*Proceedings of the Royal Society of Edinburgh*, page 1 of 38
DOI:10.1017/prm.2024.35

# Quasi-hereditary covers of Temperley–Lieb algebras and relative dominant dimension

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(Received 10 April 2023; accepted 29 February 2024)

Many connections and dualities in representation theory and Lie theory can be explained using quasi-hereditary covers in the sense of Rouquier. Recent work by the first-named author shows that relative dominant (and codominant) dimensions are natural tools to classify and distinguish distinct quasi-hereditary covers of a finite-dimensional algebra. In this paper, we prove that the relative dominant dimension of a quasi-hereditary algebra, possessing a simple preserving duality, with respect to a direct summand of the characteristic tilting module is always an even number or infinite and that this homological invariant controls the quality of quasi-hereditary covers that possess a simple preserving duality. To resolve the Temperley–Lieb algebras, we apply this result to the class of Schur algebras $S(2, d)$ and their $q$-analogues. Our second main result completely determines the relative dominant dimension of $S(2, d)$ with respect to $Q = V^{\otimes d}$, the $d$-th tensor power of the natural two-dimensional module. As a byproduct, we deduce that Ringel duals of $q$-Schur algebras $S(2, d)$ give rise to quasi-hereditary covers of Temperley–Lieb algebras. Further, we obtain precisely when the Temperley–Lieb algebra is Morita equivalent to the Ringel dual of the $q$-Schur algebra $S(2, d)$ and precisely how far these two algebras are from being Morita equivalent, when they are not. These results are compatible with the integral setup, and we use them to deduce that the Ringel dual of a $q$-Schur algebra over the ring of Laurent polynomials over the integers together with some projective module is the best quasi-hereditary cover of the integral Temperley–Lieb algebra.

*Keywords:* quasi-hereditary cover; relative dominant dimension; $q$-Schur algebra; Temperley–Lieb algebra; Frobenius twist

2020 *Mathematics Subject Classification* Primary: 16E10
Secondary: 20G43, 16G10, 16G30, 82B20

## 1. Introduction

The theory of quasi-hereditary covers, introduced in [49], gives a framework to study finite-dimensional algebras of infinite global dimension through algebras having nicer homological properties, for instance, quasi-hereditary algebras via an exact functor known as Schur functor. Quasi-hereditary covers appear naturally and are useful in algebraic Lie theory, representation, theory and homological algebra. In particular, they are in the background of Auslander's correspondence [2] and in Iyama's proof of finiteness of representation dimension [37]. Further, quasi-hereditary algebras arise quite naturally in the representation theory of algebraic groups [4, 47] and algebras of global dimension at most two are quasi-hereditary.

Schur algebras $S(n, d)$ form an important class of quasi-hereditary algebras, they provide a link between polynomial representations of general linear groups and representations of symmetric groups. Classically, when $n \geqslant d$, the Schur algebra, via the Schur functor, is a quasi-hereditary cover of the group algebra of the symmetric group $\mathcal{S}_d$. This connection is one of the versions of Schur–Weyl duality. Indeed, this formulation clarifies the connection between the representation theory of symmetric groups and the representation theory of Schur algebras, by detecting how the subcategories of modules with Weyl filtrations and of modules with dual Specht filtrations are related and how the Yoneda extension groups in these subcategories are related by the Schur functor (see also [35]). Further, this connection becomes stronger as the characteristic of the ground field increases. It was first observed in [29] that this behaviour is captured by the classical dominant dimension. However, this is not the case for all quasi-hereditary covers.

To fix this, in [10] the concepts of relative dominant dimension and relative codominant dimension with respect to a module were introduced. Further, in [10] these homological invariants were exploited to create new quasi-hereditary covers. With this, the link between Schur algebras and symmetric groups can be regarded as a special case of quasi-hereditary covers of quotients of Iwahori–Hecke algebras.

Temperley–Lieb algebras are quotients of Iwahori–Hecke algebras and they sometimes have infinite global dimension. They were introduced in [50] in the context of statistical mechanics and they were popularized by Jones, in particular, they are used to define the Jones polynomial (see [41]). However, contrary to Iwahori–Hecke algebras no Hemmer–Nakano type result was known for Temperley–Lieb algebras until now. Both classes of algebras are cellular (see e.g. [32]) and so an important property that they have in common is the existence of a simple preserving duality.

Quasi-hereditary algebras with a simple preserving duality always have even global dimension. Mazorchuk and Ovsienko have shown this fact in [45] by proving that the global dimension of a quasi-hereditary algebra with a simple preserving duality is exactly twice the projective dimension of the characteristic tilting module. Later, under much stronger conditions, the analogue result for dominant dimension was obtained in [29] by Fang and Koenig exploiting that a faithful projective–injective module is a summand of the characteristic tilting module.

The present paper has two aims. First, we will establish that the relative dominant dimension of a quasi-hereditary algebra with respect to any summand of its characteristic tilting module is always twice as large as that of the characteristic tilting module, in the case when the algebra has a simple preserving duality. In particular, this homological invariant is always even for such quasi-hereditary algebras. Further, Fang and Koenig's result can then be recovered from ours by just fixing

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the summand to be a projective–injective module. Therefore, we obtain an alternative approach to the classical case of dominant dimension without any further assumptions.

The second aim is to study classes of quasi-hereditary covers of Temperley–Lieb algebras and their link to the representation theory of Temperley–Lieb algebras. In particular, we aim to completely understand such a connection using the representation theory of $q$-Schur algebras and their Ringel duals.

## Questions to be addressed and setup

To make our results precise, we need further notation. In general, assume that $B$ is a finite-dimensional algebra over an algebraically closed field. A pair $(A, P)$ is a *quasi-hereditary cover* of $B$ if $A$ is a quasi-hereditary algebra, $P$ is a finitely generated projective $A$-module such that $B = \operatorname{End}_A(P)^{op}$, and in addition the restriction of the associated Schur functor $F := \operatorname{Hom}_A(P, -)\colon A\text{-mod} \to B\text{-mod}$ to the subcategory of finitely generated projective $A$-modules is full and faithful.

Let $\mathcal{F}(\Delta)$ be the category of $A$-modules which have a filtration by standard modules. We would like to get information on the category $\mathcal{F}(F\Delta)$ of $B$-modules which have a filtration by images under $F$ of standard modules. Specifically, we would like the functor $F$ to be faithful on $\mathcal{F}(\Delta)$ and to induce isomorphisms $\operatorname{Ext}_A^j(X, Y) \to \operatorname{Ext}_B^j(FX, FY)$ for $X, Y$ modules in $\mathcal{F}(\Delta)$. If this is the case for $0 \leqslant j \leqslant i$, then $(A, P)$ is called an $i$-$\mathcal{F}(\Delta)$ cover of $B$. The largest $n$ such that $(A, P)$ is an $n$-$\mathcal{F}(\Delta)$ cover of $B$ is called the *Hemmer–Nakano dimension* of $\mathcal{F}(\Delta)$ in [**29**]. When $n \geqslant 1$, filtration multiplicities in the category $\mathcal{F}(F\Delta)$ are well-defined, which places the Hemmer–Nakano theorem on Specht filtrations into a wider context. When $B$ is self-injective, Fang and Koenig showed that this dimension is controlled by the dominant dimension of a characteristic tilting module. In addition, they proved that if $B$ is a symmetric algebra and the quasi-hereditary cover admits a certain simple preserving duality, then the dominant dimension of a characteristic tilting module is exactly half of the dominant dimension of $A$.

Recently, in [**10**], the situation was generalized to include cases where $B$ is not necessarily self-injective. Moreover, it was proved in [**10**] that the Hemmer–Nakano dimension of $\mathcal{F}(\Delta)$ associated with a 0-$\mathcal{F}(\Delta)$ cover can be determined using the relative codominant dimension of a characteristic tilting module with respect to a certain summand of the characteristic tilting module.

The concepts of relative dominant and relative codominant dimension (see the definition below in § 2.3) and the concept of quasi-hereditary cover can be considered in an integral setup, that is, both of these concepts can be studied for Noetherian algebras which are finitely generated and projective as modules over a regular commutative Noetherian ring. In [**9**], methods were developed to reduce the computations of Hemmer–Nakano dimensions in the integral setup to the setup where the ground ring is an algebraically closed field. So, it will be enough for our purposes to concentrate on the case when the coefficient ring is an algebraically closed field.

The new approach to construct quasi-hereditary covers appears in [**10**, Theorem 5.3.1] and [**10**, Theorem 8.1.5] when applied to Schur algebras (and $q$-Schur

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algebras). The novelty is that it uses the Ringel dual of a Schur algebra, rather than a Schur algebra, and works for arbitrary parameters $n, d$.

This can, in particular, be applied to the study of Temperley–Lieb algebras. Recall that $S(n, d)$ is isomorphic to the centralizer algebra of $\mathcal{S}_d$ in the endomorphism algebra of the tensor power $(K^n)^{\otimes d}$, where $(K^n)^{\otimes d}$ affords a module structure over $\mathcal{S}_d$ by place permutation for a field $K$. Furthermore, $V^{\otimes d} := (K^n)^{\otimes d}$ is in the additive closure of a characteristic tilting module over $S(n, d)$. When $n = 2$, the centralizer of $S(2, d)$ (and its $q$-analogue) in the endomorphism algebra of the tensor power can be identified with a Temperley–Lieb algebra. Indeed, all Temperley–Lieb algebras arise in this way. Our cases of interest have a simple preserving duality, and in such a case, for this situation, we can without ambiguity interchange the concepts relative dominant dimension and relative codominant dimension.

Denote by $Q\text{-domdim}_A X$ the relative dominant dimension of an $A$-module $X$ with respect to $Q$. In this context, the following questions arise:

(1) *What is the value of* $V^{\otimes d}\text{-domdim}_{S(2,d)} T$*, where* $T$ *is a characteristic tilting module of the quasi-hereditary algebra* $S(2, d)$*? What happens to this value when we replace a Schur algebra by a q-Schur algebra?*

(2) *The Ringel duals of Schur algebras as well as Schur algebras have a simple preserving duality. Can we expect, like in the classical case (see [29, Theorem 4.3]), the equality* $V^{\otimes d}\text{-domdim}\, S(n, d) = 2 \cdot V^{\otimes d}\text{-domdim}_{S(n,d)} T$ *to hold in general?*

(3) *Can we expect the quasi-hereditary cover of the Temperley–Lieb algebra constructed in [10, Theorem 8.1.5] to be unique, in some meaningful way?*

Our goal in this paper is to provide answers to these three questions.

## Main results

Surprisingly, the answer to (2) is positive without using extra structure on $S(n, d)$ besides the quasi-hereditary structure and the existence of a simple preserving duality.

THEOREM A (see theorem 3.1). *Let* $A$ *be a quasi-hereditary algebra over a field* $K$*. Suppose that there exists a simple preserving duality* ${}^{\diamond}(-)\colon A\text{-mod} \to A\text{-mod}$*. Let* $T$ *be the characteristic tilting module of* $A$*. Assume that* $Q \in \operatorname{add}(T)$*. Then*

$$Q\text{-domdim}_A A = 2 \cdot Q\text{-domdim}_A T.$$

This means that for $A$ as in the theorem, the characteristic tilting module $T$ is a test module to compute the relative dominant dimension of $A$ with respect to direct summands of $T$. By fixing $Q$ to be a projective–injective module in this setting, this result generalizes [29, Theorem 4.3] and our methods provide a new proof to their case without using any information on $A$ being an endomorphism algebra of a faithful module over a symmetric algebra. In particular, our result also works for dominant dimension exactly zero and when $A$ is the endomorphism algebra of a faithful module over a self-injective algebra. The left-hand side of the equation in

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theorem A is also known as the faithful dimension of $Q$ in the sense of [**3**]. This means that the faithful dimension of $Q$ fully determines the connection between $\mathrm{End}_A(Q)$ and its quasi-hereditary cover formed by the Ringel dual when the $A$ is a quasi-hereditary with a simple preserving duality.

Theorem A has many applications for quasi-hereditary algebras with a simple preserving duality. For instance, it can be used to reprove a result of [**36**] which states that there exists a unique minimal direct summand of the characteristic tilting module affording a double centralizer property (see Appendix A). It can also be used to reprove that Ringel duality preserves dominant dimension in this setting.

Combining techniques of Frobenius twisted tensor products with theorem A we obtain a complete answer to (1):

THEOREM B (see also theorem 5.8 for the $q$-version). *Let $K$ be a field and let $A$ be the Schur algebra $S_K(2, d)$ and $T$ be the characteristic tilting module of $A$. Then,*

$$V^{\otimes d}\text{-domdim}_A\, A = 2 \cdot V^{\otimes d}\text{-domdim}_A\, T = \begin{cases} d, & \textit{if } \operatorname{char} K = 2 \textit{ and } d \textit{ is even}, \\ +\infty, & \textit{otherwise}. \end{cases}$$

In this case, $Q = V^{\otimes d}$ and $\mathrm{End}_A(Q)$ is the quotient of the symmetric group algebra, modulo the kernel of the action on $V^{\otimes d}$, and theorem B provides information about this quotient. When the relative dominant dimension is infinite, this quotient is Morita equivalent to the Ringel dual of $A$. When $\operatorname{char} K = 2$ and $d$ is even theorem B shows that for $d \geqslant 6$, the Hemmer–Nakano dimension is positive. Hence, in this case, the filtration multiplicities in the category $\mathcal{F}(F\Delta)$ are well-defined. These are the modules filtered by Specht modules labelled by 2-part partitions and this provides new information: in contrast, filtration multiplicities for arbitrary Specht modules are not well-defined for $p = 2$, as shown in [**35**]. Further, theorem B and its $q$-version generalizes [**7**, Theorem C (3), (4)] for Temperley–Lieb algebras. For the quantum case, the same approach works. In the quantum case, the algebra $A$ is the $q$-Schur algebra $S_{K,q}(2, d)$, which can be defined as the centralizer of the Hecke algebra $H_q(d)$ acting on $V^{\otimes d}$, again for $\dim V = 2$ and in the theorem the characteristic is replaced by the quantum characteristic. When we have $v \in K$ such that $v^2 = q$ and $\delta = -v - v^{-1}$, the Temperley–Lieb algebra $TL_{K,d}(\delta)$ is a quotient of this action. In both cases, the real difficulty lies in the case in which the characteristic (resp. quantum characteristic) is two.

For tilting modules in general, going beyond the characteristic tilting module, we can construct, via mutation, new tilting modules. In the context of Buan and Solberg's work [**3**], theorem B means that $V^{\otimes d}$ over $S(2, d)$ is either a full tilting module or there are precisely $d + 1$ indecomposable $S(2, d)$-modules $X$ making $X \oplus V^{\otimes d}$ a full tilting module.

Using theorem B together with cover theory, we obtain a positive answer to question (3) when we consider the Laurent polynomial ring over the integers as coefficient ring and $d > 2$ (see § 7 and corollary 7.5). In such a case, the (integral) Schur functor $F$ induces an exact equivalence $\mathcal{F}(\Delta) \to \mathcal{F}(F\Delta)$. The quasi-hereditary cover of the integral Temperley–Lieb algebra formed by the Ringel dual of a $q$-Schur algebra is the unique quasi-hereditary cover which induces this exact equivalence.

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If $d = 2$, the Temperley–Lieb algebra is exactly an Iwahori–Hecke algebra, but this is new information when $d > 2$.

We emphasize that the specialization of theorem A to projective–injective modules played a key role in determining the dominant dimension of Schur algebras $S(n, d)$ with $n \geqslant d$ in [**29**] (also their $q$-analogues [**30**]). It is our expectation that its use will be crucial to determine, in particular, $V^{\otimes d}$-domdim $S(n, d)$ and domdim $S(n, d)$ also in the cases $2 < n < d$ while the latter is also an open problem for $n = 2$.

This article is organized as follows: in § 2, we introduce the notation and the main properties of relative dominant dimension with respect to a module, split quasi-hereditary algebras with a simple preserving duality, and cover theory to be used throughout the paper. In § 3, we discuss elementary results on relative injective dimensions and we give the proof of theorem A. We then deduce that the dominant dimension is a lower bound for the faithful dimension of a faithful direct summand of a characteristic tilting module fixed by a simple preserving duality (see proposition 3.6). In § 4, we collect results on the quasi-hereditary structure of Schur algebras $S(2, d)$, in particular, reduction techniques and how to construct partial tilting and standard modules inductively using the Frobenius twist functor. In § 5, we compute the relative dominant dimension of $S(2, d)$ with respect to $V^{\otimes d}$ in terms of $V^{\otimes d}$-domdim$_{S(2,d)}\, T$, where $T$ is a characteristic tilting module of $S(2, d)$. In particular, we give the proof of theorem B and its $q$-analogue (see theorem 5.8). As an application, we compute all complements of the almost complete tilting module $V^{\otimes 4}$ when the ground field has characteristic two (see § 5.3). In § 6, we recall that all Temperley–Lieb algebras can be realized as the centralizer algebras of $q$-Schur algebras in the endomorphism algebra of the tensor power $V^{\otimes d}$. As a consequence, we determine the value of Hemmer–Nakano dimension of $\mathcal{F}(\Delta)$ in all cases associated with the cover of the Temperley–Lieb algebra formed by the Ringel dual of a $q$-Schur algebra. This computation is contained in corollary 6.8. In § 7, we determine the Hemmer–Nakano dimension of the abovementioned quasi-hereditary cover in the integral setup, dividing the study into two cases: the coefficient ring being or not 2-partially $q$-divisible (see § 7.1). When the coefficient ring does not have such property, we show that a quasi-hereditary cover with such coefficient ring has better properties. We conclude by addressing the problem of the uniqueness of this cover (see § 7.2). In Appendix A, we provide two applications of theorem A that involve faithful direct summands of the characteristic tilting module.

## 2. Preliminaries

### 2.1. The setting

This follows [**10**]. Throughout we fix a Noetherian commutative ring $R$ with identity, and $A$ is an $R$-algebra which is finitely generated and projective as an $R$-module. We refer to $A$ as a *projective Noetherian* $R$-algebra. The set of invertible elements of $R$ is denoted by $R^{\times}$.

We denote by $A$-mod the category of finitely generated (left) $A$-modules. Given $M \in A$-mod, we denote by $\operatorname{add}_A M$ (or just $\operatorname{add} M$) the full subcategory of $A$-mod whose modules are direct summands of a finite direct sum of copies of $M$. We also denote $\operatorname{add} A$ by $A$-proj.

The endomorphism algebra of a module $M \in A$-mod is denoted by $\operatorname{End}_A(M)$. We denote by $D_R$ or just $D$ the standard duality $\operatorname{Hom}_R(-, R) : A\text{-mod} \to A^{op}\text{-mod}$ where $A^{op}$ is the opposite algebra of $A$.

A module $M \in A\text{-mod} \cap R\text{-proj}$ is said to be $(A, R)$-*injective* if it belongs to $\operatorname{add} DA$, and we write $(A, R)\text{-inj} \cap R\text{-proj}$ for the full subcategory of $A\text{-mod} \cap R\text{-proj}$ whose modules are $(A, R)$-injective.

Furthermore, an exact sequence of $A$-modules which is split as an exact sequence of $R$-modules is said to be $(A, R)$-*exact*. In particular, an $(A, R)$-*monomorphism* is a homomorphism $f : M \to N$ that fits into an $(A, R)$-exact sequence $0 \to M \xrightarrow{f} N$.

Given a left exact covariant additive functor $G$, we say that $X$ is a $G$-*acyclic object* if $\mathrm{R}^{i>0}\, G(X) = 0$. An exact sequence $0 \to L \to X_0 \to X_1 \to \cdots$ is called a $G$-*acyclic coresolution* of $L$ if all objects $X_0, X_1, \ldots$ are $G$-acyclic. Given $X \in A\text{-mod} \cap R\text{-proj}$, we denote by $X^{\perp}$ the full subcategory

$$\{M \in A\text{-mod} \cap R\text{-proj}\colon\ \operatorname{Ext}_A^{i>0}(Z, M) = 0, \forall Z \in \operatorname{add} X\},$$

and by ${}^{\perp}X$ the full subcategory

$$\{M \in A\text{-mod} \cap R\text{-proj}\colon\ \operatorname{Ext}_A^{i>0}(M, Z) = 0, \forall Z \in \operatorname{add} X\}.$$

## 2.2. Approximations

Assume that $A$ is an $R$-algebra as above, and $Q$ is a fixed module in $A\text{-mod} \cap R\text{-proj}$.

An $A$-homomorphism $f : M \to N$ is a *left* $\operatorname{add}_A Q$-*approximation* of $M$ provided that $N$ belongs to $\operatorname{add}_A Q$, and moreover the induced map

$$\operatorname{Hom}_A(N, X) \to \operatorname{Hom}_A(M, X)$$

is surjective for every $X \in \operatorname{add}_A Q$. Dually, one defines right $\operatorname{add}_A Q$-approximations. Note that every module $M \in A$-mod has a left and a right $\operatorname{add}_A Q$-approximation.

## 2.3. Relative (co)dominant dimension with respect to a module

We recall from [**10**] the definition of relative (co)dominant dimensions.

Let $Q, X \in A\text{-mod} \cap R\text{-proj}$. If $X$ does not admit a left $\operatorname{add} Q$-approximation which is an $(A, R)$-monomorphism then the relative dominant dimension of $X$ with respect to $Q$ is zero. Otherwise, *the relative dominant dimension of $X$ with respect to $Q$*, denoted by $Q\text{-domdim}_{(A,R)}\, X$, or $Q\text{-domdim}_A X$ when $R$ is a field, is the supremum of all $n \in \mathbb{N}$ such that there is an $(A, R)$-exact sequence:

$$0 \to X \to Q_1 \to Q_2 \to \cdots \to Q_n$$

with all $Q_i \in \operatorname{add} Q$, which remains exact under $\operatorname{Hom}_A(-, Q)$.

Dually, one defines the *relative codominant dimension*, denoted by $Q\text{-codomdim}_{(A,R)}(X)$ with $Q, X$ as above: if $X$ does not admit a surjective

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right $\operatorname{add} Q$-approximation, then $Q\text{-codomdim}_{(A,R)}(X) = 0$. Otherwise it is the supremum of all $n \in \mathbb{N}$ such that there is an $(A, R)$-exact sequence:

$$Q_n \to Q_{n-1} \to \cdots \to Q_1 \to X \to 0$$

with all $Q_i \in \operatorname{add} Q$, which remains exact under $\operatorname{Hom}_A(Q, -)$.

Hence, $Q\text{-codomdim}_{(A,R)} X = DQ\text{-domdim}_{(A^{op},R)} DX$. By $Q\text{-domdim}\,(A, R)$ we mean the value $Q\text{-domdim}_{(A,R)} A$. We will write $Q\text{-codomdim}_A X$ to denote $Q\text{-codomdim}_{(A,R)} X$ when $R$ is a field.

*2.3.1. Results from [10]* We will give criteria towards finding these invariants. These depend essentially on the theorems [**10**, Theorems 3.1.1, 3.1.4] and their dual versions. We briefly describe part of it. Assume $Q \in A\text{-mod} \cap R\text{-proj}$ such that the endomorphism algebra of $Q$, $B = \operatorname{End}_A(Q)$, is a projective Noetherian $R$-algebra. The Schur functor $F = \operatorname{Hom}_A(Q, -)$ has a left adjoint $\mathcal{I} := Q \otimes_B -$. Let $\chi$ be the associated counit $\chi\colon \mathcal{I}F \to \operatorname{id}_{A\text{-mod}}$. Explicitly, if $M$ belongs to $A\text{-mod} \cap R\text{-proj}$, then $\chi_M$ is the $A$-homomorphism:

$$Q \otimes_B \operatorname{Hom}_A(Q, M) \to M, \quad \chi_M(q \otimes g) = g(q).$$

By Theorem 3.1.1 of [**10**], for a module $M \in A\text{-mod} \cap R\text{-proj}$, $Q\text{-codomdim}_{(A,R)} M \geqslant 2$ if and only if $\chi_M$ is an isomorphism. Theorem 3.1.4 of [**10**] states that $Q\text{-codomdim}_{(A,R)} \geqslant n$ for $n \geqslant 2$ if and only if $\chi_M$ is an isomorphism and $Q \otimes_B -$ is exact on the first part of a projective resolution of $\operatorname{Hom}_A(Q, M)$ as a $B$-module.

Section 3.1 of [**10**] describes several consequences; we will use especially [**10**, Corollary 3.1.5, Corollary 3.1.8, Proposition 3.1.11 and Corollary 3.1.12].

LEMMA 2.1. *Assume $M \in A\text{-mod} \cap R\text{-proj}$, and let $Q_i \in \operatorname{add} Q$. An exact sequence*

$$0 \to M \xrightarrow{\alpha_0} Q_0 \xrightarrow{\alpha_1} Q_1 \to \cdots \to Q_t$$

*remains exact under $\operatorname{Hom}_A(-, Q)$ if and only if for every factorization $Q_i \to \operatorname{im} \alpha_{i+1} \to Q_{i+1}$ of $\alpha_{i+1}$, the $(A, R)$-monomorphism $\operatorname{im} \alpha_{i+1} \to Q_{i+1}$ and $\alpha_0$ are left $\operatorname{add} Q$-approximations.*

*Proof.* See [**10**, Lemma 2.1.4]. □

In addition to the assumptions on $R$, $A$, and $Q$, in the following, we also assume that $\operatorname{Hom}_A(Q, Q) \in R\text{-proj}$.

It is crucial to compare relative dominant dimensions for end terms of a short exact sequence which remains exact under $\operatorname{Hom}_A(-, Q)$. This is completely described in [**10**, Lemma 3.1.7], part of it is as follows.

LEMMA 2.2. *Let $M \in A\text{-mod} \cap R\text{-proj}$ and consider an $(A, R)$-exact sequence*

$$0 \to M_1 \to M \to M_2 \to 0$$

*which remains exact under $\operatorname{Hom}_A(-, Q)$. Let $n = Q\text{-domdim}_A M$ and $n_i = Q\text{-domdim}_A M_i$ for $i = 1, 2$, then:*

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(a) $n \geqslant \min\{n_1, n_2\}$.

(b) *If* $n = \infty$ *and* $n_1 < \infty$ *then* $n_2 = n_1 - 1$.

COROLLARY 2.3. *Let* $M_i$ *for* $i \in I$ *be a finite set of modules in* $A\text{-mod} \cap R\text{-proj}$. *Then*

$$Q\text{-domdim}_A \left( \bigoplus_{i \in I} M_i \right) = \inf\{Q\text{-domdim}_A M_i \mid i \in I\}.$$

*Proof.* See [**10**, Corollary 3.1.8]. □

Recall ${}^{\perp}Q = \{M \in A\text{-mod} \cap R\text{-proj} \mid \mathrm{Ext}_A^{i>0}(M, Q) = 0\}$. The following is proved in [**10**, Proposition 3.1.11].

PROPOSITION 2.4. *Assume* $\mathrm{Ext}_A^{i>0}(Q, Q) = 0$, *and* $M \in {}^{\perp}Q$. *An exact sequence*

$$0 \to M \to Q_1 \to \cdots \to Q_n$$

*yields* $Q\text{-domdim}_{(A,R)}(M) \geqslant n$ *if and only if* $Q_i \in \mathrm{add}\, Q$ *and the cokernel of* $Q_{n-1} \to Q_n$ *belongs to* ${}^{\perp}Q$.

The following is an application of lemma 2.2.

COROLLARY 2.5. *Assume* $Q \in {}^{\perp}Q$. *Let* $M \in A\text{-mod} \cap R\text{-proj}$, *and consider an* $(A, R)$*-exact sequence:*

$$0 \to M \to Q_1 \to \cdots \to Q_t \to X \to 0$$

*with* $Q_i \in \mathrm{add}\, Q$ *and* $t \in \mathbb{N}$. *If* $\mathrm{Ext}_A^i(X, Q) = 0$ *for* $1 \leqslant i \leqslant t$, *then*

$$Q\text{-domdim}_{(A,R)} M = t + Q\text{-domdim}_{(A,R)} X.$$

*Proof.* See [**10**, Corollary 3.1.12]. □

### 2.4. Split quasi-hereditary algebras with duality

For the definition and general properties of split quasi-hereditary algebras we refer to [**5**, **9**–**11**, **49**]. In particular, we follow the notation of [**9**–**11**]. One of the advantages to use such setup stems from the fact that split quasi-hereditary $R$-algebras $(A, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ are exactly the algebras so that $(S \otimes_R A, \{S \otimes_R \Delta(\lambda)_{\lambda \in \Lambda}\})$ are quasi-hereditary algebras for every commutative Noetherian ring $S$ which is an $R$-algebra. Concerning the terminology, we remark the word split arises from the endomorphism algebra $\mathrm{End}_A(\Delta(\lambda))$ being isomorphic to the ground ring $R$. As it was observed in [**49**], when $(A, \{\Delta(\lambda)_{\lambda \in \Lambda}\})$ is a split quasi-hereditary $R$-algebra, the objects $T(\lambda)$ satisfying $\mathrm{add} \bigoplus_{\lambda \in \Lambda} T(\lambda) = \mathcal{F}(\tilde{\Delta}) \cap \mathcal{F}(\tilde{\nabla})$ are no longer unique, in contrast to quasi-hereditary algebras over a field. For this reason, we will say that $T$ is a characteristic tilting module of $A$ if $\mathrm{add}\, T = \mathcal{F}(\tilde{\Delta}) \cap \mathcal{F}(\tilde{\nabla})$ and $T$ is the (basic) characteristic tilting module of $A$ if $A$ is a quasi-hereditary algebra over a field and $T = \bigoplus_{\lambda \in \Lambda} T(\lambda)$.

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The following prepares the ground for quasi-hereditary covers, constructed from the Ringel dual $R(A) = \mathrm{End}_A(T)^{op}$ of a quasi-hereditary algebra $A$ with a characteristic tilting module $T$. To see that Ringel duality is well defined in the integral setup, we refer to [**10**, Subsection 2.2.3].

PROPOSITION 2.6. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary $R$-algebra with a characteristic tilting module $T$. Denote by $R(A)$ the Ringel dual $\mathrm{End}_A(T)^{op}$ of $A$. Suppose that $Q \in \mathrm{add}_A T$ is a partial tilting module. Then,*

(i) $\mathrm{Hom}_A(T, Q)$-$\mathrm{codomdim}_{(R(A),R)}\, DT = Q$-$\mathrm{domdim}_{(A,R)}\, T$.

(ii) $DQ$-$\mathrm{domdim}\,(A, R) = Q$-$\mathrm{codomdim}_{(A,R)}\, DA = Q$-$\mathrm{domdim}\,(A, R)$.

(iii) $\inf\{Q\text{-}\mathrm{codomdim}_{(A,R)}\, M : M \in \mathcal{F}(\tilde{\nabla})\} = Q$-$\mathrm{codomdim}_{(A,R)}\, T$.

*Proof.* For (i), see [**10**, Proposition 6.1.1]. For (ii), see [**10**, Corollary 3.1.5]. For (iii), see [**10**, Theorem 5.3.1]. □

Recall that $\mathrm{Hom}_A(T, DA) \simeq DT$ is a characteristic tilting module over $R(A)$. Relative codominant dimension can also be used to observe that a characteristic tilting module behaves like a flat module in $\mathcal{F}(\tilde{\Delta}_{R(A)})$.

PROPOSITION 2.7. *The functor $\mathrm{Hom}_A(T, -) : \mathcal{F}(\tilde{\nabla}) \to \mathcal{F}(\tilde{\Delta}_{R(A)})$ is an exact equivalence with quasi-inverse the exact functor $T \otimes_{R(A)} - : \mathcal{F}(\tilde{\Delta}_{R(A)}) \to \mathcal{F}(\tilde{\nabla})$.*

*Proof.* The fact that the functor $\mathrm{Hom}_A(T, -) : \mathcal{F}(\tilde{\nabla}) \to \mathcal{F}(\tilde{\Delta}_{R(A)})$ is an exact equivalence follows from Ringel duality (see e.g. [**10**, 2.2.3], or [**48**, Theorem 6] in the field case).

By proposition 2.6, $T$-$\mathrm{codomdim}_{(A,R)}\, M \geqslant T$-$\mathrm{codomdim}_{(A,R)}\, T = +\infty$ for every $M \in \mathcal{F}(\tilde{\nabla})$. By Theorem 3.1.4 of [**10**], $T \otimes_{R(A)} \mathrm{Hom}_A(T, M) \to M$ is an isomorphism and $\mathrm{Tor}^A_{i>0}(T, \mathrm{Hom}_A(T, M)) = 0$ for every $M \in \mathcal{F}(\tilde{\nabla})$. Since $\mathrm{Hom}_A(T, -) : \mathcal{F}(\tilde{\nabla}) \to \mathcal{F}(\tilde{\Delta}_{R(A)})$ is essentially surjective, the result follows. □

PROPOSITION 2.8. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra over a field $k$. Assume that there exists a simple preserving duality ${}^{\diamond}(-) : A\text{-mod} \to A\text{-mod}$. Let $T$ be the characteristic tilting module of $A$ and assume that $Q \in \mathrm{add}_A T$. Then,*

(i) ${}^{\diamond}\Delta(\lambda) \simeq \nabla(\lambda)$ *for all* $\lambda \in \Lambda$*;*

(ii) ${}^{\diamond}T(\lambda) \simeq T(\lambda)$ *for all indecomposable direct summands of* $T$*;*

(iii) $Q$-$\mathrm{domdim}_{(A,R)}\, T = Q$-$\mathrm{codomdim}_{(A,R)}\, T$.

*Proof.* (i) and (ii) follow by applying the simple preserving duality to the canonical exact sequences defining $\Delta(\lambda)$ and $T(\lambda)$, respectively. For (iii), see [**10**, Proposition 3.1.6]. □

Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra over a field $k$. The $\nabla$-*filtration dimension* of $X$, denoted by $\dim_{\mathcal{F}(\nabla)} X$, is the infimum of all $n \geqslant 0$

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such that there exists an exact sequence:

$$0 \to X \to M_0 \to \cdots \to M_n \to 0$$

with $M_0, \ldots, M_n \in \mathcal{F}(\nabla)$. Analogously, the $\Delta$-filtration dimension is defined. The $\nabla$-filtration dimensions first appeared in [**31**] in the study of cohomology of algebraic groups.

$\nabla$- and $\Delta$-filtration dimensions play a crucial role in [**25**, **45**] establishing that the global dimension of a quasi-hereditary algebra having a simple preserving duality is always an even number. For us, they are of importance due to the following result.

PROPOSITION 2.9. *Let* $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *be a split quasi-hereditary algebra over a field* $k$. *Assume that there exists a simple preserving duality* ${}^{\diamond}(-)\colon A\text{-mod} \to A\text{-mod}$. *If* $M \in A\text{-mod}$ *satisfies* $\dim_{\mathcal{F}(\Delta)} M = t < +\infty$, *then* $\operatorname{Ext}_A^{2t}(M, {}^{\diamond}M) \neq 0$.

*Proof.* See [**45**, Corollary 6]. □

### 2.5. Cover theory

The concept of a cover, and in particular, of a split quasi-hereditary cover was introduced in [**49**] to give an abstract framework to connections in representation theory like Schur–Weyl duality. Given a split quasi-hereditary algebra $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ over a commutative Noetherian ring $R$ and a finitely generated projective $A$-module $P$, let $B := \operatorname{End}_A(P)^{op}$. We say that $(A, P)$ is a *split quasi-hereditary cover* of $B$ if the restriction of the functor

$$F := \operatorname{Hom}_A(P, -)\colon A\text{-mod} \to B\text{-mod},$$

known as *Schur functor*, to $A$-proj is fully faithful. Given, in addition, $i \in \mathbb{N} \cup \{-1, 0, +\infty\}$, following the notation of [**9**], we say that $(A, P)$ is an *$i$-$\mathcal{F}(\tilde{\Delta})$ (quasi-hereditary) cover* of $B$ if the following conditions hold:

- $(A, P)$ is a split quasi-hereditary cover of $\operatorname{End}_A(P)^{op}$;
- The restriction of $F$ to $\mathcal{F}(\tilde{\Delta})$ is faithful;
- The Schur functor $F$ induces bijections $\operatorname{Ext}_A^j(M, N) \simeq \operatorname{Ext}_B^j(FM, FN)$, for every $M, N \in \mathcal{F}(\tilde{\Delta})$ and $0 \leqslant j \leqslant i$.

Here, $\mathcal{F}(\tilde{\Delta})$ denotes the resolving subcategory of $A\text{-mod} \cap R\text{-proj}$ whose modules admit a finite filtration into direct summands of direct sums of standard modules $\Delta(\lambda)$, $\lambda \in \Lambda$.

The optimal value of the quality of a cover is known as the *Hemmer–Nakano dimension*. More precisely, if $(A, P)$ is a $(-1)$-$\mathcal{F}(\tilde{\Delta})$ (quasi-hereditary) cover of $B$, the Hemmer–Nakano dimension of $\mathcal{F}(\tilde{\Delta})$ with respect to $F$ is $i \in \mathbb{N} \cup \{-1, 0, +\infty\}$ if $(A, P)$ is an $i$-$\mathcal{F}(\tilde{\Delta})$ (quasi-hereditary) cover of $\operatorname{End}_A(P)^{op}$ but $(A, P)$ is not an $(i+1)$-$\mathcal{F}(\tilde{\Delta})$ (quasi-hereditary) cover of $B$. The Hemmer–Nakano dimension of $\mathcal{F}(\tilde{\Delta})$ is denoted by $\operatorname{HNdim}_F(\mathcal{F}(\tilde{\Delta}))$.

Major tools to compute Hemmer–Nakano dimensions are classical dominant dimension and relative dominant dimensions. This can be traced back to [**29**] which was later amplified in several directions in [9] and in [10], and it is briefly summarized in the following result proved in [10, Theorem 5.3.1, Corollary 5.3.4]. Note that $\mathrm{Hom}_A(T, Q)$ is projective as a $B$-module.

THEOREM 2.10. *Let $R$ be a commutative Noetherian ring. Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary $R$-algebra with a characteristic tilting module $T$. Denote by $R(A)$ the Ringel dual $\mathrm{End}_A(T)^{op}$ of $A$. Assume that $Q \in \mathrm{add}\, T$ is a (partial) tilting module of $A$. Then, the following assertions hold.*

(a) *If $Q$-$\mathrm{codomdim}_{(A,R)}\, T \geqslant n \geqslant 2$, then $(R(A), \mathrm{Hom}_A(T, Q))$ is an $(n-2)$-$\mathcal{F}(\tilde{\Delta}_{R(A)})$ split quasi-hereditary cover of $\mathrm{End}_A(Q)^{op}$.*

(b) *Assume, in addition, that $R$ is a field. Then, $Q$-$\mathrm{codomdim}_{(A,R)}\, T \geqslant n \geqslant 2$ if and only if $(R(A), \mathrm{Hom}_A(T, Q))$ is an $(n-2)$-$\mathcal{F}(\tilde{\Delta}_{R(A)})$ split quasi-hereditary cover of $\mathrm{End}_A(Q)^{op}$.*

Let $B$ be a projective Noetherian $R$-algebra, $(A, P)$ be an $(-1)$-$\mathcal{F}(\tilde{\Delta})$ (quasi-hereditary) cover of $B$ and $(A', P')$ be an $(-1)$-$\mathcal{F}(\tilde{\Delta}')$ (quasi-hereditary) cover of $B$. We say that $(A, P)$ *is equivalent to* $(A', P')$ *as quasi-hereditary covers* if there exists an equivalence $H\colon A\text{-mod} \to A'\text{-mod}$ which restricts to an equivalence of categories between $\mathcal{F}(\tilde{\Delta})$ and $\mathcal{F}(\tilde{\Delta}')$ making the following diagram commutative:

$$\begin{array}{ccc} A\text{-mod} & \xrightarrow{\mathrm{Hom}_A(P,-)} & B\text{-mod} \\ \downarrow H & & \downarrow L \\ A'\text{-mod} & \xrightarrow{\mathrm{Hom}_{A'}(P',-)} & B\text{-mod} \end{array},$$

for some equivalence of categories $L$. The first application of uniqueness of covers goes back to [49]. Split quasi-hereditary covers with higher values of Hemmer–Nakano dimension associated with them are essentially unique. In fact, this is due to the following result which can be found in [9, Corollary 4.3.6].

COROLLARY 2.11. *Let $B$ be a projective Noetherian $R$-algebra, $(A, P)$ be a 1-$\mathcal{F}(\tilde{\Delta})$ (quasi-hereditary) cover of $B$ and $(A', P')$ be a 1-$\mathcal{F}(\tilde{\Delta}')$ (quasi-hereditary) cover of $B$. If there exists an exact equivalence $L\colon B\text{-mod} \to B\text{-mod}$ which restricts to an exact equivalence between $\mathcal{F}(\mathrm{Hom}_A(P, -)\tilde{\Delta})$ and $\mathcal{F}(\mathrm{Hom}_{A'}(P', -)\tilde{\Delta}')$, then $(A, P)$ is equivalent as split quasi-hereditary cover to $(A', P')$.*

For a more detailed exposition on cover theory and Hemmer–Nakano dimensions we refer to [9, 10].

## 3. The first main result

The aim of this section is to prove theorem A, that is, to prove the following.

THEOREM 3.1. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra over a field $k$. Suppose that there exists a simple preserving duality ${}^{\natural}(-)\colon A\text{-mod} \to A\text{-mod}$. Let $T$ be a characteristic tilting module of $A$. Assume that $Q \in \mathrm{add}\, T$. Then,*

$$Q\text{-}\mathrm{domdim}\, A = Q\text{-}\mathrm{codomdim}_A\, DA = 2 \cdot Q\text{-}\mathrm{codomdim}_A\, T = 2 \cdot Q\text{-}\mathrm{domdim}_A\, T. \tag{3.1}$$

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### 3.1. Outline of the proof

The proof will be divided in two parts: $\leqslant$, $\geqslant$. The easier part is $\geqslant$, that is, to prove that:

$$Q\text{-domdim}_A A \geqslant 2 \cdot Q\text{-domdim}_A T \text{ or equivalently}$$
$$Q\text{-codomdim}_A DA \geqslant 2 \cdot Q\text{-codomdim}_A T,$$

the latter being a particular case of lemma 3.4. For this inequality, there are two main ideas. First, we construct (using Ringel duality) an exact sequence:

$$0 \to \overline{C} \to \overline{X_n} \to \cdots \to \overline{X_1} \to DA \to 0 \tag{3.2}$$

with the following properties:

- $n = Q\text{-domdim}_A T$;
- $\overline{C} \in \mathcal{F}(\nabla)$ and $\overline{X_i} \in \operatorname{add} Q$.

Hence, (3.2) remains exact under $\operatorname{Hom}_A(T, -)$ and under $\operatorname{Hom}_A(Q, -)$. Second, we use the formula:

$$Q\text{-codomdim}_A T = \inf_{M \in \mathcal{F}(\nabla)} Q\text{-codomdim}_A M.$$

On the other hand, to establish the part $\leqslant$, that is, to prove that $Q\text{-codomdim}_A DA$ cannot be larger than $2Q\text{-codomdim}_A T = 2n$, there are two key steps:

(1) From any exact sequence $\overline{X_{n+1}} \to \overline{X_n} \to \cdots \to \overline{X_1} \to DA \to 0$ that gives $Q\text{-codomdim}_A DA \geqslant n+1$ we can infer that the kernel of $\overline{X_{n+1}} \to \overline{X_n}$, denoted here by $E$ (with $\overline{X_0} := DA$ if $n = 0$) satisfies $\operatorname{Ext}_A^{i\geqslant n+2}(X, E) = 0$ for every $X \in {}^{\perp}Q$;

(2) Proposition 2.9 provides a method to use the simple preserving duality functor to construct a module $X \in {}^{\perp}Q$ that satisfies $\operatorname{Ext}_A^{n+2}(X, E) \neq 0$ if $Q\text{-codomdim}_A DA > 2n$.

### 3.2. Relative injective dimension

Before we start with the proof of theorem A we need to discuss the concept of relative injective dimension.

Definition 3.2. *Let $\mathcal{A}$ be a full subcategory of $A$-mod. We define the $\mathcal{A}$-injective dimension of $N \in A$-mod (or the relative injective dimension of $N$ with respect to $\mathcal{A}$) as the value:*

$$\inf\{n \in \mathbb{N} \cup \{0\} \colon \operatorname{Ext}_A^{i>n}(M, N) = 0, \forall M \in \mathcal{A}\}. \tag{3.3}$$

*We denote by $\operatorname{idim}_{\mathcal{A}} N$ the $\mathcal{A}$-injective dimension of $N$. Analogously, we define the $\mathcal{A}$-projective dimension of $N \in A$-mod as the value:*

$$\inf\{n \in \mathbb{N} \cup \{0\} \colon \operatorname{Ext}_A^{i>n}(N, M) = 0, \forall M \in \mathcal{A}\}. \tag{3.4}$$

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Lemma 3.3. *Let $A$ be a projective Noetherian $R$-algebra and let $Q \in A$-mod such that $\operatorname{Ext}_A^{i>0}(Q,Q)=0$. Then, the following assertions hold.*

*(1) If there exists an exact sequence $0 \to X \to Y \to Z \to 0$ with $Y \in \operatorname{add} Q$, then $\operatorname{idim}_{{}^\perp Q} X \leqslant 1 + \operatorname{idim}_{{}^\perp Q} Z$.*

*(2) If there exists an exact sequence $0 \to X \to X_r \to \cdots \to X_1 \to Z \to 0$ with $X_1, \ldots, X_r \in \operatorname{add} Q$, then $\operatorname{idim}_{{}^\perp Q} X \leqslant r + \operatorname{idim}_{{}^\perp Q} Z$.*

*(3) If there exists an exact sequence $0 \to X \to X_r \to \cdots \to X_1 \to Z \to 0$ with $X_1, \ldots, X_r \in \operatorname{add} Q$, then for every $Y \in Q^\perp$, $\operatorname{Ext}_A^i(X,Y) \simeq \operatorname{Ext}_A^{i+r}(Z,Y)$ for all $i \in \mathbb{N}$.*

*Proof.* For each $M \in {}^\perp Q$, applying $\operatorname{Hom}_A(M,-)$ yields that $\operatorname{Ext}_A^i(M,Z) \simeq \operatorname{Ext}_A^{i+1}(M,X)$ for all $i \geqslant 1$. Hence, (i) follows. By induction and using (i), (ii) follows. Denote by $C_i$ the image of $X_{i+1} \to X_i$ for all $i = 1, \ldots, r-1$. By applying $\operatorname{Hom}_A(-,Y)$ we deduce that $\operatorname{Ext}_A^i(X,Y) \simeq \operatorname{Ext}_A^{i+1}(C_{r-1},Y) \simeq \operatorname{Ext}_A^{i+2}(C_{r-2},Y) \simeq \operatorname{Ext}_A^{i+r-1}(C_1,Y) \simeq \operatorname{Ext}_A^{i+r}(Z,Y)$. □

### 3.3. Computing relative dominant dimension of the regular module using a characteristic tilting module

Let $T$ be a characteristic tilting module of a split quasi-hereditary algebra $A$ and let $Q$ be a partial tilting module. By proposition 2.6(ii)–(iii), the relative codominant dimension of $T$ with respect to $Q$ is a lower bound to the relative dominant dimension of $A$ with respect to $Q$. In the following we will see that this lower bound can be sharpened.

Lemma 3.4. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary $R$-algebra with a characteristic tilting module $T$. Denote by $R(A)$ the Ringel dual $\operatorname{End}_A(T)^{op}$ of $A$. Suppose that $Q \in \operatorname{add} T$. Then,*

$$Q\text{-domdim}(A,R) \geqslant Q\text{-domdim}_{(A,R)}\, T + Q\text{-codomdim}_{(A,R)}\, T. \tag{3.5}$$

*Proof.* Observe that $DQ \otimes_A Q \in R$-proj (see e.g. [9, A.4.3]). By proposition 2.6(ii) and (iii), we obtain that:

$$Q\text{-domdim}_{(A,R)}\, A = Q\text{-codomdim}_{(A,R)}\, DA \geqslant Q\text{-codomdim}_{(A,R)}\, T. \tag{3.6}$$

If $Q\text{-domdim}_A T = 0$, then there is nothing more to prove. Assume that $n := Q\text{-domdim}_{(A,R)} T \geqslant 1$. By proposition 2.6(i), $\operatorname{Hom}_A(T,Q)\text{-codomdim}_{R(A)} DT = n$. That is, there exists an exact sequence:

$$0 \to C \to X_n \to \cdots \to X_1 \to DT \to 0 \tag{3.7}$$

with all $X_i \in \operatorname{add}(\operatorname{Hom}_A(T,Q))$, and so they are projective modules over $R(A)$. The modules $DT$ and $X_i$ are in $\mathcal{F}(\tilde{\Delta}_{R(A)})$, which is closed under kernels of epimorphisms and hence also $C$ is in $\mathcal{F}(\tilde{\Delta}_{R(A)})$.

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By the Ringel dual equivalence described in proposition 2.7, the functor $T \otimes_{R(A)} -$ takes (3.7) to the exact sequence:

$$0 \to \overline{C} \to \overline{X_n} \to \cdots \to \overline{X_1} \to T \otimes_{R(A)} DT \to 0, \tag{3.8}$$

and $\overline{X_i} \in \operatorname{add} Q$ since $T \otimes_{R(A)} \operatorname{Hom}_A(T, Q) = \operatorname{add} Q$. Moreover,

$$T \otimes_{R(A)} DT \simeq T \otimes_{R(A)} \operatorname{Hom}_A(T, DA) \simeq DA$$

and $\overline{C} \in \mathcal{F}(\tilde{\nabla})$. Thus, the exact sequence (3.8) remains exact under $\operatorname{Hom}_A(Q, -)$. By the dual version of corollary 2.5, we obtain that

$$Q\text{-codomdim}_{(A,R)} DA = n + Q\text{-codomdim}_{(A,R)} \overline{C}.$$

Applying again proposition 2.6(iii) gives that $Q\text{-codomdim}_{(A,R)} \overline{C} \geqslant Q\text{-codomdim}_{(A,R)} T$. □

Remark 3.5. An alternative proof to lemma 3.4 would be to repeat the steps of lemma 3.4 but only for quasi-hereditary algebras over a field. Then, since relative dominant dimension of modules with a standard filtration with respect to direct summands of a characteristic tilting module remains invariant under change of ground ring, the result would follow for split quasi-hereditary algebras with arbitrary commutative Noetherian ground ring.

Surprisingly, theorem 3.1 generalizes [**29**, Theorem 4.3] without using any techniques on symmetric algebras. In particular, for the following proof we do not need to require $A$ to be an endomorphism algebra of a generator over a symmetric algebra.

***Proof of theorem 3.1.*** By proposition 2.6(ii), proposition 2.8, and lemma 3.4, part $\geqslant$ follows. So, it remains to show that

$$Q\text{-codomdim}_A DA \leqslant 2 \cdot Q\text{-codomdim}_A T.$$

If $Q\text{-codomdim}_A T = +\infty$, then there is nothing to prove. Denote by $n$ the value $Q\text{-domdim}_A T = Q\text{-codomdim}_A T$. Assume first that $n > 0$. So, we can consider again exact sequences of the form (3.7) and (3.8). Assume, for a contradiction, that $Q\text{-codomdim}_A DA > 2n$. Hence, also $Q\text{-codomdim}_A \overline{C} > n$ according to the dual of corollary 2.5. So, there exists an exact sequence:

$$0 \to L \to \overline{X_{2n+1}} \to \overline{X_{2n}} \to \cdots \to \overline{X_{n+1}} \to \overline{C} \to 0 \tag{3.9}$$

which remains exact under $\operatorname{Hom}_A(Q, -)$ and $\overline{X_i} \in \operatorname{add} Q$, $i = n+1, \ldots, 2n+1$. In particular, $0 \to L \to \overline{X_{2n+1}} \to \cdots \to \overline{X_1} \to DA \to 0$ is an $\operatorname{Hom}_A(Q, -)$-acyclic coresolution of $L$, so it can be used to compute $\operatorname{Ext}_A^i(Q, L)$ for all $i$. Since it remains exact under $\operatorname{Hom}_A(Q, -)$ we obtain that $\operatorname{Ext}_A^{i>0}(Q, L) = 0$ and so $L \in Q^{\perp}$ and ${}^{\diamond}L \in {}^{\perp}Q$. Let $E$ be the kernel of the map $\overline{X_{n+1}} \to \overline{C}$ and consider the exact sequence:

$$0 \to E \to \overline{X_{n+1}} \to \overline{X_n} \to \cdots \to \overline{X_1} \to DA \to 0. \tag{3.10}$$

By lemma 3.3(2), $\operatorname{idim}_{{}^{\perp}Q} E \leqslant n + 1$. Since (3.10) remains exact under $\operatorname{Hom}_A(Q, -)$ we have that $E \in Q^{\perp}$. On the other hand, observe that $E$ cannot belong to $\mathcal{F}(\nabla)$

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because otherwise (3.10) would remain exact under $\operatorname{Hom}_A(T,-)$ yielding that $n < \operatorname{Hom}_A(T,Q)\text{-codomdim}_{R(A)} DT$ contradicting the definition of $n$.

So, the exact sequence $0 \to E \to \overline{X_{n+1}} \to \overline{C} \to 0$ yields that $\dim_{\mathcal{F}(\nabla)} E = 1$. Hence, $\dim_{\mathcal{F}(\Delta)} {}^{\diamond}E = 1$. By proposition 2.9, we obtain that $0 \neq \operatorname{Ext}_A^2({}^{\diamond}E, {}^{\diamond\diamond}E) \simeq \operatorname{Ext}_A^2({}^{\diamond}E, E)$. By lemma 3.3(3) on the exact sequence $0 \to {}^{\diamond}E \to {}^{\diamond}\overline{X_{n+2}} \to \cdots \to {}^{\diamond}\overline{X_{2n+1}} \to {}^{\diamond}L \to 0$ we obtain $0 \neq \operatorname{Ext}_A^2({}^{\diamond}E, E) \simeq \operatorname{Ext}_A^{n+2}({}^{\diamond}L, E)$. This contradicts $\operatorname{idim}_{{}^{\perp}Q} E$ being at most $n+1$. We will now treat the case $n = 0$. Assume, for the sake of contradiction, that $Q\text{-codomdim}_A DA \geqslant 1$, then there exists an exact sequence $0 \to L \to X_1 \to DA \to 0$ which remains exact under $\operatorname{Hom}_A(Q,-)$ and $X_1 \in \operatorname{add} Q$. Hence, $L \in Q^{\perp}$, ${}^{\diamond}L \in {}^{\perp}Q$, $\dim_{\mathcal{F}(\nabla)}(L) \leqslant 1$ and the ${}^{\perp}Q$-injective dimension of $L$ is at most one. In particular, $\operatorname{Ext}_A^2({}^{\diamond}L, L) = 0$. By proposition 2.9, we must have that $L \in \mathcal{F}(\nabla)$. But, then applying $\operatorname{Hom}_A(T,-)$ to $0 \to L \to X_1 \to DA \to 0$ yields that $\operatorname{Hom}_A(T,Q)\text{-codomdim}_{R(A)} \operatorname{Hom}_A(T, DA) \geqslant 1$ which, in turn, implies that $n = Q\text{-domdim}\, T \geqslant 1$ by proposition 2.6(i). □

The following gives a positive answer to the Conjecture 6.2.4 of [**8**], which is a special case by taking $A$ to be a Schur algebra and $Q$ the tensor space $V^{\otimes d}$.

Proposition 3.6. *Let $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ be a split quasi-hereditary algebra over a field $k$ with a simple preserving duality. Let $T$ be a characteristic tilting module of $A$. Assume that $Q \in \operatorname{add} T$ satisfies $Q\text{-domdim}_A A \geqslant 2$. Then,*

$$Q\text{-domdim}_A A = 2 \cdot Q\text{-domdim}_A T \geqslant 2\, \operatorname{domdim}_A T = \operatorname{domdim} A. \tag{3.11}$$

*Proof.* Let $P$ be a faithful projective–injective module over $A$. By assumption, $Q\text{-domdim}_A A \geqslant 2$, so by corollary 2.3 it follows that $Q\text{-domdim}_A P \geqslant 2$. Since $P$ is injective we must have that $P \in \operatorname{add} Q$. Hence, $\operatorname{Hom}_A(T,P) \in \operatorname{add} \operatorname{Hom}_A(T,Q)$.

By proposition 2.6(i),

$$\begin{aligned} Q\text{-domdim}_A T &= \operatorname{Hom}_{R(A)}(T,Q)\text{-codomdim}_{R(A)} DT \\ &\geqslant \operatorname{Hom}_A(T,P)\text{-codomdim}_{R(A)} DT \\ &= P\text{-domdim}_A T = \operatorname{domdim}_A T. \end{aligned} \tag{3.12}$$

Applying theorem 3.1 to the partial tilting modules $P$ and $Q$, the result follows. □

Example 3.7. We illustrate the proof of theorem A, using the Schur algebra $S(2,4)$ over a field of characteristic 2. Its basic algebra $A$ is isomorphic to $KQ/I$ where $Q$ is the quiver

$$2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 0 \underset{\delta}{\overset{\gamma}{\rightleftarrows}} 4$$

and $I = \langle \beta\alpha, \gamma\delta, (\alpha\beta)\delta, \gamma(\alpha\beta)\rangle$ (see [**26**, 5.6]). (Here, we label vertices consistent with later sections.) The partial order is $4 > 2 > 0$, all standard modules are uniserial, $\Delta(2)$ has composition factors labelled $2, 0$ and $\Delta(4)$ has composition factors labelled $4, 0, 2$. Then, $T(4)$ and $T(2)$ are uniserial, with

$$0 \to \Delta(4) \to T(4) \to \Delta(2) \to 0, \quad 0 \to \Delta(2) \to T(2) \to \Delta(0) \to 0$$

We take $Q = T(4) \oplus T(2)$, then the characteristic tilting module is $Q \oplus T(0)$ and $T(0)$ is the simple module $L(0)$.

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Consider the sequences in the proof of lemma 3.4. First, sequence (3.7) is the start of a minimal projective resolution of $DT$ as a module for $R(A)$. We apply $T \otimes_{R(A)} (-)$ which gives sequence (3.8). We know that it is a resolution of $D(A)$ with terms in $\operatorname{add}(Q)$ which is exact under $\operatorname{Hom}(Q, -)$, and from this information we see that it is

$$0 \to \overline{C} \simeq T(0)^2 \to \overline{X_2} \to \overline{X_1} \to D(A) \to 0$$

(where $\overline{X_1} = T(4)^3 \oplus T(2)$ and $\overline{X_2} \simeq T(2)^2$). This shows, by the dual of corollary 2.5, that

$$Q\text{-codomdim}_A D(A) = 2 + Q\text{-codomdim}_A T(0).$$

In particular, $\operatorname{Hom}_A(T, Q)$-codomdim$_{R(A)} DT = 2$. By proposition 2.6(i), $n = Q\text{-domdim}_A T = 2$. In this case, $\overline{C}$ is already too simple and we know its relative codominant dimension using the simple preserving duality, thus this is enough already to deduce that $Q\text{-codomdim}_A DA = 4$.

Assume, for the moment, that at this point we did not know the value of $Q\text{-codomdim}_A \overline{C}$. We consider now the proof of theorem 3.1. The exact sequence (3.10) is precisely:

$$0 \to \Delta(2) \to T(2) \to \overline{X_2} \to \overline{X_1} \to DA \to 0,$$

where the map $T(2) \to \overline{X_2}$ factors through $T(0)$. Hence, $E = \Delta(2)$. Step (1), that is, (3.10) together with lemma 3.3(2) gives that $\operatorname{idim}_{{}^\perp Q} \Delta(2) \leqslant 3$. Indeed, the (absolute) injective dimension $\operatorname{idim}_A \Delta(2)$ is exactly 3, so $\operatorname{idim}_{{}^\perp Q} \Delta(2)$ must be at most 3. By the proof of theorem 3.1, if $Q\text{-codomdim}_A DA > 2 \cdot 2 = 4$, then proposition 2.9 through the object ${}^\diamond L$ would give that $\operatorname{idim}_{{}^\perp Q} \Delta(2) > 3$ which does not happen.

## 4. Input from Schur algebras

The main work to prove the second main result, to determine the Hemmer–Nakano dimension of $\mathcal{F}(\Delta)$ over the quasi-hereditary cover for the Temperley–Lieb algebra, in §6 and 7, will be done for Schur algebras, and we can work over an algebraically closed field. In this section, we give an outline of the background. To keep the notation simple, we do this for the classical case.

Assume $K$ is an algebraically closed field. The Schur algebra $S = S_K(n, d)$ (or just $S(n, d)$) of degree $d$ over $K$ can be defined in different ways. One can start with the symmetric group $\mathcal{S}_d$ which acts (on the right) by place permutations on the tensor power $V^{\otimes d}$ where $V$ is an $n$-dimensional vector space. Then, the *Schur algebra* $S(n, d)$ is the endomorphism algebra $\operatorname{End}_{K\mathcal{S}_d}(V^{\otimes d})$. Analogously, the *integral Schur algebra* $S_R(2, d)$ is defined as the endomorphism algebra $\operatorname{End}_{R\mathcal{S}_d}((R^2)^{\otimes d})$ where $(R^2)^{\otimes d}$ affords a right $R\mathcal{S}_d$-module structure via place permutations with $R$ being a commutative Noetherian regular ring (of positive Krull dimension). Alternatively one can construct $S(n, d)$ via the general linear group $GL(V)$, for details see for example [**33**, 2.3] or [**16**, Section 1]. The first route shows that the endomorphism algebra of $S(n, d)$ acting on $V^{\otimes d}$ is a quotient of $K\mathcal{S}_d$. The second approach allows one to use tensor products and Frobenius twists as tools to study representations.

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The Schur algebra $S(n,d)$ is quasi-hereditary, with respect to the dominance order on the set $\Lambda^+(n,d)$ of partitions of $d$ with at most $n$ parts, which is the standard labelling set for simple modules. It has a simple preserving duality ${}^{\diamond}(-)$ (see e.g. [20, p. 83]). For each partition $\lambda$ of $d$ with at most $n$ parts, the corresponding simple module will be denoted by $L(\lambda)$. We denote the standard module with simple top $L(\lambda)$ by $\Delta(\lambda)$, then the costandard module with simple socle $L(\lambda)$ is $\nabla(\lambda) = {}^{\diamond}\Delta(\lambda)$. For background we refer to [21, 22, 27].

Of central importance for the quasi-hereditary structure is the characteristic tilting module $T$: by [48] the indecomposable modules in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ are in bijection with the weights. Write $T(\lambda)$ for the indecomposable labelled by $\lambda \in \Lambda^+(n,d)$. Then, the direct sum $T := \bigoplus_{\lambda\in\Lambda^+(n,d)} T(\lambda)$ (or a module with the same indecomposable summands) is a distinguished tilting module, known as the *characteristic tilting module* of $S$. Its endomorphism algebra $R(S) := \mathrm{End}_S(T)^{op}$ is again quasi-hereditary and $R(R(S))$ is Morita equivalent (as quasi-hereditary algebra) to $S$.

For each $\lambda$, there is an associated exact sequence:

$$0 \to \Delta(\lambda) \to T(\lambda) \to X(\lambda) \to 0 \qquad (4.1)$$

where $X(\lambda)$ has $\Delta$-filtration where only $\Delta(\mu)$ with $\mu < \lambda$ occur. We will refer to this as a standard sequence.

We follow the usual practice in algebraic Lie theory to refer to a module in $\mathrm{add}(T)$ as a tilting module, and to $T$ as a full tilting module (this will not be ambiguous here).

For the connection between Schur algebras and symmetric groups, the tensor space $V^{\otimes d}$ is of central importance. As it happens, the tensor space is a direct sum of tilting modules, and $T(\lambda)$ occurs as a summand if and only if $\lambda$ is $p$-regular (i.e. does not have $p$ equal parts). For the quantum case, $T(\lambda)$ occurs in the tensor space if and only if $\lambda$ is $\ell$-regular where $q$ is a primitive $\ell$-th root of 1. This is proved in [27, 4.2], or combining the reasoning of [27, 4.2] with [20, 2.2(1), 4.3, 4.7] respectively. Hence, the following result has become folklore.

LEMMA 4.1. *Assume that $n = 2$ and $d$ is a natural number. If $\mathrm{char}\, K \neq 2$ or $d$ is odd, then $V^{\otimes d}$ is a characteristic tilting module over $S_K(2,d)$.*

*Proof.* If $K$ has characteristic zero, then the Schur algebra $S(2,d)$ is semi-simple (see e.g. [33, (2.6)e]) and since $V^{\otimes d}$ is faithful over $S(2,d)$ it contains the regular module in its additive closure, and in particular, $V^{\otimes d}$ is a characteristic tilting module. If $K$ has positive characteristic, as discussed before $V^{\otimes d}$ is a characteristic tilting module over $S(2,d)$ if and only if all partitions of $d$ in at most 2 parts are $\mathrm{char}\, K$-regular partitions of $d$. Of course, all partitions of $d$ in at most 2 parts are $p$-regular if $p > 2$. If $d$ is odd, then there are no partitions of $d$ in exactly two equal parts. □

From now on we assume $n = 2$ and $\mathrm{char}\, K = 2$, or in the quantum case that $\ell = 2$. We also assume $d$ is even (unless specified differently).

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### 4.1. On the quasi-hereditary structure of $S(2, d)$

Let $S = S(2, d)$, and let $e = \xi_{(d)}$ be the idempotent corresponding to the largest weight (in the notation of [**33**, 3.2]). Then, $SeS$ is an idempotent heredity ideal and $S/SeS$ is isomorphic to $S(2, d-2)$ (for details see e.g. [**26**, 1.3]). Since $SeS$ is a heredity ideal corresponding to $(d)$, factoring it out is compatible with the quasi-hereditary structure. Furthermore, as it is proved in the appendix of [**22**] computing $\mathrm{Ext}^i$'s for $S/SeS$-modules is the same whether in $S$ or in $S/SeS$. In particular, $S(2, d)$-mod is the full subcategory of $S(2, d+2)$-mod consisting of modules whose composition factors are different from those appearing in the top of $Se$.

We work mostly with the restrictions of simple modules, (co)standard modules and tilting modules to $SL(2, K)$. Recall that $L(\lambda)$ and $L(\mu)$ are isomorphic as $SL(2, K)$-modules if and only if they can be regarded both as $S(2, d)$-modules for some large $d$ and they are isomorphic as $S(2, d)$-modules. This fact can be seen using the canonical surjective map of $KSL(2, K)$ onto $S(2, d)$. Since every partition $(\lambda_1, \lambda_2)$ of $d$ in at most two parts is completely determined by the value $\lambda_1 - \lambda_2$, it follows that $L(\lambda)$ and $L(\mu)$ are isomorphic as $SL(2, K)$-modules if and only if $\lambda_1 - \lambda_2 = \mu_1 - \mu_2$. Similarly, we may label the standard modules and tilting modules.

We therefore label these modules by $m = \lambda_1 - \lambda_2$ if $\lambda = (\lambda_1, \lambda_2)$ (such labellings can also be found e.g. in [**24**, Subsection 3.2]). This means that we consider Schur algebras $S = S(2, d)$, allowing degrees to vary but keeping the parity. We make the convention that we view tacitly modules for $S(2, d')$ with $d' \leqslant d$ of the same parity as modules for $S(2, d)$. We say that such a degree $d'$ is admissible for the module defined in degree $d$. With this, the weights labelling the simple modules for $S(2, d)$ are precisely all non-negative integers $m \leqslant d$ of the same parity. The dominance order when $p = 2$ and the degree is even, is the linear order.

The tilting module $T(0)$ is simple, it is the trivial module for $SL(2, K)$. As a building block, the tilting module $T(1)$ appears, which is isomorphic to the natural $SL(2, K)$-module $V$. Furthermore, $T(2) \cong V^{\otimes 2}$. For $d \geqslant 4$ we have that $V^{\otimes d}$ is the direct sum of $T(k)$ where all $T(k)$ occur for $k$ of the same parity of $d$, except that $T(0)$ does not occur when $d$ is even. (See e.g. [**27**, 4.2], it is implicitly in [**17**, 3.4].)

### 4.2. The category $\mathcal{F}(\Delta)$ and projective modules

Non-split extensions of standard modules satisfy a directedness property, that is:

$$\mathrm{Ext}^1_S(\Delta(r), \Delta(s)) \neq 0 \quad \text{implies} \quad r < s.$$

This has the following immediate consequence:

Lemma 4.2. *Every module in $\mathcal{F}(\Delta)$ has a filtration in which weights of $\Delta$-quotients increase from top to bottom.*

*Proof.* This follows for example from [**22**, Lemma 1.4], see also [**11**, B.0.6]. See [**38**], for an earlier reference. □

Of main interest for us are the indecomposable projective modules. Let $P_d(m)$ denote the indecomposable projective of $S(2, d)$ with simple quotient $L(m)$. Recall

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| | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | | | | | | | | | | | | | |
| 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | |
| 6 | 1 | . | 1 | 1 | | | | | | | | | | | | | | | | | | | | |
| 8 | 1 | . | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | |
| 10 | 1 | 1 | 1 | . | 1 | 1 | | | | | | | | | | | | | | | | | | |
| 12 | 1 | 1 | . | . | 1 | 1 | 1 | | | | | | | | | | | | | | | | | |
| 14 | 1 | . | . | . | 1 | . | 1 | 1 | | | | | | | | | | | | | | | | |
| 16 | 1 | . | . | . | 1 | . | 1 | 1 | 1 | | | | | | | | | | | | | | | |
| 18 | 1 | 1 | . | . | 1 | 1 | 1 | . | 1 | 1 | | | | | | | | | | | | | | |
| 20 | 1 | 1 | 1 | . | 1 | 1. | . | 1 | 1 | 1 | 1 | | | | | | | | | | | | | |
| 22 | 1 | . | 1 | 1 | 1 | . | . | . | 1 | . | 1 | 1 | | | | | | | | | | | | |
| 24 | 1 | . | 1 | 1 | . | . | . | . | 1 | . | 1 | 1 | 1 | | | | | | | | | | | |
| 26 | 1 | 1 | 1 | . | . | . | . | . | 1 | 1 | 1 | . | 1 | 1 | | | | | | | | | | |
| 28 | 1 | 1 | . | . | . | . | . | . | 1 | 1 | . | . | 1 | 1 | 1 | | | | | | | | | |
| 30 | 1 | . | . | . | . | . | . | . | 1 | . | . | . | 1 | . | 1 | 1 | | | | | | | | |
| 32 | 1 | . | . | . | . | . | . | . | 1 | . | . | . | 1 | . | 1 | 1 | 1 | | | | | | | |
| 34 | 1 | 1 | . | . | . | . | . | . | 1 | 1 | . | . | 1 | 1 | 1 | . | 1 | 1 | | | | | | |
| 36 | 1 | 1 | 1 | . | . | . | . | . | 1 | 1 | 1 | . | 1 | 1 | . | . | 1 | 1 | 1 | | | | | |
| 38 | 1 | . | 1 | 1 | . | . | . | . | 1 | . | 1 | 1 | 1 | . | . | . | 1 | . | 1 | 1 | | | | |
| 40 | 1 | . | 1 | 1 | 1 | . | . | . | 1 | . | 1 | 1 | . | . | . | . | 1 | . | 1 | 1 | 1 | | | |
| 42 | 1 | 1 | 1 | . | 1 | 1 | . | . | 1 | 1 | 1 | . | . | . | . | . | 1 | 1 | 1 | . | 1 | 1 | | |
| 44 | 1 | 1 | . | . | 1 | 1 | 1 | . | 1 | 1 | . | . | . | . | . | . | 1 | 1 | . | . | 1 | 1 | 1 | |
| 46 | 1 | . | . | . | 1 | . | 1 | 1 | 1 | . | . | . | . | . | . | . | 1 | . | . | . | 1 | . | 1 | 1 |

⋮

Figure 1. Decomposition matrix for $S(2,46)$ for $p=2$. The $(m,n)$-entry denotes $(\Delta(m):L(n))$, the column label is the same as the row label.

$P_d(m)$ has a $\Delta$-filtration, and that the filtration multiplicities $[P_d(m):\Delta(w)]$ are the same as the decomposition numbers. That is,

$$[P_d(m):\Delta(w)] = (\nabla(w):L(m)) = (\Delta(w):L(m)).$$

where we write $(M:L(m))$ for the multiplicity of $L(m)$ as a composition factor of the module $M$. Note this also shows that projective modules depend on the degree $d$. In this case, decomposition numbers are always 0 or 1, see [**34**, Prop. 2.2, Theorem 3.2]. We give an example in figure 1.

It follows that either $P_d(m) \cong P_{d-2}(m)$ as a module for $S(2,d)$, or else there is a non-split exact sequence:

$$0 \to \Delta(d) \to P_d(m) \to P_{d-2}(m) \to 0. \tag{4.2}$$

Namely, the top of $P_{d-2}(m)$ is $L(m)$, so there is a surjective homomorphism from $P_d(m)$ onto $P_{d-2}(m)$. Recall that $\mathcal{F}(\Delta)$ is closed under kernels of epimorphisms. By the filtration property in lemma 4.2 if this is not an isomorphism, then its kernel is a direct sum of copies of $\Delta(d)$ and there is only one since the decomposition numbers are $\leqslant 1$.

### 4.3. Twisted tensor product methods

Let $(-)^F$ denote the Frobenius twist (see [**20**, p. 64]), this is an exact functor. In our setting, that is for even characteristic, we have the following tools, due to [**17**]. Odd degrees when $p=2$ are less important. Namely, each block of $S(2,d)$ for $d$ odd is Morita equivalent to some block of some Schur algebra $S(2,x)$ with

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$x = (d-1)/2$ via the functor $\Delta(1) \otimes (-)^F$, see for example [23, Lemma 1] or [18, Section 4, Theorem].

(1) (a) Let $m = 2t$. There is an exact sequence of $S$-modules:

$$0 \to \Delta(t-1)^F \to \Delta(m) \to \Delta(t)^F \to 0$$

Taking contravariant duals gives the analogue for costandard modules.

(b) Let $m = 2t+1$, then $\Delta(m) \cong L(1) \otimes \Delta(t)^F$.
(See e.g. [6, Prop. 3.3]).

We note that this determines recursively the decomposition numbers, as input using that $\Delta(t)$ is simple and isomorphic to $L(t)$ for $t = 0, 1$, recall $p = 2$. This can also be used to show that when $p = 2$ and $d$ is even, the algebra $S(2, d)$ is indecomposable. Further, this also implies, by induction, that the decomposition numbers are always 0 and 1 when $p = 2$ and $d$ is even.

(2) We have a complete description of the indecomposable tilting modules in this case. We have already described $T(m)$ for $m \leqslant 2$. The following is due to Donkin, see [17, Example 2, p. 47].

Proposition 4.3. *Let $m = 2s$ and $m \geqslant 2$, then*

$$T(m) \cong T(2) \otimes T(s-1)^F$$

*If $m = 2s+1$, then $T(r) \cong T(1) \otimes T(s)^F$.*

This describes recursively all indecomposable tilting modules.
The following shows that filtration multiplicities $[T(m) : \Delta(w)]$ are $\leqslant 1$.

Proposition 4.4. *The $\Delta$-filtration multiplicities of indecomposable tilting modules in even degree can be computed recursively from:*

$$0 \to \Delta(2t+2) \to T(2) \otimes \Delta(t)^F \to \Delta(2t) \to 0.$$

To prove this, one may specialize [6, Prop. 3.4].
We will see below that modules $T(2) \otimes X^F$ for $X$ in $\mathcal{F}(\Delta)$ have infinite relative dominant dimension with respect to $V^{\otimes d}$. This means that we can use lemma 2.2 (from Lemma 3.1.7 of [10]) to relate the relative $V^{\otimes d}$-dominant dimension of the end terms, and this suggests a route towards the proof of our second main result.
We define a *twisted filtration* of a module $M \in \mathcal{F}(\Delta)$ to be a filtration where each quotient is isomorphic to $T(2) \otimes \Delta(t)^F$ for some $t$.

Lemma 4.5. *Let $m = 2s \geqslant 1$. Then, the tilting module $T(m)$ has a twisted filtration*

$$0 = M_k \subset M_{k-1} \subset \cdots \subset M_1 \subset M_0 = T(m)$$

*with $M_{i-1}/M_i \cong T(2) \otimes \Delta(s_i)^F$, with quotients $\Delta(2s_i)$ and $\Delta(2s_i+2)$, for $s_1 < s_2 < \cdots < s_k$.*

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*Proof.* We have $T(m) \cong T(2) \otimes T(s-1)^F$. The module $T(s-1)$ has a $\Delta$-filtration

$$N_k = 0 \subset N_{k-1} \subset \ldots \subset N_0 = T(s-1)$$

with $N_{i-1}/N_i \cong \Delta(s_i)$ and such that $s_1 < s_2 < \cdots < s_k$, by lemma 4.2. Applying the exact functor $T(2) \otimes (-)^F$ gives the claim. □

REMARK 4.6. (1) The algebra $S(2,d)$ is Ringel self-dual for $p = 2$ and $d$ even if and only if $d = 2^{n+1} - 2$ for some $n$, see [**23**, Theorem 27] and for a functorial proof see [**24**, 5.4]. In [**23**, Corollary 21] and [**24**, Section 5], the projective tilting modules (and also the injective tilting modules) have been identified for these degrees.

(2) Each $T(m)$ has a simple top, by proposition 4.3, using [**23**, Lemma 11], or [**24**, 3.4].

## 5. The relative dominant dimension of the regular module with respect to $V^{\otimes d}$

Let $S = S(2,d)$ and assume that $K$ has characteristic $p$. Recall that the indecomposable summands of $V^{\otimes d}$ are precisely the $T(\lambda)$ where $\lambda$ is a partition of $d$ with at most two parts, such that $\lambda$ does not have $p$ equal parts. Recall that we identify $\lambda$ with $m = \lambda_1 - \lambda_2$. Hence, unless $p = 2$ and $d$ is even, all indecomposable summands of $T$ occur in $V^{\otimes d}$, and then $V^{\otimes d}$-domdim$_S S = \infty$, by the following:

LEMMA 5.1. *If $V^{\otimes d}$ has all $T(\lambda)$ as direct summands, then*

$$\inf\{V^{\otimes d}\text{-domdim}_S M : M \in \mathcal{F}(\Delta)\} = +\infty.$$

*Proof.* Every module in $\mathcal{F}(\Delta)$ admits a finite $\operatorname{add} T$-coresolution (see e.g. [**48**, lemma 6] or [**15**]) which, in particular, remains exact under $\operatorname{Hom}_S(-, V^{\otimes d})$. By corollary 2.5, the result follows. □

### 5.1. The characteristic two case

Lemma 5.1 leaves us to consider $p = 2$ and $d$ even ($\neq 0$). In this case, as mentioned above, the components of $V^{\otimes d}$ are the $T(m)$ with $m \neq 0$. The standard sequence (4.1) is an $\operatorname{add} T$-approximation, this follows from a special case of proposition 2.4. In particular,

$$V^{\otimes d}\text{-domdim}_S \Delta(d) = 1 + V^{\otimes d}\text{-domdim}_S X(d). \tag{5.1}$$

THEOREM 5.2. *Let $d = 2s > 0$. We have*

$$V^{\otimes d}\text{-domdim}_S(\Delta(d)) = s + V^{\otimes d}\text{-domdim}_S(T(0)).$$

To prove this, we will use lemma 2.2 on extensions of $\Delta(t)$ by $\Delta(2+t)$.
The cases $d = 2$ and $d = 4$ are easy.

(1) For $d = 2$ we have the exact sequence $0 \to \Delta(2) \to T(2) \to \Delta(0) = T(0) \to 0$, which proves the statement of the theorem by corollary 2.5.

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(2) Let $d = 4$, we have the exact sequence $0 \to \Delta(4) \to T(4) \to \Delta(2) \to 0$. Splicing this with the sequence for $\Delta(2)$ gives the claim.

Degrees $d \geqslant 6$ need more work. The main ingredient is the observation that subquotients of the form $T(2) \otimes N^F$ with $N \in \mathcal{F}(\Delta)$ are not relevant for a minimal $V^{\otimes d}$-approximation.

Lemma 5.3. *Let $X \in S(2, s)$-mod. Assume that $X \in \mathcal{F}(\Delta)$. Then, the module $T(2) \otimes X^F$ has infinite relative dominant dimension with respect to $V^{\otimes d}$ for any even degree $d$ greater or equal to $2s + 2$.*

*Proof.* By lemma 2.2 it suffices to prove this when $X = \Delta(s)$. We proceed by induction on $s$. When $s = 0$ or $s = 1$ we see that $T(2) \otimes \Delta(s)^F$ is a summand of $V^{\otimes d}$. For the inductive step consider the exact sequence:

$$0 \to T(2) \otimes \Delta(s)^F \to T(2) \otimes T(s)^F \to T(2) \otimes X(s)^F \to 0.$$

The middle term is isomorphic to $T(2 + 2s)$. Since $X(s)$ has a filtration with quotients $\Delta(t)$ for $t < s$ it follows by induction (and lemma 2.2) that the module $T(2) \otimes X(s)^F$ has infinite $V^{\otimes d}$-dominant dimension for any even degree $d \geqslant 2s + 2$. We deduce that the module $T(2) \otimes \Delta(s)^F$ has infinite $V^{\otimes d}$-dominant dimension as well. □

***Proof of theorem 5.2***. Assume $d = 2s \geqslant 4$. By proposition 4.4, there exists $S(2, d)$-exact sequences

$$0 \to \Delta(2t + 2) \to T(2) \otimes \Delta(t)^F \to \Delta(2t) \to 0, \tag{5.2}$$

for every $0 \leqslant t \leqslant s - 1$. Moreover, they remain exact over $\mathrm{Hom}_{S(2,d)}(-, V^{\otimes d})$ since $V^{\otimes d}$ is a partial tilting module and so $\mathrm{Ext}^1_{S(2,d)}(\Delta(2t + 2), V^{\otimes d}) = 0$. By lemma 5.3, $T(2) \otimes \Delta(t)^F$ has infinite relative dominant dimension with respect to $V^{\otimes d}$ for $0 \leqslant t \leqslant s - 1$. By lemma 2.2,

$$V^{\otimes d}\text{-domdim}_{S(2,d)}\, \Delta(2t + 2) = 1 + V^{\otimes d}\text{-domdim}_{S(2,d)}\, \Delta(2t), \quad 0 \leqslant t \leqslant s - 1. \tag{5.3}$$

Hence, $V^{\otimes d}\text{-domdim}_{S(2,d)}\, \Delta(0) = s + V^{\otimes d}\text{-domdim}_{S(2,d)}\, \Delta(d)$. □

We will now determine the relative dominant dimension of $S(2, d)$ with respect to $V^{\otimes d}$. Let $P_d(m)$ be the indecomposable projective $S(2, d)$-module with homomorphic image $L(m)$. Throughout, $d$ and $m$ are even.

Lemma 5.4. *Consider a projective module $P_d(m)$ where $d$ and $m$ are even with $m < d$. Then, one of the following holds.*

(a) *The number of quotients in a $\Delta$-filtration of $P_d(m)$ is even and $P_d(m)$ has a twisted filtration.*

(b) *There is an exact sequence*

$$0 \to \Delta(d) \to P_d(m) \to P_{d-2}(m) \to 0$$

*and $P_{d-2}(m)$ has a twisted filtration.*

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*Proof.* Our strategy consists of proving that the projective module $P_d(m)$ is a quotient of a tilting module and then to combine this fact with lemma 4.5. Let $r_n := 2^{n+1} - 2$. There is a unique $n$ such that $r_{n-1} < d \leqslant r_n$. By our convention, we can view $P_d(m)$ as a module in degree $r_n$. Since it has a simple top isomorphic to $L(m)$, it is isomorphic to a quotient of $P_{r_n}(m)$.

By [**23**, Corollary 21] or [**24**, Section 5], we have the following.

(i) If $0 \leqslant m < (r_n)/2$, then $P_{r_n}(m)$ is a tilting module (in fact, it is isomorphic to $T(r_n - m)$).

(ii) For $(r_n)/2 < m \leqslant r_n$, the projective module $P_{r_n}(m)$ is a factor module of the tilting module $T(r_{n+1} - m)$.

We exploit this now. Let $\hat{m}$ be the weight as above such that $P_d(m)$ is a quotient of $T(\hat{m})$. With the notation as in lemma 4.5, since $\mathcal{F}(\Delta)$ is closed under kernels of epimorphisms there is a submodule $U \subseteq T(\hat{m})$ which has a $\Delta$-filtration, with $M_i \subseteq U \subset M_{i-1}$ where $0 < i \leqslant k$, and $P_d(m) \cong T(\hat{m})/U$.

If $U = M_i$ then we have part (a). Otherwise, $P_d(m)$ has the submodule $M_{i-1}/U$ which is isomorphic to $\Delta(2s_i)$ and $U/M_i \cong \Delta(2s_i + 2)$. Moreover, $P_d(m)/\Delta(2s_i) \simeq T(\hat{m})/M_{i-1}$ which has a twisted filtration. Since $\Delta(2s_i) \subset P_d(m)$ we deduce $2s_i \leqslant d$. Suppose we have $2s_i < d$, then $2s_i + 2 \leqslant d$. Hence, $T(\hat{m})/M_i \in S(2,d)$-mod. Since $P_d(m)$ is a quotient of the indecomposable $T(\hat{m})$, $T(\hat{m})$ has a simple top isomorphic to $L(m)$. So, the module $T(\hat{m})/M_i$ has a simple top isomorphic to $L(m)$ and is in degree $d$, and therefore must be a quotient of $P_d(m)$. In particular, we would obtain $M_i = U$. This is not so in the case considered. Therefore, $2s_i = d$ and the result follows from (4.2). □

Example 5.5. Consider figure 1, with $d = 28$. Then, $r_n = 30$ and the projective modules $P_d(m)$ for $15 < m < 28$ are as follows. We have (a) when $m = 18, 20, 22, 26$ and we have (b) when $m = 16, 24$. Note that cases $P_d(m) \cong P_{d-2}(m)$ occur in (a).

Corollary 5.6. *With the setting as in lemma 5.4,*
*if (a) occurs, then* $V^{\otimes d}$-$\mathrm{domdim}_S\, P_d(m) = +\infty$.
*If (b) occurs, then for* $d = 2s$ *we have* $V^{\otimes d}$-$\mathrm{domdim}_S\, P_d(m) = V^{\otimes d}$-$\mathrm{domdim}_S\, \Delta(d)$. *In particular,*

$$V^{\otimes d}\text{-}\mathrm{domdim}_S\, S(2,d) = V^{\otimes d}\text{-}\mathrm{domdim}_S\, \Delta(d) = (d/2) + V^{\otimes d}\text{-}\mathrm{domdim}_S\, T(0).$$

*Proof.* This follows directly from lemmas 5.4 and 2.2. □

This completes the proof of theorem B for algebraically closed fields. By [**10**, Lemma 3.2.3], the result also holds over arbitrary fields.

### 5.2. The quantum case

Remark 5.7. If $q$ is not a root of unity, then $S_{K,q}(2,d)$ is semi-simple [**20**, 4.3(7)] and $V^{\otimes d}$ being faithful is a characteristic tilting module. Otherwise, the summands of $V^{\otimes d}$ over $S_{K,q}(2,d)$ are the tilting modules labelled by the $\ell$-regular partitions of $d$ in at most two parts, where $q$ is an $\ell$-root of unity. Hence, replacing char $K$ by $\ell$

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in lemma 4.1, we obtain that $V^{\otimes d}$ is a characteristic tilting module over $S_{K,q}(2,d)$ if $q+1 \neq 0$ or $d$ is odd.

For the quantum case, it is enough to take $S = S_{K,q}(2,d)$ where $q+1=0$. In this case, everything is exactly the same as over $S(2,d)$ when $\operatorname{char} K = 2$. Namely, we may take $S_{K,q}(2,d)$ as $A_q(2,d)^*$ as it is done in [**6**, **20**], and also in [**12**, **19**]. The definition of $A_q(2)$ may be found in [**12**, p. 16]. This means that one takes the quantum group $G(2)$ as defined in [**12**] instead of $SL(2,K)$.

As it is explained in [**24**, Sections 3.1 and 3.2], we can use the same labelling for weights; in that paper the parameter $q$ is a primitive $\ell$-th root of 1 and we only need $\ell = 2$. We can regard $S_{K,q}(2,d)$ as a factor algebra of $S_{K,q}(2,d+2)$, using [**20**, Section 4.2], and therefore regard modules in degree $d$ again as modules in degree $d'$ for $d' > d$ of the same parity.

There is a Frobenius morphism from the quantum group $G(2)$ to the classical setting, hence if $\Delta(m)$ (resp. $T(m)$) is a standard module (resp. a tilting module) for the classical setting, then $\Delta(m)^F$ (resp. $T(m)^F$) is a module for the quantum group, and so are the tensor products $T(2) \otimes \Delta(m)^F$ and $T(2) \otimes T(m)^F$ modules for the quantum group. The $q$-analogues of the exact sequences in § 4.3 and proposition 4.4 exist by [**6**, Prop. 3.3 and 3.4]. See also [**24**, Proposition 3.1] (our situation of interest is recovered by fixing $l = 2$ in their setup). The $q$-analogue of proposition 4.3 can be found in [**20**, Section 3.4, p. 73, (8)].

We note that (1) of § 4.3 and the $q$-analogue imply, by induction, that all decomposition numbers are 0 or 1.

In [**12**], it is shown that this version of the $q$-Schur algebra is the same as our definition, as the endomorphism algebra of the action of the Iwahori–Hecke algebra on the tensor space $V^{\otimes d}$, see § 6. The definition of the Iwahori–Hecke algebra, as we take it is given in 6.2. In particular, we denote by $H$ the Iwahori–Hecke algebra. Their strategy in [**12**, Section 3] is to show that the action of the Iwahori–Hecke algebra on the tensor space is a comodule homomorphism (see 3.1.6 of [**12**]). The $H$-action in [**12**, 3.1.6] is not the same as ours, but it is explained in detail (see 4.4.3 of [**12**]) that the action we use also can be taken.

Hence, the arguments of § 5 remain valid in the quantum case and therefore, we obtain the following:

THEOREM 5.8. *Let $K$ be a field and fix $q = u^{-2}$ for some $u \in K$. Let $S$ be the $q$-Schur algebra $S_{K,q}(2,d)$ and $T$ be the characteristic tilting module of $S$. Then,*

$$V^{\otimes d}\text{-domdim}_S\, S = 2 \cdot V^{\otimes d}\text{-domdim}_S\, T = \begin{cases} d, & \text{if } 1+q=0 \text{ and } d \text{ is even}, \\ +\infty, & \text{otherwise}. \end{cases}$$

REMARK 5.9. One might want to know for which $m$ it is true that $V^{\otimes d}$-$\text{domdim}_S\, P_d(m)$ is finite. In principle, one can answer this, using the formula in [**34**] for decomposition numbers. Namely, this $V^{\otimes d}$-dominant dimension is finite if and only the number of $\Delta$-quotients of $P_d(m)$ is odd, that is the number of 1s in the column of $L(m)$.

EXAMPLE 5.10. We give an illustration of theorem B, we continue with example 3.7. Let $A = S(2,4)$ where $K$ has characteristic 2, then $\operatorname{add}(V^{\otimes 4}) = \operatorname{add}[T(2) \oplus T(4)]$.

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The fact that $Q$-domdim $\Delta(4) = 2 + Q$-domdim$_A\, T(0)$ for this example was already done in the proof of theorem 5.2. Case (a) of lemma 5.4 occurs for $P(2)$. Indeed, $Q$-domdim$_A\, P(2) = +\infty$ since $P(2) \simeq T(4)$. Case (b) of lemma 5.4 occurs for $P(0)$. Indeed, there exists an exact sequence $0 \to \Delta(4) \to P(0) \to T(2) \to 0$, where $Q$-domdim$_A\, T(2) = +\infty$. So, lemma 2.2 gives that $Q$-domdim$_A\, P(0) = Q$-domdim$_A\, \Delta(4)$. Therefore, $Q$-domdim $A =$ $Q$-domdim$_A\, \Delta(4) = 2 + Q$-domdim$_A\, T(0) = 2 + Q$-domdim$_A\, T = 4$.

### 5.3. An application of theorem B

In [**3**], Buan and Solberg characterize complements of almost complete (co) tilting modules. Let $Q$ be an $A$-module, then the faithful dimension of $Q$ as they define it, is equal to $Q$-domdim$_A\, A$. A partial tilting module $Q$ is *almost complete* if it is not a tilting module, but there is an indecomposable module $X$ such that $Q \oplus X$ is a tilting module. In this case, call $X$ a *complement* of $Q$. Recall that a module $M$ is a tilting module over $A$ if and only if $DM$ is a cotilting module over $A^{op}$ and $M$-domdim $A = DM$-domdim$_{A^{op}}\, A$.

Hence, dualizing [**3**, Theorem 3.6] and [**3**, Proposition 3.2] of Buan and Solberg we have the following.

THEOREM 5.11. *Let $A$ be a finite-dimensional algebra over a field.*

(1) *Let $n$ be a non-negative integer. An almost complete tilting $A$-module $Q$ has $n+1$ complements if and only if $Q$-domdim$_A\, A = n$.*

(2) *Let $Q \in A$-mod so that $Q \oplus X$ is a tilting $A$-module. If $g\colon M_0 \to X$ is a surjective minimal right* add$_A\, Q$*-approximation of $X$, then* $\ker g$ *is a complement of $Q$.*

(3) *Let $Q \in A$-mod so that $Q \oplus X$ is a tilting $A$-module. If $f\colon X \to M_0$ is an injective minimal left* add$_A\, Q$*-approximation of $X$, then* coker $f$ *is a complement of $Q$.*

Assume $A = S(2, d)$ for char$(K) = 2$ and $d$ even, then $Q = V^{\otimes d}$ is an almost complete tilting module. We have proved that $Q$-domdim $A = d$, so there should be in general $d + 1$ complements.

EXAMPLE 5.12. Assume $A = S(2, 4)$ when $p = 2$, we continue with the notation as in example 5.10. Consider $Q = V^{\otimes 4}$. We know $T(0)$ is a complement of $Q$. By computing minimal right add$(Q)$-approximations, we get the following exact sequence:

$$0 \to P(0) \xrightarrow{f_2} T(2) \oplus T(4) \xrightarrow{f_1} T(2) \xrightarrow{f_0} T(0) \to 0. \tag{5.4}$$

Namely, the kernel of the surjection $f_0 : T(2) \to \Delta(0) \cong T(0)$ is $\Delta(2)$. Take $f_1$ to be the map where the first component has image $L(0)$ and the second component has image $\Delta(2)$, this is a right add $Q$ approximation, and its kernel is isomorphic to $P(0)$. It is minimal since $P(0)$ is indecomposable. A minimal right add $Q$ approximation of $P(0)$ cannot be surjective since $P(0)$ is projective but not a direct summand of $Q$.

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By (2) of theorem 5.11 on (5.4), we get that $\ker f_0 = \Delta(2)$ and $P(0)$ are complements of $Q$. Now, applying the simple preserving duality functor to (5.4) we get an exact sequence:

$$0 \to T(0) \to T(2) \to T(2) \oplus T(4) \to I(0), \tag{5.5}$$

and surjective minimal right approximations are sent to injective minimal left approximations. Thus, the complements of $Q$ are

$$P(0),\ \Delta(2),\ T(0),\ \nabla(2),\ I(0) \to 0.$$

## 6. Temperley–Lieb algebras

These algebras were introduced in the context of statistical mechanics [**50**], and then became popular through the work of Jones. In particular, he noticed that they occur as quotients of Iwahori–Hecke algebras [**40**, **41**]. See also [**51**] for further details. We give the definition and discuss the connections with Schur algebras.

DEFINITION 6.1. *Let $R$ be a commutative ring and $\delta$ an element of $R$. The Temperley–Lieb algebra $TL_{R,d}(\delta)$ over $R$ is the $R$-algebra generated by elements $U_1, U_2, \ldots, U_{d-1}$ with defining relations, here $1 \leqslant i, j \leqslant d-1$ such that each term is defined:*

(a) $U_iU_j = U_jU_i$ $(|i-j| > 1)$,

(b) $U_i^2 = \delta U_i$,

(c) $U_iU_{i+1}U_i = U_i$, $1 \leqslant i \leqslant n-2$,

(d) $U_iU_{i-1}U_i = U_i$, $2 \leqslant i \leqslant n-1$.

It can be viewed as a diagram algebra, with a very extensive literature, but we will not give details since we do not use diagram calculations.

### 6.1. The classical case

We will start by considering the class of Temperley–Lieb algebras of the form $TL_{R,d}(-2)$ which can be viewed as quotients of group algebras of the symmetric group.

LEMMA 6.2. *There is a surjective algebra homomorphism $\Phi : R\mathcal{S}_d \to TL_{R,d}(-2)$ taking the generator $T_i = (i\ i+1)$ of $\mathcal{S}_d$ to $U_i + 1$ for $1 \leqslant i \leqslant d-1$.*

*Proof.* Recall that the group algebra $R\mathcal{S}_d$ is generated by the $T_i$ subject to the relations:

(a) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$,

(b) $T_iT_j = T_jT_i$ $(|i-j| > 1)$,

(c) $T_i^2 = 1$,

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for $1 \leqslant i, j \leqslant d-1$ such that each factor is defined. To show that the map is well-defined one has to check that it preserves these relations; this is straightforward. It is clear that $\Phi$ is surjective, noting that $U_i = \Phi(T_i - T_i^2)$. □

The following description of the kernel of $\Phi$ goes back to [42, p. 364].

THEOREM 6.3. *For each $i = 1, 2, \ldots, d-2$ define*

$$x_i := T_i T_{i+1} T_i - T_i T_{i+1} - T_{i+1} T_i + T_i + T_{i+1} - 1 \in R\mathcal{S}_d.$$

*Let $I$ be the ideal of $R\mathcal{S}_d$ generated by the $x_i$ for $1 \leqslant i \leqslant d-2$. Then, there is an exact sequence*

$$0 \to I \to R\mathcal{S}_d \xrightarrow{\Phi} TL_{R,d}(-2) \to 0$$

*Proof.* One checks that $\Phi(x_i) = 0$ for each $i = 1, \ldots, d-2$. So, we have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\Phi & \longrightarrow & R\mathcal{S}_d & \xrightarrow{\Phi} & TL_{R,d}(-2) & \longrightarrow & 0 \\ & & {\scriptstyle\iota}\uparrow & & {\scriptstyle id_{R\mathcal{S}_d}}\uparrow & & {\scriptstyle\pi}\uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & R\mathcal{S}_d & \longrightarrow & R\mathcal{S}_d/I & \longrightarrow & 0 \end{array},$$

where $\pi$ maps the image of $T_i$ in $R\mathcal{S}_d/I$ to $\Phi(T_i) = U_i + 1$, and $\iota$ is the inclusion map. Consider $\pi' : TL_{R,d}(-2) \to R\mathcal{S}_d/I$ defined by taking $U_i$ to the image of $T_i - 1$ in $R\mathcal{S}_d/I$. One checks that $\pi'$ preserves the defining relations for $TL_{R,d}(-2)$, so that it is a well-defined map. Finally,

$$\pi'(\pi(T_i + I)) = T_i + I, \quad \pi(\pi'(U_i)) = U_i.$$

Therefore, $\ker \Phi = I$. □

It is nowadays widely known that Temperley–Lieb algebras can be viewed as the centralizer algebras of quantum groups $\mathfrak{sl}_2$ in the endomorphism algebra of a tensor power and this goes back to the work of Martin [44] and Jimbo [39]. Recall that over $R$, the Schur algebra $S_R(2, d)$ is defined as the endomorphism algebra $\mathrm{End}_{R\mathcal{S}_d}((R^2)^{\otimes d})$, where $(R^2)^{\otimes d}$ affords a right $R\mathcal{S}_d$-module structure via place permutation. In order to relate the Temperley–Lieb algebra to the Schur algebra, we need a suitable action of the Temperley–Lieb algebra on the tensor space. It is as follows.

THEOREM 6.4. *Let $V$ be a free $R$-module of rank 2. Then, $V^{\otimes d}$ is a module over $\Lambda = TL_{R,d}(-2)$ where $U_i$ acts as $\mathrm{id}_V^{\otimes(i-1)} \otimes \tau \otimes \mathrm{id}_V^{\otimes(d-i-1)}$. Here, $\tau$ is the endomorphism of $V^{\otimes 2}$ defined by*

$$\tau(v_1 \otimes v_2) = v_2 \otimes v_1 - v_1 \otimes v_2$$

*(for $v_1, v_2 \in V$). Moreover, there is an algebra isomorphism*

$$\Lambda \to \mathrm{End}_{S_R(2,d)}(V^{\otimes d})^{op}.$$

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*Proof.* We know that $R\mathcal{S}_d$ acts by place permutations on $V^{\otimes d}$ and we can view this as a right action. With this, $T_i - 1$ acts as $U_i$ on the space $V^{\otimes d}$. This shows that it factors through $\Lambda$. In particular, to show that $\Lambda \to \operatorname{End}_{S(2,d)}(V^{\otimes d})^{op}$ is surjective it is enough to check that the canonical map $RS_d \to \operatorname{End}_{S(2,d)}(V^{\otimes d})^{op}$ is surjective. But this follows from classical Schur–Weyl duality (see e.g. [**43**]). In fact, this can be seen in the following way: let $K$ be a field, then the canonical map $K\mathcal{S}_d \to \operatorname{End}_{S(2,d)}(V^{\otimes d})^{op}$ fits in the following commutative diagram:

$$\begin{array}{ccc} K\mathcal{S}_d & \longrightarrow & \operatorname{End}_{S(2,d)}(V^{\otimes d})^{op} \\ & {\scriptstyle\psi}\searrow \quad \nearrow{\scriptstyle\phi} & \\ & \operatorname{End}_{S(d,d)}((K^d)^{\otimes d})^{op} & \end{array}$$

Here, $\psi$ is surjective because domdim $S_K(d,d) \geqslant 2$ and $\phi$ is surjective by [**27**, 1.7] and [**20**, 4.7] because $(K^d)^{\otimes d}$ is a projective–injective module over $S(d,d)$. Observe that $\operatorname{End}_{S_R(2,d)}(V^{\otimes d})^{op} \in R$-proj (see e.g. [**9**, Proposition A.4.3, Corollary A.4.4]) and it has a base change property (see e.g. [**9**, Corollary A.4.6]). In particular, $R(\mathfrak{m}) \otimes_R \operatorname{End}_{S_R(2,d)}(V^{\otimes d})^{op} \simeq \operatorname{End}_{S_{R(\mathfrak{m})}(2,d)}(V^{\otimes d})^{op}$ for every maximal ideal $\mathfrak{m}$ of $R$. By the above discussion, the maps $R(\mathfrak{m})S_d \to \operatorname{End}_{S_{R(\mathfrak{m})}(2,d)}(V^{\otimes d})^{op}$ are surjective for every maximal ideal $\mathfrak{m}$ of $R$. Now, by Nakayama's lemma the map $RS_d \to \operatorname{End}_{S_R(2,d)}(V^{\otimes d})^{op}$ is surjective.

It remains to show that the action of $\Lambda$ is injective. Let $\sum_i a_i U_i \in \Lambda$ acting as zero on $V^{\otimes d}$. The action of $\sum_i a_i U_i$ in $y_k$, defined as the basis element $e_1 \otimes \cdots \otimes e_1 \otimes e_2 \otimes e_1 \otimes \cdots \otimes e_1$ where $e_2$ appears in position $k+1$, yields that $a_k = a_{k+1} = 0$. This concludes the proof. □

### 6.2. The $q$-analogue

We shall now discuss the general case of theorem 6.4 and its importance for all Temperley–Lieb algebras.

Let $R$ be a commutative Noetherian ring with an invertible element $u \in R$. We fix a natural number $d$, and we set $q := u^{-2}$. We take the *Iwahori–Hecke algebra* $H = H_{R,q}(d)$ to be the $R$-algebra with basis $\{\widetilde{T}_w \mid w \in \mathcal{S}_d\}$ with relations:

$$\widetilde{T}_w\widetilde{T}_s = \begin{cases} \widetilde{T}_{ws} & \text{if } l(ws) = l(w) + 1 \\ (u - u^{-1})\widetilde{T}_w + \widetilde{T}_{ws} & \text{otherwise.} \end{cases}$$

Here, $s$ runs through the set of transpositions $S = \{(i\ i+1) \mid 1 \leqslant i < d\}$ in $\mathcal{S}_d$, and $l(w)$ is the usual length for $w \in \mathcal{S}_d$, that is the minimal number of transpositions needed in a factorization of $w$.

This presentation corresponds to the presentation used in [**12–14**, **20**] by

$$\widetilde{T}_w = (-u)^{l(w)} T_w.$$

The algebra $H$ can also be defined by the braid relations, together with $\widetilde{T}_s^2 = (u - u^{-1})\widetilde{T}_s + 1$, that is

$$(\widetilde{T}_s - u)(\widetilde{T}_s + u^{-1}) = 0 \quad (s \in S).$$

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Lemma 6.5. *Let $\delta = -u - u^{-1}$. Then, there is a surjective algebra homomorphism*

$$\Phi : H_{R,q}(d) \to TL_{R,d}(\delta)$$

*taking the generator $\widetilde{T}_i := \widetilde{T}_{(i\ i+1)}$ to $U_i + u$ for $1 \leqslant i \leqslant d-1$.*

*Proof.* We must show that $\Phi$ is well-defined, that is it respects the relations of the Hecke algebra. It is clearly surjective. We work with the presentation via the braid relations, together with $(\widetilde{T}_i - u)(\widetilde{T}_i + u^{-1}) = 0$ (for $1 \leqslant i < d-1$). With the definition given, $\Phi(\widetilde{T}_i + u^{-1}) = U_i + u + u^{-1} = U_i - \delta$ and $\Phi(\widetilde{T}_i - u) = U_i$. Hence, we have $\Phi(\widetilde{T}_i + u^{-1})\Phi(\widetilde{T}_i - u) = (U_i - \delta)U_i = 0$.

To check the braid relations, we compute:

$$(U_i + u)(U_{i+1} + u)(U_i + u) = U_i + (u\delta)U_i + u(U_iU_{i+1} + U_{i+1}U_i) + 2u^2U_i + u^2U_{i+1} + u^3.$$

The coefficient of $U_i$ is equal to $u^2$. With this, the expression is symmetric in $i, i+1$ and is therefore equal to $(U_{i+1} + u)(U_i + u)(U_{i+1} + u)$. □

We want to determine the kernel of $\Phi$.

Theorem 6.6. *For each $i = 1, 2, \ldots, d-2$ define*

$$x_i := \widetilde{T}_i\widetilde{T}_{i+1}\widetilde{T}_i - u\widetilde{T}_i\widetilde{T}_{i+1} - u\widetilde{T}_{i+1}\widetilde{T}_i + u^2\widetilde{T}_i + u^2\widetilde{T}_{i+1} - u^3 \in H.$$

*Let $I$ be the ideal of $H_{R,q}(d)$ generated by the $x_i$ for $1 \leqslant i \leqslant d-2$. Fix $\delta = -u - u^{-1}$, then there is an exact sequence*

$$0 \to I \to H_{R,q}(d) \xrightarrow{\Phi} TL_{R,d}(\delta) \to 0.$$

*Proof.* From the proof of lemma 6.5 we see that

$$\Phi(\widetilde{T}_i)\Phi(\widetilde{T}_{i+1})\Phi(\widetilde{T}_i) = u^2(U_i + U_{i+1}) + u(U_{i+1}U_i + U_iU_{i+1}) + u^3$$

and with this one gets that $\Phi(x_i) = 0$ for all $i$. Hence, $I \subseteq \ker(\Phi)$. Analogously to the proof of theorem 6.3 replacing the map $\pi'$ with the map $TL_{R,d}(\delta) \to H_{R,q}(d)$ defined by taking $U_i$ to the image of $\widetilde{T}_i - u$ in $H_{R,q}(d)/I$ one proves equality. □

Let $V^{\otimes d}$ be the free $R$-module of rank $n$ over $R$ (later we will take $n = 2$). Then, $V^{\otimes d}$ is a right $H$-module, which can be regarded as a deformation of the place permutation action of $\mathcal{S}_d$. Denote by $I(n, d)$ the set of maps $\{1, \ldots, d\} \to \{1, \ldots, n\}$ and by $i_j$ the image $i(j)$. If $\mathbf{i} \in I(n, d)$ labels the basis element $e_{\mathbf{i}} = e_{i_1} \otimes \cdots \otimes e_{i_d}$ of $V^{\otimes d}$ and $s = (t\ t+1) \in S$ we write $e_{\mathbf{i}} \cdot s$ for the basis element obtained by interchanging $e_{i_t}$ and $e_{i_{t+1}}$. Then

$$e_{\mathbf{i}} \cdot \widetilde{T}_s := \begin{cases} e_{\mathbf{i}} \cdot s & i_t < i_{t+1} \\ ue_{\mathbf{i}} & i_t = i_{t+1} \\ (u - u^{-1})e_{\mathbf{i}} + e_{\mathbf{i}} \cdot s & i_t > i_{t+1} \end{cases}$$

Focusing on the Temperley–Lieb algebra, we take $n = 2$. Recall that the $q$-Schur algebra $S_q(2, d)$ is the endomorphism algebra $\mathrm{End}_H(V^{\otimes d})$ via the action as above.

THEOREM 6.7. *The $H$-module structure on $V^{\otimes d}$ factors through $\Phi : H \to \Lambda = TL_{R,d}(\delta)$, where $\delta = -u - u^{-1}$. Hence, $U_s$ acts as:*

$$e_{\mathbf{i}}U_s := \begin{cases} e_{\mathbf{i}s} - ue_{\mathbf{i}} & i_t < i_{t+1} \\ ue_{\mathbf{i}} - ue_{\mathbf{i}} & i_t = i_{t+1} \\ (-u)^{-1}e_{\mathbf{i}} + e_{\mathbf{i}s} & i_t > i_{t+1} \end{cases}$$

*where $s = (t\ t+1) \in S$. Moreover, there is an algebra isomorphism*

$$\Lambda \to \mathrm{End}_{S_{R,q}(2,d)}(V^{\otimes d})^{op}.$$

*Proof.* The first statement follows by checking that the elements $x_i$ act as zero on $V^{\otimes d}$.

The element $\widetilde{T}_i - u$ acts exactly as the action of $U_i$ in $V^{\otimes d}$, so the canonical map $H_{R,q}(d) \to \mathrm{End}_{S_{R,q}(2,d)}(V^{\otimes d})^{op}$ factors through $\Lambda$, that is, there is an algebra homomorphism $\Lambda \to \mathrm{End}_{S_{R,q}(2,d)}(V^{\otimes d})^{op}$. The same argument as the one given in theorem 6.4 works in this case replacing the Schur algebra by the $q$-Schur algebra and the group algebra of the symmetric group by the Iwahori–Hecke algebra. The injectivity follows again by considering the action of the elements in $\Lambda$, acting as zero on $V^{\otimes d}$, on the elements $y_k$ defined in the exactly same way as in theorem 6.4. □

Theorem 6.7 places $V^{\otimes d}$ in a central position in the representation theory of Temperley–Lieb algebras where it plays a role similar to that played by $(R^n)^{\otimes d}$ in the study of the representation theory of symmetric groups via Schur algebras. In fact, Theorem 8.1.5 of [**10**] specializes to the following.

COROLLARY 6.8. *Let $K$ be a field and fix $q = u^{-2}$ for some element $u \in K^{\times}$. Let $T$ be a characteristic tilting module of $S$ and let $R(S)$ be the Ringel dual of $S := S_{K,q}(2,d)$ over a field $K$.*

*Then, $(R(S), \mathrm{Hom}_S(T, V^{\otimes d})$ is a $(V^{\otimes d}\text{-domdim}_S\, T - 2)$-$\mathcal{F}(\Delta_{R(S)})$ quasi-hereditary cover of $TL_{K,d}(-u - u^{-1})$, where $\Delta_{R(S)}$ denotes the set of standard modules over $R(S)$. Moreover, the following assertions hold:*

(i) *If $q + 1 \neq 0$ or $d$ is odd, then $TL_{K,d}(-u - u^{-1})$ is the Ringel dual of $S_{K,q}(2,d)$, and in particular, it is a split quasi-hereditary algebra over $K$;*

(ii) *If $q + 1 = 0$ and $d$ is even, then $(R(S), \mathrm{Hom}_S(T, V^{\otimes d})$ is a $\left(\frac{d}{2} - 2\right)$-$\mathcal{F}(\Delta_{R(S)})$ quasi-hereditary cover of $TL_{K,d}(0)$ and $\mathrm{HNdim}_F\, \mathcal{F}(\Delta_{R(S)}) = \frac{d}{2} - 2$. In particular, the Schur functor*

$$F := \mathrm{Hom}_{R(S)}(\mathrm{Hom}_S(T, V^{\otimes d}), -) \colon R(S_{K,q}(2,d))\text{-}\mathrm{mod} \to TL_{K,d}(0)\text{-mod}$$

*induces bijections*

$$\mathrm{Ext}^i_{R(S)}(M,N) \simeq \mathrm{Ext}^i_{TL_{K,d}(0)}(FM, FN), \quad \forall M, N \in \mathcal{F}(\Delta_{R(S)}),$$
$$0 \leqslant i \leqslant \frac{d}{2} - 2.$$

*Proof.* The result follows from theorem 5.8, [**10**, Theorem 8.1.5] and [**10**, Theorem 6.0.1]. □

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## 7. Uniqueness of the quasi-hereditary cover of $TL_{R,d}(\delta)$

In corollary 6.8, we have constructed a quasi-hereditary cover of $TL_{K,q}(\delta)$ using the Ringel dual of a $q$-Schur algebra. We will argue now that it is the best quasi-hereditary cover of $TL_{K,d}(\delta)$ if $d > 2$ (in the sense that there is no other quasi-hereditary cover of the Temperley–Lieb algebra which satisfies (7.3)). For that, going to the integral case is helpful. Assume that $R$ is a commutative Noetherian ring. Let $u$ be an invertible element of $R$ and fix $q = u^{-2}$. If $d = 1, 2$ then the Temperley–Lieb algebra $TL_{R,q}(-u - u^{-1})$ coincides with the Iwahori–Hecke algebra $H_{R,q}(d)$ and so this case was dealt with in [**9**, Subsection 7.2].

Assume from now on that $d > 2$. Combining theorem 6.7 with Theorem 8.1.5 of [**10**] we obtain the following:

Corollary 7.1. *Let $R$ be a commutative Noetherian ring. Fix an element $u \in R^\times$ and $q = u^{-2}$. Let $T$ be a characteristic tilting module of $(S_{R,q}(2,d), \{\Delta(\lambda)_{\lambda\in\Lambda^+(2,d)}\})$. Denote by $R(S)$ the Ringel dual of $(S_{R,q}(2,d)$, $\{\Delta(\lambda)_{\lambda\in\Lambda^+(2,d)}\})$, that is, $R(S) = \operatorname{End}_{S_{R,q}(2,d)}(T)^{op}$.*

*Then, $(R(S), \operatorname{Hom}_{S_{R,q}(2,d)}(T, V^{\otimes d}))$ is a $(V^{\otimes d}\text{-domdim}_{S_{R,q}(2,d),R}\, T - 2)$-$\mathcal{F}(\tilde{\Delta}_{R(S)})$ split quasi-hereditary cover of $TL_{R,d}(-u - u^{-1})$.*

In the following, we will write $R(S)$ to denote the Ringel dual $\operatorname{End}_{S_{R,q}(2,d)}(T)^{op}$. Denote by $F_{R,q}$ the Schur functor associated with the quasi-hereditary cover constructed in corollary 7.1. The aim now is to compute $\operatorname{HNdim}_{F_{R,q}} \mathcal{F}(\tilde{\Delta}_{R(S)})$ and in particular to determine $V^{\otimes d}\text{-domdim}_{S_{R,q}(2,d),R}\, T$ in terms of the ground ring $R$.

Theorem 7.2. *Let $R$ be a commutative Noetherian ring. Fix an element $u \in R^\times$ and $q = u^{-2}$. Let $T$ be a characteristic tilting module of $(S_{R,q}(2,d)$, $\{\Delta(\lambda)_{\lambda\in\Lambda^+(2,d)}\})$. Then,*

$$V^{\otimes d}\text{-domdim}_{(S_{R,q}(2,d),R)}\, T = \begin{cases} \dfrac{d}{2}, & \text{if } 1+q \notin R^\times \text{ and dis even} \\ +\infty, & \text{otherwise} \end{cases}.$$

*Proof.* The algebra $S_{R,q}(2,d)$ has the base change property: $S \otimes_R S_{R,q}(2,d) \simeq S_{S,1_S\otimes q}(2,d)$ as $S$-algebras for every commutative ring $S$ which is an $R$-algebra and the standard modules of $S_{S,1_S\otimes q}(2,d)$ are of the form $S \otimes_R \Delta(\lambda)$, $\lambda \in \Lambda^+(n,d)$ (see e.g. [**11**, Subsection 3.3, Section 5]). Hence, the result follows from theorem 5.8, [**9**, Propositions A.4.7, A.4.3], and [**10**, Theorem 3.2.5]. □

### 7.1. Hemmer–Nakano dimension of $\mathcal{F}(\tilde{\Delta}_{R(S)})$

Similarly to the classical case (see also [**9**]), there are two cases to be considered.

Following [**9**], the commutative Noetherian ring $R$ is called 2-*partially $q$-divisible* if $1 + q \in R^\times$ or $1 + q = 0$.

Theorem 7.3. *Let $R$ be a local regular 2-partially $q$-divisible (commutative Noetherian) ring, where $q = u^{-2}$, $u \in R^\times$. Let $T$ be a characteristic tilting module of*

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$S_{R,q}(n,d)$. *Then,*

$$\mathrm{HNdim}_{F_{R,q}}\,\mathcal{F}(\tilde{\Delta}_{R(S)}) = V^{\otimes d}\text{-domdim}_{(S_{R,q}(2,d),R)}\,T - 2. \tag{7.1}$$

*Proof.* By corollary 7.1, $\mathrm{HNdim}_{F_{R,q}}\,\mathcal{F}(\tilde{\Delta}_{R(S)}) \geqslant V^{\otimes d}\text{-domdim}_{(S_{R,q}(2,d),R)}\,T - 2$.

If $V^{\otimes d}\text{-domdim}_{(S_{R,q}(2,d),R)}\,T = +\infty$, then $d$ is odd, and then there is nothing to prove. Assume that it is finite. By theorem 7.2, $V^{\otimes d}\text{-domdim}_{(S_{R,q}(2,d),R)}\,T = \frac{d}{2}$. In particular, $d$ is even and $1+q \notin R^{\times}$. Hence, $1+q$ must be zero. Therefore,

$$Q(R)\otimes_R V^{\otimes d}\text{-domdim}_{(S_{Q(R),q}(2,d),Q(R))}\,Q(R)\otimes_R T = \frac{d}{2},$$

where $Q(R)$ is a quotient field of $R$.

By [**10**, Corollary 5.3.6], $\mathrm{HNdim}_{Q(R)\otimes_R F_{R,q}}\,\mathcal{F}(Q(R)\otimes_R \Delta_{R(S)})$ cannot be higher than $V^{\otimes d}\text{-domdim}_{(S_{R,q}(2,d),R)}\,T - 2$. It follows that

$$\begin{aligned} V^{\otimes d}\text{-domdim}_{(S_{R,q}(2,d),R)}\,T - 2 &= \mathrm{HNdim}_{Q(R)\otimes_R F_{R,q}}\,\mathcal{F}(Q(R)\otimes_R \Delta_{R(S)}) \\ &\geqslant \mathrm{HNdim}_{F_{R,q}}\,\mathcal{F}(\tilde{\Delta}_{R(S)}). \end{aligned}$$

□

Theorem 7.4. *Let $R$ be a local regular commutative Noetherian ring which is not a 2-partially $q$-divisible commutative ring, where $q = u^{-2}$, $u \in R^{\times}$. Let $T$ be a characteristic tilting module of $S_{R,q}(n,d)$. Then,*

$$\mathrm{HNdim}_{F_{R,q}}\,\mathcal{F}(\tilde{\Delta}_{R(S)}) = V^{\otimes d}\text{-domdim}_{(S_{R,q}(2,d),R)}\,T - 1. \tag{7.2}$$

*Proof.* By corollary 7.1, if $V^{\otimes d}\text{-domdim}_{(S_{R,q}(2,d),R)}\,T$ is infinite, then there is nothing to show. So, assume that $V^{\otimes d}\text{-domdim}_{(S_{R,q}(2,d),R)}\,T$ is finite. By theorem 7.2, $d$ is even and $1+q \notin R^{\times}$. By assumption, $1+q \neq 0$, otherwise $R$ would be a 2-partially $q$-divisible ring. It follows that

$$Q(R)\otimes_R V^{\otimes d}\text{-domdim}_{(Q(R)_{\otimes} RS_{R,q}(2,d),R)}\,Q(R)\otimes_R T$$

is infinite by theorem 7.2. The result for $d=2$ follows from [**9**, Theorem 7.2.7]. Assume that $d \geqslant 4$. By corollary 7.1 and [**10**, Theorem 3.2.5], $\mathrm{HNdim}_{F_{R(\mathfrak{m}),q_{\mathfrak{m}}}}\,\mathcal{F}(R(\mathfrak{m})\otimes_R \Delta_{R(S)}) \geqslant 0$ and $\mathrm{HNdim}_{Q(R)\otimes_R F_{R,q}}\,\mathcal{F}(Q(R)\otimes_R \Delta_{R(S)}) = +\infty$, where $\mathfrak{m}$ is the unique maximal ideal of $R$, and $q_{\mathfrak{m}}$ is the image of $q$ in $R/\mathfrak{m}$. By [**9**, Theorem 5.0.9], $\mathrm{HNdim}_{F_{R,q}}\,\mathcal{F}(\tilde{\Delta}_{R(S)}) \geqslant \frac{d}{2} - 1$. The Hemmer–Nakano dimension cannot be higher because similarly to the proof of Theorem 7.2.7 of [**9**] there exists a prime ideal of height one $\mathfrak{p}$ such that $1+q \in \mathfrak{p}$. Hence, $Q(R/\mathfrak{p})\otimes_R V^{\otimes d}\text{-domdim}_{S_{Q(R/\mathfrak{p}),q_{\mathfrak{p}}}(2,d),Q(R/\mathfrak{p})}$ is exactly $\frac{d}{2}$, where $q_{\mathfrak{p}}$ denotes the image of $q$ in $R/\mathfrak{p} \subset Q(R/\mathfrak{p})$. The result follows from [**9**, Theorem 5.1.1]. □

### 7.2. Uniqueness

In this part, assume that $R = \mathbb{Z}[x, x^{-1}]$ and fix $q = x^{-2}$. Assume that $d > 2$. By [**9**, Proposition 5.0.3] and theorem 7.4, $\mathrm{HNdim}_{F_{R,q}}\,\mathcal{F}(\tilde{\Delta}_{R(S)}) \geqslant \frac{d}{2} - 1$. In particular, the Schur functor $F_{R,q}$ induces an exact equivalence:

$$\mathcal{F}(\tilde{\Delta}_{R(S)}) \to \mathcal{F}(F_{R,q}\tilde{\Delta}_{R(S)}). \tag{7.3}$$

Corollary 7.5. *$(R(S), \mathrm{Hom}_{S_{R,q}(2,d)}(T, V^{\otimes d}))$ is the unique split quasi-hereditary cover of $TL_{R,d}(-x-x^{-1})$ satisfying property (7.3), where $T$ is a characteristic tilting module of $S_{R,q}(2,d)$ and $R(S)$ denotes the Ringel dual of $S_{R,q}(2,d)$. In particular, $TL_{R,d}(-x-x^{-1})$ is a split quasi-hereditary algebra over $R$ if and only if $d$ is odd.*

*Proof.* The first statement follows from corollary 2.11 together with [**9**, Proposition 5.0.3] and theorem 7.4. For the second statement see for example [**9**, Proposition A.4.7] or [**10**, Theorem 6.0.1] together with theorem 7.2). □

As a consequence, when $d$ is odd, the Temperley–Lieb algebra $TL_{R,d}(-x-x^{-1})$ is exactly a Ringel dual of $S_{\mathbb{Z}[x,x^{-1}],q}(2,d)$.

**Acknowledgements**
Part of the collaboration was done during a research visit of the second-named author to the University of Stuttgart, and the authors are grateful for the support given. Some results were obtained during the first-named author's visit to the MPIM at Bonn. The authors would like to thank Steffen Koenig for his comments on an earlier version of this manuscript. The authors are grateful to the anonymous referees for their constructive input and their suggestions to improve the exposition of this manuscript.

## Appendix A. More applications of theorem A

In this section, we give two applications of theorem 3.1 showing how it can be a useful tool to understand other problems.

### Appendix A.1. Minimality of faithful direct summands of the characteristic tilting module

In this subsection, we prove that there exists a unique minimal direct summand of the characteristic tilting module, $Q$, affording a double centralizer property. Here, minimal means that every other direct summand of the characteristic tilting module affording a double centralizer property has $Q$ as a direct summand. This fact was first established in [**36**] using a different concept of relative dominant dimension; however, our proof is shorter knowing theorem A. We also refer to [**36**, **46**] for previous background on this question.

Let $M$ be an $A$-module and $B$ its endomorphism algebra, then $M$ satisfies the *double centralizer property* if the canonical algebra homomorphism $A \to \mathrm{End}_B(M)$ is an isomorphism. There are several characterizations in the literature for this property. If $A$ is a finite-dimensional algebra over a field, $M$ satisfies the double centralizer property if and only if $M$-domdim $A \geqslant 2$ (see e.g. [**1**, Corollary 2.4] or [**10**, Lemma 5.1.2, Theorem 3.1.1, Corollary 3.1.5]).

*A.1.1. Existence* Let $A$ be a split quasi-hereditary algebra (over a field) with a simple preserving duality. Let $T$ be the characteristic tilting module of $A$ and denote by $R(A)$ the Ringel dual $\mathrm{End}_A(T)^{op}$ of $A$. We use properties as summarized in § 3.

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Let $P_0(DT)$ be the projective cover of $DT$ as $R(A)$-module. Fix $T_0 := T \otimes_{R(A)} P_0(DT) \in \operatorname{add}_A T$. We show first that $T_0$-domdim $A \geqslant 2$. The projective cover of $DT$ gives the exact sequence

$$0 \to \Omega(DT) \to P_0(DT) \to DT \to 0 \tag{A.1}$$

of $R(A)$-modules. Recall that $DT$ is the characteristic tilting module of $R(A)$, hence is in $\mathcal{F}(\Delta_{R(A)})$, and so is $P_0(DT)$. Since $\mathcal{F}(\Delta_{R(A)})$ is closed under kernels of epimorphisms, $\Omega(DT) \in \mathcal{F}(\Delta_{R(A)})$. Therefore, by proposition 2.7, applying $T \otimes_{R(A)} -$ to (A.1) gives the exact sequence

$$0 \to T \otimes_{R(A)} \Omega(DT) \to T_0 \to T \otimes_{R(A)} DT \to 0. \tag{A.2}$$

Moreover, $T \otimes_{R(A)} \Omega(DT) \in \mathcal{F}(\nabla)$ and $T \otimes_{R(A)} DT \simeq DA$, so the exact sequence (A.2) remains exact under $\operatorname{Hom}_A(T_0, -)$. That is, $T_0$-codomdim$_A\, DA \geqslant 1$. By theorem 3.1, $T_0$-domdim $A \geqslant 2$. So, $T_0$ affords a double centralizer property.

*A.1.2. Uniqueness and minimality* Pick another $Q \in \operatorname{add}_A T$ satisfying $Q$-domdim $A \geqslant 2$. By theorem 3.1, $Q$-domdim$_A\, T \geqslant 1$. Applying proposition 2.6(i) shows that

$$\operatorname{Hom}_A(T, Q)\text{-codomdim}_{R(A)}\, DT \geqslant 1.$$

Therefore, the projective cover of $DT$ is in the additive closure of $\operatorname{Hom}_A(T, Q)$. Therefore,

$$T_0 \in \operatorname{add}_A T \otimes_{R(A)} \operatorname{Hom}_A(T, Q) = \operatorname{add}_A Q.$$

Theorem A.1. *Let $A$ be a split quasi-hereditary algebra (over a field) with a simple preserving duality. Then, the following assertions hold*:

(1) *$T_0$ is the unique minimal direct summand of the characteristic tilting module affording a double centralizer property.*

(2) *For every $Q \in \operatorname{add}_A T$ affording a double centralizer property, the following holds:*

$$Q\text{-domdim}_A\, A \geqslant T_0\text{-domdim}_A\, A \geqslant 2.$$

*Proof.* Part (1) was proved in the above discussion. By proposition 2.6(i),

$$\begin{aligned} Q\text{-domdim}_A\, T &= \operatorname{Hom}_A(T, Q)\text{-codomdim}_{R(A)}\, DT \geqslant T_0\text{-codomdim}_{R(A)}\, DT \\ &= T_0\text{-domdim}_A\, T \end{aligned}$$

where the inequality holds since $T_0 \in \operatorname{add}_A Q$. By theorem 3.1, part (2) follows. □

## Appendix A.2. Invariance of dominant dimension under Ringel duality

Given a split quasi-hereditary algebra $A$ (over a field) with a simple preserving duality, there are currently at least two methods to see that its dominant dimension remains invariant under Ringel duality.

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Indeed, in such a setup, the Ringel dual of $A$ also has a simple preserving duality (see e.g. [**29**, Proposition 2.4, Lemma 3.2(1), (2)]) and so the Ringel dual of $A$ also has a faithful projective–injective if $A$ has one. Moreover, in such a setup, the Ringel dual of $A$ has a faithful projective–injective if and only if $A$ has one because the Ringel dual of the Ringel dual of $A$ is up to Morita equivalence $A$ again. Since the Ringel dual of $A$ and $A$ are derived equivalent, Theorem 5.5 of [**28**] states that $A$ and the Ringel dual of $A$ have the same dominant dimension.

Under the addition of the assumption that $A$ has positive dominant dimension so that the endomorphism algebra of the faithful projective–injective module is a symmetric algebra, Fang and Koenig proved in [**29**, Theorem 4.3] that the dominant dimension of $A$ is equal to the dominant dimension of the Ringel dual of $A$.

In the rest of this section, we propose a new proof of this fact.

THEOREM A.2. *Let* $(A, \{\Delta(\lambda)_{\lambda\in\Lambda}\})$ *be a split quasi-hereditary algebra over a field* $k$*. Suppose that there exists a simple preserving duality* ${}^{\diamond}(-)\colon A\text{-mod} \to A\text{-mod}$*. Then*

$$\operatorname{domdim} A = \operatorname{domdim} R(A),$$

*where* $R(A)$ *denotes the Ringel dual of* $A$*.*

*Proof.* Let $T$ be the basic characteristic tilting module of $A$. As discussed above, $A$ has positive dominant dimension if and only if the Ringel dual of $A$ has positive dominant dimension. Assume that $A$ has positive dominant dimension and assume that $P$ is a faithful projective–injective $A$-module. Then, $\operatorname{Hom}_A(T, P)$ is a partial tilting over $R(A)$ since $P$ is injective over $A$ and it is projective since $P$ is partial tilting over $A$. Thus, $\operatorname{Hom}_A(T, P)$ is a projective–injective module since each partial tilting module is self-dual under the simple preserving duality functor (see e.g. [**45**, Section 2]).

Then, by theorem 3.1 and proposition 2.6,

$$\begin{aligned}\operatorname{domdim} A &= 2\operatorname{domdim}_A T = 2P\text{-}\operatorname{domdim}_A T = 2\operatorname{Hom}_A(T, P)\text{-}\operatorname{codomdim}_{R(A)} DT\\ &= \operatorname{domdim} R(A).\end{aligned}$$

□

## References

1 M. Auslander and Ø. Solberg. Relative homology and representation theory. III: cotilting modules and Wedderburn correspondence. *Commun. Algebra* **21** (1993), 3081–3097.

2 M. Auslander. *Representation dimension of Artin algebras. With the assistance of Bernice Auslander*. Queen Mary College Mathematics Notes, p. 179 (London: Queen Mary College, 1971).

3 A. B. Buan and Ø. Solberg. Relative cotilting theory and almost complete cotilting modules. In *Algebras and Modules II. Eighth International Conference on Representations of Algebras, Geiranger, Norway, 4–10 August, 1996*, pp. 77–92 (Providence, RI: American Mathematical Society, 1998).

4 E. Cline, B. Parshall and L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.* **391** (1988), 85–99.

5 E. Cline, B. Parshall and L. Scott. Integral and graded quasi-hereditary algebras I. *J. Algebra* **131** (1990), 126–160.

https://doi.org/10.1017/prm.2024.35 Published online by Cambridge University Press

6 A. Cox. $\mathrm{Ext}^1$ for Weyl modules for $q$-$GL(2,k)$. *Math. Proc. Cambridge Philos. Soc.* **124** (1998), 231–251.

7 K. Coulembier and R. Zhang. Borelic pairs for stratified algebras. *Adv. Math.* **345** (2019), 53–115.

8 T. Cruz. *Algebraic analogues of resolution of singularities, quasi-hereditary covers and Schur algebras*. Ph.D. thesis (University of Stuttgart, 2021). doi:10.18419/opus-11835

9 T. Cruz. On Noetherian algebras, Schur functors and Hemmer–Nakano dimensions. arXiv:2208.00291v1 (2022).

10 T. Cruz. On split quasi-hereditary covers and Ringel duality. arXiv:2210.09344 (2022).

11 T. Cruz. Cellular Noetherian algebras with finite global dimension are split quasi-hereditary. *J. Algebra Appl.* (2023). doi:10.1142/S0219498824501627

12 R. Dipper and S. Donkin. Quantum $GL_n$. *Proc. Lond. Math. Soc.* **63** (1991), 165–211.

13 R. Dipper and G. James. The $q$-Schur algebra. *Proc. Lond. Math. Soc.* **59** (1989), 23–50.

14 R. Dipper and G. James. $q$-Tensor space and $q$-Weyl modules. *Trans. Am. Math. Soc.* **327** (1991), 251–282.

15 S. Donkin. Finite resolutions of modules for reductive algebraic groups. *J. Algebra* **101** (1986), 473–488.

16 S. Donkin. On Schur algebras and related algebras. II. *J. Algebra* **111** (1987), 354–364.

17 S. Donkin. On tilting modules for algebraic groups. *Math. Z.* **212** (1993), 39–60.

18 S. Donkin. On Schur algebras and related algebras. IV: the blocks of the Schur algebras. *J. Algebra* **168** (1994), 400–429.

19 S. Donkin. Standard homological properties for quantum $GL_n$. *J. Algebra* **181** (1996), 235–266.

20 S. Donkin. *The q-Schur algebra*. London Mathematical Society Lecture Note Series, Vol. 253 (Cambridge: Cambridge University Press, 1998).

21 V. Dlab and C. M. Ringel. Quasi-hereditary algebras. *Ill. J. Math.* **33** (1989), 280–291.

22 V. Dlab and C. M. Ringel. The module theoretical approach to quasi-hereditary algebras. In *Representations of Algebras and Related Topics. Proceedings of the Tsukuba International Conference, held in Kyoto, Japan, 1990*, pp. 200–224 (Cambridge: Cambridge University Press, 1992).

23 K. Erdmann and A. Henke. On Ringel duality for Schur algebras. *Math. Proc. Cambridge Philos. Soc.* **132** (2002), 97–116.

24 K. Erdmann and S. Law. Torsion pairs and Ringel duality for Schur algebras. *Algebra Represent. Theory* **26** (2023), 411–432.

25 K. Erdmann and A. E. Parker. On the global and $\nabla$-filtration dimensions of quasi-hereditary algebras. *J. Pure Appl. Algebra* **194** (2004), 95–111.

26 K. Erdmann. Schur algebras of finite type. *Quart. J. Math. Oxford Ser.* **44** (1993), 17–41.

27 K. Erdmann. Symmetric groups and quasi-hereditary algebras. In *Finite Dimensional Algebras and Related Topics. Proceedings of the NATO Advanced Research Workshop on Representations of Algebras and Related Topics. Ottawa, Canada, 10–18 August, 1992*, pp. 123–161 (Dordrecht: Kluwer Academic Publishers, 1994).

28 M. Fang, W. Hu and S. Koenig. On derived equivalences and homological dimensions. *J. Reine Angew. Math.* **770** (2021), 59–85.

29 M. Fang and S. Koenig. Schur functors and dominant dimension. *Trans. Am. Math. Soc.* **363** (2011), 1555–1576.

30 M. Fang and H. Miyachi. Hochschild cohomology and dominant dimension. *Trans. Am. Math. Soc.* **371** (2019), 5267–5292.

31 E. M. Friedlander and B. J. Parshall. Cohomology of Lie algebras and algebraic groups. *Am. J. Math.* **108** (1986), 235–253.

32 J. J. Graham and G. I. Lehrer. Cellular algebras. *Invent. Math.* **123** (1996), 1–34.

33 J. A. Green. *Polynomial representations of $GL_n$*. Lecture Notes in Mathematics, vol. 830 (Berlin, New York: Springer-Verlag, 1980).

34 A. E. Henke. The Cartan matrix of the Schur algebra $S(2,r)$. *Arch. Math. (Basel)* **76** (2001), 416–425.

35 D. J. Hemmer and D. K. Nakano. Specht filtrations for Hecke algebras of type $A$. *J. Lond. Math. Soc., Ser. II* **69** (2004), 623–638.

36 J. Hu and Z. Xiao. Tilting modules, dominant dimensions and Brauer–Schur–Weyl duality. *Trans. Am. Math. Soc., Ser. B* **8** (2021), 823–848.

37 O. Iyama. Finiteness of representation dimension. *Proc. Am. Math. Soc.* **131** (2003), 1011–1014.

38 J. C. Jantzen. Darstellungen halbeinfacher Gruppen und ihrer Frobenius–Kerne. *J. Reine Angew. Math.* **317** (1980), 157–199.

39 M. Jimbo. A $q$-analogue of $U(\mathfrak{gl}(N+1))$. Hecke algebra, and the Yang–Baxter equation. *Lett. Math. Phys.,* **11** (1986), 247–252.

40 V. F. R. Jones. Index for subfactors. *Invent. Math.* **72** (1983), 1–25.

41 V. F. R. Jones. A polynomial invariant for knots via von Neumann algebras. *Bull. Am. Math. Soc., New Ser.* **12** (1985), 103–111.

42 V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. Math.* **126** (1987), 335–388.

43 S. König, I. H. Slungård and C. Xi. Double centralizer properties, dominant dimension, and tilting modules. *J. Algebra* **240** (2001), 393–412.

44 P. P. Martin. On Schur–Weyl duality, $A_n$ Hecke algebras and quantum $sl(N)$ on $\otimes^{n+1}\mathbb{C}^N$. *Int. J. Mod. Phys. A* **7** (1992), 645–673.

45 V. Mazorchuk and S. Ovsienko. Finitistic dimension of properly stratified algebras. *Adv. Math.* **186** (2004), 251–265.

46 V. Mazorchuk and C. Stroppel. Projective–injective modules. Serre functors and symmetric algebras. *J. Reine Angew. Math.* **616** (2008), 131–165.

47 B. Parshall and L. Scott. Derived categories, quasi-hereditary algebras, and algebraic groups. *Carlton Univ. Lect. Notes Math.* **3** (1988), 1–104.

48 C. M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.* **208** (1991), 209–223.

49 R. Rouquier. $q$-Schur algebras and complex reflection groups. *Moscow Math. J.* **8** (2008), 119–158.

50 H. V. N. Temperley and E. H. Lieb. Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem. *Proc. R. Soc. Lond. Ser. A* **322** (1971), 251–280.

51 B. W. Westbury. The representation theory of the Temperley–Lieb algebras. *Math. Z.* **219** (1995), 539–565.