface singularity*, C. R. Math. Acad. Sci. Paris **334** (2002), 1001–1106.
- Pop04 P. Popescu-Pampu, *On the analytical invariance of the semigroups of a quasi-ordinary hypersurface singularity*, Duke Math. J. **124** (2004), 67–104.
- Rem94 R. Remmert, *Local theory of complex spaces*, in *Several complex variables, VII*, Encyclopaedia of Mathematical Sciences, vol. 74 (Springer, Berlin, 1994), 7–96.
- Spi90 M. Spivakovskiy, *Sandwiched singularities and desingularization of surfaces by normalized Nash transformations*, Ann. of Math. (2) **131** (1990), 411–491.
- Tei76 B. Teissier, *Résolution simultanée: II - Résolution simultanée et cycles évanescents*. *Séminaire sur les singularités des surfaces*, 1976–1977. Talk:9.
- Tei82 B. Teissier, *Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney*, in *Algebraic geometry (La Rábida, 1981)*, Lecture Notes in Mathematics, vol. 961 (Springer, Berlin, 1982), 314–491.
- Val05 G. Valette, *Lipschitz triangulations*, Illinois J. Math. **49** (2005), 953–979.
- Zar39 O. Zariski, *The reduction of the singularities of an algebraic surface*, Ann. of Math. (2) **40** (1939), 639–689.

André Belotto da Silva andre.belotto@imj-prg.fr

IMJ-PRG, CNRS 7586, Université de Paris, Institut de Mathématiques de Jussieu Paris Rive Gauche, 75013 Paris, France

Lorenzo Fantini lorenzo.fantini@polytechnique.edu

Centre de Mathématiques Laurent Schwartz, Ecole Polytechnique and CNRS, Institut Polytechnique de Paris, 91120 Palaiseau, France

Anne Pichon anne.pichon@univ-amu.fr

Aix Marseille Univ, CNRS, I2M, Marseille, France