Probability in the Engineering and Informational Sciences, **14**, 2000, 81–99. Printed in the U.S.A.

ASYMPTOTIC EVALUATION OF BLOCKING PROBABILITIES IN A HIERARCHICAL CELLULAR MOBILE NETWORK

Stefan Verwijmeren
Universiteit van Amsterdam
Department of Operations Research and Management
1018 WB Amsterdam, The Netherlands

Michel Mandjes*
Bell Labs/Lucent Technologies
Murray Hill, New Jersey 07974

Richard J. Boucherie
Universiteit van Amsterdam
Department of Operations Research and Management
1018 WB Amsterdam, The Netherlands

This paper investigates blocking probabilities obtained from multidimensional truncated Poisson distributions. For blocking probabilities typically arising in layered cellular mobile communications networks, the large deviations results of Gazdzicki et al. [9] are extended to state spaces determined by multiple constraints. The results yield asymptotically exact expressions that provide an accurate approximation of probabilities up to 1%, which considerably extends the applicability of large deviations results and enables efficient approximation of blocking probabilities for realistic mobile communications networks.

1. INTRODUCTION

1.1. Motivation

Truncated multidimensional Poisson distributions frequently arise in the study of (mobile) telecommunications networks and are typically of the form

*This work was done while this author was at KPN Research, 2260 AK Leidschendam, The Netherlands.

 0269-9648/00 $12.50

https://doi.org/10.1017/S0269964800141075 Published online by Cambridge University Press

$$\pi(\mathbf{m}) = G^{-1}\prod_{j=0}^{d}\frac{\nu_j^{m_j}}{m_j!}, \qquad \mathbf{m} = (m_0,\ldots,m_d) \in S, \qquad G = \sum_{\mathbf{m}\in S}\prod_{j=0}^{d}\frac{\nu_j^{m_j}}{m_j!}. \qquad \textbf{(1.1)}$$

Often, the state space, S, is determined by a matrix constraint

$$S = \{\mathbf{m} : A\mathbf{m} \le \mathbf{C}\} \subset \mathbb{N}_0^{d+1}, \qquad \textbf{(1.2)}$$

in which A is a $p \times (d+1)$ matrix and $\mathbf{C}$ is a p-vector, where p is the number of constraints. For example, a loss network [10] with p links, where link α comprises C_α circuits and a call on route j uses $A_{\alpha j}$ circuits from link α, has a state-space representation (1.2). Then, m_j is the number of calls in progress on route j, and ν_j is the load offered to route j. Alternatively, a state space of the form (1.2) emerges in a mobile communications network [3,4,12] consisting of $d+1$ cells, where a set of cells α shares C_α channels. A call in cell j simultaneously blocks (due to interference) channels in neighboring cells in the set α determined by $A_{\alpha j} > 0$. Here, ν_j is the load offered to cell j, and m_j is the number of calls in progress in cell j.

Relevant performance measures can be obtained in closed form from the distribution (1.1). For example, the probability that an additional call in cell j of a mobile communications network (or on route j of a loss network) cannot be accepted due to capacity restrictions (fresh-call blocking probability) can be expressed as the summation of π over a part of the boundary of the state space:

$$P\{B_j\} = \frac{\displaystyle\sum_{\mathbf{m}\in T_j}\prod_{k=0}^{d}(\nu_k^{m_k}/m_k!)}{\displaystyle\sum_{\mathbf{m}\in S}\prod_{k=0}^{d}(\nu_k^{m_k}/m_k!)}, \qquad \textbf{(1.3)}$$

with

$$T_j := \{\mathbf{m} : A\mathbf{m} \le \mathbf{C}, A(\mathbf{m}+\mathbf{e}_j) \not\le \mathbf{C}\}, \qquad \textbf{(1.4)}$$

where $\mathbf{e}_j$ denotes the jth unit vector. Despite the explicit expression (1.3), numerical evaluation of the fresh-call blocking probability (e.g., via recursive methods [6] or generating function methods [5]) is often extremely time-consuming, as the size of the state space (and of the set T_j) grows exponentially fast with the size of the network. As an alternative, Monte Carlo summation [7,9] can be applied to obtain an estimate of the blocking probability (1.3). Especially for smaller blocking probabilities, methods based on importance sampling can be used to improve on the efficiency of such methods (see, e.g., [3,14]). However, the resulting methods are still very demanding with respect to their required computation time.

As an alternative, asymptotic evaluation of blocking probabilities might lead to a fast and accurate approximation. For heavily or moderately loaded networks (determined by $A\boldsymbol{\nu} \not\le \mathbf{C}$, $\boldsymbol{\nu} = (\nu_0,\ldots,\nu_d)$), approximations based on the central limit theorem (normal approximation, reduced load methods [10]) might be used. For lightly loaded networks ($A\boldsymbol{\nu} < \mathbf{C}$), such methods might not lead to satisfactory approximations. In that domain, methods based on Cramér's theorem [15] and the

Bahadur–Rao estimate for deviations of the sample mean [1] are more applicable. Roughly, after scaling $\boldsymbol{\nu} \to n\boldsymbol{\nu}$ and $\mathbf{C} \to n\mathbf{C}$ $(n \to \infty)$, the blocking probability $P\{B_j\}$ can be approximated as

$$P\{B_j\} \sim \eta(n) e^{-nI(\mathbf{x}_0)} \qquad (n \to \infty),$$

where $\eta(n)$ is a subexponential function, with $\lim_{n\to\infty}(1/n)\log\eta(n) = 0$, and $I(\mathbf{x}_0)$ is the large deviations rate function (evaluated in the state $\mathbf{x}_0$ determined by T_j) that can be explicitly obtained for blocking probabilities resulting from multidimensional Poisson distributions [see (2.7)]. For small loss probabilities ($P\{B_j\} \sim 10^{-9}$), the large deviations rate function usually provides an accurate approximation: In that range the exponential $e^{-nI(\mathbf{x}_0)}$ dominates the expression. For larger loss probabilities, this is no longer the case, and additional information on $\eta(n)$ is required. In a one-dimensional setting, Bahadur and Rao [1] provided an explicit expression of the function $\eta(n)$, and for a loss network with a single restriction on the state space ($S = \{\mathbf{m} : \sum_i m_i \le C\}$), Gazdzicki et al. [8] obtained a similar result. In general, for state spaces determined by multiple restrictions, an explicit evaluation of $\eta(n)$ leading to a satisfactory approximation of $P\{B_j\}$ is not yet available in the literature. Such results require special structure on the state space S. The state space of a layered mobile communications network has a special structure that enables asymptotic evaluation of $\eta(n)$ to approximate blocking probabilities as large as $P\{B_j\} \sim 10^{-2}$. The contribution of this paper is an explicit analysis of blocking probabilities for such networks, thus extending the applicability of large-deviations-based methods to also evaluate moderate blocking probabilities.

1.2. Background for Modeling Assumptions

Capacity for wireless communications is severely limited. Therefore, the area covered by providers of wireless services is divided into cells, transmissions in each cell use a part of the spectral capacity, and this capacity is reused in cells that are sufficiently far away to avoid interference. Reuse of capacity substantially increases the network capacity. This capacity can be further increased using hierarchical network structures, where microcells are placed in areas with a higher than average density of communications. Microcells cover a small area and do not give rise to interference problems. Thus, a layered or hierarchical cellular mobile network consists of macrocells and microcells; a number of microcells is contained in the coverage area of a macrocell. Both macrocells and microcells allocate a number of channels. Channels in the microcells can only be used by that particular microcell, yet a prespecified part of the channels in the macrocell can be used by any microcell underneath that macrocell. Under the assumptions yielding a truncated Poisson distribution for the number of calls in the cells, blocking in different macrocells with underlying microcells can be treated separately; see [3]. Therefore, in the following, we provide a detailed description of a single macrocell. Let cell 0 denote the macrocell, and cell j, $j = 1, \ldots, d$, the underlying microcells. Cell i has c_i channels available. In addition, the macrocell has $c_{(h)}$ channels that can be shared by the underlying microcells. Let

m_j denote the number of calls in cell j, $j = 0, \ldots, d$. The restrictions on the state space can then be summarized as

$$\sum_{j \in \alpha} m_j \leq \sum_{j \in \alpha} c_j + c_{(h)}, \qquad \alpha \in D, \tag{1.5}$$

where $D = \{$all possible combinations of cells $j \in \{0, 1, \ldots, d\}\}$. This gives a total of $p = 2^{d+1} - 1$ restrictions. Restrictions (1.5) ensure that the channels $c_{(h)}$ can be used only once, because each possible combination of microcells and the overlapping macrocell is included. The set of restrictions (1.5) can be written as $A\mathbf{m} \leq \mathbf{C}$, where A is a 0–1 matrix, and $C_\alpha = \sum_{j \in \alpha} c_j + c_{(h)}$. For layered mobile communications networks, besides the fresh-call blocking probability $P\{B_j\}$ as given in (1.3), determining the probability that a new call generated in cell j cannot be accepted due to lack of capacity, one of the most important performance measures is the handover blocking probability. A handover occurs when a user moves from cell i to a neighboring cell j and leaves the area where a channel from cell i can be used at sufficient quality. Then, the call is "handed over" from a channel in cell i to a channel in the neighboring cell j. If cell j has no available channels, the call is blocked and therefore interrupted. Obviously, handover blocking should be avoided in practical networks as the service degradation due to interruption of existing calls is severer than that due to fresh-call blocking. The handover blocking probability for a call moving from cell i to cell j is given by [3]:

$$P\{B_{ij}\} = \frac{\displaystyle\sum_{\mathbf{m} \in T_{ij}} \prod_{k=0}^{d} (\nu_k^{m_k}/m_k!)}{\displaystyle\sum_{\mathbf{m} \in U_i} \prod_{k=0}^{d} (\nu_k^{m_k}/m_k!)}, \tag{1.6}$$

with

$$T_{ij} := \{\mathbf{m} : A(\mathbf{m} + \mathbf{e}_i) \leq \mathbf{C}, A(\mathbf{m} + \mathbf{e}_j) \not\leq \mathbf{C}\}, \qquad U_i := \{\mathbf{m} : A(\mathbf{m} + \mathbf{e}_i) \leq \mathbf{C}\}. \tag{1.7}$$

Application of the results of this paper is not restricted to cellular mobile networks. It is the structure of the state space (1.5) that determines applicability of the results. For example, overflow models [16] and retrial queues [2] can give rise to a similar structure of the state space and blocking probabilities.

1.3. Contribution of the Paper

The analysis presented in this paper requires the special structure of the state space as determined by (1.5). Besides the 0–1 structure of A, an important property of the state space $S = \{\mathbf{m} \in \mathbb{N}_0^{d+1} : A\mathbf{m} \leq \mathbf{C}\}$ is that the faces of the polytope determined by $A\mathbf{m} \leq \mathbf{C}$ cannot be orthogonal. This enables us to conclude that call loss (fresh-call and handover blocking) is determined by at most two constraints. This is a crucial observation that allows us to generalize the results of Gazdzicki et al. [8] to state

https://doi.org/10.1017/S0269964800141075 Published online by Cambridge University Press

spaces determined by multiple constraints as described by (1.5). At both constraints, again due to the 0–1 structure of A, evaluation of blocking probabilities reduces to a one-dimensional problem for which Petrov [13] has provided integral and local limit theorems that determine $\eta(n)$ up to $O(n^{-3/2})$. Additional terms can be obtained for $\eta(n)$; for example, invoking results of Bahadur and Rao [1], but results of $O(n^{-3/2})$ provide a sufficiently accurate approximation for blocking probabilities up to 10^{-2}, as is illustrated by numerical tests.

The organization of this paper is as follows: Section 2 provides preliminary results related to the asymptotic regime. Our large deviations approximation is developed in Section 3, and its accuracy is illustrated in Section 4.

2. PRELIMINARIES

Consider a truncated multivariate Poisson distribution (1.1) at state space (1.2) determined by restrictions (1.5). For this distribution, we are interested in blocking probabilities as expressed in (1.3) and (1.6) under the assumption that the network is lightly loaded (i.e., that $A\boldsymbol{\nu} < \mathbf{C}$). In fact, to avoid technical problems, for handover blocking probabilities we will assume that

$$\nu_i < c_i, \qquad i = 0, \ldots, d. \tag{2.1}$$

This assumption restricts the results. However, for $\nu_i \geq c_i$, the blocking probabilities will be too large for the asymptotics to provide accurate results. Therefore, assumption (2.1) is not a restriction on the range of applicability of our results.

The first step enabling the approximation of the blocking probabilities (1.3) and (1.6) is multiplying both numerator and denominator by $\prod_k e^{-\nu_k}$. Expression (1.3) obtained for the fresh-call blocking probabilities in cell j can then be written as the ratio of two multidimensional Poisson probabilities:

$$P\{B_j\} = \frac{\sum_{\mathbf{m}\in T_j} \prod_{k=0}^{d} (\nu_k^{m_k}/m_k!)e^{-\nu_k}}{\sum_{\mathbf{m}\in S} \prod_{k=0}^{d} (\nu_k^{m_k}/m_k!)e^{-\nu_k}}. \tag{2.2}$$

As a consequence, both numerator and denominator can be separately evaluated using multivariate Poisson distributions. Here, Monte Carlo simulation has been proposed in the literature. For example, direct methods estimating numerator and denominator can be applied; the Harvey–Hills method [9] is a more efficient Monte Carlo technique using the fact that $T_j \subset S$ to estimate the conditional probability $P\{T_j|S\}$. For smaller blocking probabilities, importance sampling techniques have been developed. Ross and Wang [14] provide a heuristic method that shifts the parameter $\boldsymbol{\nu}$ of the multidimensional Poisson distribution toward the boundary by a factor of roughly 10%. Boucherie and Mandjes [3] present an importance sampling method based on large deviations theory. As the denominator corresponds to an event that occurs with large probability, the denominator is estimated via direct

https://doi.org/10.1017/S0269964800141075 Published online by Cambridge University Press

Monte Carlo summation. The numerator is estimated via an exponentially twisted density, which can be shown to be an asymptotically optimal change of measure (also see [11]). In the following, we will improve on this method, showing that the numerator can be approximated efficiently using large deviations theory. The denominator can be efficiently estimated via other methods. Therefore, we will focus our attention on the multidimensional Poisson probability for the sets T_j and T_{ij}; recall (1.4) and (1.7). These probabilities can be estimated efficiently by scaling of input and capacity, a procedure frequently used in analysis of blocking probabilities in large circuit-switched networks (see, e.g., [10]).

Scaling of load and capacity, $\nu_i \to n\nu_i$ and $C_i \to nC_i$, for $n \to \infty$, obviously also influences the sets T_j, T_{jk}, U_j, and S; recall (1.2), (1.4), and (1.7). Applying the scaling to these sets refines the raster but leaves the area unaffected. Let $T_{j,n}$, $T_{jk,n}$, $U_{j,n}$, and S_n denote the sets obtained from T_j, T_{jk}, U_j, and S, respectively, by replacing $C_i \to nC_i$ and refining the grid by substituting $\mathbf{m}_n = \mathbf{m}/n$ instead of $\mathbf{m}$. Let $\overline{T}_j$, $\overline{T}_{ij}$, $\overline{U}_i$, and $\overline{S}$ be the limiting sets for $n \to \infty$. Then, from the expressions for $\overline{T}_j$, $\overline{T}_{ij}$, $\overline{U}_i$, and $\overline{S}$, as obtained in [3],

$$
\begin{aligned}
S_n &= \{\mathbf{m}_n : \mathbf{m}_n = \mathbf{m}/n,\, A\mathbf{m}_n \le \mathbf{C}\} \\
\to \overline{S} &= \{\mathbf{x} : A\mathbf{x} \le \mathbf{C}\}, \\
T_{j,n} &= \{\mathbf{m}_n : \mathbf{m}_n = \mathbf{m}/n,\, A\mathbf{m}_n \le \mathbf{C}, A(\mathbf{m}_n + \mathbf{e}_j/n) \not\le \mathbf{C}\} \\
\to \overline{T}_j &= \overline{S} \cap \cup_i \{\mathbf{x} : (A\mathbf{x})_i = C_i,\, a_{ij} = 1\}, \qquad (2.3) \\
T_{jk,n} &= \{\mathbf{m}_n : \mathbf{m}_n = \mathbf{m}/n, A(\mathbf{m}_n + \mathbf{e}_j/n) \le \mathbf{C}, A(\mathbf{m}_n + \mathbf{e}_k/n) \not\le \mathbf{C}\} \\
\to \overline{T}_{jk} &= \overline{S} \cap \cup_i \{\mathbf{x} : (A\mathbf{x})_i = C_i,\, a_{ij} = 0,\, a_{ik} = 1\}, \qquad (2.4) \\
U_{j,n} &= \{\mathbf{m}_n : \mathbf{m}_n = \mathbf{m}/n,\, A(\mathbf{m}_n + \mathbf{e}_j/n) \le \mathbf{C}\} \\
\to \overline{U}_j &= \overline{S}.
\end{aligned}
$$

For multidimensional Poisson random variates, scaling is motivated by the observation that the sum of i.i.d. Poisson random variates is again a Poisson random variate. Applying scaling $\nu_i \to n\nu_i$ and $C_i \to nC_i$, the numerator (and denominator) of (2.2) can now be interpreted as the distribution of the sum of n independent Poisson($\boldsymbol{\nu}$) random variates. Let $X(\boldsymbol{\nu})$ denote a multidimensional Poisson random variate with mean $\boldsymbol{\nu}$. Then, $P\{B_j\} = P\{X(\boldsymbol{\nu}) \in T_j\}/P\{X(\boldsymbol{\nu}) \in S\}$. The asymptotics of the denominator $P\{X(\boldsymbol{\nu}) \in S\}$ are trivial. In the following, we will focus on the asymptotics of the numerator $P\{X(\boldsymbol{\nu}) \in T_j\}$. As $\boldsymbol{\nu}$ is finite, the central limit theorem yields that $X(n\boldsymbol{\nu})/n \to \boldsymbol{\nu}$ for $n \to \infty$ almost surely. For deviations from the sample mean, Cramér's theorem (applied to the sequence $X^{(i)}(\boldsymbol{\nu})$, $i = 1, \ldots, n$, of independent Poisson($\boldsymbol{\nu}$) random variates) states that (under some mild conditions; see [15]), for $\boldsymbol{\nu} \notin T_j$,

$$P\{X(n\boldsymbol{\nu})/n \in T_{j,n}\} = \eta(n) e^{-nI(\mathbf{x}_0)} \qquad (n \to \infty), \qquad (2.5)$$

with $I(\mathbf{x}_0)$ the large deviations rate function of X evaluated in the optimum $\mathbf{x}_0$ over $\overline{T}_j$ (see (2.7)), and $\eta(n)$ a subexponential function, with $\lim_{n\to\infty}(1/n)\log\eta(n) = 0$.

https://doi.org/10.1017/S0269964800141075 Published online by Cambridge University Press

The (multidimensional) large deviations rate function of a Poisson random variable with mean $\nu_0,\ldots,\nu_d$ can be calculated easily (see [15, p. 13]). Let $M_i(\cdot)$ be the moment generating function of a Poisson random variable with mean ν_i:

$$M_i(\theta_i) = \exp[\nu_i(e^{\theta_i}-1)]. \tag{2.6}$$

Then,

$$I(\mathbf{x}) = \sup_{\theta}\left(\sum_{i=0}^{d}(\theta_i x_i - \log M_i(\theta_i))\right) = \sum_{i=0}^{d}\left(x_i \log\frac{x_i}{\nu_i} - x_i + \nu_i\right). \tag{2.7}$$

In Cramér's theorem, for obtaining $\mathbf{x}_0$, the large deviations rate function must be minimized over $\overline{T}_j$:

$$\mathbf{x}_0 = \operatorname*{argmin}_{\mathbf{x}\in\overline{T}_j} I(\mathbf{x}). \tag{2.8}$$

Asymptotics based on Cramér's theorem (2.5) usually exploit the observation that the exponential factor $\exp[-nI(\mathbf{x}_0)]$ dominates the expression and, therefore, $\exp[-nI(\mathbf{x}_0)]$ is used as the approximation of (2.5); that is, one approximates $\eta(n)\sim 1$. This usually yields sufficient accuracy for probabilities of the order 10^{-6}–10^{-10} (as in ATM networks). In cellular mobile networks, however, typical blocking probabilities are in the order of 10^{-2}–10^{-3}. In this regime, the subexponential function $\eta(n)$ contributes significantly to (2.5), which requires more accurate asymptotics for $\eta(n)$. In the following, we develop the function $\eta(n)$ up to $O(n^{-3/2})$.

Bahadur and Rao [1] explicitly evaluate the function $\eta(n)$ in a one-dimensional setting via an integral limit theorem. Although this yields sufficient accuracy for our approximation, this result is difficult to apply in our context. Furthermore, in deriving our approximation, we require both an integral limit theorem and a local limit theorem. Petrov [13] gives these results (in a one-dimensional setting) of sufficient accuracy and in an easily applicable form.

THEOREM 2.1 (Petrov [13]): *Let $X^{(1)}, X^{(2)},\ldots$ be a sequence of i.i.d. discrete random variables with finite expectation. Let $M(\cdot)$ denote the moment generating function of $X^{(1)}$, and $J(x) = \sup_\theta(\theta x - \log M(\theta))$ the large deviations rate function of $X^{(1)}$. Then, for all $\epsilon > 0$,*

$$P\{X^{(1)}+\cdots+X^{(n)} = nx\} = \frac{1}{\sigma(\tau)\sqrt{2\pi n}}\exp[-nJ(x)]\left[1+O\left(\frac{1}{n}\right)\right] \tag{2.9}$$

and

$$P\{X^{(1)}+\cdots+X^{(n)} \ge nx\} = \frac{1}{\sigma(\tau)\sqrt{2\pi n}(1-e^{-\tau})} \times \exp[-nJ(x)]\left[1+O\left(\frac{1}{n}\right)\right], \tag{2.10}$$

https://doi.org/10.1017/S0269964800141075 Published online by Cambridge University Press

as $n \to \infty$ uniformly in x in the range of $X^{(1)}$ such that $EX^{(1)} + \epsilon \le x$, where τ is the unique real root of $M'(\tau)/M(\tau) = x$ and $\sigma^2(\tau) = [M''(\tau)/M(\tau)] - x^2$.

In our derivation below, application of Petrov's result requires the moment generating function, M_p, and large deviations rate function, J_p, of a Poisson(ν) random variable, and M_b and J_b, of a binomial(n,p) random variable:

$$M_p(\nu,\theta) = \exp[\nu(e^\theta - 1)], \qquad J_p(\nu,x) = x\log\frac{x}{\nu} - x + \nu, \tag{2.11}$$

$$M_b(n,p,\theta) = (pe^\theta + 1 - p)^n,$$

$$J_b(n,p,x) = x\log\frac{x}{p} + (n-x)\log\left(\frac{n-x}{1-p}\right) - n\log n. \tag{2.12}$$

This provides us with the necessary tools to derive results concerning the blocking probabilities in a hierarchical cellular network.

3. ASYMPTOTICS OF BLOCKING PROBABILITIES

This section develops an approximation algorithm for the probabilities on $\overline{T}_j$ and on $\overline{T}_{jk}$, the numerator of the blocking probabilities (1.3) and (1.6). Due to the special structure of the matrix A as expressed by (1.5), Theorem 3.1 shows that in the most likely point where blocking occurs at most two restrictions are tight. As a consequence, it is sufficient to consider only these restrictions rather than the whole set $\overline{T}$, which is an essential step in this paper. This greatly simplifies the approximation of the large deviations probability. Furthermore, again due to the special structure of A, if two restrictions are tight, it is sufficient to perform only a single optimization of the large deviations rate function at both constraints, a result shown in Theorem 3.5.

Before continuing with the results, we introduce some notation. The matrix $A = (a_{ij})$ of (1.5) determines the state spaces S and $\overline{S}$, and the parts of the boundary determining blocking via a number of restrictions that can be expressed as hypersurfaces. To this end, let

$$r_i = \{\mathbf{x} : (A\mathbf{x})_i = C_i\},$$

$$r_{ij} = \{\mathbf{x} : (A\mathbf{x})_i = C_i,\ a_{ij} = 1\},$$

$$r_{ijk} = \{\mathbf{x} : (A\mathbf{x})_i = C_i,\ a_{ij} = 0,\ a_{ik} = 1\},$$

$$\bar{r}_i = \{\mathbf{x} : (A\mathbf{x})_i \le C_i\},$$

$$\bar{r}_{ij} = \{\mathbf{x} : (A\mathbf{x})_i \le C_i,\ a_{ij} = 1\},$$

$$\bar{r}_{ijk} = \{\mathbf{x} : (A\mathbf{x})_i \le C_i,\ a_{ij} = 0,\ a_{ik} = 1\}.$$

For each j (resp. jk), the boundary hyperplanes r_{ij} (resp. r_{ijk}) are disjoint. In contrast, for fixed i, these hyperplanes may coincide for different j (resp. jk). The sets where blocking occurs can now be written as (recall (2.3) and (2.4))

$$\bar{T}_j = \bar{S} \cap \bigcup_i r_{ij},$$

$$\bar{T}_{jk} = \bar{S} \cap \bigcup_i r_{ijk}.$$

With a slight abuse of notation, we will state that $r_{ij} \in \bar{T}_j$ and $r_{ijk} \in \bar{T}_{jk}$.

The hypersurface r_i corresponds to the ith row of the matrix A. For $C_{(h)} > 0$, due to the special structure of the matrix A, in the boundary of $\bar{S}$ hypersurfaces (determined by the rows of A) cannot be orthogonal [obviously the boundary hypersurfaces corresponding to rows (1,0,0,…,0) and (0,1,0,…,0) are orthogonal, but these surfaces do not intersect in any $\bar{T}$]. Thus, if the intersection of r_i and r_j forms a boundary hypersurface of $\bar{S}$, then the normal vectors of r_i and r_j cannot be orthogonal. We will refer to this as the *nonorthogonality property* of the boundary hypersurfaces.

We will use $\bar{T}$ (resp. T) as generic notation for $\bar{T}_j$ and $\bar{T}_{jk}$ (resp. T_j and T_{jk}) and r for the restrictions determining $\bar{T}$. Results will be provided and proven for $\bar{T}$ if this does not lead to confusion. Otherwise, the results will be proven for $\bar{T}_j$. In all cases, for the other sets $\bar{T}_{jk}$, the proofs are similar.

Theorem 3.1 shows that in a point $\mathbf{x}_0$ where blocking occurs, exactly one restriction of $\bar{T}$ and at most one restriction not of $\bar{T}$ have zero slack (are tight).

THEOREM 3.1: *Let* $\mathbf{x}_0 \in \operatorname{argmin}_{\mathbf{x}\in\bar{T}} I(\mathbf{x})$. *Then, there exists a unique* $r^* \in \bar{T}$ *such that* $\mathbf{x}_0 \in r^*$. *Moreover, there is at most one* r', $r' \notin \bar{T}$, *such that* $\mathbf{x}_0 \in r'$. *Furthermore, for fresh-call blocking in cell* j, *if* $r^* = \bigcap_{l\in D^*} r_{il}$, *where* $D^* = \{t : a_{it} = 1 \text{ in } r^*\}$, *then* $r' = \bigcap_{l\in D'} r_{il}$, *where* $D' = D^* \backslash \{j\}$. *Similarly, for handover blocking from cell* k *to cell* j, *if* $r^* = \bigcap_{l\in D^*} r_{ikl}$, *where* $D^* = \{t : a_{it} = 1 \text{ in } r^*\}$, *then* $D' = D^* \backslash \{j\}$.

PROOF: Consider $\bar{T} = \bar{T}_j$ and define $\bar{H}_j = \bigcap_i \bar{r}_{ij}$ and $H_j = \bigcup_i r_{ij}$.

Observe that $I(\mathbf{x})$ of (2.7) is continuous and strictly convex in all coordinates. Therefore, the level sets $L(g) = \{\mathbf{x} : I(\mathbf{x}) \le g\}$ are convex, and $L(g) \subset L(g')$ for $g < g'$. Furthermore, as $\boldsymbol{\nu} = \operatorname{argmin}_{\mathbf{x}\in\bar{S}} I(\mathbf{x}) \in \operatorname{int}(\bar{S})$, there is a unique g^* such that $L(g^*) \subset \bar{H}_j$, $L(g^*) \cap H_j \neq \emptyset$, and, for all $\epsilon > 0$, $L(g^* + \epsilon) \not\subset \bar{H}_j$.

If $L(g^*) \cap \bar{T} \neq \emptyset$, then $L(g^*) \cap \bar{T} = \operatorname{argmin}_{\mathbf{x}\in\bar{T}} I(\mathbf{x})$. Obviously, $\mathbf{x}_0 \in r_{ij}$ for at least one i, and the normal vectors, $\nabla I(\mathbf{x}_0)$, of $I(\mathbf{x})$ in $\mathbf{x}_0$ and of r_{ij} coincide. Now, assume that $\mathbf{x}_0 \in r_{ij} \cap r_{i'j}$, for $i \neq i'$. Again, the normal vectors of $I(\mathbf{x}_0)$ and r_{ij} and of $I(\mathbf{x}_0)$ and $r_{i'j}$ must coincide. Obviously, for $i \neq i'$, this cannot be the case, as the normal vectors of r_{ij} and $r_{i'j}$ are distinct.

If $L(g^*) \cap \bar{T} = \emptyset$, then there exists a unique $g' > g^*$ such that, for all $\epsilon > 0$, $L(g' - \epsilon) \cap \bar{T} = \emptyset$ and $\emptyset \neq L(g') \cap \bar{T} = \operatorname{argmin}_{\mathbf{x}\in\bar{T}} I(\mathbf{x})$. If in $\mathbf{x}_0 \in \operatorname{argmin}_{\mathbf{x}\in\bar{T}} I(\mathbf{x})$, the normal vector of $I(\mathbf{x}_0)$ and of r_{ij} for some i coincide, then r_{ij} must be the unique boundary hyperplane in $\bar{T}$ such that $\mathbf{x}_0 \in r_{ij}$. Otherwise, $\mathbf{x}_0$ lies on the boundary of $\bar{T}$

https://doi.org/10.1017/S0269964800141075 Published online by Cambridge University Press

[i.e., $\mathbf{x}_0 \in \overline{T} \cap \{\mathbf{x} : x_j = c_j\}$]. Let $\mathbf{x}_0 \in r_{ij}$. There exists a unique hyperplane through $\mathbf{x}_0$ with normal $\nabla I(\mathbf{x}_0)$. This hyperplane separates $L(g')$ and $r_{ij} \cap \overline{T}$ and contains the line $r_{ij} \cap \{\mathbf{x} : x_j = c_j\}$. Now, assume that $\mathbf{x}_0 \in r_{ij} \cap r_{i'j}$, for $i \neq i'$. As r_{ij} and $r_{i'j}$ intersect in $\overline{T}$, it must be that there exists a k such that $a_{kj} = 1$ in both rows i and i' of the matrix A [recall (1.5)]. The normal of the hyperplane separating $L(g')$ and $r_{ij} \cap r_{i'j} \cap \overline{T}$ must then have a nonpositive kth coordinate, and thus $(\nabla I(\mathbf{x}_0))_k \leq 0$. As $\boldsymbol{\nu} \in \text{int}(\overline{S})$, the form of $I(\mathbf{x})$ [recall (2.7)] clearly leads to a contradiction. [$(d/dx_k)\, I(\mathbf{x}) < 0$ if and only if $x_k < \nu_k$, placing $\boldsymbol{\nu}$ outside $\overline{S}$.] Thus, $\mathbf{x}_0$ must be contained in a unique $r^* \in \overline{T}$.

The same argument also shows that there exists at most one $r' \notin \overline{T}$ such that $\mathbf{x}_0 \in r'$. To this end, note that if $\mathbf{x}_0 \in \text{int}(\overline{T})$, then there cannot be an r', $r' \notin \overline{T}$, such that $\mathbf{x}_0 \in r'$. If $\mathbf{x}_0 \in \overline{T} \cap \{\mathbf{x} : x_j = c_j\}$, then $\mathbf{x}_0 \in \text{int}(r^* \cap \overline{T} \cap \{\mathbf{x} : x_j = c_j\})$; that is, $\mathbf{x}_0$ cannot be located on the endpoint of this line segment, for this would imply that $\mathbf{x}_0$ is also contained in two distinct $r \in \overline{T}$. This line segment is contained in exactly one $r' \notin \overline{T}$.

It remains to determine r'. Two restrictions that intersect in a line in $\overline{T}$ have exactly one element a_{ij} that is different. As $r^* \in \overline{T}$ and $r' \notin \overline{T}$, it must be that the element $\{j\}$ of D^* is not contained in D', which completes the proof. ■

Theorem 3.1 allows for $\text{argmin}_{\mathbf{x} \in \overline{T}} I(\mathbf{x})$ to contain multiple points. As $I(\mathbf{x})$ is continuous and strictly convex in all coordinates, the elements $\mathbf{x} \in \text{argmin}_{\mathbf{x} \in \overline{T}} I(\mathbf{x})$ must be contained in *distinct* boundary hyperplanes. Obviously, in general, $\text{argmin}_{\mathbf{x} \in \overline{T}} I(\mathbf{x})$ will contain exactly one element.

Theorem 3.1 is most interesting for $c_{(h)} > 0$. If $c_{(h)} = 0$ (the case of fixed channel allocation), then $\overline{S}$ is a box (boundary hyperplanes are orthogonal) and $\mathbf{x}_0$ is contained in the interior of a single boundary hyperplane. As a consequence, for each cell, blocking can be separately (and independently) analyzed.

Assumption (2.1) plays an important role in the above result. For handover blocking from cell k to cell j, the boundary of $\overline{T}_{kj}$ is $\overline{T}_{kj} \cap (\{\mathbf{x} : x_k = c_k\} \cup \{\mathbf{x} : x_j = c_j\})$. Assumption (2.1) excludes the possibility that blocking occurs on $\overline{T}_{kj} \cap \{\mathbf{x} : x_k = c_k\}$. In most cases, this set does not lead to additional complications: Results similar to those presented in the sequel can be derived, taking into account that $D' = D^* \cup \{k\}$. However, it is also possible for $\mathbf{x}_0$ to have $(\mathbf{x}_0)_j = c_j$, and $(\mathbf{x}_0)_k = c_k$; $\mathbf{x}_0$ is then contained in multiple r', $r' \notin \overline{T}_{kj}$, which makes the formulation of our results and the proofs thereof much more cumbersome.

The immediate consequence of Theorem 3.1 is that in order to approximate the blocking probability, it suffices to consider only the restrictions that apply with equality in $\mathbf{x}_0$. This is a consequence of the factor $\exp[-nI(\mathbf{x})]$ in Crameŕ's theorem (2.5). This is formalized in the following lemma. Here, $r_{ij,n}$ (resp. $r_{ijk,n}$) denotes boundary hyperplanes of $T_{j,n}$ (resp. $T_{jk,n}$); recall (2.3) and (2.4). For example, $r_{ij,n} = \{\mathbf{m}_n : \mathbf{m}_n = \mathbf{m}/n, A\mathbf{m}_n \leq \mathbf{C}, (A\mathbf{m}_n + \mathbf{e}_j/n) \not\leq \mathbf{C}, a_{ij} = 1\} \to r_{ij}$ $(n \to \infty)$. As before, we will denote r_n (resp. T_n) to indicate $r_{ij,n}$ and $r_{ijk,n}$ (resp. $T_{j,n}$ and $T_{jk,n}$) and r_n^* and r_n' denote the hyperplanes r_n such that $r_n^* \to r^*$ and $r_n' \to r'$ (e.g., the boundary hyperplanes containing $\mathbf{x}_0$; recall their definition in Theorem 3.1).

LEMMA 3.2: *If one restriction has zero slack in* $\mathbf{x}_0$, *then*

$$P\{X(n\boldsymbol{\nu}) \in T_n\}/P\{X(n\boldsymbol{\nu}) \in r_n^*\} \to 1.$$

If two restrictions have zero slack, then

$$P\{X(n\boldsymbol{\nu}) \in T_n\}/P\{X(n\boldsymbol{\nu}) \in r_n^* \cap r_n'\} \to 1.$$

PROOF: In the first case, $\mathbf{x}_0$ simply lies in the interior of one of the boundary planes of the simplex $A\mathbf{x} \leq \mathbf{C}$, as was found in the proof of Theorem 3.1. We abbreviate $P\{A_n\} := P\{X(n\boldsymbol{\nu}) \in A_n\}$. First, notice that

$$P\{S_n \cap r_n^*\} = P\{r_n^*\} - P\{S_n^c \cap r_n^*\}.$$

As $\mathbf{x}_0$ lies on the interior of $S_n \cap r_n^*$, the probability $P\{S_n^c \cap r_n^*\}$ vanishes exponentially (in n), with decay rate $I(\mathbf{x}) > I(\mathbf{x}_0)$, whereas $P\{r_n^*\}$ vanishes at rate $I(\mathbf{x}_0)$. (This is due to Petrov's Theorem 2.1, where we apply that the exponential decay dominates the polynomial decay.) Then, notice that

$$P\{T_{j,n}\} = P\left\{S_n \cap \bigcup_i r_{ij,n}\right\} = P\{S_n \cap r_n^*\} + P\left\{S_n \cap \left(\bigcup_i r_{ij,n} \backslash r_n^*\right)\right\}.$$

The first probability on the right hand side decays at rate $I(\mathbf{x}_0)$, whereas the second vanishes at a larger rate. We arrive at the stated.

In the second case, the above proof can be copied, with one exception:

$$P\{S_n \cap r_n^*\} = P\{r_n' \cap r_n^*\} - P\{S_n^c \cap r_n' \cap r_n^*\}.$$

The first probability on the right-hand side tends exponentially to zero at rate $I(\mathbf{x}_0)$; the second goes at a higher rate. This is due to the fact that both r^* and r' apply with equality in $\mathbf{x}_0$. ∎

To simplify the proof of Theorem 3.5, the main result of this section that determines the asymptotics of the blocking probabilities, we now introduce some notation and provide two lemmas. Lemma 3.3 proves that the entries $(\mathbf{x}_0)_i$ and ν_i are proportional as long as $i \in D'$. Lemma 3.4 determines the asymptotics of a binomial random variable.

Define

$$X' := \sum_{i \in D'} X_i(n\boldsymbol{\nu}), \qquad X^* := \sum_{i \in D^*} X_i(n\boldsymbol{\nu}),$$

$$\nu' := \sum_{i \in D'} \nu_i, \qquad \nu^* := \sum_{i \in D^*} \nu_i,$$

$$C' := \sum_{i \in D'} c_i + c_{(h)}, \qquad C^* := \sum_{i \in D^*} c_i + c_{(h)}.$$

Let $f(n) \sim g(n)$ denote $f(n)/g(n) \to 1$ as $n \to \infty$.

https://doi.org/10.1017/S0269964800141075 Published online by Cambridge University Press

LEMMA 3.3: *If in* $\mathbf{x}_0$ *a single restriction is tight, then*

$$(\mathbf{x}_0)_i = \frac{C^*}{\nu^*}\,\nu_i, \qquad i \in D^*.$$

Otherwise, when two restrictions are tight, then

$$(\mathbf{x}_0)_i = \frac{C'}{\nu'}\,\nu_i, \quad i \in D', \qquad (\mathbf{x}_o)_i = \frac{C^* - C'}{\nu^* - \nu'}\,\nu_i, \quad i \in D^* \backslash D'.$$

PROOF: Based on Theorem 3.1, $\mathbf{x}_0$ is the argmin of

$$\sum_{i \in D^*} x_i \log \frac{x_i}{\nu_i} - x_i + \nu_i$$

under

$$\sum_{i \in D^*} x_i = C^* \quad \text{and} \quad \sum_{i \in D'} x_i = C'$$

or, equivalently,

$$\sum_{i \in D^* \backslash D'} x_i = C^* - C' \quad \text{and} \quad \sum_{i \in D'} x_i = C'.$$

Lagrangian optimization shows that $(\mathbf{x}_0)_i/\nu_i$ must be constant at $D^*\backslash D'$ and at D'. Summing over the appropriate i (e.g., $i \in D'$) yields, introducing $\kappa = (\mathbf{x}_0)_i/\nu_i$, $i \in D'$,

$$C' = \sum_{i \in D'} x_i = \sum_{i \in D'} \kappa \nu_i = \kappa \nu'$$

(i.e., $\kappa = C'/\nu'$), which completes the proof. ∎

LEMMA 3.4: *Let* Z_n *be binomially* (nk, p) *distributed, for some* $k \in \mathbb{N}$. *Then, for any* $y < kp$,

$$P\left\{\frac{Z_n}{n} < y\right\} \sim \frac{1}{\sigma_J(\tau)\sqrt{2\pi n}[1 - \exp(\tau)]} \exp[-nJ_b(k, p, y)]$$

for

$$\tau = \log\left(\frac{1-p}{p}\,\frac{y}{k-y}\right) \quad \textit{and} \quad \sigma_b^2(\tau) = \frac{y(k-y)}{k}.$$

PROOF: The proof is a matter of invoking Petrov's Theorem 2.1, where it should be noticed that Z_n is distributed as the convolution of n i.i.d. binomial(k, p) random variables and that we consider $Z_n/n < y$ instead of $Z_n/n \geq y$, as in (2.10). The

https://doi.org/10.1017/S0269964800141075 Published online by Cambridge University Press

moment generating function of one of those random variables is $M_b(k,p,\theta) = (pe^\theta + 1 - p)^k$, leading to the following expression for τ:

$$e^\tau = \left(\frac{1-p}{p}\right)\left(\frac{y}{k-y}\right).$$

Inserting this in (2.10) yields, after tedious calculations, the desired result. ■

We are now ready to formulate our main result. As a consequence of Theorem 3.1, for asymptotics of $P\{X(n\boldsymbol{\nu}) \in T_n\}$ we need to consider two cases only: either, in $\mathbf{x}_0$, a single restriction $r^* \in \overline{T}$ is tight or, in $\mathbf{x}_0$, two restrictions, $r^* \in \overline{T}$ and $r' \notin \overline{T}$, are tight. The first case is basically covered by the results of Gazdzicki et al. [8]. The second case, covering all remaining cases, cannot be concluded from these results. The result for this case exploits the structure of the state space, which implies that $P\{X(n\boldsymbol{\nu}) \in T_n\}$ can be seen to be determined by a Poisson distribution at r^* and a binomial distribution at r' conditional on $X(n\boldsymbol{\nu}) \in r^*$. The asymptotics for both events can be combined to require only a single optimization step for the original large deviations rate function (2.7) to produce $I(\mathbf{x}_0)$. This optimization step can, in fact, be avoided, as is demonstrated in Corollary 3.6.

THEOREM 3.5: *If one restriction has zero slack in* $\mathbf{x}_0$, *then*

$$P\{X(n\boldsymbol{\nu}) \in T_n\} \sim \frac{\exp[-nI(\mathbf{x}_0)]}{\sqrt{2\pi nC^*}}. \tag{3.1}$$

If two restrictions have zero slack, *then*

$$P\{X(n\boldsymbol{\nu}) \in T_n\} \sim \frac{\exp[-nI(\mathbf{x}_0)]}{2\pi n\sigma_J(\tau)\sqrt{C^*}[1-\exp(\tau)]}, \tag{3.2}$$

with

$$e^\tau = \left(\frac{C}{C^*-C}\right)\left(\frac{\nu^*-\nu}{\nu}\right) \quad \textit{and} \quad \sigma_b^2(\tau) = \frac{C(C^*-C)}{C^*}.$$

PROOF: We consider both cases separately.

The *first* case (3.1) is a direct application of Petrov's result (2.9), after applying Lemma 3.2 and observing that $P\{X(n\boldsymbol{\nu}) \in r_n^*\} = P\{X^* = nC^*\}$. Here, we have used that $X(n\boldsymbol{\nu})$ is multivariate Poisson, implying that $X_i(n\boldsymbol{\nu}) \in \mathbb{N}_0$. For $X(n\boldsymbol{\nu}) \in r_n^*$ [i.e., $\sum_{i\in D^*} X_i(n\boldsymbol{\nu}) \le nC^*$ and $\sum_{i\in D^*} X_i(n\boldsymbol{\nu}) + 1 \not\le nC^*$], it must be that $\sum_{i\in D^*} X_i(n\boldsymbol{\nu}) = nC^*$ as $nC^* \in \mathbb{N}_0$.

It is left to calculate $\sigma(\tau)$. As $M(\tau) = \exp[\nu^*(e^\tau - 1)]$, the root τ of $M'(\tau)/M(\tau) = C^*$ is $\tau = \log(C^*/\nu^*)$ and

$$\sigma_b^2(\tau) = \frac{M''(\tau)}{M(\tau)} - \left(\frac{M'(\tau)}{M(\tau)}\right)^2 = C^*.$$

The *second* case (3.2) is proven as follows. With Lemma 3.2, we may consider

$$P\{X(n\boldsymbol{\nu}) \in \{r_n^* \cap \bar{r}_n'\}\} = P\{X(n\boldsymbol{\nu}) \in \bar{r}_n' | X(n\boldsymbol{\nu}) \in r_n^*\} P\{X(n\boldsymbol{\nu}) \in r_n^*\}. \quad \textbf{(3.3)}$$

We derive the asymptotics of both probabilities on the right-hand side of (3.3) separately. The first one can be rewritten as $P\{X' \le nC' | X^* = nC^*\}$. Notice that $(X' | X^* = nC^*)$ is distributed binomially with parameters nC^* and ν'/ν^*. By applying Lemma 3.4, we get that the asymptotics read

$$\frac{\exp[-nJ_b(C^*, \nu'/\nu^*, C')]}{\sqrt{2\pi n}\sigma_b(\tau)[1 - \exp(\tau)]}. \quad \textbf{(3.4)}$$

The asymptotics of the second probability in (3.3) is—analogously to the first case—equal to

$$\frac{\exp[-nJ_p(\nu^*, C^*)]}{\sqrt{2\pi n C^*}}. \quad \textbf{(3.5)}$$

Calculations yield that

$$J_b(C^*, \nu'/\nu^*, C') + J_p(\nu^*, C^*) = C' \log\left(\frac{C'}{\nu'}\right) + (C^* - C') \log\left(\frac{C^* - C'}{\nu^* - \nu'}\right) + \nu^* - C^*.$$

Invoking Lemma 3.3, we see that this equals

$$\sum_{i \in D^*} \left((\mathbf{x}_0)_i \log \frac{(\mathbf{x}_0)_i}{\nu_i} - (\mathbf{x}_0)_i + \nu_i \right).$$

Multiplying (3.4) and (3.5) yields (3.2). ■

The observation that, in the optimum, $(\mathbf{x}_0)_i$ and ν_i for all $i \in D^* \backslash D'$ and D' are proportional gives useful additional information, which can be used to quickly solve the optimization problem. As shown in Corollary 3.6, it is possible to explicitly find the values of the components of $\mathbf{x}_0$. This implies that the minimum of the nonlinear function $I(\mathbf{x})$ over $\overline{T}$ can be found through some simple calculations, avoiding numerical methods to solve the optimization problem (2.8). This has a considerable effect on the speed of the method, and large problems can be easily handled.

Corollary 3.6: *Consider fresh-call blocking in cell j or handover blocking from cell k to cell j. In the* argmin $\mathbf{x}_0$ *of* $I(\mathbf{x})$ *over* $\overline{T}_j$ *or* $\overline{T}_{kj}$,

- *either* $(\mathbf{x}_0)_j > C_j$ *and* $(\mathbf{x}_0)_i = (C^*/\nu^*)\,\nu_i$, *for* $i \in D^*$, *and* $(\mathbf{x}_0)_i = \nu_i$ *for* $i \notin D^*$
- *or* $(\mathbf{x}_0)_j = C_j$, *and* $(\mathbf{x}_0)_i = (C'/\nu')\,\nu_i$, *for* $i \in D'$, *and* $(\mathbf{x}_0)_i = \nu_i$ *for* $i \notin D^*$.

Proof: Given the constraints r^* and r' that contain $\mathbf{x}_0$, the components $(\mathbf{x}_0)_i$ for which $a_{ji} = 0$ can be independently minimized. This obviously yields $(\mathbf{x}_0)_i = \nu_i$ for $i \notin D^*$. Lemma 3.3 completes the proof. ■

https://doi.org/10.1017/S0269964800141075 Published online by Cambridge University Press

The results of this section can be summarized in an easily applicable algorithm that takes very limited computational effort to approximate the probability on *T*, as it only requires substitution of the appropriate terms in Corollary 3.6 and Theorem 3.5. Observe that this result depends heavily on Theorem 3.1.

ALGORITHM 3.7: *Asymptotical evaluation of the probability on T in a hierarchical cellular mobile communications network*:

1. *Minimize* (2.7) *over* $\bar{T}$ *by using Theorem* 3.6, *which yields* $I(\mathbf{x}_0)$.
2. *Identify the restrictions with zero slack in* $\mathbf{x}_0$.
3. *The restriction that is in* $\bar{T}$ *is* r^*; *if there is a second restriction*, *call it* r'.
4. *Apply Theorem* 3.5 *to calculate the asymptotics*.

4. NUMERICAL EXAMPLES

As an illustration of the method, this section contains two numerical examples. In the first example, fixed channel allocation is considered, as this allows us to compare the results of our approximation with exact results to evaluate the accuracy of the asymptotics. Second, an example of a typical city center where microcells are placed to increase the network capacity is treated. The results indicate that not only blocking probabilities up to 1% are estimated accurately but also that our method provides insight into whether blocking is due to local effects (a single cell is overloaded) or due to global effects (multiple cells are simultaneously overloaded).

4.1. Fixed Channel Allocation

This first example illustrates the accuracy of the method in case of fixed channel allocation ($c_{(h)} = 0$), a special case for which the blocking probabilities can easily be obtained in exact form from the Erlang loss formula. It also provides insight into the regime under which the method based on large deviations techniques can be used. Table 4.1 presents the exact blocking probabilities and those estimated using the method developed in this paper. When the exact probability exceeds 5%, the method proposed in this paper loses accuracy; for smaller probabilities, it performs very well. In the results presented in this section, the term in the denominator of (2.2) has been approximated by 1. Using Theorem 3.5 on $P\{S^C\}$ will lead to more accurate estimates of the larger probabilities. However, one should keep in mind that large deviations is not a technique to estimate such large probabilities and, hence, such results should be handled very carefully.

The column "Simple LD" gives the results obtained by using Cramérs result with $\eta(n) = 1$ (i.e., the values of $e^{-nI(\mathbf{x}_0)}$, the result standardly used to approximate small loss probabilities). It shows that use of the correction terms is important for achieving accurate estimates of the loss probability. The column "Exact" presents the exact value of the loss probability, and the column "$P\{B_j\}$" gives the result from our approximation.

https://doi.org/10.1017/S0269964800141075 Published online by Cambridge University Press

Table 4.1. Fixed Channel Allocation; Accuracy of Approximation

Cell	ν_i	c_i	Exact	$P\{B_j\}$	Simple LD
0	12	14	1.17×10^{-1}	9.10×10^{-2}	8.54×10^{-1}
1	10	14	5.68×10^{-2}	5.24×10^{-2}	4.91×10^{-1}
2	8	14	1.72×10^{-2}	1.70×10^{-2}	1.60×10^{-1}
3	6	14	2.23×10^{-3}	2.24×10^{-3}	2.10×10^{-2}
4	4	14	5.64×10^{-5}	5.67×10^{-5}	5.32×10^{-4}
5	2	14	2.54×10^{-8}	2.56×10^{-8}	2.40×10^{-7}
6	1	7	7.30×10^{-5}	7.39×10^{-5}	4.90×10^{-4}
7	2	7	3.44×10^{-3}	3.48×10^{-3}	2.31×10^{-2}
8	3	7	2.19×10^{-2}	2.19×10^{-2}	1.45×10^{-1}
9	4	7	6.27×10^{-2}	6.03×10^{-2}	4.00×10^{-1}
10	5	7	1.21×10^{-1}	1.06×10^{-1}	7.01×10^{-1}

In mobile communications, loss probabilities are typically smaller than 2%. The method developed in this paper proves to be accurate when estimating such small probabilities in the case of fixed channel allocation.

4.2. City Center

Consider a city center with a load of about 50 Erlang/km^2, which will frequently occur in the near future. Assume one macrocell (cell 0) and seven microcells (cells $1,\ldots,7$) have to carry the load. Some cells contain two frequencies (14 channels), others one (7 channels). The macrocell has two frequencies available, and four of the channels from the macrocell can be used by the microcells (i.e., $c_{(h)} = 4$). Tables 4.2 and 4.3 summarize the results. In Table 4.2, not only the fresh-call loss probability

Table 4.2. Fresh Call Blocking in Cell j ($c_{(h)} = 4$)

Cell	ν_j	c_j	$P\{B_j\}$	Constraint r^*	No. of Constraints
0	7	10	7.14×10^{-3}	1 0 0 0 0 0 0 0	1
1	10	14	7.12×10^{-3}	0 1 0 0 0 0 0 0	1
2	9	14	2.91×10^{-3}	0 0 1 0 0 0 0 0	1
3	8	14	1.03×10^{-3}	0 0 0 1 0 1 0 0	1
4	7	14	5.08×10^{-4}	0 1 0 0 1 0 0 0	2
5	5	7	8.30×10^{-3}	0 0 0 0 0 1 0 0	1
6	4	7	2.91×10^{-3}	0 0 0 0 0 1 1 0	1
7	3	7	8.56×10^{-4}	0 1 0 0 0 1 0 1	1

https://doi.org/10.1017/S0269964800141075 Published online by Cambridge University Press

TABLE 4.3. Handover Blocking ($c_{(h)} = 4$)

	0	1	2	3	4	5	6	7
0	0	7.12×10^{-3}	2.91×10^{-3}	1.03×10^{-3}	5.08×10^{-4}	8.30×10^{-3}	2.91×10^{-3}	8.56×10^{-4}
1	7.14×10^{-3}	0	2.91×10^{-3}	1.03×10^{-3}	3.79×10^{-4}	8.30×10^{-3}	2.91×10^{-3}	8.58×10^{-4}
2	7.14×10^{-3}	7.12×10^{-3}	0	1.03×10^{-3}	5.08×10^{-4}	8.30×10^{-3}	2.91×10^{-3}	8.56×10^{-4}
3	7.14×10^{-3}	7.12×10^{-3}	2.91×10^{-3}	0	5.08×10^{-4}	8.30×10^{-3}	2.91×10^{-3}	8.56×10^{-4}
4	7.14×10^{-3}	7.12×10^{-3}	2.91×10^{-3}	1.03×10^{-3}	0	8.30×10^{-3}	2.91×10^{-3}	8.56×10^{-4}
5	7.14×10^{-3}	7.12×10^{-3}	2.91×10^{-3}	8.56×10^{-4}	5.08×10^{-4}	0	2.42×10^{-3}	6.81×10^{-4}
6	7.14×10^{-3}	7.12×10^{-3}	2.91×10^{-3}	1.03×10^{-3}	5.08×10^{-4}	8.30×10^{-3}	0	8.56×10^{-4}
7	7.14×10^{-3}	7.12×10^{-3}	2.91×10^{-3}	1.03×10^{-3}	5.08×10^{-4}	8.30×10^{-3}	2.91×10^{-3}	0

but also the constraint (presented by a row of A) on which blocking occurs and the number of constraints with zero slack in $\mathbf{x}_0$ are listed. Some qualitative insight into the network can be gained from these results:

- Blocking in cells 0, 1, 2, and 5 is local; that is, r^* corresponds to those cells only.
- Blocking in cells 3, 4, 6, and 7 has a more global nature; for example, blocking in cell 7 occurs when cells 1, 5, and 7 are simultaneously using the overflow channels $c_{(h)}$.
- Blocking in cell 4 involves two constraints, which implies that blocking occurs because *other* cells (in this case cell 1) are using all available overflow channels.

Table 4.3 presents the handover blocking probabilities from cell k (row) to cell j (column). These results and a comparison with the results for fresh-call blocking provide additional insight into the behavior of the network:

- The handover blocking probability to cells 0, 1, 2, and 5 is equal for all cells and equals the fresh-call blocking probability for these cells, a direct consequence of the fact that fresh-call blocking in these cells is local (the constraint r^* for fresh-call blocking is contained in $\overline{T}_j \cap \overline{T}_{kj}$).
- For other cells, the handover blocking probability is lower for the entries $l \neq j$ in r^* (e.g., $P\{B_{53}\} < P\{B_3\}$, as $5 \in D^*$ for fresh-call blocking in cell 3). The handover blocking probability is lower than the fresh-call blocking probability for cells that do not block locally. The difference lies in the combined blocking of cells for fresh-call blocking which does not yield handover blocking. In particular, when fresh-call blocking in cell 3 is also caused by calls in cell 5 using the overflow channels, then a call moving from cell 5 to cell 3 is less likely to be blocked, as a call using an overflow channel can take its channel along from cell 5 to cell 3.

In practice the load in the microcells will be quite similar. This implies that blocking more often will have a global character. Therefore, the constraint on which handover blocking is most likely to occur will differ more often from that of fresh-call blocking, leading to lower blocking probabilities for handover blocking. Also, when blocking has a global character, assigning one extra frequency to a particular cell will increase the network capacity less than it would have in the case that blocking occurs locally. However, if this cell is a cause for any type of blocking (appears in each constraint in Table 4.2), network performance can be increased considerably by assigning an extra frequency to this cell. From this observation, extra capacity would be most likely to increase network performance when added to cells 5 or 1.

5. CONCLUSION

This paper has presented an asymptotic approximation of blocking probabilities obtained from multivariate Poisson probabilities at state spaces typically arising in

https://doi.org/10.1017/S0269964800141075 Published online by Cambridge University Press

layered cellular mobile communications networks. The results generalize the large deviations results of Gazdzicki et al. [8] to state spaces with multiple constraints and extend applicability of large deviations approximations to probabilities of the order of 1%. A numerically efficient method that identifies bottleneck constraints and computes blocking probabilities has been proved.

Acknowledgments

The research of S. Verwijmeren has been supported by a one-year KPN Research grant, contract No. 2044400. The research of R. J. Boucherie is supported by the Technology Foundation STW, Applied Science Division of NWO and the Technology Programme of the Ministry of Economic Affairs, The Netherlands.

References

1. Bahadur, R.R. & Ranga Rao, R. (1960). On deviations of the sample mean. *Annals of Mathematical Statistics* 31: 1015–1027.
2. Borst, S.C., Boucherie, R.J., & Boxma, O.J. (1999). ERMR: A generalised equivalent random method for overflow systems with repacking. In P. Key and D. Smith (eds.), *Teletraffic Engineering in a Competitive World*. *Proceedings of ITC-16*, pp. 313–323.
3. Boucherie, R.J. & Mandjes, M. (1998). Estimation of performance measures for product form cellular mobile communications networks. *Telecommunication Systems* 10: 321–354.
4. Boucherie, R.J. & Van Dijk, N.M. (1999). On a queueing network model for cellular mobile telecommunications networks. *Operations Research* (in press).
5. Choudhury, G.L., Leung, K.K., & Whitt, W. (1995). An algorithm to compute blocking probabilities in multi-rate multi-class multi-resource loss models. *Advances in Applied Probability* 27: 1104–1143.
6. Dziong, Z. & Roberts, J.W. (1987). Congestion probabilities in a circuit-switched integrated services network. *Performance Evaluation* 7: 267–284.
7. Everitt, D. & Manfield, D. (1989). Performance analysis of cellular mobile communications systems with dynamic channel assignment. *IEEE Journal on Selected Areas in Communications* 7(8): 1172–1180.
8. Gazdzicki, P., Lambadaris, I., & Mazumdar, R.R. (1993). Blocking probabilities for large multirate Erlang loss systems. *Advances in Applied Probability* 25: 997–1009.
9. Harvey, C. & Hills, C.R. (1979). Determining grades of service in a network. *9th International Teletraffic Conference*.
10. Kelly, F.P. (1991). Loss networks. *Annals of Applied Probability* 1: 319–378.
11. Mandjes, M. (1997). Fast simulation of blocking probabilities in loss networks. *European Journal of Operational Research* 101: 393–404.
12. Pallant, D.L. & Taylor, P.G. (1995). Modeling handovers in cellular mobile networks with dynamic channel allocation. *Operations Research* 43(1): 33–42.
13. Petrov, V.V. (1965). On the probabilities of large deviations for sums of independent random variables. *Theory of Probability and its Applications* 10(2): 287–298 (translated by A.R. Kraiman).
14. Ross, K. & Wang, J. (1992). Monte Carlo summation applied to product form loss networks. *Probability in the Engineering and Informational Sciences* 6: 323–348.
15. Shwartz, A. & Weiss, A. (1995). *Large deviations for performance analysis*. London: Chapman & Hall.
16. Wolff, R.W. (1989). *Stochastic modeling and the theory of queues*. Englewood Cliffs, NJ: Prentice-Hall.