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Exact Distance Colouring in Trees

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For an integer $q \geqslant 2$ and an even integer d, consider the graph obtained from a large complete q-ary tree by connecting with an edge any two vertices at distance exactly d in the tree. This graph has clique number $q+1$, and the purpose of this short note is to prove that its chromatic number is $\Theta\big((d \log q)/\log d\big)$. It was not known that the chromatic number of this graph grows with d. As a simple corollary of our result, we give a negative answer to a problem of van den Heuvel and Naserasr, asking whether there is a constant C such that for any odd integer d, any planar graph can be coloured with at most C colours such that any pair of vertices at distance exactly d have distinct colours. Finally, we study interval colouring of trees (where vertices at distance at least d and at most cd, for some real $c > 1$, must be assigned distinct colours), giving a sharp upper bound in the case of bounded degree trees.

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1. Introduction

Given a metric space X and some real $d > 0$, let $\chi(X, d)$ be the minimum number of colours in a colouring of the elements of X such that any two elements at distance exactly d in X are assigned distinct colours. The classical Hadwiger–Nelson problem asks for the value of $\chi(\mathbb{R}^2, 1)$, where $\mathbb{R}^2$ is the Euclidean plane. It is known that $5 \leqslant \chi(\mathbb{R}^2, 1) \leqslant 7$ [1],

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and since the Euclidean plane $\mathbb{R}^2$ is invariant under homothety, $\chi(\mathbb{R}^2, 1) = \chi(\mathbb{R}^2, d)$ for any real $d > 0$. Let $\mathbb{H}^2$ denote the hyperbolic plane. Kloeckner [3] proved that $\chi(\mathbb{H}^2, d)$ is at most linear in d (the multiplicative constant was recently improved by Parlier and Petit [6]), and observed that $\chi(\mathbb{H}^2, d) \geqslant 4$ for any $d > 0$. He raised the question of determining whether $\chi(\mathbb{H}^2, d)$ grows with d or can be bounded independently of d. As noticed by Kahle (see [3]), it is not known whether $\chi(\mathbb{H}^2, d) \geqslant 5$ for some real $d > 0$. Parlier and Petit [6] recently suggested studying infinite regular trees as a discrete analogue of the hyperbolic plane. Note that any graph G can be considered as a metric space (whose elements are the vertices of G and whose metric is the graph distance in G), and in this context $\chi(G, d)$ is precisely the minimum number of colours in a vertex colouring of G such that vertices at distance d apart are assigned different colours. Note that $\chi(G, d)$ can be equivalently defined as the chromatic number of the *exact dth power* of G, that is, the graph with the same vertex-set as G in which two vertices are adjacent if and only if they are at distance exactly d in G.

Let T_q denote the infinite q-regular tree. Parlier and Petit [6] observed that when d is odd, $\chi(T_q, d) = 2$, and proved that when d is even, $q \leqslant \chi(T_q, d) \leqslant (d+1)(q-1)$. A similar upper bound can also be deduced from the results of van den Heuvel, Kierstead and Quiroz [2], while the lower bound is a direct consequence of the fact that when d is even, the clique number of the exact dth power of T_q is q (note that it does not depend on d). In this short note, we prove that when $q \geqslant 3$ is fixed,

$$\frac{d\log(q-1)}{4\log(d/2)+4\log(q-1)} \leqslant \chi(T_q, d) \leqslant (2+o(1))\frac{d\log(q-1)}{\log d},$$

where the asymptotic $o(1)$ is in terms of d. A simple consequence of our main result is that for any even integer d, the exact dth power of a complete binary tree of depth d is of order $\Theta(d/\log d)$ (while its clique number is equal to 3).

The following problem (attributed to van den Heuvel and Naserasr) was raised in [4] (see also [2] and [5]).

Problem 1.1 (Problem 11.1 in [4]). *Is there a constant C such that for every odd integer d and every planar graph G we have $\chi(G, d) \leqslant C$?*

We will show that our result on large complete binary trees easily implies a negative answer to Problem 1.1. More precisely, we will prove that the graph U_3^d obtained from a complete binary tree of depth d by adding an edge between any two vertices with the same parent gives a negative answer to Problem 1.1 (in particular, for odd d, the chromatic number of the exact dth power of U_3^d grows as $\Theta(d/\log d)$). We will also prove that the exact dth power of a specific subgraph Q_3^d of U_3^d grows as $\Omega(\log d)$. Note that U_3^d and Q_3^d are outerplanar (and thus planar) and chordal (see Figure 2).

Kloeckner [3] proposed the following variant of the original problem. For a metric space X, an integer d and a real $c > 1$, we let $\chi(X, [d, cd])$ denote the smallest number of colours in a colouring of the elements of X such that any two elements of X at

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distance at least d and at most cd apart have distinct colours. Considering as above the natural metric space defined by the infinite q-regular tree T_q, Parlier and Petit [6] proved that

$$q(q-1)^{\lfloor cd/2\rfloor-\lfloor d/2\rfloor} \leqslant \chi(T_q,[d,cd]) \leqslant (q-1)^{\lfloor cd/2+1\rfloor}(\lfloor cd\rfloor+1).$$

We will show that

$$\chi(T_q,[d,cd]) \leqslant \frac{q}{q-2}(q-1)^{\lfloor cd/2\rfloor-d/2+1}+cd+1,$$

which implies that the lower bound of Parlier and Petit [6] (which directly follows from a clique size argument) is asymptotically sharp.

2. Exact distance colouring

Throughout the paper, we assume that the infinite q-regular tree T_q is rooted in some vertex r. This naturally defines the children and descendants of a vertex and the parent and ancestors of a vertex distinct from r. In particular, given a vertex u, we define the ancestors $u^0,u^1,\dots$ of u inductively as follows: $u^0=u$, and for any i such that u^i is not the root, u^{i+1} is the parent of u^i. With this notation, u^d can be equivalently defined as the ancestor of u at distance d from u (if such a vertex exists). For a given vertex u in T_q, the *depth* of u, denoted by depth(u), is the distance between u and r in T_q. For a vertex v and an integer ℓ, we define $L(v,\ell)$ as the set of descendants of v at distance exactly ℓ from v in T_q.

We first prove an upper bound on $\chi(T_q,d)$.

Theorem 2.1. *For any integer $q\geqslant 3$, any even integer d, and any integer $k\geqslant 1$ such that $k(q-1)^{k-1}\leqslant d$, we have*

$$\chi(T_q,d) \leqslant (q-1)^k+(q-1)^{\lfloor k/2\rfloor}+\frac{d}{k}+1.$$

In particular, $\chi(T_q,d)\leqslant d+q+1$, and when q is fixed and d tends to infinity,

$$\chi(T_q,d) \leqslant (2+o(1))\frac{d\log(q-1)}{\log d}.$$

Proof. A vertex of T_q distinct from r and whose depth is a multiple of k is said to be a *special vertex*. Let v be a special vertex. Every special vertex u distinct from v such that $u^k=v^k$ is called a *cousin* of v. Note that v has at most $q(q-1)^{k-1}-1$ cousins (at most $(q-1)^k-1$ if $v^k\neq r$). A special vertex u is said to be a *relative* of v if u is either a cousin of v, or u has the property that u and v^k have the same depth and are at distance at most k apart in T_q. Two vertices a,b at distance at most k apart and at the same depth must satisfy $a^{\lfloor k/2\rfloor}=b^{\lfloor k/2\rfloor}$, and so the number of vertices u such that u and v^k have the same depth and are at distance at most k apart in T_q is $(q-1)^{\lfloor k/2\rfloor}$. It follows that if $v^k=r$, then v has at most $q(q-1)^{k-1}-1$ relatives and otherwise v has at most $(q-1)^k+(q-1)^{\lfloor k/2\rfloor}-1$ relatives.

The first step is to define a colouring C of the special vertices of T_q. This will be used later to define the desired colouring of T_q, that is, a colouring such that vertices of T_q at distance d apart are assigned distinct colours (in this second colouring, the special vertices will not retain their colour from C).

We greedily assign a colour $C(v)$ to each special vertex v of T_q as follows. We consider the vertices of T_q in a breadth-first search starting at r, and for each special vertex v we encounter, we assign to v a colour distinct from the colours already assigned to its relatives, and from the set of ancestors v^{ik} of v, where $2 \leqslant i \leqslant d/k+1$ (there are at most d/k such vertices). Note that if $v^k = r$, the number of colours forbidden for v is at most $q(q-1)^{k-1}-1$, and if $v^k \neq r$ the number of colours forbidden for v is at most $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + d/k - 1$. Since $k(q-1)^{k-1} \leqslant d$, in both cases v has at most $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + d/k - 1$ forbidden colours, therefore we can obtain the colouring C by using at most $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + d/k$ colours.

For any special vertex v, the set of descendants of v at distance at least $d/2-k$ and at most $d/2-1$ from v is denoted by $K(v,k)$. We now define the desired colouring of T_q as follows: for each special vertex v, all the vertices of $K(v,k)$ are assigned the colour $C(v)$. Finally, all the vertices at distance at most $d/2-1$ from r are coloured with a single new colour (note that any two vertices in this set lie at distance less than d apart). The resulting vertex-colouring of T_q is called c. Note that c uses at most $(q-1)^k + (q-1)^{\lfloor k/2 \rfloor} + d/k + 1$ colours, and indeed every vertex of T_q gets exactly one colour.

We now prove that vertices at distance d apart in T_q are assigned distinct colours in c. Assume for the sake of contradiction that two vertices x and y at distance d apart were assigned the same colour. Then the depth of both x and y is at least $d/2$. We can assume by symmetry that the difference t between the depth of x and the depth of y is such that $0 \leqslant t \leqslant d$, since otherwise they would be at distance more than d. Let u be the unique (special) vertex of T_q such that $x \in K(u,k)$ and let v be the unique (special) vertex such that $y \in K(v,k)$. By the definition of our colouring c, we have $C(u) = C(v)$. Note that u and v are distinct; indeed, otherwise x and y would not be at distance d in T_q. Assume first that u and v have the same depth. Then since u and x (resp. v and y) are distance at least $d/2-k$ apart, u and v are cousins (and thus relatives), which contradicts the definition of the vertex-colouring C. We may therefore assume that the depths of u and v are distinct. Moreover, since u and v are special vertices, we may assume that their depths differ by at least k. In particular, u lies deeper than v in T_q.

First assume that the depths of u and v differ by at least $2k$. Then v is not an ancestor of u in T_q. Indeed, for otherwise we would have $v = u^{ik}$ for some integer $2 \leqslant i \leqslant d/k+1$, which would contradict the definition of C. This implies that the distance between x and y is at least $d/2-k+d/2-k+2k+2 = d+2$, which is a contradiction. Therefore, we can assume that the depths of u and v differ by precisely k. Since v is not a relative of u, we have that $v \neq u^k$ and the distance between u^k and v is more than k. Moreover, since u and x (resp. v and y) are at distance at least $d/2-k$ apart, this implies that the distance between x and y is more than $d/2-k+k+k+d/2-k = d$, a contradiction. Thus c is a proper colouring.

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By taking $k = 1$ we obtain a colouring c using at most $(q-1)^1 + (q-1)^{\lfloor 1/2 \rfloor} + d/1 + 1 = q + d + 1$ colours, and by taking

$$k = \left\lfloor \frac{\log d - \log\log d + \log\log(q-1)}{\log(q-1)} \right\rfloor,$$

we obtain a colouring c using at most

$$\frac{d\log(q-1)}{\log d} + \sqrt{\frac{d\log(q-1)}{\log d}} + \frac{d\log(q-1)}{\log d - \log\log d + \log\log(q-1) - \log(q-1)} + 1$$
$$= (2+o(1))\frac{d\log(q-1)}{\log d}$$

colours. □

For $k = 1$, the proof above can be optimized to show that $\chi(T_q, d) \leqslant q + d/2$ (by simply noting that vertices at even depth and vertices at odd depth can be coloured independently). Since we are mostly interested in the asymptotic behaviour of $\chi(T_q, d)$ (which is of order $O(d/\log d)$), we omit the details.

We now prove a simple lower bound on $\chi(T_q, d)$. Let T_q^d be the rooted complete $(q-1)$-ary tree of depth d, with root r. Note that each node has $q-1$ children, so this graph is a subtree of T_q.

Theorem 2.2. *For any integer $q \geqslant 3$ and any even d,*

$$\chi(T_q^d, d) \geqslant \log_2\left(\frac{d}{4} + q - 1\right).$$

Proof. Consider any colouring of T_q^d with colours $1, 2, \ldots, C$, such that vertices at distance precisely d apart have distinct colours. For any vertex v at depth at most $d/2 + 1$ in T_q^d, the set of colours appearing in $L(v, d/2 - 1)$ is denoted by S_v. Observe that if v and w have the same parent, then S_v and S_w are disjoint, since for any $x \in L(v, d/2 - 1)$ and $y \in L(w, d/2 - 1)$, x and y are at distance d.

Fix some vertex u at depth at most $d/2$ in T_q^d and some child v of u.

Claim 2.3. *For any integer $1 \leqslant k \leqslant \mathrm{depth}(u)/2$, there is a colour of $S_{u^{2k-1}}$ that does not appear in S_v.*

To see that Claim 2.3 holds, observe that in the subtree of T_q^d rooted in u^k, there is a vertex of $L(u^{2k-1}, d/2 - 1)$ at distance d from all the elements of $L(v, d/2 - 1)$. The colour of such a vertex does not appear in S_v, therefore Claim 2.3 holds.

In particular, Claim 2.3 implies that all the sets

$$\{S_{u^{2k-1}} \mid 1 \leqslant k \leqslant d/4\} \quad \text{and} \quad \{S_w \mid w \text{ is a child of } u\}$$

are pairwise distinct. Since there are $d/4 + q - 1$ such sets, we have $d/4 + q - 1 \leqslant 2^C$ and therefore $C \geqslant \log_2(d/4 + q - 1)$, as desired. □

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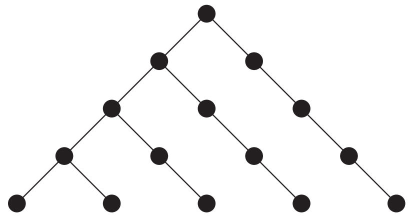

Figure 1. The graph P_3^4.

It was observed by Stéphan Thomassé that the proof of Theorem 2.2 only uses a small fraction of the graph T_q^d. Consider for simplicity the case $q = 3$, and define P_3^d as the graph obtained from a path $P = v_0, v_1, \ldots, v_d$ on d edges, by adding, for each $1 \leqslant i \leqslant d$, a path on i edges ending at v_i (see Figure 1). This graph is an induced subgraph of T_q^d and the proof of Theorem 2.2 directly shows the following.[1]

Corollary 2.4. *For any even integer d, $\chi(P_3^d, d) \geqslant \log_2(d+8) - 2$.*

The proof of Theorem 2.2 can be refined to prove the following better estimate for T_q^d, showing that the upper bound of Theorem 2.1 is (asymptotically) tight within a constant multiplicative factor of 8.

Theorem 2.5. *For any integer $q \geqslant 3$ and every even integer $d \geqslant 2$,*

$$\chi(T_q^d, d) \geqslant \frac{d \log(q-1)}{4\log(d/2) + 4\log(q-1)}.$$

Proof. Consider any colouring of T_q^d with colours $1, 2, \ldots, C$, such that vertices at distance precisely d apart have distinct colours. We perform a random walk $v_0, v_1, \ldots, v_d$ in T_q^d as follows: we start with $v_0 = r$, and for each $i \geqslant 1$, we choose a child of v_i uniformly at random and set it as v_{i+1}. Note that the depth of each vertex v_i is precisely i.

From now on we fix a colour $c \in \{1, \ldots, C\}$. For any vertex v of T_q^d, the set of vertices contained in the subtree of T_q^d rooted in v is denoted by V_v, and we set

$$A_v = \{\text{depth}(u) \mid u \in V_v \text{ and } u \text{ has colour } c\}.$$

When $v = v_i$, for some integer $0 \leqslant i \leqslant d$, we write A_i instead of A_{v_i}.

Claim 2.6. *Assume that for some even integers i and j with $2 \leqslant i < j \leqslant d$, and for some vertex v at depth $(i+j-d)/2$, the set A_v contains both i and j. Then v has precisely one child u such that A_u contains i and j, and moreover all the children w of v distinct from u are such that A_w contains neither i nor j.*

[1] Stéphan Thomassé noticed that this can also be deduced from the fact that the vertices at depth at least $d/2$ and at most d in the exact dth power of P_3^d induce a *shift graph*.

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To see that Claim 2.6 holds, simply note that $(i+j-d)/2 < i < j$ and if two vertices u_1, u_2 coloured c are respectively at depths i and j, and their common ancestor is v, then they are at distance d in T_q^d (which contradicts the fact that they were assigned the same colour). Indeed, the distance of u_1 to v is $i-(i+j-d)/2$ and the distance of u_2 to v is $j-(i+j-d)/2$. This proves the claim.

We now define a family of graphs $(G_k)_{0\leqslant k\leqslant d/2}$ as follows. For any $0 \leqslant k \leqslant d/2$, the vertex-set $V(G_k)$ of G_k is the set $A_k \cap 2\mathbb{N} \cap (d/2, d]$, and two (distinct) even integers $i, j \in A_k$ are adjacent in G_k if and only if $(i+j-d)/2 < k$. For each $0 \leqslant k \leqslant d/2$ we define the *energy* $\mathcal{E}_k$ of G_k as follows:

$$\mathcal{E}_k = \sum_{i\in V(G_k)} (q-1)^{\deg(i)},$$

where $\deg(i)$ denotes the degree of the vertex i in G_k.

Note that each graph G_k depends on the (random) choice of $v_1, v_2, \ldots, v_k$.

Claim 2.7. *For any* $0 \leqslant k \leqslant d/2-1$, $\mathbb{E}(\mathcal{E}_{k+1}) \leqslant \mathbb{E}(\mathcal{E}_k)$.

Assume that $v_1, v_2, \ldots, v_k$ (and therefore also G_k) are fixed. Observe that G_{k+1} is obtained from G_k by possibly removing some vertices and adding some edges. Thus, $\mathcal{E}_{k+1}$ can be larger than $\mathcal{E}_k$ only if G_{k+1} contains edges that are not in G_k. Therefore, it suffices to consider the contributions of those pairs of non-adjacent vertices in G_k which could become adjacent in G_{k+1} (since these correspond to pairs i, j with $k = (i+j-d)/2$, these pairs are pairwise disjoint), and prove that these contributions are, in expectation, equal to 0. Fix a pair of even integers $i < j$ in $V(G_k)$ with $k = (i+j-d)/2$ (and note that i and j are not adjacent in G_k). By Claim 2.6, either v_{k+1} is such that A_{k+1} contains i and j (this event occurs with probability $1/(q-1)$), or A_{k+1} contains neither i nor j (with probability $1-1/(q-1)$). As a consequence, for any $i < j$ in $V(G_k)$ with $k = (i+j-d)/2$, with probability $1/(q-1)$ we add the edge ij in G_{k+1} and with probability $1-1/(q-1)$ we remove vertices i and j from G_{k+1}. This implies that for any $i, j \in V(G_k)$, $i < j$, with $k = (i+j-d)/2$, with probability $1/(q-1)$ we have contribution at most

$$(q-1)^{\deg(i)+1} + (q-1)^{\deg(j)+1} - (q-1)^{\deg(i)} - (q-1)^{\deg(j)} = (q-2)((q-1)^{\deg(i)} + (q-1)^{\deg(j)})$$

to $\mathcal{E}_{k+1}$ (where deg refers to the degree in G_k) and with probability $1-1/(q-1)$ we have a contribution of at most $-(q-1)^{\deg(i)} - (q-1)^{\deg(j)}$ to $\mathcal{E}_{k+1}$. Thus, the expected contribution of such a pair i, j is at most

$$\frac{1}{q-1}(q-2)((q-1)^{\deg(i)} + (q-1)^{\deg(j)}) - \frac{q-2}{q-1}((q-1)^{\deg(i)} + (q-1)^{\deg(j)}) = 0.$$

Summing over all such pairs i, j, we obtain $\mathbb{E}(\mathcal{E}_{k+1}) \leqslant \mathbb{E}(\mathcal{E}_k)$. This proves Claim 2.7.

Since $2 \leqslant i < j \leqslant d$, we have

$$\frac{i+j-d}{2} \leqslant \frac{d}{2} - 1,$$

and in particular it follows that $G_{d/2}$ is a (possibly empty) complete graph, whose number of vertices is denoted by $\omega \geqslant 0$. Note that the energy $\mathcal{E}$ of a complete graph on ω vertices

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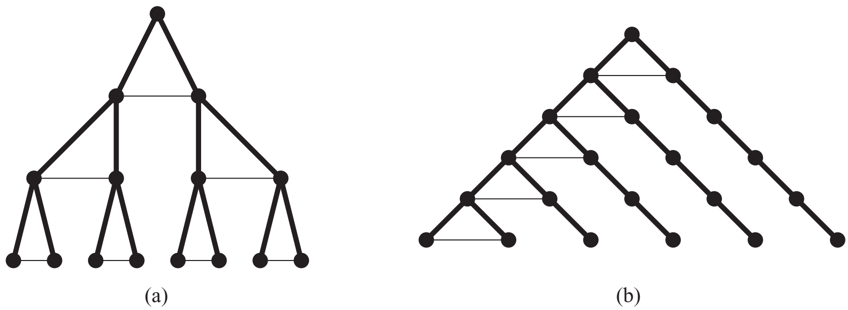

Figure 2. The graphs U_3^3 (a) and Q_3^5 (b). The bold edges represent the original copies of T_3^3 and P_3^5, respectively.

is equal to $\omega(q-1)^{\omega-1}$, while the energy $\mathcal{E}_0$ of G_0 is equal to $|A_0 \cap 2\mathbb{N} \cap (d/2, d]| \leqslant d/4$. For a vertex $u \in L(r, d/2)$, let $\omega_u = |A_u \cap 2\mathbb{N} \cap (d/2, d]|$ (this is the number of distinct even depths at which a vertex coloured c appears in the subtree of height $d/2$ rooted in u). It follows from Claim 2.7 that the average of $\omega_u(q-1)^{\omega_u-1}$, over all vertices $u \in L(r, d/2)$, is at most $d/4$. Let a be the average of ω_u, over all vertices $u \in L(r, d/2)$. By Jensen's inequality and the convexity of the function $x \mapsto x(q-1)^{x-1}$ for $x \geqslant 0$, we have that $a(q-1)^{a-1} \leqslant d/4$, and thus

$$a \leqslant \frac{\log(d/2)}{\log(q-1)} + 1.$$

Note that a depends on the colour c under consideration (to make this more explicit, let us now write a_c instead of a). Since there are $d/4$ even depths between depth $d/2$ and depth d, there is a colour $c \in \{1, \dots, C\}$ such that $a_c \cdot C \geqslant d/4$ and thus

$$C \geqslant \frac{d}{4a_c} \geqslant \frac{d \log(q-1)}{4\log(d/2) + 4\log(q-1)},$$

as desired. □

We now explain how the results proved above give a negative answer to Problem 1.1. Let U_3^d (resp. Q_3^d) be obtained from T_3^d (resp. P_3^d) by adding an edge uv for any pair of vertices u, v having the same parent. Note that for any d, U_3^d and Q_3^d are outerplanar (and thus planar) and chordal, and Q_3^d has pathwidth 2 (U_3^3 and Q_3^5 are depicted in Figure 2) and the original copies of T_3^d and P_3^d are spanning trees of U_3^d and Q_3^d, respectively. In the remainder of this section, whenever we write T_3^d, we mean *the original copy of* T_3^d *in* U_3^d.

Observe that for any two vertices u and v distinct from the root of T_3^d, u and v are at distance d in T_3^d if and only if they are at distance $d-1$ in U_3^d (since the depth of T_3^d is d, the fact that u and v differ from the root and are at distance d apart implies that none of the two vertices is an ancestor of the other). The same property holds for Q_3^d and P_3^d. As a consequence, for any odd integer d, $\chi(U_3^{d+1}, d)$ and $\chi(T_3^{d+1}, d+1)$ differ by at most one, and $\chi(Q_3^{d+1}, d)$ and $\chi(P_3^{d+1}, d+1)$ also differ by at most one. Using this observation, we immediately obtain the following corollary of Theorem 2.5 and Corollary 2.4, which gives a negative answer to Problem 1.1.

Corollary 2.8. *For any odd integer d,*

$$\chi(U_3^{d+1}, d) \geqslant \frac{(d+1)\log(2)}{4\log((d+1)/2) + 4\log(2)} - 1 \quad \textit{and} \quad \chi(Q_3^{d+1}, d) \geqslant \log_2(d+8) - 3.$$

The graphs U_3^{d+1} and its exact dth power have $n = 2^{d+2}$ vertices, and thus the chromatic number of the exact dth power of U_3^{d+1} grows as

$$\Omega\left(\frac{\log n}{\log\log n}\right).$$

The graphs Q_3^{d+1} and its exact dth power have $n = \binom{d+2}{2}$ vertices, and thus the chromatic number of the exact dth power of Q_3^{d+1} grows as $\Omega(\log n)$. It is not difficult (using Theorem 2.1 for U_3^{d+1}) to show that these bounds are asymptotically tight.

It was recently proved by Quiroz [8] that if G is a chordal graph of clique number at most $t \geqslant 2$, and d is an odd number, then $\chi(G, d) \leqslant \binom{t}{2}(d+1)$. By Corollary 2.8, the graph U_3^d shows that this is asymptotically best possible (as d tends to infinity), up to a $\log d$ factor.

3. Interval colouring

For an integer d and a real $c > 1$, recall that $\chi(T_q, [d, cd])$ denotes the smallest number of colours in a colouring of the vertices of T_q such that any two vertices of T_q at distance at least d and at most cd apart have distinct colours. Parlier and Petit [6] proved that

$$q(q-1)^{\lfloor cd/2 \rfloor - \lfloor d/2 \rfloor} \leqslant \chi(T_q, [d, cd]) \leqslant (q-1)^{\lfloor cd/2+1 \rfloor}(\lfloor cd \rfloor + 1).$$

In this final section, we prove that their lower bound (which is proved by finding a set of vertices of this cardinality that are pairwise at distance at least d and at most cd apart in T_q) is asymptotically tight.

Theorem 3.1. *For any integers $q \geqslant 3$ and d and any real $c > 1$,*

$$\chi(T_q, [d, cd]) \leqslant \frac{q}{q-2}(q-1)^{\lfloor cd/2 \rfloor - d/2 + 1} + cd + 1.$$

Proof. The proof is similar to the proof of Theorem 2.1. We consider any ordering $e_1, e_2, \ldots$ of the edges of T_q obtained from a breadth-first search starting at r. Then, for any $i = 1, 2, \ldots$ in order, we assign a colour $c(e_i)$ to the edge e_i as follows. Let $e_i = uv$, with u being the parent of v, and let $\ell = \lfloor cd/2 \rfloor - d/2$. We assign to uv a colour $c(uv)$ distinct from the colours of all the edges xy (with x being the parent of y) such that x is at distance at most ℓ from u^k (where k is the minimum of ℓ and the depth of u), or x is an ancestor of u at distance at most cd from u (and y lies on the path from u to x). There are at most

$$cd + \sum_{j=0}^{\ell} q(q-1)^j \leqslant \frac{q}{q-2}(q-1)^{\ell+1} + d - 1$$

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such edges, so we can colour all the edges following this procedure by using a total of at most $q/(q-2)(q-1)^{\ell+1}+cd$ colours.

As in the proof of Theorem 2.1, we now define our colouring of the vertices of T_q as follows: first colour all the vertices at distance at most $d/2-1$ from r with a new colour that does not appear on any edge of T_q, then for each vertex v with parent u, we colour all the vertices of $L(v, d/2-1)$ with colour $c(uv)$. In this vertex-colouring, at most $q/(q-2)(q-1)^{\ell+1}+cd+1$ colours are used.

Assume that two vertices s and t, at distance at least d and at most cd apart, were assigned the same colour. This implies that $c(s^{d/2-1}s^{d/2}) = c(t^{d/2-1}t^{d/2})$. Assume without loss of generality that the depth of s is at least the depth of t, and consider first the case where $t^{d/2-1}$ is an ancestor of s. Then $t^{d/2}$ is an ancestor of $s^{d/2}$ at distance at most cd from $s^{d/2}$ (and $t^{d/2-1}$ lies on the path from $s^{d/2}$ to $t^{d/2}$), which contradicts the definition of our edge-colouring c. Thus, we can assume that $t^{d/2-1}$ is not an ancestor of s. This implies that $t^{d/2-1}t^{d/2}$ lies on the path between s and t, and therefore $t^{d/2}$ is at distance at most $\ell = \lfloor cd/2 \rfloor - d/2$ from the ancestor of $s^{d/2}$ at distance ℓ from $s^{d/2}$ (or simply from r, if the depth of $s^{d/2}$ is at most ℓ). Again, this contradicts the definition of our colouring c. We obtained a colouring of the vertices of T_q with at most $q/(q-2)(q-1)^{\ell+1}+cd+1$ colours in which each pair of vertices at distance at least d and at most cd apart have distinct colours, as desired. □

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