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VISCOUS PROFILES OF VORTEX PATCHES

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Abstract We deal with the incompressible Navier–Stokes equations with vortex patches as initial data. Such data describe an initial configuration for which the vorticity is discontinuous across a hypersurface. We give an asymptotic expansion of the solutions in the vanishing viscosity limit which exhibits an internal layer where the fluid vorticity has a sharp variation. This layer moves with the flow of the Euler equations.

1. Introduction and overview of the results

In this paper we deal with the equation

$$\partial_t v^\nu + v^\nu \cdot \nabla v^\nu + \nabla p^\nu = \nu \Delta v^\nu, \tag{1}$$

with the incompressibility condition

$$\operatorname{div} v^\nu = 0, \tag{2}$$

where the spatial variable x is in $\mathbb{R}^d$ for $d = 2$ or 3, where v^ν and p^ν respectively denote the velocity and the pressure of a fluid and where $\nu \geqslant 0$ is the viscosity coefficient. The derivative fields $v^\nu \cdot \nabla$ and Δ are taken with respect to x and applied componentwise to the vector-valued function v^ν, whereas ∇p^ν denotes the gradient (also with respect to x) of the scalar function p^ν. Equations (1), (2) are Newton's laws for a homogeneous fluid (with constant density). They are the **Euler** equations when $\nu = 0$ and the **Navier–Stokes** equations when $\nu > 0$.

The study of these equations is naturally affected by the function space in which initial data is provided, and in which solutions are sought. In this paper we will consider as initial data some **vortex patches** which are basically fluid configurations where the vorticity

$$\omega^\nu := \operatorname{curl} v^\nu \tag{3}$$

is initially discontinuous across a hypersurface of $\mathbb{R}^d$.

The problem was initially considered for the Euler equations in two dimensions. The vorticity ω^0 is then scalar and a natural example of discontinuous vorticity is the characteristic function of a bounded domain. In this case the existence and uniqueness of a solution of the Euler equations in the more general case of an initial vorticity which is a bounded function with compact support were proved by Yudovich in [88]. The corresponding velocity field is log-Lipschitz and admits a bicontinuous flow $\mathcal{X}^0$ which transports the vorticity. As a consequence, in the case of a vortex patch as initial data, the vorticity ω^0 at time t remains a vortex patch relative to a domain which is homeomorphic to the initial domain. However if we only use that the initial vorticity is a bounded function with compact support, we can only get that the smoothness of Yudovich's flow $\mathcal{X}^0$ is exponentially decreasing: it is in $C^{\exp -\alpha t}$ where α depends on the initial vorticity (cf. for example [10], Theorem 7.26). Therefore in the case of a vortex patch, Yudovich's approach only provides that the boundary of a vortex patch is in $C^{\exp -\alpha t}$. The aim is to establish how the smoothness of the boundary of the patch really evolves. Numerical experiments of Zabusky in [123] suggested that singularities of the boundary of the patches would develop, presumably in finite time, whereas the ones of Buttke [19] suggested a loss of smoothness. Majda in [96] studied theoretically the evolution of the boundary of piecewise constant vortex patches, by a contour dynamic approach, announcing local-in-time existence and conjecturing that there are smooth initial curves such that the curve becomes non-rectifiable in finite time. Constantin and Titi [40] studied a quadratic approximation of the equation governing the evolution of the boundary for which Alinhac [7] found some evidence of finite time breakdown. As a consequence it was very surprising when a proof of the global-in-time persistence of the initial $C^{s+1,r}$ smoothness of the boundary was given, by Chemin in [32] (see also his earlier local-in-time results in [30, 26], his proceedings work [29, 27, 28] and his recent survey [35]).

There were numerous works after Chemin's results, in particular some extensions to the three-dimensional case, that we will consider here. In three dimensions constant patches of vorticity are not a good pattern; instead we will consider:

Definition 1.1 (vortex patches). Let a compact connected hypersurface Γ_0 in the Hölder class $C^{s+1,r}$ where s is in the set $\mathbb{N}$ of all natural numbers including 0 and $0 < r < 1$ be given. This means that there exists a function $\varphi_0 \in C^{s+1,r}(\mathbb{R}^3; \mathbb{R})$ such that an equation for Γ_0 is given by $\Gamma_0 = \{\varphi_0 = 0\}$, with $\nabla\varphi_0 \neq 0$ in a neighborhood of Γ_0. According to the Jordan–Brouwer theorem, $\mathbb{R}^3 \setminus \Gamma_0$ has two distinct connected components. One of them is bounded (the 'interior')—we will denote it as $\mathcal{O}_{0,+}$—and the other one (the 'exterior') is unbounded—we will denote it as $\mathcal{O}_{0,-}$. We assume that $\mathcal{O}_{0,\pm} = \{\pm\varphi_0 > 0\}$. We will consider as the initial velocity a divergence free vector field v_0 in $L^2(\mathbb{R}^3)$ whose vorticity $\omega_0 := \text{curl } v_0$ is in the Hölder space $C^{s,r}_c(\mathcal{O}_{0,\pm})$, that is a vorticity which is compactly supported in $\mathbb{R}^3$ and which is $C^{s,r}$ on each side of Γ_0.

To avoid any confusion let us recall the definition of the Hölder spaces:

Definition 1.2 (Hölder spaces). For an open subset $\mathcal{O}$ of $\mathbb{R}^d$, for s in $\mathbb{N}$ and $0 < r < 1$, the Hölder space $C^{s,r}(\mathcal{O})$ is the set of the functions of class $C^s(\mathcal{O})$ such that

$$\|u\|_{C^{s,r}(\mathcal{O})} := \sup_{|\alpha| \leqslant s} \left(\|\partial^\alpha u\|_{L^\infty(\mathcal{O})} + \sup_{x \neq y \in \mathcal{O}} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^r} \right) < +\infty,$$

and $C^{s,r}_c(\mathcal{O})$ is the subset of the functions in $C^{s,r}(\mathcal{O})$ which are compactly supported in the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ in $\mathbb{R}^d$.

The main goal of this paper is to obtain an expansion for the solutions of the Navier–Stokes equations in the vanishing viscosity limit. It is useful to re-examine first the case of the Euler equations.

1.1. The inviscid case

We first gather from the literature the following compendium of results regarding the inviscid case:

Theorem 1.1. *For initial velocities v_0 as described in Definition 1.1 the following hold true:*

(1) **Existence and uniqueness.** *There exist $T > 0$ and a unique solution*

$$v^0 \in L^\infty([0,T]; \mathrm{Lip}(\mathbb{R}^3)) \cap \mathrm{Lip}([0,T]; L^2(\mathbb{R}^3))$$

to the **Euler** *equations:*

$$\partial_t v^0 + v^0 \cdot \nabla v^0 = -\nabla p^0, \tag{4}$$

$$\mathrm{div}\; v^0 = 0, \tag{5}$$

with v_0 as initial velocity. From now on we denote as $\mathcal{D}$ the vector field

$$\mathcal{D} := \partial_t + v^0 \cdot \nabla, \tag{6}$$

and as $\mathcal{X}^0$ the flow of particle trajectories defined by the differential equation $\partial_t \mathcal{X}^0(t,x) = v^0(t, \mathcal{X}^0(t,x))$ with initial data $\mathcal{X}^0(0,x) = x$.

(2) **Propagation of smoothness of the vorticity.** *Moreover for each $t \in [0,T]$ the vorticity*

$$\omega^0 := curl\, v^0 \tag{7}$$

is $C^{s,r}_c(\mathcal{O}_\pm(t))$ where $\mathcal{O}_\pm(t)$ are respectively the domains transported by the flow at time t starting from $\mathcal{O}_{0,\pm}$ at time t, that is $\mathcal{O}_\pm(t) := \mathcal{X}^0(t, \mathcal{O}_{0,\pm})$.

(3) **Propagation of smoothness of the boundary.** *For each $t \in [0,T]$ the boundary $\Gamma(t) := \mathcal{X}^0(t, \Gamma_0) = \partial\mathcal{O}_+(t) = \partial\mathcal{O}_-(t)$ is $C^{s+1,r}$.*

(4) **Propagation of smoothness of the level function.** *For each $t \in [0,T]$ the boundary $\Gamma(t)$ is given by the equation $\Gamma(t) = \{\varphi^0(t,\cdot) = 0\}$, where*

$$\varphi^0 \in L^\infty([0,T]; C^{1,r}(\mathbb{R}^3)) \cap L^\infty([0,T]; C^{s+1,r}(\mathcal{O}_\pm(t)))$$

verifies

$$\mathcal{D}\varphi^0 = 0, \tag{8}$$

$$\varphi^0|_{t=0} = \varphi_0. \tag{9}$$

Moreover $\mathcal{O}_\pm(t) = \{\pm\varphi^0(t,\cdot) > 0\}$ and there exists $\eta > 0$ such that for $0 \leqslant t \leqslant T$, and x such that for $|\varphi^0(t,x)| < \eta$ the vector $n(t,x) := \nabla_x \varphi^0(t,x)$ satisfies $n(t,x) \neq 0$.

(5) **The incompressible Rankine–Hugoniot condition.** *For each* $t \in [0, T]$ *the function* $(\omega^0 \cdot n)(t, \cdot)$ *is* $C^{0,r}$ *on* $\{|\varphi^0(t, \cdot)| < \eta\}$.

(6) **Smoothness in time.** *Finally the internal boundary* $\Gamma(t)$ *is analytic with respect to* $t \in [0, T]$ *and the restrictions on each side of the boundary of the flow* $\mathcal{X}^0$ *are also analytic with respect to time with values in* $C^{s+1,r}$.

Remark 1.1. The notation $L^\infty([0, T]; C^{s+1,r}(\mathcal{O}_\pm(t)))$ is slightly improper since the domain $\mathcal{F}(t)$ depends on t. One should more precisely think of u as the section of a vector bundle. However, since we think that there should not be any ambiguity, we will retain this notation in the sequel.

Remark 1.2. Equation (4) means that the curl of the left side is identically zero, or equivalently, that the left side is the gradient of a scalar tempered distribution.

Remark 1.3. When $v \in L^2(\mathbb{R}^3)$ satisfies the equations div $v = 0$ and curl $v = \omega$ with $\omega \in L^\infty_c(\mathbb{R}^3)$ the Biot–Savart law holds:

$$v(x) = \int_{\mathbb{R}^3} \omega(y) \wedge \nabla F(x-y) dy \quad \text{where } F(x) := -1/(4\pi|x|). \tag{10}$$

Conversely if v satisfies (10) with $\omega \in L^\infty_c(\mathbb{R}^3)$ then $v \in L^2(\mathbb{R}^3)$.

Remark 1.4. We will give a proof of Theorem 1.1 which relies on the vorticity formulation of the Euler equations and some norms based on L^∞. The condition $v^0 \in \mathrm{Lip}([0, T]; L^2(\mathbb{R}^3))$ in Theorem 1.1 imposes uniqueness.

Remark 1.5. We stress that the vector n is not a unit vector even if it is so at $t = 0$, since it is stretched when time proceeds according to the equation

$$\mathcal{D}n = -{}^t(\nabla v^0) \cdot n. \tag{11}$$

Chemin's proof of the two-dimensional case uses vorticity smoothness with respect to the vector fields tangential to the boundary of the patch. These vector fields move with the fluid, and their own smoothness is therefore linked to the smoothness of the fluid velocity. The idea of using regularity properties with respect to a family of vector fields originates in Hörmander's Fourier operator theory (see Hörmander [83] for a comprehensive expository). In a nonlinear setting it goes back to the work of Bony—cf. [14]—and to the work of Alinhac [3] and Chemin [25] for the case of non-smooth vector fields.

Remark 1.6. This approach even allows us to deal with more general cases since it could apply for example to initial data which are irregular with respect not just to one hypersurface $\{\varphi^0(t, .) = 0\}$ but to the whole foliation of the hypersurfaces $\{\varphi^0(t, .) = a\}$ for $a \in \mathbb{R}$.

In [12] Bertozzi and Constantin succeeded in recovering global-in-time persistence of the $C^{s+1,r}$ smoothness of the boundary in the special case of constant vortex patches by the contour dynamics approach.

The persistence of piecewise smoothness of the vorticity (Hölder regularity up to the boundary) was proved later—first by Depauw for the case $s = 0$ in [51, 50] and by Huang [85] for the general case s in $\mathbb{N}$ with a Lagrangian approach.

Chemin's approach was extended to the three-dimensional case $d = 3$ by Gamblin and Saint-Raymond in [63], but it is only a short time result since in the three-dimensional case the vorticity ω^0 is stretched along particle trajectories according to the formula $\omega^0(t, \mathcal{X}^0(t, x)) = \omega_0(x) \cdot \nabla_x \mathcal{X}^0(t, x)$ which solves the equation

$$\mathcal{D}\omega^0 = \omega^0 \cdot \nabla v^0, \tag{12}$$

$$\omega^0|_{t=0} = \omega_0. \tag{13}$$

A rough estimate of the vorticity stretching is given by

$$\|\omega^0(t)\|_{L^\infty} \leqslant e^{\int_0^t \|v^0\|_{\mathrm{Lip}}(s) ds} \, \|\omega_0\|_{L^\infty}. \tag{14}$$

Of course in some specific situations vorticity stretching vanishes, so the existence and regularity results are in fact global in time, i.e. $T > 0$ can be taken arbitrarily large as in the two-dimensional case. For instance a particular situation is the one of an axisymmetric initial velocity, as considered by Gamblin and Saint-Raymond (and others). We will not specifically consider this case here.

Indeed the statement of the three-dimensional case in Theorem 1.1 is not strictly given in [62]; roughly speaking, Gamblin and Saint-Raymond deal with tangential smoothness in the case $s = 0$. The tangential smoothness in the general case $s \in \mathbb{N}$ was alluded to in the comment (ii) of § 1.d of Gamblin and Saint-Raymond and rigorously proved by Zhang and Qiu in the couple of papers [125, 124]. The persistence of piecewise $C^{0,r}$ smoothness (the case $s = 0$) was proved by Huang in [86] by means of a Lagrangian approach (see also [55] § 3.1). We did not find any proof of the persistence of higher order piecewise $C^{s,r}$ smoothness in the literature. This is why we will give a few details of the whole proof of the three-dimensional case including a proof of the persistence of piecewise $C^{s,r}$ smoothness (in the Eulerian framework of Chemin, Gamblin and Saint-Raymond).

Theorem 1.1 gathers the results that we will need, but many other works deal with close issues. Let us mention some papers by Chemin and Danchin on vortex patches with singular boundary in two dimensions. By using the pseudo-locality of the Paley–Littlewood theory and transport equations, by means of log-Lipschitz velocities they show that if the initial boundary is regular ($C^{0,r}$ in Chemin's works [33, 34], chapter 9, generalized into $C^{s,r}$ in Danchin's paper [45]) apart from a closed subset; this remains regular for all time apart from for the closed subset transported by the flow. When the singularity of the boundary is a cusp, the corresponding velocity is Lipschitz, which allows Danchin (cf. [48, 47, 44]) to prove the global stability of the cusp with conservation of the order (actually he gives a global result of persistence of conormal regularity with respect to vector fields vanishing at a singular point, which generalizes the structure of a cusp). On the other hand, in [38] several numerical simulations, based on an adaptive multi-scale algorithm using wavelet interpolation, show that corners are unstable, each either immediately becoming a cusp, in the case of an initial sharp corner, or immediately becoming flat, in the case of an initial obtuse corner.

In [51] (see also [50]) Depauw addresses the case where the domain of the fluid is a bounded subset of $\mathbb{R}^2$ with smooth boundary. When the boundary of the vortex patch is away from the domain boundary, it will remain so under the evolution, and Chemin's result remains valid without restriction. When the vortex patch is tangent to the material boundary (for transverse intersection there is a counterexample of Bahouri and Chemin), the author proves that if the initial vortex patch is of class $C^{1,r}$ there exists a unique solution in this class of vortex patches at least up to some time $T_* > 0$. He also proves local-in-time existence and uniqueness for several mutually tangent vortex patches in $\mathbb{R}^2$.

In [55] Dutrifoy proves local-in-time existence and uniqueness in the case where the domain of the fluid is a bounded subset of $\mathbb{R}^3$ with smooth boundary, when the boundary of the vortex patch is away from the domain boundary, without restriction, and when the vortex patch is tangent to the material boundary, under a technical condition. In this latter case, his method allowed completing the previously mentioned two-dimensional local-in-time result of Depauw to a global one (with a slight loss of smoothness).

Finally the two-dimensional result of Depauw was recently extended to global-in-time results by Huang in [87] without loss of smoothness.

1.2. The viscous case

Let us now consider the Navier–Stokes equations. The smoothing effect of the viscosity term $\nu \Delta v^\nu$ is well-known: it is even crucial in both Leray's and Kato's existence theories, and several papers (see for example [36, 104, 53, 64, 23]) analyze it precisely. Here we will show a *conormal* smoothing of the initial vorticity discontinuity into a layer of width $\sqrt{\nu t}$ around the hypersurface $\{\varphi^0(t, .) = 0\}$ where the discontinuity has been transported at the time t by the flow of the Euler equations. Hence the fluid vorticity ω^ν depends—locally—on an extra 'fast' scale: $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$ (cf. § 4.1) and will be described by an expansion of the form

$$\omega^\nu(t, x) \sim \Omega \left(t, x, \frac{\varphi^0(t, x)}{\sqrt{\nu t}} \right), \tag{15}$$

where the *viscous profile* $\Omega(t, x, X)$ admits some limits when $X \to \pm\infty$.

The idea of associating a viscous profile with an inviscid discontinuity seems to date back to Rankine [105] and is widely known for when the singularity is a shock, as for instance in compressible fluid mechanics (see the recent achievements by Guès, Métivier, Williams and Zumbrun in [75, 74, 72, 73, 70]). However since they are characteristic and conservative, the vortex patches are very different from the shocks of the compressible fluid mechanics (which are non-characteristic and dissipative; cf. for instance [103]). We therefore would like to be precise that we borrow the words 'viscous profile' from the setting of shock profiles but that our setting is quite different. For instance, the extra scales involved are not the same.

Still we hope that the approach developed here can be extended to some other setting where smoothing of characteristic conormal singularities occurs, including jump discontinuities of the velocity gradient in compressible fluid mechanics, which are studied for the inviscid case in [6, 5, 4, 2, 110, 102, 101], in meteorology (cf. [24]) or for the domain walls in ferromagnetism (cf. [78]). Another challenging issue is the case of contact discontinuities in compressible Navier–Stokes theory, where a similar diffusive behavior appears. However, the contact problem, studied in various degrees of generality by for instance Liu and Xin [95], Rousset [108], and Bianchini and Bressan [13], has been treated so far in one spatial dimension only, so vorticity has up to now not directly entered the picture.

Let us go back to the vortex patches. We will now describe the construction of the viscous profile Ω involved in the expansion (15). We will look for a viscous profile Ω of the form

$$\Omega(t, x, X) = \omega^0(t, x) + \tilde{\Omega}(t, x, X),$$

where $\tilde{\Omega}(t, x, X)$ denotes a perturbation local with respect to the extra scale X such that

$$\lim_{X\to\pm\infty} \tilde{\Omega}(t, x, X) = 0. \tag{16}$$

Hence the Navier–Stokes vorticities $\omega^\nu(t, x)$ will be described by an expansion of the form

$$\omega^\nu(t, x) \sim \omega^0(t, x) + \tilde{\omega}^\nu(t, x) \quad \text{where } \tilde{\omega}^\nu(t, x) := \tilde{\Omega}\left(t, x, \frac{\varphi^0(t, x)}{\sqrt{\nu t}}\right).$$

The dependence of the perturbation $\tilde{\omega}^\nu$ on $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$ encodes the 'conormal self-similarity' of the layer. Pragmatically, the consequences of the condition (16) on the profile $\tilde{\Omega}(t, x, X)$ at the level of the function $\tilde{\omega}^\nu$ are threefold:

(1) For any $(t, \nu) \in (0, T) \times \mathbb{R}_+^*$, $\tilde{\omega}^\nu(t, x) \to 0$ when $\varphi^0 \to \pm\infty$. This was actually our motivation for imposing the condition (16) on the profile $\tilde{\Omega}(t, x, X)$: it sounds natural that the viscous layer is confined to the neighborhood of the hypersurface where the inviscid discontinuity occurs.

(2) For any $t \in (0, T)$, for any $x \in \mathbb{R}^d \setminus \Gamma(t)$, $\tilde{\omega}^\nu(t, x) \to 0$ when $\nu \to 0^+$. This consequence is directly linked with another strong underlying motivation for this work that is the issue of the inviscid limit of the Navier–Stokes equation being the Euler ones. The 'strength' of this inviscid limit (that is, the functional space where it holds) depends not only on the presence or absence of material boundaries but also on the smoothness of the initial data. Basically the smoother the initial data, the stronger the convergence. For smooth data the Navier–Stokes solutions are *regular* perturbations of the corresponding Euler solutions in the inviscid limit (see for instance Swann [117] and Kato [90, 91]) and converge, say, in any Hölder spaces, with a rate of order νt. We also quote here Masmoudi [98], for a slight improvement. At the other end, in the two-dimensional case when vortex sheets are prescribed as initial data, one only knows the weak L^2 convergence (cf. [49]). The vortex patches

are an intermediate case, first studied in two dimensions by Constantin and Wu in [41]. In [1], Abidi and Danchin found the optimal rate in $L^\infty([0,T];L^2(\mathbb{R}^d))$, and recently [98] extended this result to the three-dimensional case. We also refer the reader to the papers of Hmidi [81, 82] for the study of two-dimensional vortex patches (including ones with singular boundary). Here we describe what happens locally, which reveals the optimal estimates of the convergence rates in any spaces as a simple byproduct.

(3) For any $(x,\nu) \in \mathbb{R}^d \times \mathbb{R}^*_+$, $\tilde{\omega}^\nu(t,x) \to 0$ when $t \to 0^+$. This yields that the Navier–Stokes vorticities ω^ν have the same initial value as the Euler one ω^0. Let us mention here that the analysis can be simplified if, on the contrary, we allow ourselves to choose the initial data for the Navier–Stokes vorticities ω^ν since there exist some well-prepared data for which the viscous smoothing is already taken into account (cf. § 7.3).

We will argue (cf. § 4.2) that the corresponding expansion of the velocity is of the form

$$v^\nu(t,x) \sim v^0(t,x) + \sqrt{\nu t}\, V\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right), \tag{17}$$

with $V(t,x,X)$ satisfying

$$V(t,x,X) \to 0 \quad \text{when } X \to \pm\infty. \tag{18}$$

We will be led (cf. § 4.3) to consider for the profile $V(t,x,X)$ the *linear* partial differential equation

$$\mathcal{L}V = 0 \tag{19}$$

where the differential operator L is given by

$$\mathcal{L} = \mathcal{E} - t(\mathcal{D} + A) \tag{20}$$

where $\mathcal{D}$ is the vector field in (6), and $\mathcal{E}$ and A are some operators of respective orders 2 and 0 acting formally on functions $V(t,x,X)$ as follows:

$$\mathcal{E}V := a\partial_X^2 V + \frac{X}{2}\partial_X V - \frac{1}{2}V, \tag{21}$$

$$AV := V \cdot \nabla_x v^0 - 2\frac{(V \cdot \nabla_x v^0)\cdot n}{a} n. \tag{22}$$

Here a denotes a function in the space

$$\mathcal{B} := L^\infty([0,T]; C^{0,r}(\mathbb{R}^d)) \cap L^\infty([0,T]; C^{s,r}(\mathcal{O}_\pm(t))) \tag{23}$$

such that

$$\inf_{[0,T]\times\mathbb{R}^d} a = c > 0 \tag{24}$$

and such that $a = |n|^2$ when $|\varphi^0| < \eta$.

We expect a continuous transition of the viscous fluid velocity v^ν and of the viscous fluid vorticity ω^ν (these are the Rankine–Hugoniot conditions), instead of

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the discontinuity of the inviscid vorticity ω^0. These continuity conditions would be translated into the following Dirichlet–Neumann-type transmission conditions for the profile $V(t,x,X)$ on the internal boundary $\{X=0\}$ (cf. § 4.3.5): for any $(t,x)\in(0,T)\times\mathbb{R}^3$,

$$[V]=0 \quad \text{and} \quad [\partial_X V]=-\frac{n\wedge(\omega^0_+-\omega^0_-)}{a}, \tag{25}$$

where the brackets denote the jump discontinuity across $\{X=0\}$, that is $[f(t,x,X)]:=f|_{X=0^+}-f|_{X=0^-}$ and $\omega^0_\pm$ are some well-chosen extensions of $\omega^0|_{\mathcal{O}_\pm(t)}$.

The transmission conditions (25) are normal for the operator $\mathcal{E}$ which is elliptic with respect to X thanks to the condition (24). Actually we point out here that because of its unbounded coefficient X, the operator $\mathcal{E}$ does not strictly enter in the classical theory of elliptic operators (with t,x as parameters through the coefficient a). However we will see that it shares their main features, at least for our purposes. For instance omitting, to simplify, the dependence on t (which is here only a parameter), we have the following result (cf. § 5.1):

Proposition 1.1. *For any $f\in L^2(\mathbb{R}^3;H^{-1}(\mathbb{R}))$ and $g\in L^2(\mathbb{R}^3)$ there is exactly one solution $V\in L^2(\mathbb{R}^3;H^1(\mathbb{R}))$ of the equation $\mathcal{E}V=f$ with the transmission conditions $[V]=0$ and $[\partial_X V]=g$ across $\{X=0\}$.*

One can verify that this is actually the kind of equation that arises for the layer created in the very particular case of stationary two-dimensional circular vortex patches (see [1]). In this case, because of the symmetry, there is neither convection nor stretching, and the norm of the normal vector is conserved, so the profile equation is simply an ODE, whose solutions involve a Gaussian function.

Of course the full equation (19) is much more intricate. Roughly speaking, for $t>0$ equation (19) is hyperbolic in t,x and parabolic in t,X; but it degenerates for $t=0$ precisely into the previous elliptic equation. However we will show that equation (19), with the transmission conditions (25) and the conditions (18) at infinity, is well-posed. We stress that since the hypersurface $\{t=0\}$ is characteristic for the operator L, no initial condition at $t=0$ has to be prescribed for equation (19). We will use here an L^2 setting, for two reasons. First, with a view to future extensions we want to give a claim that one hopes is robust. In particular it has been well-known since [18] that in (multi-dimensional) compressible fluid mechanics the inviscid system should be tackled in L^2-type spaces. This aim of robustness is also the reason for choosing to put the emphasis on the velocity in this presentation, more than on the vorticity. The second reason for an L^2 setting is linked to the degeneracy at $t=0$ of equation (19), which leads to the existence of parasite solutions. For instance if we look for solutions V not depending on X and neglect the term involving A, equation (19) simplifies to the Fuchsian differential equation $t\partial_t V=-\frac{V}{2}$, which admits an infinity of solutions, i.e. $V(t)=\frac{C}{\sqrt{t}}$, for $C\in\mathbb{R}$. However only one is in $L^2(0,T)$, corresponding to $C=0$; and we expect that the scaling is relevant enough for us to have a solution with $L^2(0,T)$ smoothness, even in the case of the full equation (19). Let us give a precise statement:

denoting as E_1 the space

$$E_1 := L^2\big((0,T)\times\mathbb{R}^3; H^1(\mathbb{R})\big),$$

and as E_1' its topological dual space, we will prove:

Theorem 1.2. *For any $f \in E_1'$, for any $g \in L^2((0,T)\times\mathbb{R}^3)$ there exists exactly one solution $V(t,x,X) \in E_1$ of $\mathcal{L}V = f$ with the transmission conditions $([V],[\partial_X V]) = (0,g)$ on $\Gamma := (0,T)\times\mathbb{R}^3\times\{0\}$. In addition the function $\sqrt{t}\|V(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^3\times\mathbb{R})}$ is continuous on $(0,T)$.*

The equation $\mathcal{L}V = f$ is satisfied in the sense of distributions on both sides, $\mathcal{U}_\pm := (0,T)\times\mathbb{R}^3\times\mathbb{R}^*_\pm$, of the hypersurface Γ. Since V is in E_1 the jump discontinuity $[V]$ is in $L^2(\Gamma)$. The sense given to the jump of the derivative $[\partial_X V]$ is actually a part of the problem. The idea is to give some sense by using the equation put into a weak form thanks to Green's formula. We will explain this in detail in § 5.2.

In the case of the transmission conditions (25), the source terms are orthogonal to n. It is then possible to use the uniqueness part of the previous theorem to prove that the function $V(t,x,X)\cdot n(t,x)$ vanishes identically. This orthogonality condition is consistent with the incompressibility condition (see § 5.4) and with the linearity of the equation (see § 4.3.2).

We are now interested in the smoothness of the solution V given by Theorem 1.2. It is judicious to look again at the associated elliptic problem first. We will prove that the solution inherits the smoothness with respect to the usual variables t,x from the coefficients; and these are piecewise smooth with respect to the fast variable X. To be more precise, let us denote as p-$\mathcal{S}(\mathbb{R})$ the space of the functions $f(X)$ whose restrictions to the half-lines $\mathbb{R}_\pm$ are in the Schwartz space of rapidly decreasing functions, and as $\mathcal{A}$ the space (of the functions $f(t,x,X)$)

$$\mathcal{A} := L^\infty\big((0,T); C^{0,r}\big(\mathbb{R}^3, p\text{-}\mathcal{S}(\mathbb{R})\big)\big) \cap L^\infty\big([0,T]; C^{s,r}\big(\mathcal{O}_\pm(t), p\text{-}\mathcal{S}(\mathbb{R})\big)\big).$$

In § 5.1 we will prove:

Theorem 1.3. *The solution $V(t,x,X)$ of the equation $\mathcal{E}V = 0$ with the transmission conditions (25) is in $\mathcal{A}$.*

The main idea of the proof is to use a spectral localization with respect to x, which is here a parameter. The point is that this process is compatible with some classical elliptic arguments used to get smoothness with respect to X.

We will be able to prove the same for the full equation (19):

Theorem 1.4. *The solution $V(t,x,X)$ of equation (19) with the transmission conditions (25) is in $\mathcal{A}$.*

In contrast to the case for Theorem 1.2, we will use here the particular properties of equation (19) through point 6 of Theorem 1.1. To explain this, let us define for any Fréchet space E of functions depending on t,x and possibly on X the space

$$E_{\mathcal{D}} := \left\{ f \in E / \exists C > 0 \Big/ \left(\frac{\mathcal{D}^k f}{C^k k!}\right)_{k\in\mathbb{N}} \text{ is bounded in } E \right\}.$$

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Thanks to point 6 of Theorem 1.1, we will be able to construct (see § 4.3.5) the extensions $\omega_\pm^0$ of the vorticities and the function a in the space $\mathcal{B}_{\mathcal{D}}$. As a consequence we will actually prove in § 5.3 that V is in $\mathcal{A}_{\mathcal{D}}$.

The vorticity profile Ω in the expansion (15) is then constructed as

$$\Omega(t,x,X) := \omega_\pm^0(t,x) + n(t,x) \wedge \partial_X V(t,x,X) \quad \text{for } \pm X > 0. \tag{26}$$

If piecewise smoothness of the initial data is sufficient, it is possible to continue the expansion with respect to νt of the solutions of the Navier–Stokes equations. At the extreme limit, if the initial data is piecewise smooth on each side of the interface $\{\varphi^0 = 0\}$—that is if $s = +\infty$—then it is possible to write a complete formal asymptotic expansion of the vorticity of the form

$$\omega^\nu(t,x) \sim \sum_{j\geqslant 0} \sqrt{\nu t}^j \, \Omega^j \left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right), \tag{27}$$

where the first profile Ω^0 is the one previously constructed: $\Omega^0 := \Omega$.

This vorticity expansion corresponds to some expansions of the velocity and of the pressure of the form

$$v^\nu(t,x) = v^0(t,x) + \sum_{j\geqslant 1} \sqrt{\nu t}^j \, V^j \left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right) + O(\sqrt{\nu t}^\infty), \tag{28}$$

$$p^\nu(t,x) = p^0(t,x) + \sum_{j\geqslant 2} \frac{\sqrt{\nu t}^j}{t} \, P^j \left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right) + O(\sqrt{\nu t}^\infty) \tag{29}$$

where the profile V^1 is the one constructed in the previous section: $V^1 := V$.

The other profiles V^j and P^j, for $j \geqslant 2$, will be chosen such that substituting

$$v_a^\nu(t,x) = v^0(t,x) + \sum_{j\geqslant 1} \sqrt{\nu t}^j \, V^j \left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right), \tag{30}$$

$$p_a^\nu(t,x) = p^0(t,x) + \sum_{j\geqslant 2} \frac{\sqrt{\nu t}^j}{t} \, P^j \left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right) \tag{31}$$

for v^ν, p^ν leads to arbitrarily small errors in the Navier–Stokes equations.

More precisely we will consider the following reformulation of (1), (2):

$$\partial_t v^\nu + v^\nu \cdot \nabla v^\nu + \nabla p^\nu = \nu \Delta v^\nu \quad \text{on } \mathcal{O}_\pm, \tag{32}$$

$$\operatorname{div} v^\nu = 0 \quad \text{on } \mathcal{O}_\pm, \tag{33}$$

$$[v^\nu] = 0, \tag{34}$$

$$[\partial_n v^\nu] = 0, \tag{35}$$

$$[p^\nu] = 0, \tag{36}$$

where $\mathcal{O}_\pm$ denote the space–time domains

$$\mathcal{O}_\pm := \{(t,x) \in (0,T) \times \mathbb{R}^3 / x \in \mathcal{O}_\pm(t)\},$$

and

$$\partial\mathcal{O} := \{(t,x) \in (0,T) \times \mathbb{R}^3 / x \in \partial(\mathcal{O}_\pm(t))\}$$

the interface between them, and the brackets denote here the jump across $\{\varphi^0(t,.) = 0\}$, that is $[f] := f|_{\varphi^0=0^+} - f|_{\varphi^0=0^-}$.

Let us recall that the conditions (34)–(36) are the Rankine–Hugoniot conditions associated with the Navier–Stokes equations. Indeed if $v^\nu_\pm$ and $p^\nu_\pm$ are in $C^\infty((0,T); C^\infty(\mathcal{O}_\pm(t)))$, satisfy (1), (2) on $\mathcal{O}_\pm$ and are such that $v^\nu_\pm$, $\operatorname{curl} v^\nu_\pm$ and $p^\nu_\pm$ have the same values on $\partial\mathcal{O}$, then v^ν and p^ν, defined by $v^\nu = v^\nu_\pm$ and $p^\nu = p^\nu_\pm$ on $\mathcal{O}_\pm$, are in $C^\infty((0,T) \times \mathbb{R}^3)$ and satisfy (1), (2) on $(0,T) \times \mathbb{R}^3$. We point out that for a piecewise smooth velocity vector field $v^\nu_\pm$ which is continuous across $\partial\mathcal{O}$, the requirement that $\operatorname{curl} v^\nu_\pm$ is also continuous across $\partial\mathcal{O}$ is equivalent to the requirement that $\partial_n((v^\nu_\pm)_{\tan})$ is continuous across $\partial\mathcal{O}$, where $u_{\tan}$ denotes the tangential part of a vector field u, defined by $u_{\tan} := u - \frac{1}{a}(u \cdot n)n$.

On formally substituting v^ν_a and p^ν_a respectively for v^ν and p^ν, we get

$$\partial_t v^\nu_a + v^\nu_a \cdot \nabla v^\nu_a + \nabla p^\nu_a - \nu \Delta v^\nu_a = \sum_{j\geqslant 1} \frac{\sqrt{\nu t}^j}{t} F^j_a\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right) \quad \text{on } \mathcal{O}_\pm, \tag{37}$$

$$\operatorname{div} v^\nu_a = \sum_{j\geqslant 1} \sqrt{\nu t}^j F^j_b\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right) \quad \text{on } \mathcal{O}_\pm, \tag{38}$$

$$[v^\nu_a] = \sum_{j\geqslant 2} \sqrt{\nu t}^j [V^j], \tag{39}$$

$$[\partial_n (v^\nu_a)_{\tan}] = \sum_{j\geqslant 2} \sqrt{\nu t}^j [a\partial_X V^j_{\tan} + \partial_n V^{j-1}_{\tan}], \tag{40}$$

$$[p^\nu_a] = \sum_{j\geqslant 2} \frac{\sqrt{\nu t}^j}{t} [P^j], \tag{41}$$

where, here, the notation $[U]$, where U denotes a function which depends on t, x and X, stands for

$$[U] := \tilde{U}|_{X=0^+,\varphi^0=0^+} - \tilde{U}|_{X=0^+,\varphi^0=0^+}.$$

In § 6 we will construct the profiles V^j and P^j, for $j \geqslant 2$, such that the resulting profiles in the right hand sides of (37)–(41) vanish, that is such that $F^j_a = 0$ and $F^j_b = 0$ for any $j \geqslant 1$, and $[V^j] = [a\partial_X V^j_{\tan} + \partial_n V^{j-1}_{\tan}] = 0$ and $[P^j] = 0$ for any $j \geqslant 2$.

We plan to address the stability of these expansions in a forthcoming paper. More precisely we believe that there exists $\nu_0 > 0$ such that for $0 < \nu < \nu_0$ for all $k \in \mathbb{N}$ for any $(t,x) \in (0,T) \times \mathbb{R}^3$,

$$\omega^\nu(t,x) = \sum_{j=0}^{k} \sqrt{\nu t}^j \,\Omega^j\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right) + \sqrt{\nu t}^{k+1} \omega^\nu_R, \tag{42}$$

where $(\sqrt{\nu t}^{s'+r'} \|\omega^\nu_R\|_{L^\infty((0,T);C^{s',r'}(\mathbb{R}^3))})_{0<\nu<\nu_0}$ is bounded for any $s' \in \mathbb{N}$, for any $r' \in (0,1)$.

Remark 1.7. Regarding the transmission aspect of our strategy we refer the reader to [113], in the setting of the approximation of semi-linear symmetric hyperbolic systems of PDEs by the vanishing viscosity method. We also mention [76, 77] by Guès and Rauch, in the context of internal waves for semi-linear symmetric hyperbolic systems, and [79] by Guès and Williams, in the context of viscous shocks profiles.

2. A compendium on the Littlewood–Paley theory

In this section we gather some usual results of the Littlewood–Paley theory that we will need. We begin with the case of the whole space for which we refer the reader to, for example, the books [10, 119] for a much more detailed expository.

2.1. Dyadic decomposition

We first recall the existence of a smooth dyadic partition of unity: there exist two radial functions φ and χ valued in the interval $[0,1]$, belonging respectively to $C_c^\infty(B(0,4/3))$ and to $C_c^\infty(C(3/4,8/3))$ (where $B(0,4/3)$ and $C(3/4,8/3)$ denote respectively the ball $B(0,4/3) := \{\|\xi\|_{\mathbb{R}^3} < 4/3\}$ and the shell $C(3/4,8/3) := \{3/4 < \|\xi\|_{\mathbb{R}^3} < 8/3\}$; and $C_c^\infty(U)$ denotes the space of the smooth functions whose support is a compact included in U) such that

$$\forall \xi \in \mathbb{R}^3, \quad \chi(\xi) + \sum_{j\geqslant 0} \varphi(2^{-j}\xi) = 1,$$

$$|j-j'| \geqslant 2 \Rightarrow \ \operatorname{supp} \varphi(2^{-j}\cdot) \cap \operatorname{supp} \varphi(2^{-j'}\cdot) = \emptyset,$$

$$j \geqslant 1 \Rightarrow \operatorname{supp} \chi(2^{-j}\cdot) \cap \operatorname{supp} \varphi(2^{-j}\cdot) = \emptyset.$$

The Fourier transform $\mathcal{F}$ is defined on the space of integrable functions $f \in L^1(\mathbb{R}^3)$ by

$$\mathcal{F}f := \int_{\mathbb{R}^3} e^{-x.\xi} f(x)dx$$

and extended in an automorphism of the space $\mathcal{S}'(\mathbb{R}^3)$ of the tempered distributions, which is the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ of rapidly decreasing functions. We will use the non-homogeneous Littlewood–Paley decomposition (in $\mathcal{S}'(\mathbb{R}^3)$) $Id = \sum_{j\geqslant -1} \Delta_j$, where the so-called dyadic blocks Δ_j correspond to the Fourier multipliers: $\Delta_{-1} := \chi(\nabla)$ and $\Delta_j := \varphi(2^{-j}\nabla)$ for $j \geqslant 0$, that is

$$\Delta_{-1}u(x) := \int_{\mathbb{R}^3} \tilde{h}(y)u(x-y)dy \quad \text{and} \quad \Delta_j u(x) := 2^{3j}\int_{\mathbb{R}^3} \tilde{h}(2^j y)u(x-y)dy \quad \text{for } j \geqslant 0, \quad (43)$$

where $h := \mathcal{F}^{-1}\varphi$ and $\tilde{h} := \mathcal{F}^{-1}\chi$. We also introduce the low frequency cutoff operator $S_j := \sum_{k\leqslant j-1} \Delta_j$.

2.2. Besov spaces

We now recall the definition of the Besov spaces $B^\lambda_{p,q}$ on the whole space $\mathbb{R}^3$ which are, for $\lambda \in \mathbb{R}$ (the smoothness index), $p, q \in [1,+\infty]$ (respectively the integral exponent and

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the sum exponent), some Banach spaces defined by

$$B^{\lambda}_{p,q}(\mathbb{R}^3) := \{f \in \mathcal{S}'(\mathbb{R}^3) / \|f\|_{B^{\lambda}_{p,q}(\mathbb{R}^3)} := \|(2^{j\lambda} \|\Delta_j f\|_{L^p(\mathbb{R}^3)})_{j\geqslant -1}\|_{l^q} < \infty\}.$$

These spaces do not depend on the choice of the dyadic partition above.

When $p = q = +\infty$ the Besov spaces $B^{\lambda}_{p,q}$ are simply the Hölder–Zygmund spaces C^{λ}_{*} and in particular for $\lambda \in \mathbb{R}_+ \setminus \mathbb{N}$ they coincide with the $C^{s,r}$ spaces of the introduction in the sense that $B^{\lambda}_{\infty,\infty}(\mathbb{R}^3) = C^{s,r}(\mathbb{R}^3)$ where s is the entire part of λ and $r := \lambda - s$. We will use the short notation $\|\cdot\|_{\lambda}$ for $\|\cdot\|_{B^{\lambda}_{\infty,\infty}(\mathbb{R}^3)}$ and $\|\cdot\|_{s,r}$ for $\|\cdot\|_{B^{s+r}_{\infty,\infty}(\mathbb{R}^3)}$.

It is worth mentioning that the space $L^{\infty}(\mathbb{R}^3)$ is continuously embedded in the Besov space $B^{0}_{\infty,\infty}(\mathbb{R}^3)$:

$$L^{\infty}(\mathbb{R}^3) \hookrightarrow B^{0}_{\infty,\infty}(\mathbb{R}^3). \tag{44}$$

Conversely for $\lambda > 0$ the spaces $B^{\lambda}_{\infty,\infty}(\mathbb{R}^3)$ are continuously embedded in the space $L^{\infty}(\mathbb{R}^3)$, and the L^{∞} norm can also be estimated by the following logarithmic interpolation inequality:

$$\|f\|_{L^{\infty}(\mathbb{R}^3)} \lesssim L(\|f\|_{B^{0}_{\infty,\infty}(\mathbb{R}^3)}, \|f\|_{B^{\lambda}_{\infty,\infty}(\mathbb{R}^3)}), \tag{45}$$

where we define for a and b strictly positive $L(a,b) := a \ln(e + \frac{a}{b})$—which is increasing with respect to both a and b. Here, and in the remainder of the paper, we use the notation $\lesssim$ to avoid writing meaningless constants.

We also recall the way in which Fourier multipliers act on Besov spaces: if f is a smooth function such that for any multi-index α there exists an integer $m \in \mathbb{N}$ such that

$$\forall \xi \in \mathbb{R}^3, \quad |\partial^{\alpha} f(\xi)| \leqslant C_{\alpha}(1 + |\xi|)^{m-|\alpha|}, \tag{46}$$

then for all $\lambda \in \mathbb{R}$ and $p, q \in [1, +\infty]$ the operator $f(\nabla)$ is continuous from $B^{\lambda}_{p,q}$ to $B^{\lambda-m}_{p,q}$. In particular if χ is a function in $C^{\infty}_{0}(\mathbb{R}^3; \mathbb{R})$, positive and equal to 1 near 0, and Λ is the Fourier multiplier associated with the Fourier symbol $(\chi(\zeta) + |\zeta|^2)^{\frac{1}{2}}$, then we get a one-parameter group of elliptic operators Λ^t, for $t \in \mathbb{R}$, continuous from $B^{\lambda}_{p,q}(\mathbb{R}^3)$ to $B^{\lambda+t}_{p,q}(\mathbb{R}^3)$.

2.3. Bony's paraproduct

When v and w are two Hölder distributions, we denote by $T_v w$ Bony's paraproduct of w by v:

$$T_v w := \sum_{j\geqslant 1} S_{j-1} v \cdot \Delta_j w \tag{47}$$

for which we have the following tame estimates:

$$\|T_v w\|_{\lambda} \lesssim \|v\|_{L^{\infty}} \|w\|_{\lambda} \quad \text{for } \lambda \in \mathbb{R}, \tag{48}$$

$$\|(v - T_v) w\|_{\lambda} \lesssim \|v\|_{\lambda} \|w\|_{L^{\infty}} \quad \text{for } \lambda > 0. \tag{49}$$

We will also use the following commutator estimate (cf. [10] Lemma 2.92): if f is a smooth function, homogeneous, and of degree m away from a neighborhood of 0 then

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the commutator $[T_a, f(\nabla)] := T_a f(\nabla) - f(\nabla) T_a$ between a paraproduct T_a and the Fourier multiplier $f(\nabla)$ may be estimated for any $\lambda \in \mathbb{R}$ and for any $r \in (0,1)$ by

$$\|[T_a, f(\nabla)]u\|_{\lambda-m+r} \lesssim \|a\|_r \, \|u\|_\lambda. \tag{50}$$

We also mention the following useful estimate for commutators of the form $R_j := [v^0 \cdot \nabla, \Delta_j] f$ (cf. [10] Lemma 2.93):

$$\sup_{j \geqslant -1} 2^{j\lambda} \, \|R_j\|_{L^p} \lesssim \|\nabla v^0\|_{B^{d/p}_{p,\infty} \cap L^\infty} \|f\|_{B^\lambda_{p,\infty}} \quad \text{for } 0 < \lambda < 1 + d/p, \tag{51}$$

$$\sup_{j \geqslant -1} 2^{j\lambda} \, \|R_j\|_{L^p} \lesssim \|\nabla v^0\|_{L^\infty} \|f\|_{B^\lambda_{p,\infty}} + \|\nabla f\|_{L^p} \, \|\nabla v^0\|_{B^{\lambda-1}_{p,\infty}} \quad \text{for } 0 < \lambda. \tag{52}$$

2.4. Transport estimates

We recall the following transport estimates, where $r \in (0,1)$ and $\mathcal{D}$ is the vector field defined in (6). For $s = -1$ or $s = 0$, there exists a positive constant C such that for any function $f \in C([0,T], C^{s,r}(\mathbb{R}^3))$, for any $t \in [0,T]$,

$$\|f(t)\|_{s,r} \, e^{-CV(t)} \leqslant \|f(0)\|_{s,r} + \int_0^t \|(\mathcal{D}f)(\tau)\|_{s,r} \, e^{-CV(\tau)} d\tau, \tag{53}$$

where

$$V(t) := \int_0^t \|(\nabla v^0)(\tau)\|_{L^\infty(\mathbb{R}^3)} d\tau,$$

and for any $s \geqslant 1$, there exists a positive constant C such that for any function $f \in C([0,T], C^{s,r}(\mathbb{R}^3))$, for any $t \in [0,T]$,

$$\|f(t)\|_{s,r} e^{-CV_{s-1,r}(t)} \leqslant \|f(0)\|_{s,r} + \int_0^t \|(\mathcal{D}f)(\tau)\|_{s,r} e^{-CV_{s-1,r}(\tau)} d\tau, \tag{54}$$

where

$$V_{s-1,r}(t) := \int_0^t \|(\nabla v^0)(\tau)\|_{s-1,r} d\tau.$$

We refer the reader to, for example, [10] Theorem 3.11 for a proof of the estimates (53), (54) by spectral localization, that is by applying the dyadic block Δ_j to the equation and then making use of an energy method, taking care of the commutators thanks to Bony's paraproduct.

2.5. Besov spaces on Lipschitz domains

When Ω is a Lipschitz domain we define the Besov spaces $B^\lambda_{p,q}(\Omega)$ (see for instance [118]) as the set of restrictions of all elements of $B^\lambda_{p,q}(\mathbb{R}^3)$ in the sense of $\mathcal{D}'(\Omega)$, the space of the distributions on Ω. In other words, the spaces $B^\lambda_{p,q}(\Omega)$ consist of exactly those distributions $f \in \mathcal{D}'(\Omega)$ which have extensions belonging to $B^\lambda_{p,q}(\mathbb{R}^3)$. Endowed with the quotient space norms, they become Banach spaces.

We will make use of extension operators. We give the following nice general result.

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Theorem 2.1 (Rychkov [109]). *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ with a bounded boundary. There exists a so-called universal extension operator that is a linear operator ext which maps $B^s_{p,q}(\Omega)$ continuously into itself for any $s \in \mathbb{R}$, $0 < p, q \leqslant \infty$, and satisfies $(ext, u)|_\Omega = u$.*

We may use this result to give intrinsic characterizations of Besov spaces on Lipschitz domains. Dispa [52] has shown in particular that—as in the case of the whole space—the Besov spaces $B^\lambda_{\infty,\infty}(\Omega)$ still coincide with the Hölder spaces: for $\lambda \in \mathbb{R}_+ \setminus \mathbb{N}$, $B^\lambda_{\infty,\infty}(\Omega) = C^{s,r}(\Omega)$ where s is the entire part of λ and $r := \lambda - s$.

We will also use the following result concerning characteristic functions as pointwise multipliers in Besov spaces.

Theorem 2.2 (Frazier and Jawerth [56]). *The characteristic function χ_Ω of a Lipschitz domain $\Omega \subset \mathbb{R}^n$ with a bounded boundary is a pointwise multiplier in $B^s_{p,q}$ (that is the map $u \mapsto \chi_\Omega u$ is bounded from $B^s_{p,q}(\mathbb{R}^n)$ into itself) if and only if max $((\frac{1}{p}-1), n(\frac{1}{p}-1)) < s < \frac{1}{p}$.*

Let us stress that this theorem says in particular that the characteristic function of a Lipschitz domain is a pointwise multiplier in the Hölder spaces $C^{-1,r}$, with $0 < r < 1$.

3. On the proof of Theorem 1.1

In this section we give a unified proof of the three-dimensional case of Theorem 1.1 in the Eulerian framework of Chemin, Gamblin and Saint-Raymond including the persistence of piecewise $C^{s,r}$ smoothness.

3.1. Existence from *a priori* estimates

We first recall that local existence and uniqueness of solutions of the Euler equations when the initial velocity field is assumed to be in $C^{s+1,r}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ (that is, without a hypersurface of discontinuity) were proved by Chemin in the paper [31]. Moreover Bahouri and Dehman in [11] prove that the criterion obtained by Beale, Kato and Majda (singularities will not form in the flow as long as there is no rapid accumulation of vorticity, i.e., as long as the integral, in time, of the L^∞ norm of vorticity remains bounded) remains valid in this setting. Actually Gamblin and Saint-Raymond begin their paper [62] by collecting these results, so we also refer the reader to [62] Theorem 2.8 (uniqueness) and Theorem 2.9 (existence). We also refer the reader to the book [10], Theorems 7.1 and 7.20, for a complete expository concerning local existence and uniqueness of solutions of the Euler equations in supercritical Besov spaces.

In order to construct the solutions of Theorem 1.1 we proceed by regularization of the initial data, taking limits thanks to some *a priori* estimates—uniform with respect to the regularization parameter—on the smooth solutions given by the results above. We refer the reader to [62] Proposition 5.2 for this passage to the limit, and we now focus on the way to get the *a priori* estimates. Actually we will study precisely in the following sections very carefully what happens in the passage to the limit when we regularize the Euler equations into slightly viscous Navier–Stokes equations.

We will use the vorticity formulation (12), (13) where the velocity v^0 is deduced from the vorticity ω^0 by using the Biot–Savart law (10).

3.2. The first smoothness property of the boundary

As a direct application of the estimate (53) with $s = 0$ we have the following estimate for the function φ^0 which satisfies the transport equations (8), (9)

$$\|\varphi^0(t)\|_{0,r} \leqslant \|\varphi_0\|_{0,r}\, e^{CV(t)}. \tag{55}$$

3.3. Initial conormal vector fields

We first observe that at the initial time $t = 0$ the vector fields

$$w_0^1 := \begin{bmatrix} 0 \\ -\partial_3\varphi_0 \\ \partial_2\varphi_0 \end{bmatrix}, \quad w_0^2 := \begin{bmatrix} \partial_3\varphi_0 \\ 0 \\ -\partial_1\varphi_0 \end{bmatrix}, \quad w_0^3 := \begin{bmatrix} -\partial_2\varphi_0 \\ \partial_1\varphi_0 \\ 0 \end{bmatrix}$$

are tangential to the foliation of hypersurfaces given by φ_0. As in our case we deal with only one initial hypersurface $\{\varphi_0 = 0\}$ of singularity, we also add the vector fields

$$w_0^4 := \begin{bmatrix} \partial_3(\chi x_3) \\ 0 \\ -\partial_1((1-\chi)x_3) \end{bmatrix}, \quad w_0^5 := \begin{bmatrix} -\partial_2(\chi x_1) \\ \partial_1((1-\chi)x_1) \\ 0 \end{bmatrix}$$

to the previous triplet, where χ is a smooth function compactly supported and identically equal to 1 when $|\varphi^0(t,.)| < \eta$. We therefore get a set $W_0 := \{w_0^i/1 \leqslant i \leqslant 5\}$ of five divergence free vector fields, in $C^{s,r}(\mathbb{R}^3)$ and tangential to the boundary Γ_0 of the initial vortex patch: $w_0^i \cdot n = 0$ for any $1 \leqslant i \leqslant 5$ on ΓO_0. Moreover this set is maximal in the sense that it satisfies

$$[W_0] := \inf_{\mathbb{R}^3} \sup_{1\leqslant i\neq j\leqslant 5} |w_0^i \times w_0^j| > 0.$$

We now illustrate that the initial vorticity ω_0 has 'good' derivatives in the directions that are tangential to the boundary of the patch. To do this we first set up the notion of conormal derivatives. For $u \in L^\infty(\mathbb{R}^3)$ and for $1 \leqslant i \leqslant 5$ we define the distributions

$$w_0^i \dagger u := \operatorname{div}(w_0^i \otimes u) = \sum_{j=1}^{3} \partial_j(w_{0,j}^i \cdot u),$$

where we denote by $w_{0,j}^i$, for $j = 1, 2, 3$, the components of the vector field w_0^i. Since the w_0^i are divergence free, it holds that $w_0^i \dagger u = w_0^i \cdot \nabla u$ when this last quantity is defined. To express iterated conormal derivatives we will define for $k \in \mathbb{N}$, for smooth enough vector field u,

$$(w_0\dagger)^\beta u := w_0^{\beta_1} \dagger (w_0^{\beta_2} \dagger (\ldots (w_0^{\beta_k} \dagger u)) \ldots),$$

where $\beta := (\beta_1, \ldots, \beta_k) \in \{1, \ldots, 5\}^k$. We are going to prove

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Proposition 3.1. *The* kth*-order conormal derivatives of the initial vorticity for any integer* k *between* 1 *and* s*, that is the vector fields* $(w_0\dagger)^\beta \omega_0$ *for* $\beta := (\beta_1, \ldots, \beta_k) \in \{1, \ldots, 5\}^k$*, are in the space* $C^{-1,r}(\mathbb{R}^3)$*.*

Proof. To do this we first use Rychkov's extension Theorem 2.1 to get from the restrictions $\omega_0|_{\mathcal{O}_{0,\pm}}$ on $\mathcal{O}_{0,\pm}$ two functions $\omega_{0,\pm}$ both in $C^{s,r}(\mathbb{R}^3)$. Since the initial vorticity may be decomposed thanks to these extensions and to the characteristic functions $\chi_{\mathcal{O}_{0,\pm}}$ of $\mathcal{O}_{0,\pm}$ it is sufficient to show that the $(w_0\dagger)^\beta(\omega_{0,\pm} \cdot \chi_{\mathcal{O}_{0,\pm}})$ are in $C^{-1,r}(\mathbb{R}^3)$. But since the vector fields w_0^i are tangential to the patch, they commute with the characteristic functions $\chi_{\mathcal{O}_{0,\pm}}$. Moreover thanks to Theorem 2.2 the characteristic functions $\chi_{\mathcal{O}_{0,\pm}}$ are some pointwise multipliers in the Hölder spaces $C^{-1,r}$, so the question of the $C^{-1,r}(\mathbb{R}^3)$ regularity of the $(w_0\dagger)^\beta \omega_0$ finally reduces to that of the $(w_0\dagger)^\beta \omega_{0,\pm}$ which is deduced from the paraproduct rules (48), (49). □

3.4. Time-dependent conormal vector fields

Since the vorticity satisfies the transport-stretching equation (12) we expect the tangential smoothness to be conserved. However the boundary of the patch moves with the flow, so the notion of tangential smoothness depends itself on the solution. Indeed we define time-dependent conormal vector fields w^i via the following formula, which imitates the dynamics of the vorticity ω^0:

$$w^i(t,x) := (w_0^i \cdot \nabla_x \mathcal{X}^0)(t, (\mathcal{X}^0(t,\cdot))^{-1}(x)), \tag{56}$$

so they verify the equations

$$\mathcal{D} w^i = w^i \cdot \nabla v^0, \tag{57}$$

$$w^i|_{t=0} = w_0^i. \tag{58}$$

By using a straightforward estimate we can show that for any t the set $W(t)$ of the vector fields $w^i(t)$ remains maximal. Indeed (cf. [62] Corollary 4.3) they satisfy the estimate

$$[W(t)]^{-1} \leqslant e^{C\int_0^t \|v^0\|_{\mathrm{Lip}(\mathbb{R}^3)}} [W_0]^{-1}. \tag{59}$$

The vector fields w^i remain divergence free when time proceeds.

Let us define for any circular permutation of (i,j,k) of $(1,2,3)$ the vector $u^i := w^j \wedge w^k$. It emerges that the vector n satisfies

$$n(t, \mathcal{X}^0(t,x)) = \sum_{1 \leqslant i \leqslant 3} \frac{\partial_i \varphi_0(x)}{|\nabla_x \varphi_0(x)|} u^i(t, \mathcal{X}^0(t,x)). \tag{60}$$

In particular the vector fields $w^i(t)$ remain conormal to the moving boundary $\Gamma(t)$ when time proceeds: $w^i \cdot n = 0$ for any $1 \leqslant i \leqslant 5$ on $\Gamma(t)$.

The conormal vector fields satisfy the transport-stretching equation (57) so we expect their smoothness to persist as long as we have good estimates of the velocity. In order to prove this we need a Lipschitz estimate as well as conormal and piecewise estimates of the velocity. The next section is devoted to the method used to deduce such estimates from those for the vorticity.

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3.5. Static estimates

Let us start with the Lipschitz estimate of the velocity. As the Marcinkiewicz–Calderon–Zygmund theorem fails to give a Lipschitz estimate of the velocity simply by $\|\omega^0\|_{L^\infty \cap L^2}$, we need to consider a larger norm in the right hand side. But using normal derivatives (i.e. along n) would lead to estimates non-uniform with respect to the understood regularization parameter (the vorticity is expected to be only bounded not discontinuous across the patch boundary). There is an argument for overcoming this difficulty which dates back to Chemin in the two-dimensional case (see [35, 29, 27, 28, 26]). He used the ellipticity of the div–curl system to establish a Lipschitz estimate of the velocity field with respect to the L^∞ norm and to conormal derivatives only of the vorticity. Moreover one can manage to involve the conormal derivatives only through a log, which is helpful in a subsequent Gronwall lemma since an inequality of the form $y' \leqslant y \log y$ does not lead to a blowup. For the three-dimensional case Gamblin and Saint-Raymond (cf. [62] Lemma 3.5) also succeed in getting rid of the derivatives normal to the boundary, proving:

Proposition 3.2. *The velocity v^0 (deduced from the vorticity ω^0 by using the Biot–Savart law (10)) satisfies*

$$\|v^0\|_{Lip} \lesssim \|\omega^0\|_{L^2 \cap L^\infty} + L(\|\omega^0\|_{L^\infty}, \|\omega^0\|_{C^{0,r}_{co}}), \tag{61}$$

where L is the function defined in (45) and the conormal $C^{0,r}_{co}$ norm is denoted as

$$\|u\|_{C^{0,r}_{co}} := \|u\|_{L^\infty} + [W]^{-1} \sum_{1 \leqslant i \leqslant 5} (\|w^i\|_{0,r} \|u\|_{L^\infty} + \|w^i \dagger u\|_{-1,r}).$$

Let us stress that all the norms are relative to the whole space $\mathbb{R}^3$. It is useful to recall the proof for the sequel.

Proof. First $\|v^0\|_{L^\infty}$ can be estimated by $\|v^0\|_{L^\infty} \lesssim \|\omega^0\|_{L^\infty \cap L^2}$ thanks to an appropriate splitting into the near and far field in the integral formula of the Biot–Savart law. Let us introduce a function χ in $C^\infty_0(\mathbb{R}^3)$, positive, and equal to 1 near 0, and $\chi(\nabla)$, the highly smoothing corresponding Fourier multiplier. Let $\lambda(\zeta) := (\chi(\zeta) + |\zeta|^2)^{\frac{1}{2}}$ and $\Lambda := \lambda(\nabla)$, the corresponding elliptic Fourier multiplier. We split v^0 into a low frequencies part and a high frequencies part as follows:

$$v^0 := v^0_{LF} + v^0_{HF} \quad \text{with } v^0_{LF} := \chi(\nabla)\Lambda^{-2}\, v^0 \quad \text{and} \quad v^0_{HF} := (1 - \chi(\nabla)\Lambda^{-2})v^0. \tag{62}$$

Since the low frequencies part can simply be estimated by

$$\|\partial_j v^0_{LF}\|_{-1,r} \lesssim \|v^0\|_{L^\infty} \tag{63}$$

we are left to estimate the high frequencies part $\|\partial_j v^0_{HF}\|_{L^\infty}$. We first use the relations $1 - \chi(\nabla)\Lambda^{-2} = -\Lambda^{-2}\Delta$ and $-\Delta v^0 = \operatorname{curl}\omega^0$ to get

$$\|\partial_j v^0_{HF}\|_{L^\infty} \leqslant \sum_{1 \leqslant k \leqslant 3} \|\Lambda^{-2}\, \partial_j \partial_k \omega^0\|_{L^\infty}. \tag{64}$$

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Gamblin and Saint-Raymond (cf. [62] Lemma 3.5) succeed in writing

$$\partial_j \partial_k = \Delta a_{j,k} + \sum_{l,i,p} \partial_l \partial_p b^{l,i}_{j,k} w^i_p, \tag{65}$$

where the functions $a_{j,k}$ and $b^{l,i}_{j,k}$ are built by partitioning of unity from local expressions of the form

$$a_{j,k} = \frac{(w^m \times w^n)_j (w^m \times w^n)_k}{|w^m \times w^n|^2}, \tag{66}$$

$$b^{l,i}_{j,k} = \frac{P^{l,i}_{j,k}(w^m, w^n)}{|w^m \times w^n|^2}, \tag{67}$$

where the $P^{l,i}_{j,k}(w^m, w^n)$ are homogeneous polynomials of w^m, w^n of degree 7 and $w^m \times w^n$ does not vanish, such that

$$\|a_{j,k}\|_{L^\infty} \leqslant 1 \quad \text{and} \quad \|b^{l,i}_{j,k}\|_{C^{0,r}} \leqslant C(\|[W(t)]^{-1}. \sum_{1 \leqslant i \leqslant 5} \|w^i\|_{0,r})^{19}.$$

We then rewrite the function $\Lambda^{-2}\, \partial_j \partial_k \omega^0$ in (64) as

$$\Lambda^{-2}\, \partial_j \partial_k \omega^0 = f_1 + f_2 \quad \text{where}\, f_1 := -(1 - \chi(\nabla)\Lambda^{-2}) a_{j,k} \omega^0 \quad \text{and}$$
$$f_2 := \Lambda^{-2}(\partial_j \partial_k - \Delta a_{j,k})\omega^0. \tag{68}$$

Since $\|f_1\|_{L^\infty} \lesssim \|\omega^0\|_{L^\infty}$ we are left with the estimate of $\|f_2\|_{L^\infty}$. We then apply the logarithmic interpolation inequality:

$$\|f_2\|_{L^\infty} \lesssim L(\|f_2\|_{B^0_{\infty,\infty}(\mathbb{R}^3)}, \|f_2\|_{0,r}) \tag{69}$$

and observe that

$$\|f_2\|_{B^0_{\infty,\infty}(\mathbb{R}^3)} \lesssim \|\omega^0\|_{L^\infty},$$

$$\|f_2\|_{0,r} \lesssim \left(\|[W(t)]^{-1} \sum_{1 \leqslant i \leqslant 5} \|w^i\|_{0,r} \right)^{19} \|\omega^0\|_{C^{0,r}_{co}}$$

which allows us to conclude the estimate (61). □

In the same vein we have that conormal estimates on the vorticity imply conormal estimates on the velocity. More precisely it holds that

$$\|w^i \dagger v^0\|_{0,r} \lesssim \|w^i \dagger \omega^0\|_{-1,r} + \|v^0\|_{\mathrm{Lip}} \|w^i\|_{0,r}. \tag{70}$$

Proof. To prove the estimate (70), we make use of Bony's paraproduct, writing

$$w^i \dagger v^0 = (w^i - T_{w^i}) \cdot \nabla v^0 + T_{w^i} \cdot \nabla v^0, \tag{71}$$

where $T_{w^i} \cdot \nabla$ denotes the paradifferential operator $\sum_{1 \leqslant j \leqslant 3} T_{w^i_j} \partial_j$. We estimate the first term by $\|v^0\|_{\mathrm{Lip}} \|w^i\|_{0,r}$ thanks to the paraproduct rule (49). Then to estimate the second term we once again split the velocity v^0 into low and high frequencies parts as in (62). On one hand, we have the inequality $\|T_{w^i} \cdot \nabla v^0_{LF}\|_{0,r} \lesssim \|v^0\|_{\mathrm{Lip}} \|w^i\|_{0,r}$. On the other

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hand, $T_{w^i} \cdot \nabla v^0_{HF}$ is a combination, with coefficients valued in $\{-1, 0, 1\}$, of the terms $T_{w^i} \cdot \nabla \Lambda^{-2}\, \partial_k \omega^0$. To control these last terms, we commute:

$$T_{w^i} \cdot \nabla \Lambda^{-2}\, \partial_k \omega^0 = \sum_{1 \leqslant j \leqslant 3} [T_{w^i_j}, \Lambda^{-2}\, \partial_k \partial_j] \omega^0 \\ + \Lambda^{-2}\, \partial_k w^i \dagger \omega^0 + \Lambda^{-2}\, \partial_k \mathrm{div}\, ((T_{w^i} - w^i) \otimes \omega^0). \tag{72}$$

Now the first term in the right hand side of (72) can be estimated by $\|v^0\|_{\mathrm{Lip}} \|w^i\|_{0,r}$ thanks to the commutator estimate (50) with $m = \lambda = 0$. The second one is estimated simply by $\|w^i \dagger \omega^0\|_{-1,r}$ and the third one by $\|v^0\|_{\mathrm{Lip}} \|w^i\|_{0,r}$ thanks to the paraproduct rule (49). □

3.6. $C^{0,r}$ dynamic estimates

With the previous velocity estimates in hand we can now prove a persistence result of vorticity smoothness. Actually Gamblin and Saint-Raymond (cf. [62] Proposition 4.1) prove the following estimate:

$$\|w^i(t)\|_{0,r} + \frac{\|(w^i \dagger \omega^0)(t)\|_{-1,r}}{\|\omega^0(t)\|_{L^\infty}} \leqslant e^{CV(t)} \left(\|w^i_0\|_{0,r} + \frac{\|w^i_0 \dagger \omega_0\|_{-1,r}}{\|\omega_0\|_{L^\infty}} \right), \tag{73}$$

where $V(t)$ denotes $V(t) := \int_0^t \|v^0\|_{\mathrm{Lip}}$ and where the constant C depends only on r. All the norms above are relative to the whole space $\mathbb{R}^3$.

Proof. We apply the transport estimate (53) with $s = 0$ to the conormal vector fields $f = w^i$, which satisfy the transport-stretching equations (57), (58), and we estimate $\|w^i \cdot \nabla v\|_{0,r}$ by (70). This yields

$$\|w^i(t)\|_{0,r}\, e^{-CV(t)} \leqslant \|w^i_0\|_{0,r} + \int_0^t (\|w^i \dagger \omega^0\|_{-1,r} + \|v^0\|_{\mathrm{Lip}} \|w^i\|_{0,r})(\tau) e^{-CV(\tau)} d\tau. \tag{74}$$

By using equations (12), (13) for the vorticity ω^0 and equations (57), (58) for the conormal vector fields w^i we next observe that the field $w^i \dagger \omega^0$ involved in the inequality above satisfies the following Cauchy problem:

$$\mathcal{D}(w^i \dagger \omega^0) = w^i \dagger (\omega^0 \cdot \nabla v^0), \quad \text{with } (w^i \dagger \omega^0)|_{t=0} = w^i_0 \dagger \omega_0. \tag{75}$$

We apply the estimate (53) with $s = -1$ to the field $f = w^i \dagger \omega^0$:

$$\|w^i \dagger \omega^0(t)\|_{-1,r}\, e^{-CV(t)} \leqslant \|w^i_0 \dagger \omega_0\|_{-1,r} + \int_0^t \|w^i \dagger (\omega^0 \cdot \nabla v^0)(\tau)\|_{-1,r} e^{-CV(\tau)} d\tau. \tag{76}$$

Let us recall that the initial data $\|w^i_0 \dagger \omega_0\|_{-1,r}$ are estimated by Proposition 3.1. Now to estimate the integral we first notice that

$$w^i \dagger (\omega^0 \cdot \nabla v^0) := \mathrm{div}\, (\omega^0 \otimes (w^i \dagger \omega^0)) + \mathrm{div}\, (\zeta \otimes v^0) \tag{77}$$

with $\zeta := \mathrm{div}\, (\omega^0 \otimes w^i - w^i \otimes \omega^0)$. Then we make use of the paraproduct to commute the divergence operator in the first term of the right hand side of (77):

$$\mathrm{div}\, (\omega^0 \otimes (w^i \dagger \omega^0)) = [\partial_k, T_{w^i \dagger \omega_0}] \omega^0_k + \partial_k \big((w^i \dagger \omega_0 - T_{w^i \dagger \omega_0}) \omega^0_k \big),$$

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the sum over k being understood. This gives, thanks to the estimate (49) and (50),

$$\|\mathrm{div}\,(\omega^0 \otimes (w^i \dagger \omega^0))\|_{-1,r} \lesssim \|w^i \dagger v^0\|_{0,r}\|\omega^0\|_{L^\infty}.$$

Now to estimate the second term in the right hand side of (77) we first notice that ζ is divergence free, so

$$\|\mathrm{div}\,(\zeta \otimes v^0)\|_{-1,r} \lesssim \|v^0\|_{\mathrm{Lip}}\|\zeta\|_{-1,r}.$$

Since the tame estimates (48) and (49) yield

$$\|\zeta\|_{-1,r} \lesssim \|w^i \dagger v^0\|_{0,r} + \|\omega^0\|_{L^\infty}\|w^i\|_{0,r}$$

we finally get—using one more time the estimate by (70)—the following estimate of the conormal derivative of the stretching term:

$$\|w^i \dagger (\omega^0 \cdot \nabla v^0)\|_{-1,r} \lesssim \|v^0\|_{\mathrm{Lip}}(\|w^i \dagger \omega^0\|_{-1,r} + \|\omega^0\|_{L^\infty}\|w^i\|_{0,r}). \tag{78}$$

Now we observe that for any $0 \leqslant s \leqslant t \leqslant T$ the vorticity ω^0 obeys the following L^∞ estimate:

$$e^{V(s)-V(t)} \leqslant \frac{\|\omega^0(s)\|_{L^\infty}}{\|\omega^0(t)\|_{L^\infty}} \leqslant e^{V(t)-V(s)}. \tag{79}$$

Combining the estimates (76), (78) and (79) leads to

$$\begin{aligned}\frac{\|w^i \dagger \omega^0(t)\|_{-1,r}}{\|\omega(t)\|_{L^\infty}}\, e^{-CV(t)} \leqslant \frac{\|w_0^i \dagger \omega_0\|_{-1,r}}{\|\omega_0\|_{L^\infty}} + \int_0^t \|v^0(\tau)\|_{\mathrm{Lip}}(\|w^i(\tau)\|_{0,r} \\ + \frac{\|w^i \dagger \omega^0(\tau)\|_{-1,r}}{\|\omega(\tau)\|_{L^\infty}})e^{-CV(\tau)}d\tau.\end{aligned} \tag{80}$$

We add the estimates (80) and (74) and conclude by a Gronwall argument. □

3.7. Gamblin and Saint-Raymond's result

Putting together the estimates (14), (59) and (61) (Notice that the growth of the vorticity support is controlled by the norm of v^0 in $L^\infty([0,T];\mathrm{Lip}(\mathbb{R}^3))$ which allows us to estimate the L^2 norm of the vorticity) and (73), Proposition 3.1 and, thanks to the relation (60), one then concludes that there exist $T > 0$ and a solution $v^0 \in L^\infty([0,T];\mathrm{Lip}(\mathbb{R}^3))$ to the Euler equations such that for each $t \in (0,T)$ the boundary $\Gamma(t)$ is $C^{1,r}$, and can be given by the equation $\Gamma(t) = \{\varphi^0(t,\cdot) = 0\}$, where $\varphi^0 \in L^\infty([0,T];C^{1,r}(\mathbb{R}^3))$ verifies (8), (9). In addition $\mathcal{O}_\pm(t) = \{\pm\varphi^0(t,\cdot) > 0\}$ and there exists $\eta > 0$ such that for $|\varphi^0| < \eta$, $t \leqslant T$, it holds that $\nabla\varphi^0 \neq 0$. We refer the reader to [62] Theorem 2.8 for a proof of the uniqueness of the solution to the Euler equations in $L^\infty([0,T];\mathrm{Lip}(\mathbb{R}^3)) \cap \mathrm{Lip}([0,T];L^2(\mathbb{R}^3))$. The flow map $\mathcal{X}^0$ of particle trajectories is defined as the solution of the differential equation $\partial_t\mathcal{X}^0(t,x) = v^0(t,\mathcal{X}^0(t,x))$ with initial data $\mathcal{X}^0(0,x) = x$. It is for any $t \in [0,T]$ a C^1 volume-preserving diffeomorphism. Finally since the vorticity is divergence free and is tangentially regular, the normal component of the vorticity $(\omega^0 \cdot n)(t,\cdot)$ is continuous, including across $\Gamma(t) = \{\varphi^0(t,\cdot) = 0\}$.

3.8. Iterated tangential regularity

In this section we briefly explain how to prove the propagation of the higher order smoothness—that is the $C^{s+1,r}$ smoothness—of the boundary $\Gamma(t)$ (point 3 of Theorem 1.1). This was done by Zhang and Qiu in the couple of papers [125, 124], by adapting the method used by Chemin in the two-dimensional case in [30].

By local inversion there exists a parameterization of the initial boundary Γ_0 given by a function $\mathbf{x}_0(\tau_1, \tau_2) \in C^{s+1,r}(S^1 \times S^1; \mathbb{R}^3)$. Then for each $t \in [0, T]$ the boundary $\Gamma(t)$ is given by the function

$$\mathbf{x}(t, \tau_1, \tau_2) := \mathcal{X}^0(t, \mathbf{x}_0(\tau_1, \tau_2)), \tag{81}$$

and our goal in this section is to show that this function $\mathbf{x}$ is in $L^\infty([0, T]; C^{s+1,r}(S^1 \times S^1; \mathbb{R}^3))$, which means that the boundary $\Gamma(t)$ is $L^\infty([0, T]; C^{s+1,r})$.

Since for $k = 1, 2$ the vector field $\partial_{\tau_k} \mathbf{x}_0$ is tangent to Γ_0 at the point $\mathbf{x}_0(\tau_1, \tau_2)$ and since the set of the vector fields $\{w_0^1, w_0^2, w_0^3\}$ is maximal on Γ_0 there exist some functions

$$a^k(\tau_1, \tau_2) := (a_1^k(\tau_1, \tau_2), a_2^k(\tau_1, \tau_2), a_3^k(\tau_1, \tau_2)) \in C^{s,r}(S^1 \times S^1, \mathbb{R}^3)$$

such that

$$\partial_{\tau_k} \mathbf{x}_0(\tau_1, \tau_2) = \sum_{i=1}^{3} a_i^k(\tau_1, \tau_2) w_0^i(\mathbf{x}_0(\tau_1, \tau_2)).$$

We apply the derivatives $\partial_\tau^\alpha := \partial_{\tau_1}^{\alpha_1} \partial_{\tau_2}^{\alpha_2}$ to the function $\mathbf{x}$ in (81) for $\alpha := (\alpha_1, \alpha_2) \in \mathbb{N} \times \mathbb{N}$ with $|\alpha| := \alpha_1 + \alpha_2 \leqslant s + 1$. We obtain

$$\partial_\tau^\alpha \mathbf{x}(t, \tau_1, \tau_2) = \sum_{l=1}^{|\alpha|} \sum_{\beta \in \{1,2,3\}^l} a_\beta(\tau_1, \tau_2) \Big((w_0 \dagger)^\beta \mathcal{X}^0 \Big)(t, \mathbf{x}_0(\tau_1, \tau_2))$$

with the functions a_β in $C^{0,r}(S^1 \times S^1, \mathbb{R})$. Going back to the definition of the w^i in (56) we get by iteration that the iterated derivatives $((w_0 \dagger)^\beta \mathcal{X}^0)(t, \mathbf{x}_0(\tau_1, \tau_2))$ can be transformed into the iterated derivatives $((w\dagger)^{\beta'} w^{\beta_l})(t, \mathbf{x}(t, \tau_1, \tau_2))$, where we define $\beta := (\beta', \beta_l)$ with $\beta' := (\beta_1, \ldots, \beta_{l-1})$. Thus it suffices to prove that for any $\beta' := (\beta_1, \ldots, \beta_{l-1}) \in \{1, 2, 3\}^l$ with $|\beta'| \leqslant s$ and for any $i \in \{1, 2, 3\}$ the function $(w\dagger)^{\beta'} w^i$ is in $L^\infty([0, T]; C^{0,r}(\mathbb{R}^3))$.

Since each $w^i \dagger$ commutes with $\mathcal{D}$ the $(w\dagger)^{\beta'} w^i$ satisfy

$$\mathcal{D}(w\dagger)^{\beta'} w^i = (w\dagger)^\beta v^0, \quad \text{with } (w\dagger)^{\beta'} w^i|_{t=0} = (w_0 \dagger)^{\beta'} w_0^i,$$

where $\beta := (\beta', i)$. Applying the transport estimate (54) with $s = 0$ yields for any $t \in [0, T]$

$$\|(w\dagger)^{\beta'} w^i(t)\|_{0,r} \leqslant \Big(\|(w_0 \dagger)^{\beta'} w_0^i\|_{0,r} + \int_0^t \|(w\dagger)^\beta v^0(\tau)\|_{0,r} e^{-CV(\tau)} d\tau \Big) e^{CV(t)}. \tag{82}$$

Of course $V(t) := \int_0^t \|\nabla v^0\|_{L^\infty(\mathbb{R}^3)} ds$ is now under control. Moreover the initial conormal vector fields w_0^i are in $C^{s,r}$, so for $|\beta'| \leqslant s$ the paraproduct rules yield that the initial data $(w_0\dagger)^{\beta'} w_0^i$ is in $C^{0,r}$. As a consequence the estimate (82) now simply reads

$$\|(w\dagger)^{\beta'} w^i(t)\|_{0,r} \lesssim \big(1+\int_0^t \|(w\dagger)^{\beta} v^0(\tau)\|_{0,r} d\tau\big). \tag{83}$$

Now we can estimate the right hand side of (83) by following the approach of Gamblin and Saint-Raymond mentioned in § 3.5, proving on one hand that iterated conormal regularity for the vorticity implies iterated conormal regularity for the velocity and on the other hand that iterated conormal regularity for the vorticity is preserved when time proceeds. For the first step we proceed by iteration on $|\beta'| \leqslant s$. We have

$$\|(w\dagger)^{\beta} v^0\|_{0,r} \lesssim \|(w\dagger)^{\beta} \omega^0\|_{-1,r} + \|(w\dagger)^{\beta'} w^i\|_{0,r},$$

where the quantities which have been already estimated in the previous steps are omitted (cf. [124] p. 387). Now in order to prove the second step we apply the derivatives $(w\dagger)^{\beta}$ to equation (12). We get

$$\mathcal{D}(w\dagger)^{\beta}\omega^0 = (w\dagger)^{\beta}\omega^0 \nabla v^0, \quad \text{with } ((w\dagger)^{\beta}\omega^0)|_{t=0} = (w_0\dagger)^{\beta}\omega_0.$$

We use again the transport estimate (54) but this time with $s=-1$, which yields for any $t \in [0,T]$

$$\|(w\dagger)^{\beta}\omega^0(t)\|_{-1,r} \lesssim \|(w_0\dagger)^{\beta}\omega_0\|_{-1,r} + \int_0^t \|(w\dagger)^{\beta}\omega^0\nabla v^0(\tau)\|_{-1,r} d\tau. \tag{84}$$

Let us recall that the estimation of the initial data is performed in Proposition 3.1. The estimation of the integral term is done in [124], p. 388. It can be seen as an extension in the setting of iterated conormal derivatives of the estimate (78) of the conormal derivatives of the stretching term. It reads as follows:

$$\|(w\dagger)^{\beta}\omega^0\nabla v^0\|_{-1,r} \lesssim \|(w\dagger)^{\beta}\omega^0\|_{-1,r} + \|(w\dagger)^{\beta'} w^i\|_{0,r}.$$

Plugging this in the inequality (84) yields an inequality of the form

$$\|(w\dagger)^{\beta}\omega^0(t)\|_{-1,r} \lesssim 1 + \int_0^t \|(w\dagger)^{\beta}\omega^0\|_{-1,r} + \|(w\dagger)^{\beta'} w^i\|_{0,r},$$

so a Gronwall-type argument leads to

$$\|(w\dagger)^{\beta}\omega^0\|_{-1,r} \lesssim \|(w\dagger)^{\beta'} w^i\|_{0,r}.$$

Now the estimate (83) reads

$$\|(w\dagger)^{\beta'} w^i(t)\|_{0,r} \lesssim 1 + \int_0^t \|(w\dagger)^{\beta'} w^i\|_{0,r}(\tau) d\tau,$$

summations over β' and i being understood. Hence applying a Gronwall argument to the previous estimate yields that the $(w\dagger)^{\beta'} w^i$ are in $L^\infty([0,T]; C^{0,r}(\mathbb{R}^3))$. As a consequence, for any α with $|\alpha| \leqslant s+1$ we get that $\partial_\tau^\alpha \mathbf{x}$ is in $L^\infty([0,T]; C^{0,r}(\mathbb{R}^3))$, which proves that the boundary $\Gamma(t)$ is $L^\infty([0,T]; C^{s+1,r})$.

3.9. Piecewise transport estimates

In this section we give some piecewise estimates for the transport equations of the form

$$\mathcal{D}f := \partial_t f + v^0 \cdot \nabla f = g, \quad \text{with } f|_{t=0} = f_0, \tag{85}$$

where f_0, v^0 and g are assumed to be given, the two latter being time dependent. We will use from now on the short notation $C^{k,r}_{\pm}(t)$ for $C^{k,r}(\mathcal{O}_{\pm}(t))$.

Proposition 3.3. *There hold the following piecewise estimates:*

$$\|f(t)\|_{C^{0,r}_{\pm}(t)} \leqslant C(t)\big(\|f_0\|_{C^{0,r}_{\pm}(0)} + \int_0^t \|g(\tau)\|_{C^{0,r}_{\pm}(\tau)} e^{-CV_{\pm}(\tau)} d\tau\big) e^{CV_{\pm}(t)}, \tag{86}$$

$$\|f(t)\|_{C^{k,r}_{\pm}(t)} \leqslant C(t)\big(\|f_0\|_{C^{k,r}_{\pm}(0)} + \int_0^t \|g(\tau)\|_{C^{k,r}_{\pm}(\tau)} e^{-CV^{k,r}_{\pm}(\tau)} d\tau\big) e^{CV^{k,r}_{\pm}(t)}, \quad \text{for } k \geqslant 1, \tag{87}$$

with

$$V_{\pm}(t) := \int_0^t \|v^0\|_{\mathrm{Lip}(\mathcal{O}_{\pm}(s))} ds, \quad V^{k,r}_{\pm}(t) := \int_0^t \|v^0\|_{C^{k,r}_{\pm}(s)} ds, \tag{88}$$

and where the constant $C(t)$ depends only on the sup for s over $[0,T]$ of the Lipschitz norm of the domain $\mathcal{O}_{+}(s)$.

Proof. To fix the idea let us consider the case of the estimate on $\mathcal{O}_{+}$. We make use of Theorem 2.1 to get from the restrictions $f_0|_{\mathcal{O}_{0,+}}$ and $g|_{\mathcal{O}_{+}(t)}$ of the initial data f_0 and of the source term g some extensions $f_{0,+}$ and g_{+} defined in the whole space $\mathbb{R}^3$, with, for any $k \geqslant 0$,

$$\|f_{0,+}\|_{C^{k,r}(\mathbb{R}^3)} \leqslant C\|f_0|_{\mathcal{O}_{0,+}}\|_{C^{k,r}_{0,+}}, \quad \|g_{+}\|_{C^{k,r}(\mathbb{R}^3)} \leqslant C(t)\|g|_{\mathcal{O}_{+}(t)}\|_{C^{k,r}_{+}(t)}, \tag{89}$$

where the constant C and $C(t)$ depend only on the Lipschitz norm of the domains $\mathcal{O}_{0,+}$ and $\mathcal{O}_{+}(t)$. To extend the velocity field v^0 we distinguish two cases. When $k=0$ we make use of the McShane–Whitney extension theorem (cf. McShane [99] and Whitney [120]) to get a Lipschitz extension v^0_{+} defined in the whole space $\mathbb{R}^3$ of the restriction $v^0|_{\mathcal{O}_{+}(t)}$. When $k \geqslant 1$ we use again the Rychkov extension Theorem 2.1 to get from the restriction $v|_{\mathcal{O}_{+}(t)}$ an extension v^0_{+} defined in the whole space $\mathbb{R}^3$, with

$$\|v^0_{+}\|_{C^{k,r}(\mathbb{R}^3)} \leqslant C(t)\|v^0|_{\mathcal{O}_{+}(t)}\|_{C^{k,r}(\mathcal{O}_{+}(t))}. \tag{90}$$

Then we apply the estimates (53) and (54) to the transport equation

$$\partial_t f_{+} + v^0_{+} \cdot \nabla f_{+} = g_{+}, \quad \text{with } f_{+}|_{t=0} = f_{0,+}. \tag{91}$$

This yields for $k=0$

$$\|f_{+}(t)\|_{C^{k,r}(\mathbb{R}^3)} \leqslant \big(\|f_{0,+}\|_{C^{k,r}(\mathbb{R}^3)} + \int_0^t \|g_{+}(\tau)\|_{C^{k,r}(\mathbb{R}^3)} e^{-C\tilde{V}_{+}(\tau)} d\tau\big) e^{C\tilde{V}_{+}(t)}, \tag{92}$$

where $\tilde{V}_{+}(t) := \int_0^t \|v^0_{+}\|_{\mathrm{Lip}(\mathbb{R}^3)} ds$ and for any $k \geqslant 1$

$$\|f_{+}(t)\|_{C^{k,r}(\mathbb{R}^3)} \leqslant \big(\|f_{0,+}\|_{C^{k,r}(\mathbb{R}^3)} + \int_0^t \|g_{+}(\tau)\|_{C^{s,r}(\mathbb{R}^3)} e^{-C\tilde{V}^{k,r}_{+}(\tau)} d\tau\big) e^{C\tilde{V}^{k,r}_{+}(t)}, \tag{93}$$

where $\tilde{V}^{k,r}_{+}(t) := \int_0^t \|v^0_{+}\|_{C^{k,r}(\mathbb{R}^3)} ds$.

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Next we observe that since $\mathcal{O}_+(t)$ is the domain transported by the flow at time t starting from $\mathcal{O}_{+,0}$, the transport equation (85) is well-posed in $\cup_{t\in(0,T)}\{t\}\times\mathcal{O}_+(t)$ without any boundary condition. By restriction of the transport equation (91), the function $f_+|_{\mathcal{O}_+(t)}$ also satisfies the transport equation (85) in $\mathcal{O}_{+,0}$. Now we can conclude that $f_+|_{\mathcal{O}_+(t)} = f|_{\mathcal{O}_+(t)}$ by uniqueness and it suffices to use the estimates (89), (90) to conclude. □

3.10. Propagation of piecewise regularity

In this section we investigate the propagation of piecewise regularity for the Euler equations. Indeed we finish proving parts 2, 4 and 5 of Theorem 1.1. For the convenience of the reader we recall here the decomposition (68):

$$\Lambda^{-2}\,\partial_j\partial_k\omega^0 = f_1 + f_2 \quad \text{where } f_1 := (1-\chi(\nabla)\Lambda^{-2})a_{j,k}\omega^0 \text{ and}$$
$$f_2 := \Lambda^{-2}\sum_{l,i,m}\partial_l\partial_p b^{l,i}_{j,k} w^i_p\omega^0.$$

Let us denote by Ω the collection of the w^i and of the coefficients $a_{j,k}$ and $b^{l,i}_{j,k}$ which appear in the decomposition above. For any $m\in\mathbb{N}^*$ we introduce the set Op_m (respectively the set $\tilde{O}p_m$) the collection of the operators T (respectively $\tilde{T}$) of the form

$$Tf := \Omega_0\partial_{i_1}\Omega_1\partial_{i_2}\ldots\partial_{i_m}\Omega_m f \quad (\text{respectively } \tilde{T}f := (w^{i_1}\dagger)\Omega_1(w^{i_2}\dagger)\Omega_2\ldots(w^{i_m}\dagger)\Omega_m f).$$

Our strategy is to prove by iteration for $0\leqslant k\leqslant s$ that

$$\text{For } 1\leqslant i\leqslant 5,\quad w^i\in L^\infty([0,T];C^{k,r}_\pm(t)) \quad\text{and}\quad w^i\dagger\omega^0\in L^\infty([0,T];C^{k-1,r}_\pm(t)), \tag{94}$$
$$v^0\in L^\infty([0,T];C^{k+1,r}_\pm(t)), \tag{95}$$
$$\text{For } |\alpha|\leqslant k,\quad \partial^\alpha\nabla v^0 = S\omega^0 + \sum_{m\leqslant k}T_m\omega^0 + \Lambda^{-2}\,\partial_j\sum_{m\leqslant k+1}\tilde{T}_m\omega^0, \tag{96}$$

where S is infinitely smoothing, and T_m (respectively $\tilde{T}_m$) is in Op_m (respectively $\tilde{O}p_m$).

Remark 3.1. Of course the estimate (95) implies that ω^0 is in $L^\infty([0,T];C^{k,r}_\pm(t))$. Still in order to prove the estimate (95) we will use transport features of the vorticity and it will turn out that it is useful to get some extra static estimates for deducing piecewise estimates on ∇v^0 from piecewise estimates on ω^0. For an operator the property of acting continuously between some spaces of piecewise regularity is usually referred to as the 'transmission property' or the 'transmission condition'; cf. [15, 16, 67, 107]. In particular the references above prove that the operator $T:\omega\mapsto\nabla v$ satisfies the transmission property across smooth boundaries. In the present case the smoothness of the boundary is limited and we need to estimate carefully how the smoothness of the boundary is involved in the constant of continuity of the operator T above in $C^{0,r}(\mathcal{O}_\pm(t))$. In particular, as a byproduct, what follows will extend the previous results, giving that for an open subset $\mathcal{O}_+$ of class $C^{s+1,r}$ the operator T is bounded from $C^{s,r}(\mathcal{O}_\pm)$ into itself. Actually the analysis would be complicated here by the time-dependent setting. The key point is the decomposition (96) which basically allows one to write any derivatives of the velocity as the sum of a smooth term (the low frequencies part), a local term involving derivatives of the vorticity, which allows a direct piecewise estimate, and a term which despite being nonlocal involves (iterated) conormal derivatives (up to commutators) which behave better (than normal derivatives) when estimated in the whole space.

Let us start the proof by iteration of (94)–(96). Regarding the step $k=0$ we already know that the assertions concerning the w^i and the $w^i \dagger \omega^0$—the estimates (94)—are satisfied from the estimates (73) (which actually even give that the w^i and the $w^i \dagger \omega^0$ are respectively in $L^\infty([0,T]; C^{0,r}(\mathbb{R}^3))$ and $L^\infty([0,T]; C^{-1,r}(\mathbb{R}^3)))$ whereas the decomposition (96) reduces to the decomposition (68) proved in § 3.5 and recalled above. As a consequence, to conclude the step $k=0$ it only remains to prove the estimate (95). In order to prove it we apply the piecewise transport estimate (86) to the vorticity, which satisfies equation (12). Since the functions $V_\pm$ are under control by the previous sections we get an estimate of the following form:

$$\|\omega^0\|_{C^{0,r}_\pm(t)} \lesssim 1 + \int_0^t \|\omega^0 \cdot \nabla v^0\|_{C^{0,r}_\pm(\tau)} d\tau. \tag{97}$$

Now let us use the decomposition (68) to get some extra static estimates to deduce piecewise estimates on ∇v^0 from piecewise estimates on ω^0. From the low/high frequencies splitting of the § 3.5 estimates (63), (64) the task reduces to proving that the operators $\Lambda^{-2}\, \partial_j\, \partial_k$ act continuously in $C^{0,r}_\pm(t)$. Indeed—since the operator $\chi(\nabla)\Lambda^{-2}$ is smoothing—we get

$$\|f_1\|_{C^{0,r}_\pm(t)} \lesssim \|a_{j,k}\|_{C^{0,r}_\pm(t)}.\|\omega^0\|_{C^{0,r}_\pm(t)}.$$

Now we use the local representation of the coefficients $a_{j,k}$ given in (66) to estimate the norms $\|a_{j,k}\|_{C^{0,r}}$ by the norms $\|w^i\|_{C^{0,r}}$ and $[W]$ which have already been estimated (cf. § 3.6 and estimate (59)). In the same way as for the $a_{j,k}$, that is by using the local representation of the coefficients $b^{l,i}_{j,k}$ given in (67), we can estimate the norms $\|b^{l,i}_{j,k}\|_{C^{0,r}}$ by some quantities already under control. As a consequence we simply estimate $\|f_2\|_{C^{0,r}_\pm(t)}$ by $\|f_2\|_{C^{0,r}(\mathbb{R}^3)}$ what reduces to the estimate of $\|w^i \dagger \omega^0\|_{C^{-1,r}(\mathbb{R}^3)}$ which were already controlled in § 3.6 estimate (73). Finally we get $\|\nabla v^0\|_{C^{0,r}_\pm(t)} \lesssim \|\omega^0\|_{C^{0,r}_\pm(t)}$ and since we have the tame estimates

$$\|\omega^0 \cdot \nabla v^0\|_{C^{0,r}_\pm(t)} \lesssim \|\omega^0\|_{C^{0,r}_\pm(t)} \cdot \|\nabla v^0\|_{L^\infty} + \|\omega^0\|_{L^\infty} \cdot \|\nabla v^0\|_{C^{0,r}_\pm(t)},$$

we infer the following piecewise estimate of the stretching term:

$$\|\omega^0 \cdot \nabla v^0\|_{C^{0,r}_\pm(t)} \lesssim \|\omega^0\|_{C^{0,r}_\pm(t)}.$$

Then by applying a Gronwall argument to (97) we conclude that the vorticity ω^0 is in $L^\infty([0,T]; C^{0,r}_\pm(t))$. Using the static estimates once more yields the estimate (95) and the proof of the step $k=0$ is complete.

At this point we can use that the normal component of the vorticity $(\omega^0 \cdot n)(t,\cdot)$ including across $\Gamma(t) = \{\varphi^0(t,\cdot) = 0\}$ is continuous (cf. § 3.7) to get part 5 of Theorem 1.1: for each $t \in [0,T]$ the function $(\omega^0 \cdot n)(t,\cdot)$ is $C^{0,r}$ on $\{|\varphi^0(t,\cdot)| < \eta\}$.

Let us now assume that (94)–(96) hold for $0 \leqslant k \leqslant s-1$. We are going to prove that (94)–(96) hold at the order $k+1$. In order to do so, we first apply the piecewise transport estimate (87) to the conormal vector fields $f = w^i$ with $g = w^i \cdot \nabla v^0$ as respective right hand sides (we recall that the conormal vector fields w^i satisfy equations (57), (58)). Since the functions $V_\pm^{k+1,r}$ are under control by the previous step we get an estimate of the following form:

$$\|w^i\|_{C_\pm^{k+1,r}(t)} \lesssim 1 + \int_0^t \|w^i \cdot \nabla v^0\|_{C_\pm^{k+1,r}(\tau)} d\tau. \tag{98}$$

We consider the extensions (to the whole space) $w_\pm^i$ and $v_\pm^0$ given by Theorem 2.1 from the restrictions $w^i|_{\mathcal{O}_\pm}$ and $v^0|_{\mathcal{O}_\pm(t)}$ on each side of the patch of w^i and v^0. We denote by $\omega_\pm^0$ the curl of $v_\pm^0$. We have therefore

$$\|w^i \cdot \nabla v^0\|_{C_\pm^{k+1,r}(t)} \lesssim \|w_\pm^i \cdot \nabla v_\pm^0\|_{C^{k+1,r}(\mathbb{R}^3)}.$$

Now that we are dealing with functions in the full space we can use the paraproduct. Indeed we can adapt the estimate (70) into the following one (with only modifications of the indexes in the proof of (70)):

$$\|w_\pm^i \dagger v_\pm^0\|_{k+1,r} \lesssim \|w_\pm^i \dagger \omega_\pm^0\|_{k,r} + \|v_\pm^0\|_{\mathrm{Lip}} \|w_\pm^i\|_{k+1,r}.$$

We bound $\|v_\pm^0\|_{\mathrm{Lip}}$ thanks to the embedding (44) and then we use Theorem 2.1 to get

$$\|w^i\|_{C_\pm^{k+1,r}(t)} \lesssim 1 + \int_0^t \|w^i\|_{C_\pm^{k+1,r}(\tau)} + \|w^i \dagger \omega^0\|_{C_\pm^{k,r}(\tau)} d\tau. \tag{99}$$

To estimate the conormal derivatives $w^i \dagger \omega^0$ in $C_\pm^{k,r}$ we apply the transport estimate (87) to (75). At the initial time the conormal derivatives $w_0^i \dagger \omega_0$ are in $C_\pm^{k,r}$, and the amplification factor involves only $V_\pm^{k,r}$ so we infer an estimate of the form

$$\|w^i \dagger \omega^0\|_{C_\pm^{k,r}(t)} \lesssim 1 + \int_0^t \|w^i \dagger (\omega^0 \cdot \nabla v^0)\|_{C_\pm^{k,r}(\tau)} d\tau. \tag{100}$$

Now to estimate the integrals above we proceed as follows. We consider once again the extensions $w_\pm^i$ and $v_\pm^0$ such that

$$\|w^i \dagger (\omega^0 \cdot \nabla v^0)\|_{C_\pm^{k,r}} \lesssim \|w_\pm^i \dagger (\omega_\pm^0 \cdot \nabla v_\pm^0)\|_{C^{k,r}}.$$

Now we recall the decomposition (77) which for the extensions reads

$$w_\pm^i \dagger (\omega_\pm^0 \cdot \nabla v_\pm^0) := \operatorname{div}(\omega_\pm^0 \otimes (w_\pm^i \dagger \omega_\pm^0)) + \operatorname{div}(\zeta_\pm \otimes v_\pm^0) \tag{101}$$

with $\zeta_\pm := \operatorname{div}(\omega_\pm^0 \otimes w_\pm^i - w_\pm^i \otimes \omega_\pm^0)$. The first term of the right hand side of (101) can be estimated by proceeding in the same way as in § 3.6. This yields

$$\|\operatorname{div}(\omega_\pm^0 \otimes (w_\pm^i \dagger \omega_\pm^0))\|_{C^{k,r}} \lesssim \|w_\pm^i \dagger v_\pm^0\|_{C^{k+1,r}} \|\omega_\pm^0\|_{L^\infty}.$$

Now for the second term we have the following tame estimates:

$$\|\operatorname{div}(\zeta_\pm \otimes v_\pm^0)\|_{C^{k,r}} \lesssim \|w_\pm^i \dagger \omega_\pm^0\|_{C^{k,r}} + \|w_\pm^i\|_{C^{k+1,r}(\mathbb{R}^3)}.$$

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Thus using the continuity properties of the extension operator the estimate (100) now becomes

$$\|w^i \dagger \omega^0\|_{C^{k,r}_{\pm}(t)} \lesssim 1 + \int_0^t \|w^i \dagger \omega^0\|_{C^{k,r}_{\pm}(\tau)} + \|w^i\|_{C^{k+1,r}(\mathcal{O}_{\pm}(\tau))}. \tag{102}$$

Combining this with the estimate (99) and up to a Gronwall argument, we conclude that the conormal vector fields w^i and the conormal derivatives of the vorticity $w^i \dagger \omega^0$ are respectively in $L^\infty([0,T]; C^{k+1,r}_{\pm}(t))$ and in $L^\infty([0,T]; C^{k,r}_{\pm}(t))$. Thus we have already proved that the estimate (94) holds at the order $k+1$.

We are now going to show that ω^0 is in $L^\infty([0,T]; C^{k+1,r}_{\pm}(t))$. In order to prove this we apply the estimate (87) to get

$$\|\omega^0\|_{C^{k+1,r}_{\pm}(t)} \lesssim 1 + \int_0^t \|\omega^0 \cdot \nabla v^0\|_{C^{k+1,r}_{\pm}(\tau)} d\tau. \tag{103}$$

Now we deduce from Definition 1.2 and from Leibniz's rule the piecewise tame estimates

$$\|ab\|_{C^{k,r}_{\pm}(t)} \lesssim \|a\|_{C^{k,r}_{\pm}(t)} \|b\|_{C^{k-1,r}_{\pm}(t)} + \|a\|_{C^{k-1,r}_{\pm}(t)} \|b\|_{C^{k,r}_{\pm}(t)}. \tag{104}$$

This yields

$$\|\omega^0 \cdot \nabla v^0\|_{C^{k+1,r}_{\pm}(t)} \lesssim 1 + \|\nabla v^0\|_{C^{k+1,r}_{\pm}(t)} + \|\omega^0\|_{C^{k+1,r}_{\pm}(t)}. \tag{105}$$

We apply the spatial derivatives to the decomposition (96) at order k and we eliminate the double derivatives thanks to the identity (65) to get the decomposition (96) at order $k+1$. Now let us remark that thanks to the local representation (66), (67) we can estimate the $C^{k+1,r}_{\pm}(t)$ norm of the coefficient Ω by a power of the $C^{k+1,r}_{\pm}(t)$ norms of the w^i which have been previously controlled. We also know from § 3.8 that the iterated conormal derivatives $(w\dagger)^\beta \omega^0$ are in $L^\infty\big([0,T]; C^{-1,r}_{\pm}(t)\big)$. As a consequence we get

$$\|\nabla v^0\|_{C^{k+1,r}_{\pm}(t)} \lesssim 1 + \|\omega^0\|_{C^{k+1,r}_{\pm}(t)}. \tag{106}$$

Plugging this in the estimate (105) and then the resulting estimate in the estimate (103) yields

$$\|\omega^0\|_{C^{k+1,r}_{\pm}(t)} \lesssim 1 + \int_0^t \|\omega^0\|_{C^{k+1,r}_{\pm}(\tau)} d\tau,$$

so ω^0 is in $L^\infty([0,T]; C^{k+1,r}_{\pm}(t))$. Using again the previous static estimate we infer that the estimate (95) holds at order $k+1$. The iteration can be done and part 2 of Theorem 1.1 is therefore proved.

Using the relation (60) we next infer that n is in $L^\infty([0,T]; C^{s,r}_{\pm}(t))$ which together with the tangential estimates gives that φ^0 is in $L^\infty([0,T]; C^{s+1,r}_{\pm}(t))$, and we finish proving part 4 of Theorem 1.1.

3.11. Analyticity

In this section we say a few words about the final statement regarding time analyticity (part 6 of Theorem 1.1). Actually the study of smoothness in time of the boundary was

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already done by Chemin in the pioneering work [30]. He incorporates the material field $\mathcal{D} := \partial_t + v^0 \cdot \nabla$ into its conormal fields and carries out iterated conormal derivations. In [31] he proves the smoothness with respect to time of classical solutions (v_0 in $C^{1,r}$) in any dimensions and of Yudovich's solutions (with bounded vorticity) in two dimensions. This result was extended to analyticity by Serfati [111] (see also the doctoral thesis) in the case of classical solutions and in the present case of vortex patches, thanks to Lagrangian methods (by considering the Euler equations as an ODE for the flow). Let us mention papers [61, 60] which recover by an Eulerian approach the result by Serfati of time analyticity of classical solutions and prove Gevrey-3 smoothness in time of Yudovich solutions. Finally we mention that such results also hold in the case of solid boundaries: the paper [92] of Kato proves the C^∞ smoothness in time for classical solutions in a smooth bounded domain in any dimension; the paper [65] extends Kato's result to analyticity and also proves the analyticity of the motion of a body immersed into a perfect incompressible fluid with a $C^{1,r}$ initial velocity.

4. Looking for a profile problem

In this section we look for an expansion for the solutions of the Navier–Stokes equations (1), (2), with an initial velocity v_0 as described in Definition 1.1, which describes as well as possible their behavior with respect not only to the variables t, x but also to the viscosity coefficient ν. §§ 4.1–4.3 give the heuristic of the derivation of the profile problem—equations (19)–(25)—mentioned in the introductory part, § 1.2. More precisely, in § 4.1 we will identify the inner fast scale as $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$. This means that the initial discontinuity of the vorticity is smoothed out into a layer of size $\sqrt{\nu t}$ around the hypersurface $\{\varphi^0 = 0\}$ where the inviscid discontinuity occurs. In § 4.2 we pay attention to the expected order of amplitudes of the velocity and pressure profiles. We will see that it is natural to associate with a vorticity expansion of the form

$$\omega^\nu(t,x) \sim \Omega\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right)$$

a velocity expansion of the form

$$v^\nu(t,x) \sim v^0(t,x) + \sqrt{\nu t}\, \tilde{V}\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right),$$

and a pressure expansion of the form

$$p^\nu(t,x) \sim p^0(t,x) + \nu t \tilde{P}\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right).$$

In § 4.3 we look for the profile equations. We choose there to deal with the velocity formulation of the Navier–Stokes equations.

4.1. The inner fast scale

The goal of this section is to explain how we identify the inner fast scale as $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$.

4.1.1. A highly simplified model. To identify the inner fast scale we first look at the one-dimensional scalar heat equation: $\partial_t \omega^\nu = \nu \partial_x^2 \omega^\nu$, which plays here the role of a 'baby model' for the NS equations. We prescribe as initial data a discontinuous vorticity: $\omega^\nu|_{t=0} = 1_{\mathbb{R}_+}$. In the inviscid case $\nu = 0$—which stands for (highly) simplified Euler equations—the solution is simply equal to the initial data $\omega^0(t, \cdot) := 1_{\mathbb{R}_+}$ for any time, whereas for $\nu > 0$ and $t > 0$ one can explicitly compute the solutions ω^ν by convolution:

$$\omega^\nu(t, x) := \Omega\left(\frac{x}{\sqrt{\nu t}}\right) \quad \text{where } \Omega(X) := \frac{1}{\sqrt{\pi}} \int_{-\frac{X}{2}}^{\infty} e^{-y^2} dy. \tag{107}$$

Hence the initial discontinuity of the vorticity is smoothed out into a layer of size $\sqrt{\nu t}$ where there occurs—smoothly—the transition between the values 0 and 1. It is useful to rewrite the ω^ν as

$$\omega^\nu(t, x) := \omega^0(t, x) + \tilde{\Omega}_\pm\left(\frac{x}{\sqrt{\nu t}}\right) \quad \text{when } \pm x > 0,$$

where

$$\tilde{\Omega}_\pm(X) := \frac{1}{\sqrt{\pi}} \int_{-\frac{X}{2}}^{\mp\infty} e^{-y^2} dy \quad \text{when } \pm X > 0.$$

One then sees the 'viscous' solutions ω^ν as the sum of the 'inviscid' solution ω^0 plus a 'double initial (internal) boundary layer' $\tilde{\Omega}_\pm$ which satisfies the double ODE

$$\partial_X^2 \tilde{\Omega}_\pm + \frac{X}{2} \partial_X \tilde{\Omega}_\pm = 0 \quad \text{when } \pm X > 0, \tag{108}$$

obeying the continuity conditions for ω^ν and $\partial_x \omega^\nu$ at the internal boundary $x = X = 0$ (for $t > 0$):

$$\omega^0|_{x=0^+} + \tilde{\Omega}_+|_{X=0^+} = 1 - 1/2 = 0 + 1/2 = \omega^0|_{x=0^-} + \tilde{\Omega}_-|_{X=0^-}, \tag{109}$$

$$\partial_X \tilde{\Omega}_+|_{X=0^+} = \frac{1}{2\sqrt{\pi}} = \partial_X \tilde{\Omega}_-|_{X=0^-} \tag{110}$$

and the vanishing

$$\tilde{\Omega}_\pm(X) \to 0 \quad \text{in the limits } X \to \pm\infty. \tag{111}$$

These last limits correspond both to the limits $t > 0$, $x \to \pm\infty$ and the limits $\pm x > 0$, $\nu t \to 0^+$ (we recall that X is the placeholder for $\frac{x}{\sqrt{\nu t}}$). Conversely the two second-order elliptic equations (108) with the four 'normal' boundary conditions (109)–(111) (the last one contains two conditions) determine uniquely the profiles $\tilde{\Omega}_\pm$.

4.1.2. The inner scale in the NS equations. Of course the case of the NS equations is really much more complicated than the previous baby model. In particular the inviscid discontinuity moves: Theorem 1.1 states that the inviscid discontinuity occurs at the hypersurface $\{\varphi^0(t, .) = 0\}$ given by the eikonal equations (8), (9) associated with the particle derivative $\mathcal{D} := \partial_t + v^0 \cdot \nabla_x$. Therefore we are led to consider the inner fast scale $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$ and we expect that in the case of vortex patches as initial data, the solutions ω^ν of

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NS can be described by an expansion of the form

$$\omega^\nu(t,x) \sim \omega^0(t,x) + \tilde{\Omega}\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right), \tag{112}$$

where $\tilde{\Omega}(t,x,X)$ denotes a local perturbation such that

$$\lim_{X\to\pm\infty} \tilde{\Omega}(t,x,X) = 0. \tag{113}$$

Actually we will consider profiles even rapidly decreasing at infinity.

Remark 4.1. At this point experts in mathematical geometric optics should argue that the fast scale $\frac{\varphi^0(t,x)}{\sqrt{\nu}}$ should be more intuitive, since applying the particle derivative $\mathcal{D}$ to a function $\Omega(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu}})$ produces the singular term

$$\frac{1}{\sqrt{\nu}} \mathcal{D}\varphi^0.(\partial_X \Omega)|_{X=\frac{\varphi^0(t,x)}{\sqrt{\nu}}}$$

which fortunately vanishes thanks to the eikonal equation.

The point is that such a choice of inner scale does not lead to totally satisfactory results. Let us show that on our baby model. If one looks for a representation of the solutions ω^ν of the form

$$\omega^\nu(t,x) := \omega^0(t,x) + \boldsymbol{\Omega}_\pm\left(t, \frac{x}{\sqrt{\nu}}\right) \quad \text{when } \pm x > 0, \tag{114}$$

one then sees that the 'double (internal) boundary layer' $\boldsymbol{\Omega}_\pm$ has this time to satisfy the parabolic equation

$$\partial_t \boldsymbol{\Omega}_\pm = \partial_X^2 \boldsymbol{\Omega}_\pm \quad \text{when } \pm X > 0, \tag{115}$$

with the same boundary conditions:

$$\boldsymbol{\Omega}_+|_{X=0^+} - \boldsymbol{\Omega}_-|_{X=0^-} = -1, \tag{116}$$

$$\partial_X \boldsymbol{\Omega}_+|_{X=0^+} - \partial_X \boldsymbol{\Omega}_-|_{X=0^-} = 0, \tag{117}$$

$$\boldsymbol{\Omega}_\pm(X) \to 0 \quad \text{in the limits } X \to \pm\infty. \tag{118}$$

Since we prescribe the same initial data for ω^ν as for ω^0, one has to prescribe zero initial data for the layers:

$$\boldsymbol{\Omega}_\pm|_{t=0} = 0, \tag{119}$$

so the condition of compatibility between the transmission condition (116) and the initial condition (119) on the 'corner' $\{t = X = 0\}$ is not satisfied even at order zero, which removes any hope for smoothness as regards the profiles $\boldsymbol{\Omega}_\pm$.

Now let us stress that if the 'phase' $\psi(t,x) := \frac{\varphi^0(t,x)}{\sqrt{t}}$ chosen in the asymptotic expansions (112) does not satisfy exactly the eikonal equation but rather satisfies $\mathcal{D}\psi = -\frac{1}{2t}\psi$, then applying the particle derivative $\mathcal{D}$ to a function

$$\omega^\nu(t,x) := \Omega\left(t, x, \frac{\psi(t,x)}{\sqrt{\nu}}\right) \tag{120}$$

produces the term

$$-\frac{1}{2t}(X\partial_X\ \Omega)|_{X=\frac{\varphi^0(t,x)}{\sqrt{\nu t}}} \tag{121}$$

which is no longer singular (with respect to ν). The point is that in the present setting of a localized profile Ω the derivative $X\partial_X$ in (121) does not cause any difficulty (note that such a term even appears in our baby model; see (108)), in particular because the prefactor $\frac{1}{t}$ echoes that in the term

$$\frac{|n(t,x)|^2}{t}\partial_X^2\ \Omega|_{X=\frac{\varphi^0(t,x)}{\sqrt{\nu t}}} \tag{122}$$

which is the larger one produced by applying the Laplace operator $\nu\triangle_x$ to the function $\omega^\nu(t,x)$ in (121). Let us recall that the vector $n(t,x)$ above is defined in Theorem 1.1.

Of course this additional derivative $X\partial_X$ is more problematic in the traditional context of periodic oscillations of geometric optics.

4.2. Amplitudes

We now pay attention to the expected order of amplitudes of velocity and pressure profiles. In the full plane the Biot–Savart law has Fourier symbol $(-\frac{\xi}{|\xi|^2}\wedge\cdot)$. It is a pseudo-local operator of order -1, so we expect that the velocity v^ν given by the Navier–Stokes equations can be described by an asymptotic expansion of the form

$$v^\nu(t,x)\sim v^0(t,x)+\sqrt{\nu t}\tilde{V}\left(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right), \tag{123}$$

where the profile $\tilde{V}(t,x,X)$ is also expected to satisfy

$$\lim_{X\to\pm\infty}\tilde{V}(t,x,X)=0. \tag{124}$$

Arguably, since the Euler velocity v^0 is Lipschitz we expect its viscous perturbations v^ν to be uniformly Lipschitz, which gives support to the ansatz (123). Plugging (112) and (123) into the relations (3), taking into account (7) and setting the leading order terms equal leads to

$$n\wedge\partial_X\tilde{V}=\tilde{\Omega}. \tag{125}$$

Hence the vorticity profile $\tilde{\Omega}$ has to satisfy the orthogonality condition

$$\tilde{\Omega}\cdot n=0. \tag{126}$$

This condition is not a surprise: since w^0 is divergence free, $w^0\cdot n$ is continuous (cf. part 5 of Theorem 1.1), so no (large amplitude) layer is expected on the normal component of the vorticity.

Now the pressure p^ν can be recovered from the velocity v^ν by applying the operator divergence to equation (1) which yields the Laplace problem

$$\triangle_x p^\nu=-\sum_{1\leqslant i,j\leqslant 3}(\partial_i v_j^\nu)(\partial_j v_i^\nu). \tag{127}$$

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If the velocity v^ν satisfies the expansion (123), the right hand side of (127) should admit an expansion of the form

$$\triangle_x p^\nu \sim \triangle_x p^0 + \tilde{F}\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right),$$

where the function $\tilde{F}$ satisfies

$$\lim_{X\to\pm\infty} \tilde{F}(t,x,X) = 0.$$

Since the Laplacian is of order -2 we are led to consider a perturbation of order νt on the pressure:

$$p^\nu(t,x) \sim p^0(t,x) + \nu t \tilde{P}\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right), \tag{128}$$

where—once again—the fast scale $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$ is expected to be a local inner scale, so

$$\lim_{X\to\pm\infty} \tilde{P}(t,x,X) = 0. \tag{129}$$

4.3. Looking for a profile problem

Now that we have an intuition of the amplitude of the profiles, we look for the profile equations. We choose here to deal with the velocity formulation of the NS equations, which is believed to be a more robust method (with a view to future adaptation to the compressible case for instance). We proceed in several steps. In § 4.3.1 we plug the ansatz into the velocity equation, setting the leading order terms equal. We then pay attention to the divergence free condition which leads to a crucial observation in § 4.3.2. In § 4.3.3 we get rid of the pressure in the velocity profile equation. As the vector field n may vanish, away from the patch boundary, it is useful to modify the resulting equation in order to avoid a degeneracy of the order. This will be done in § 4.3.4. In § 4.3.5 we study the transmission conditions on the inner interface $X = 0$. We will use several times the following Leibniz formulas, where we define $\mathbf{\Phi}^\nu(t,x) := (t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}})$. For any smooth function $h(t,x,X)$ it holds that

$$\partial_t\left[(\nu t)^{\frac{j}{2}}\; h\circ \mathbf{\Phi}^\nu\right] = (\nu t)^{\frac{j}{2}}\;\left(\frac{\partial_t\varphi^0}{\sqrt{\nu t}}\partial_X h - \frac{1}{2t}X\partial_X h + \frac{j}{2t}h + \partial_t h\right)\circ \mathbf{\Phi}^\nu, \tag{130}$$

$$\nabla_x\left[h\circ \mathbf{\Phi}^\nu\right] = \left(\frac{1}{\sqrt{\nu t}}(\partial_X h)n + \nabla_x h\right)\circ \mathbf{\Phi}^\nu \tag{131}$$

and

$$\nu t\triangle_x\left[h\circ \mathbf{\Phi}^\nu\right] = \left(|n|^2\partial_X^2 h + \sqrt{\nu t}(\triangle\varphi^0\; \partial_X h + 2n\cdot\nabla_x\partial_X h) + \nu t\;\triangle_x h\right)\circ \mathbf{\Phi}^\nu. \tag{132}$$

4.3.1. The velocity equation. We plug the ansatz (123) and (128) into equation (1), setting the leading order terms, which are of order $\sqrt{\nu t}^0$, equal:

$$\mathcal{D}v^0 + \nabla p^0 + \mathcal{D}\varphi^0\cdot\partial_X\tilde{V} = 0. \tag{133}$$

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This is satisfied since the velocity v^0 satisfies the Euler equations and φ^0 satisfies the eikonal equation (8). At the following order $\sqrt{\nu t}$, we get the equality

$$\mathcal{D}\tilde{V} + \tilde{V} \cdot n\, \partial_X \tilde{V} + \tilde{V} \cdot \nabla_x v^0 + \partial_X \tilde{P}\, n = \frac{1}{t} \left(|n|^2 \partial_X^2 \tilde{V} + \frac{X}{2} \partial_X \tilde{V} - \frac{1}{2} \tilde{V} \right). \tag{134}$$

4.3.2. Incompressible transparency. We now pay attention to the divergence free condition. Plugging the ansatz (123) into equation (2), retaining the terms at order $\sqrt{\nu t}^0$ and taking into account that the velocity v^0 given by Euler is divergence free, leads to the orthogonality equation $n \cdot \partial_X \tilde{V} = 0$, which by integration with the condition (124) leads to the condition

$$n \cdot \tilde{V} = 0. \tag{135}$$

An important consequence of the condition (135) is to kill the second term in (134) which is the only nonlinear one. Here lies an analogy with the WKB theory of the propagation of high frequency oscillations for hyperbolic systems (see for instance [103]). The condition (135) can be seen as a polarization of the singularity on the components tangential to the 'phase' φ^0. Then the vanishing of the Burgers-like term in (134) can be interpreted as a transparency property: the self-interaction of the singularity vanishes because this latter is characteristic for a field which is linearly degenerate (actually this concept belongs to the hyperbolic theory but the incompressible limit is reminiscent of this fact).

4.3.3. Getting rid of the pressure. Equation (134) involves both $\tilde{V}$ and $\tilde{P}$. However the pressure in the NS equations is not truly an unknown but can be recovered from the velocity (as recalled in (127)) so we expect the same to hold for the profiles. One way to proceed is to project normally equation (134), taking into account equation (11) for $n(t, x)$ and using the condition (135), to get

$$\partial_X \tilde{P} := -2 \frac{(\tilde{V} \cdot \nabla_x v^0) \cdot n}{|n|^2}. \tag{136}$$

For $\pm X > 0$, we integrate between X and $\pm\infty$, taking the condition at infinity (129) into account, to find

$$\tilde{P} := -2 \int_X^{\pm\infty} \frac{(\tilde{V} \cdot \nabla_x v^0) \cdot n}{|n|^2}. \tag{137}$$

Remark 4.2. Another way to proceed is to make explicit the term $\tilde{F}$ which occurred in the expansion (128) when we were discussing the amplitude of the pressure layers by means of the Laplace problem satisfied by the viscous pressure p^ν. Taking into account the condition (135), we get

$$\tilde{F} = -2(\partial_X \tilde{V} \cdot \nabla_x v^0) \cdot n. \tag{138}$$

Plugging, on the other hand, the expansion (128) in the left hand side of (128) yields

$$|n|^2 \partial_X^2 \tilde{P} = -2(\partial_X \tilde{V} \cdot \nabla_x v^0) \cdot n \tag{139}$$

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and we recover (137) by integrating twice with vanishing conditions at infinity for $\tilde{P}$ and $\partial_X \tilde{P}$.

It is interesting to note that though the second method involves one more derivative, it has the advantage of involving fewer terms. Furthermore we do not need to combine with the equation for n.

Remark 4.3. We note that the profile $\tilde{P}$ is in general discontinuous at $X=0$. Actually this discontinuity is compensated by another pressure profile which depends only on t, x. This profile will be constructed in § 6. On the other hand we will construct a velocity profile $\tilde{V}$ that is continuous, including at $X=0$, so $\partial_X \tilde{P}$ will also be continuous including at $X=0$.

We now use equation (136) to get rid of the pressure profile in equation (134). This yields

$$\check{\mathcal{L}}\tilde{V} := |n|^2 \partial_X^2 \tilde{V} + \frac{X}{2}\partial_X \tilde{V} - \frac{1}{2}\tilde{V} - t\left(\mathcal{D}\tilde{V} + \tilde{V}\cdot\nabla_x v^0 - 2\frac{(\tilde{V}\cdot\nabla_x v^0)\cdot n}{|n|^2}n\right) = 0. \quad (140)$$

4.3.4. Avoiding a far-field degeneracy. The vector field n may vanish, away from the patch boundary; hence so may the coefficient in front of the leading order in equation (140). To remedy this we consider a function a in the space

$$\mathcal{B} := L^\infty([0,T]; C^{0,r}(\mathbb{R}^3)) \cap L^\infty([0,T]; C^{s,r}(\mathcal{O}_\pm(t)))$$

satisfying the condition $\inf_{[0,T]\times\mathbb{R}^3} a =: c > 0$ and such that $a = |n|^2$ when $|\varphi^0| < \eta$, and we consider for the profile $V(t,x,X)$ the *linear* partial differential equation $\mathcal{L}V = 0$ where the differential operator $\mathcal{L}$ is given by $\mathcal{L} := \mathcal{E} - t(\mathcal{D}+A)$ where $\mathcal{E}$ and A are some operators of respective orders 2 and 0 acting formally on functions $V(t,x,X)$ as follows:

$$\mathcal{E}V := a\partial_X^2 V + \frac{X}{2}\partial_X V - \frac{1}{2}V \quad \text{and} \quad AV := V\cdot\nabla_x v^0 - 2\frac{(V\cdot\nabla_x v^0)\cdot n}{a}n.$$

Roughly speaking, equation (140) is now hyperbolic in t, x and parabolic in t, X, for $t>0$, and degenerates into an elliptic equation in X for $t=0$.

Remark 4.4. The substitution of a in place of $|n|^2$ is quite harmless since

$$\check{\mathcal{L}}V - \mathcal{L}V = (|n|^2 - a)\partial_X^2 V + tC(t,x)V \quad (141)$$

where $C(t,x)$ is a matrix such that $C(t,x)V := 2(V\cdot\nabla_x v^0)\cdot n(\frac{1}{|n|^2} - \frac{1}{a})n$. Therefore the right hand side of (141) vanishes for $|\varphi^0(t,x)| < \eta$. On the other hand, for $|\varphi^0(t,x)| > \eta$, the third argument of $V(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu t}})$ tends to ∞ as $\sqrt{\nu t}$ tends to 0, so, because of the vanishing condition (124), the right hand side of (141) will be small.

4.3.5. Transmission conditions. Of course we hope to extirpate from the previous equation a non-trivial solution. To this purpose an important point is that we look for a solution $\tilde{V}$ with an X-derivative $\partial_X \tilde{V}$ discontinuous in $X=0$. Actually because of the parabolic nature of the Navier–Stokes equations, we expect that v^ν and ω^ν

are continuous including through $\varphi^0 = 0$ (these are the Rankine–Hugoniot conditions associated with the problem), which lead to the transmission conditions: $\tilde{V}$ and $\omega^0 + \tilde{\Omega}$ should be continuous, which (taking into account the equalities (125), (126) and (135)) is equivalent to the transmission conditions: $\tilde{V}$ and $n \wedge \omega^0 - |n|^2 \partial_X \tilde{V}$ should be continuous. More precisely this means *a priori* that

$$\tilde{V}|_{X=0^+,\varphi^0=0^+} - \tilde{V}|_{X=0^-,\varphi^0=0^-} = 0, \tag{142}$$

$$|n|^2 \partial_X \tilde{V}|_{X=0^+,\varphi^0=0^+} - |n|^2 \partial_X \tilde{V}|_{X=0^-,\varphi^0=0^-} = -(n \wedge \omega^0|_{\varphi^0=0^+} - n \wedge \omega^0|_{\varphi^0=0^-}). \tag{143}$$

Since X is the placeholder for $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$ the function $\tilde{V}(t,x,X)$ needs to be defined only when X and $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$ share the same sign. However it is useful to look for a profile $V(t,x,X)$ defined for (t,x,X) in the whole domain $\mathcal{U} := (0,T) \times \mathbb{R}^3 \times \mathbb{R}$. As a consequence we will actually look at the following transmission conditions: for any $(t,x) \in (0,T) \times \mathbb{R}^3$: $[V] = 0$ and $[\partial_X V] = -\frac{n \wedge (\omega^0_+ - \omega^0_-)}{a}$, where the brackets denote the jump discontinuity $[V] = V|_{X=0^+} - V|_{X=0^-}$ across $\{X = 0\}$ and where the $\omega^0_\pm$ are functions in $L^\infty\Big((0,T); C^{s,r}\big(\mathbb{R}^3\big)\Big)$ such that $\omega^0_\pm|_{\mathcal{O}_\pm(t)} = \omega^0$. More precisely we recall from the compendium (to be precise, part 6 of Theorem 1.1) that the restrictions of the flow χ^0 of the Euler solution on each side of the boundary are analytic with respect to time with values in $C^{s+1,r}$. Thanks to Theorem 2.1 there exist some extensions $\chi^0_\pm$ analytic on $(0,T)$ with values in $C^{s+1,r}(\mathbb{R}^3)$ of the restriction $\chi^0|_{\mathcal{O}_{0,\pm}}$. We define the corresponding velocities $v^0_\pm$ by $v^0_\pm(t,.) := (\partial_t \chi^0_\pm)(t, (\chi^0_\pm)^{-1}(t,.))$ and the corresponding vorticities $\omega^0_\pm$ by $\omega^0_\pm := \operatorname{curl} v^0_\pm$. As a consequence, with the notation of the introduction, $\omega^0_\pm$ are in $\mathcal{B}_{\mathcal{D}}$.

The profile equations (19)–(25) announced in the introductory part § 1.2 are therefore derived.

5. Solving the profile problem

In this section we study the profile equations (19)–(25). In § 5.1 we study the problem obtained when setting formally $t = 0$ in the profile problem. In § 5.2 we prove the existence and uniqueness of the layer profile in an L^2 setting, and we prove the smoothness properties of this solution in § 5.3. The other properties of the profile are proved in § 5.4 which achieves the proof of Theorem 1.4 given in the introduction.

5.1. At the initial time

We expect the initial values $V_0(x,X) := V(0,x,X)$ to satisfy the problem obtained on formally setting $t = 0$ in equations (19)–(25), i.e.,

$$a_0 \partial_X^2 V_0 + \frac{X}{2} \partial_X V_0 - \frac{1}{2} V_0 = 0 \quad \text{when } \pm X > 0, \tag{144}$$

$$V_0|_{X=0^+} - V_0|_{X=0^-} = 0, \tag{145}$$

$$a_0 \partial_X V_0|_{X=0^+} - a_0 \partial_X V_0|_{X=0^-} = g_0. \tag{146}$$

The functions $a_0(x)$ and $g_0(x)$ which denote respectively the initial value of a and that of $n \wedge (\omega^0_+ - \omega^0_-)$ are in $C^{0,r}(\mathbb{R}^3) \cap C^{s,r}(\mathcal{O}_{\pm,0})$.

Proposition 5.1. *There exist a unique couple of solutions*

$$V_0(x,X) \in C^{0,r}\big(\mathbb{R}^3; p\text{-}\mathcal{S}(\mathbb{R})\big) \cap C^{s,r}\big(\mathcal{O}_{\pm,0}; p\text{-}\mathcal{S}(\mathbb{R})\big) \tag{147}$$

which satisfy the problem (144), (145), (146). Moreover

$$V_0(x,X)\cdot n_0(x) = 0 \quad \textit{for} \ \pm X > 0. \tag{148}$$

Proof. We first reduce the transmission conditions to the homogeneous case by defining the functions $\tilde{V}_0$ by

$$\tilde{V}_0 := V_0 \pm \frac{1}{2}\frac{g_0}{a_0}(e^{\mp X} - e^{\mp 2X}) \quad \text{when} \ \pm X > 0, \tag{149}$$

so the problem (144), (145), (146) is turned into the following one (dropping the tilde and the index 0):

$$a_0\partial_X^2 V + \frac{X}{2}\partial_X V - \frac{1}{2}V = f \quad \text{when} \ \pm X > 0, \tag{150}$$

$$V|_{X=0^+} - V|_{X=0^-} = 0, \tag{151}$$

$$\partial_X V|_{X=0^+} - \partial_X V|_{X=0^-} = 0, \tag{152}$$

with $f \in C^{0,r}\big(\mathbb{R}^3; H^{-1}(\mathbb{R})\big) \cap C^{0,r}\big(\mathbb{R}^3; p\text{-}\mathcal{S}(\mathbb{R})\big) \cap C^{s,r}\big(\mathcal{O}_{\pm,0}; H^{-1}(\mathbb{R})\big) \cap C^{s,r}\big(\mathcal{O}_{\pm,0}; p\text{-}\mathcal{S}(\mathbb{R})\big)$.

Because of the unbounded coefficient X in equation (144), the previous problem does not enter in the classical theory of elliptic problems (with x as parameter). To remedy this we introduce a cutoff. We consider $\sigma > 0$ and a smooth function χ_σ such that $\chi_\sigma(X) = X$ for $|X| < \sigma$, $\chi_\sigma(X) = 3\sigma/2$ for $|X| > 2\sigma$ and $\|\chi_\sigma'\|_{L^\infty(\mathbb{R})} < 1$. We will work with the modified equation

$$a_0\partial_X^2 V + \frac{\chi_\sigma(X)}{2}\partial_X V - \frac{1}{2}V = f \quad \text{when} \ \pm X > 0. \tag{153}$$

The variational formulation of the problem (153), (151), (152) reads: for $f \in L^2(\mathbb{R}^3; H^{-1}(\mathbb{R}))$ find $V \in L^2(\mathbb{R}^3; H^1(\mathbb{R}))$ such that $B_\sigma(V,W) = -\langle f, W\rangle$ for all $W \in L^2(\mathbb{R}^3; H^1(\mathbb{R}))$ where $\langle ., .\rangle$ denotes the bracket of duality between $L^2(\mathbb{R}^3, H^{-1}(\mathbb{R}))$ and $L^2(\mathbb{R}^3; H^1(\mathbb{R}))$, and $B_\sigma(V,W)$ is the following bilinear form on $L^2(\mathbb{R}^3; H^1(\mathbb{R})) \times L^2(\mathbb{R}^3; H^1(\mathbb{R}))$:

$$B_\sigma(V,W) := \int_{\mathbb{R}^3\times\mathbb{R}} a_0\partial_X V\cdot\partial_X W - \frac{\chi_\sigma(X)}{2}\partial_X V\cdot W + \frac{1}{2}V\cdot W. \tag{154}$$

Since the bilinear form B_σ is continuous (thanks to the cutoff χ_σ) and coercive:

$$B_\sigma(V,V) = \int_{\mathbb{R}^3\times\mathbb{R}} a_0|\partial_X V|^2 + \frac{1}{2}\left(1 + \frac{\chi_\sigma'(X)}{2}\right)|V|^2, \tag{155}$$

we infer from the Lax–Milgram theorem that there exists a unique weak/variational solution of problem (153), (151), (152).

We now turn our attention to regularity, restricting ourselves for brevity to establishing *a priori* estimates. First multiplying equations (153) by V and integrating

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(only) in X yields for any $x \in \mathbb{R}^3$

$$\int_{\mathbb{R}} a_0 |\partial_X V|^2 + \frac{1}{2}\left(1 + \frac{\chi_\sigma'(X)}{2}\right)|V|^2 dX = -\langle\langle f, W \rangle\rangle \tag{156}$$

where $\langle\langle ., . \rangle\rangle$ denotes this time the bracket of duality between $H^{-1}(\mathbb{R})$ and $H^1(\mathbb{R})$, so we get

$$\|V\|_{L^\infty(\mathbb{R}^3;H^1(\mathbb{R}))} \lesssim \|f\|_{L^\infty(\mathbb{R}^3;H^{-1}(\mathbb{R}))}. \tag{157}$$

We follow the same strategy as in § 3.9. By Rychkov's Theorem 2.1 there exist some extensions $a_{\pm,0}$ in $C^{s,r}(\mathbb{R}^3)$ and

$$f_\pm \in C^{s,r}\big(\mathbb{R}^3; H^{-1}(\mathbb{R})\big) \cap C^{s,r}\big(\mathbb{R}^3; p\text{-}\mathcal{S}(\mathbb{R})\big)$$

of the restrictions of a_0 and f to $\mathcal{O}_{\pm,0}$. Then we obtain estimates of the solutions $V_\pm$ of the problems

$$a_{\pm,0}\partial_X^2 V_\pm + \frac{\chi_\sigma(X)}{2}\partial_X V_\pm - \frac{1}{2}V_\pm = f_\pm \quad \text{when } \pm X > 0, \tag{158}$$

$$V_\pm|_{X=0^+} - V_\pm|_{X=0^-} = 0, \tag{159}$$

$$\partial_X V_\pm|_{X=0^+} - \partial_X V_\pm|_{X=0^-} = 0, \tag{160}$$

for x running in the full range $\mathbb{R}^3$, through a Fourier analysis. Finally we observe that $V|_{\mathcal{O}_{\pm,0}}$ and $V_\pm$ satisfy all of equations (153), (151), (152) for x in $\mathcal{O}_{\pm,0}$. Proceeding as in step 3 we get that they are equal.

We have therefore reduced to the case where the functions a_0 and g_0 are respectively in $C^{s,r}(\mathbb{R}^3)$ and $C^{s,r}\big(\mathbb{R}^3; H^{-1}(\mathbb{R})\big) \cap C^{s,r}\big(\mathbb{R}^3; p\text{-}\mathcal{S}(\mathbb{R})\big)$. We are going to prove by iteration for $-1 \leqslant l \leqslant s$ that

$$\|V\|_{C^{l,r}(\mathbb{R}^3;H^1(\mathbb{R}))} \lesssim \|f\|_{C^{l,r}(\mathbb{R}^3;H^{-1}(\mathbb{R}))}.$$

To do this we make use of a spectral localization with respect to x; that is we apply the operators Δ_j to equations (153) to get for $j \geqslant -1$ the equations

$$a_0\partial_X^2 \Delta_j V + \frac{\chi_\sigma}{2}\partial_X \Delta_j V - \frac{1}{2}\Delta_j V = \Delta_j f + |[a_0, \Delta_j]\partial_X^2 V \tag{161}$$

for $\pm X > 0$ and to the interface condition (151), (152) to get at $X = 0$

$$\begin{cases} \Delta_j V|_{X=0^+} - \Delta_j V|_{X=0^-} = 0, \\ \Delta_j \partial_X V|_{X=0^+} - \Delta_j \partial_X V|_{X=0^-} = 0. \end{cases} \tag{162}$$

We want to show that

$$\sup_{j\geqslant -1} 2^{j(l+r)} \|\Delta_j V\|_{L^\infty(\mathbb{R}^3;H^1(\mathbb{R}))} < \infty.$$

To do this we multiply equation (161) by $\Delta_j V$, we integrate in X over $\mathbb{R}_\pm$, and we sum the two resulting equations, noticing that the boundary term produced by the integration by parts of the sum of the respective first terms vanishes. We thus get for any $x \in \mathbb{R}^3$ the identity

$$\int_{\mathbb{R}} a_0 |\partial_X \Delta_j V|^2 + \frac{3}{4} |\Delta_j V|^2 = -\langle\langle \Delta_j f + |[a_0, \Delta_j] \partial_X^2 V, \Delta_j V \rangle\rangle.$$

Now we also have

$$|[a_0, \Delta_j] \partial_X V|_{X=0^+} = |[a_0, \Delta_j] \partial_X V|_{X=0^-}$$

so

$$\int_{\mathbb{R}} a_0 |\partial_X \Delta_j V|^2 + \frac{1}{4} |\Delta_j V|^2 \lesssim \| \Delta_j f \|^2_{H^{-1}(\mathbb{R})} + I_j(x), \tag{163}$$

where $I_j(x)$ denotes

$$I_j(x) := \int_{\mathbb{R}} |\partial_X \Delta_j V| \cdot |[a_0, \Delta_j] \partial_X V|.$$

We will prove the following commutator estimate:

Lemma 5.1. *For any $x \in \mathbb{R}^3$ it holds that*

$$sup_{j \geqslant -1}\ 2^{2j(l+r)} I_j(x) \lesssim \| \partial_X V \|_{C^{l,r}\left(\mathbb{R}^3; L^2(\mathbb{R})\right)} \cdot \| \partial_X V \|_{C^{l-1,r} \cap L^\infty \left(\mathbb{R}^3; L^2(\mathbb{R})\right)}.$$

Let us assume Lemma 5.1 for a while and infer from the estimate (163) that

$$\| V \|_{C^{l,r}\left(\mathbb{R}^3; H^1(\mathbb{R})\right)} \lesssim \| f \|_{C^{l,r}\left(\mathbb{R}^3; L^2(\mathbb{R})\right)} + \| \partial_X V \|_{C^{l-1,r} \cap L^\infty \left(\mathbb{R}^3; L^2(\mathbb{R})\right)}$$

so that starting with the case $l = -1$—which is a consequence of the estimate (157)—the iteration can be done until we get $V \in C^{s,r}(\mathbb{R}^3; H^1(\mathbb{R}))$.

Proof of Lemma 5.1. We will consider only $j > 0$, the case $j = 0$ corresponding to minor modifications of notation and being actually easier. We make use of the paraproduct, writing

$$[a_0, \Delta_j] \partial_X V = [T_{a_0}, \Delta_j] \partial_X V + (a_0 - T_{a_0}) \Delta_j \partial_X V - \Delta_j (a_0 - T_{a_0}) \partial_X V,$$

so $I_j = I_j^1 + I_j^2 + I_j^3$ where

$$\begin{aligned} I_j^1(x) &:= \int_{\mathbb{R}} |\partial_X \Delta_j V| \cdot |[T_{a_0}, \Delta_j] \partial_X V|, \\ I_j^2(x) &= \int_{\mathbb{R}} |\partial_X \Delta_j V| \cdot |(a_0 - T_{a_0}) \Delta_j \partial_X V|, \\ I_j^3(x) &= \int_{\mathbb{R}} |\partial_X \Delta_j V| \cdot |\Delta_j (a_0 - T_{a_0}) \partial_X V|. \end{aligned}$$

Referring to the definitions (47) and (43) we have

$$\begin{aligned} [T_{a_0}, \Delta_j] \partial_X V &= \sum_{k \geqslant 1} [S_{k-1} a_0, \Delta_j] \Delta_k \partial_X V \\ &= \sum_{k \geqslant 1, |k-j| \leqslant 5} 2^{3j} \int_{\mathbb{R}^3} (S_{k-1} a_0(x) - S_{k-1} a_0(y)) \tilde{h}(2^j (x-y)) \Delta_k \partial_X V(y) dy. \end{aligned}$$

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Hence

$$|[T_{a_0}, \Delta_j]\partial_X V| \leqslant 2^{-j} \sum_{k\geqslant 1, |k-j|\leqslant 5} \|S_{k-1}a_0\|_{Lip} \int_{\mathbb{R}^3} g(2^j(x-y))|\Delta_k \partial_X V(y)|2^{3j} dy$$

where g denotes the function $g(x) := |x| \cdot |\tilde{h}(x)|$ which is in $L^1(\mathbb{R}^3)$. Using the Fubini theorem we get that for any $x \in \mathbb{R}^3$

$$I_j^1(x) \lesssim \sum_{k\geqslant 1, |k-j|\leqslant 5} 2^{-j} \int_{\mathbb{R}^3} g(2^j(x-y)) \left(\int_{\mathbb{R}} |\partial_X \Delta_j V(x,X)| \cdot |\partial_X \Delta_k V(y,X)| dX \right) 2^{3j} dy.$$

We now use the Cauchy–Schwarz inequality to get for any $x \in \mathbb{R}^3$

$$\begin{aligned} I_j^1(x) &\lesssim \sum_{k\geqslant 1, |k-j|\leqslant 5} 2^{-j} \|\partial_X \Delta_j V(x,.)\|_{L^2(\mathbb{R})} \int_{\mathbb{R}^3} g(2^j(x-y)) \|\partial_X \Delta_k V(y,.)\|_{L^2(\mathbb{R})} 2^{3j} dy \\ &\lesssim \sum_{k\geqslant 1, |k-j|\leqslant 5} 2^{-j} \|\partial_X \Delta_j V(x,.)\|_{L^2(\mathbb{R})} \|\partial_X \Delta_k V(y,.)\|_{L^\infty(\mathbb{R}^3, L^2(\mathbb{R}))} \end{aligned}$$

and hence

$$\sup_{j\geqslant -1} 2^{2j(l+r)} I_j^1(x) \lesssim \|\partial_X V\|_{C^{l,r}(\mathbb{R}^3;L^2(\mathbb{R}))} \cdot \|\partial_X V\|_{C^{l-1,r}(\mathbb{R}^3;L^2(\mathbb{R}))}$$

Similar bounds for I_j^2 and I_j^3 can be obtained, mixing once again classical paradifferential arguments with a Fubini argument. □

Proceeding as previously we can prove that

$$\|V\|_{C^{s,r}(\mathbb{R}^3;H^1(\mathbb{R}))} \lesssim \|f\|_{C^{s,r}(\mathbb{R}^3;p-L^2(\mathbb{R}))}. \tag{164}$$

Then, proceeding by induction we obtain that for all k and l in $\mathbb{N}$, $V_{k,l,0} := X^k \partial_X^l V \in C^{s,r}\left(\mathbb{R}^3; p - H^1(\mathbb{R})\right)$. This yields the estimate (147).

We now let σ go to infinity.

Finally it suffices to use a uniqueness argument based on equality (156). □

5.2. Existence and uniqueness of the layer profile

Let us now study the time-dependent equation (19). We denote by $\mathcal{U}$ the domain $\mathcal{U} := (0,T) \times \mathbb{R}^3 \times \mathbb{R}$, by $\mathcal{U}_\pm$ the restrictions $\mathcal{U}_\pm := (0,T) \times \mathbb{R}^3 \times \mathbb{R}_\pm$ on each side of the boundary $\Gamma := (0,T) \times \mathbb{R}^3 \times \{0\}$ and by A, $\tilde{\mathcal{L}}$ and $\mathcal{L}$ the operators of respective orders 0, 1 and 2 acting formally on functions $V(t,x,X)$ as follows:

$$\begin{aligned} AV &:= V \cdot \nabla_x v^0 - 2\frac{(V \cdot \nabla_x v^0)\, n}{a} n \\ \tilde{\mathcal{L}}V &:= \frac{X}{2}\partial_X V - \frac{1}{2}V - t(\partial_t V + v^0 \cdot \nabla_x V + AV), \\ \mathcal{L}V &:= a\partial_X^2 V + \tilde{\mathcal{L}}V. \end{aligned}$$

The profile problem now reads as follows:

$$\mathcal{L}V = f \quad \text{on } \mathcal{U}_\pm, \qquad ([V], [\partial_X V]) = (0, g) \quad \text{on } \Gamma, \tag{165}$$

the decreasing of V for large X being encoded in the choice of the space $E_1 := L^2((0,T)\times\mathbb{R}^3; H^1(\mathbb{R}))$.

Theorem 5.1. *For any $f \in E_1'$, for any $g \in L^2((0,T)\times\mathbb{R}^3)$ there exists exactly one solution $V \in E_1$ of (165). In addition the function $\sqrt{t}\|V(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^3\times\mathbb{R})}$ is continuous on $(0,T)$.*

The first equation of (165) is satisfied in the sense of distributions on both $\mathcal{U}_\pm$ and the sense given to the jump conditions is explained in the proof below.

Proof. We consider $\sigma > 0$ and a smooth function χ_σ such that $\chi_\sigma(X) = X$ for $|X| < \sigma$, $\chi_\sigma(X) = 3\sigma/2$ for $|X| > 2\sigma$ and $\|\chi_\sigma'\|_{L^\infty(\mathbb{R})} < 1$. We will work with the modified operators $\tilde{\mathcal{L}}_\sigma := \tilde{\mathcal{L}} + \frac{\chi_\sigma(X)-X}{2}\partial_X V$ and $\mathcal{L}_\sigma V := a\partial_X^2 V + \tilde{\mathcal{L}}_\sigma V$ whose coefficients are bounded. Furthermore, the coefficients of the first-order part are Lipschitz.

We now explain the meaning of the jump conditions in equation (165). Since V is in E_1, the jump $[V]|_\Gamma$ is in $L^2(\Gamma)$. To give a sense to the jump of the derivative $[\partial_X V]$ we will use the equation. For any V in the space $E_2 := \{V \in C_0(\mathcal{U})/V|_{\mathcal{U}_\pm} \in C^\infty\}$ and W in $H^1(\mathcal{U})$ we have, integrating by parts, the following Green identity:

$$\sum_\pm \int_{\mathcal{U}_\pm} \mathcal{L}_\sigma V \cdot W = -\int_{\mathcal{U}} a\partial_X V\cdot\partial_X W + \int_{\mathcal{U}} \tilde{\mathcal{L}}_\sigma^* W\cdot V - \int_\Gamma a[\partial_X V]\cdot W - T\int_{\tilde{\Gamma}} W\cdot V \quad (166)$$

where $\tilde{\Gamma} := \{T\}\times\mathbb{R}^3\times\mathbb{R}$ and where $\tilde{L}_\sigma^*$ denotes the operator (the adjoint of $\tilde{\mathcal{L}}_\sigma$)

$$\tilde{\mathcal{L}}_\sigma^* V := -\frac{\chi_\sigma(X)}{2}\partial_X V - \frac{1}{2}(1+\chi_\sigma'(X))V + t(\mathcal{D}V + (1+\operatorname{div} v^0 - A)V). \quad (167)$$

In fact, less smoothness is needed. Let us introduce the Hilbert space $E_4 := \{V \in E_1/L_\sigma V \in H^{-1}(\mathcal{U})\}$ endowed with the norm $\|V\|_{E_4} := \|V\|_{E_1} + \|\mathcal{L}_\sigma V\|_{H^{-1}(\mathcal{U})}$. Thanks to a classical lemma of Friedrichs [57], the space E_2 is dense in E_4.

Lemma 5.2. *The map*

$$V \in E_2 \mapsto \tau := \begin{cases} a[\partial_X V] & \text{on } \Gamma \\ TV & \text{on } \tilde{\Gamma} \end{cases} \quad (168)$$

extends uniquely to a continuous linear map from E_4 to $H^{-\frac{1}{2}}(\Gamma\cup\tilde{\Gamma})$ and Green's identity (166) is still valid for any couple (V,W) in $E_4\times H^1(\mathcal{U})$ in the generalized sense that

$$\langle \mathcal{L}_\sigma V, W\rangle_{H^{-1}(\mathcal{U}),H^1(\mathcal{U})} = \int_{\mathcal{U}} (V\cdot\tilde{L}_\sigma^* W - a\partial_X V\cdot\partial_X W) - \langle \tau, W|_{\Gamma\cup\tilde{\Gamma}}\rangle_{H^{-\frac{1}{2}}(\Gamma\cup\tilde{\Gamma}),H^{\frac{1}{2}}(\Gamma\cup\tilde{\Gamma})}. \quad (169)$$

Proof. Let V be in E_2 and $\tilde{W}$ be in $H^{\frac{1}{2}}(\Gamma\cup\tilde{\Gamma})$. There exists a function W in $H^1(\mathcal{U})$ such that $W|_{\Gamma\cup\tilde{\Gamma}} = \tilde{W}$. From Green's identity (166) we infer that

$$\left|\int_{\Gamma\cup\tilde{\Gamma}} \tau\tilde{W}\right| \leqslant C\|V\|_{E_4}\|W\|_{H^1(\mathcal{U})} \leqslant C\|V\|_{E_4}\|\tilde{W}\|_{H^{\frac{1}{2}}(\Gamma\cup\tilde{\Gamma})}.$$

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Hence by the Hahn–Banach theorem we get the existence of a continuous extension, which is unique because of the density stated above. □

We therefore have given a meaning to the problem (165). This meaning can seem weak, but the next result says that it is actually strong. We introduce the space $H^{1,2}(\mathcal{U})$ of the functions $V \in H^1(\mathcal{U})$ such that $\partial_X^2 V|_{\mathcal{U}_\pm}$ are in L^2.

Lemma 5.3. *If $V \in E_4$ satisfies the jump conditions $[V] = 0$ and $[\partial_X V] = g$ on Γ in the sense given by Lemma 5.2 then there exists a sequence V^ε in $H^{1,2}(\mathcal{U})$ converging to V in E_4 and a sequence g^ε converging to g in $L^2((0,T) \times \mathbb{R}^3)$ such that $[V^\varepsilon] = 0$ and $[\partial_X V^\varepsilon] = g^\varepsilon$ on Γ.*

Proof. As this kind of process is very classical—see for instance Rauch [106]—we only briefly sketch the proof. The idea is to construct the sequence V^ε by convoluting in the variables t, x only to preserve the jump conditions, to use the Friedrichs lemma to prove the convergence in E_4 and then to gain the extra X derivative, that is to prove that the V^ε are in $H^{1,2}(\mathcal{U})$, thanks to the equation. □

We will now prove uniqueness as a consequence of the following estimate: for any function V in E_1 satisfying

$$\mathcal{L}_\sigma V = f \text{ when on } \mathcal{U}_\pm, [V] = 0 \text{ and } [\partial_X V] = g \text{ on } \Gamma, \tag{170}$$

it holds that

$$\|V\|_{E_1} \lesssim \|f\|_{E_1'} + \|g\|_{L^2(\mathbb{R}^3 \times \mathbb{R})}. \tag{171}$$

Let us first see the case where V is in $H^{1,2}(\mathcal{U})$. Then using Green's identity (166) with $V = W$ and density we get that

$$\int_{\mathcal{U}} |V|^2 + |\partial_X V|^2 + T \int_{\tilde{\Gamma}} |V|^2 \lesssim \int_{\mathcal{U}} t|V|^2 + |f \cdot V| + \int_{\Gamma} |g \cdot V|,$$

actually not only for T but for any t in $(0, T)$, so using Gronwall's lemma we get

$$\|V\|_{E_1} + \sqrt{T}\|V(T, \cdot, \cdot)\|_{L^2(\mathbb{R}^3 \times \mathbb{R})} \lesssim \|f\|_{E_1'} + \|g\|_{L^2(\mathbb{R}^3 \times \mathbb{R})}. \tag{172}$$

Now to deal with the general case we introduce the Hilbert space $E_3 := \{V \in E_1 / L_\sigma V \in E_1'\}$ endowed with the norm $\|V\|_{E_3} := \|V\|_{E_1} + \|L_\sigma V\|_{E_1'}$. From its definition (167) we see that $\tilde{\mathcal{L}}_\sigma^* V$ is in E_1' whenever $V \in E_3$. Estimate (171) will be deduced from

Lemma 5.4. *The map*

$$(V, W) \in E_2 \times E_2 \mapsto \rho := \begin{cases} a[\partial_X V] \cdot W & \text{on } \Gamma \\ TV \cdot W & \text{on } \tilde{\Gamma} \end{cases}$$

extends uniquely to a continuous bilinear map from $E_3 \times E_3$ to $\mathrm{Lip}(\Gamma \cup \tilde{\Gamma})'$ and Green's identity (166) is still valid for any couple V, W in $E_3 \times E_3$ in the generalized sense that

$$\langle \mathcal{L}_\sigma V, W \rangle_{E_1', E_1} = -\int_{\mathcal{U}} a \partial_X V \cdot \partial_X W + \langle \tilde{\mathcal{L}}_\sigma^* W, V \rangle_{E_1', E_1} - \langle \rho, 1 \rangle_{\mathrm{Lip}(\Gamma \cup \tilde{\Gamma})', \mathrm{Lip}(\Gamma \cup \tilde{\Gamma})}. \tag{173}$$

Proof. Now let V, W be in E_3 and let φ be in $\mathrm{Lip}(\Gamma \cup \tilde{\Gamma})$ where there exists a function $\mathbf{\Phi}$ in $\mathrm{Lip}(\mathcal{U})$ such that $\mathbf{\Phi}|_{\Gamma\cup\tilde{\Gamma}} = \varphi$ and such that $\|\mathbf{\Phi}\|_{\mathrm{Lip}(\mathcal{U})} \leqslant \|\varphi\|_{\mathrm{Lip}(\Gamma\cup\tilde{\Gamma})}$ with c independent of φ. From Green's identity (166) we infer that

$$\int_\Gamma \rho\varphi = -\int_{\mathcal{U}} \mathcal{L}_\sigma V \cdot \varphi W - \int_{\mathcal{U}} a\partial_X V \cdot \partial_X \varphi W + \int_{\mathcal{U}} \tilde{L}_\sigma^* \varphi W \cdot V,$$

so

$$\left|\int_\Gamma \rho\varphi\right| \leqslant \|\mathbf{\Phi}\|_{\mathrm{Lip}(\mathcal{U})}.\|\mathcal{L}_\sigma V\|_{E_1'}.\|W\|_{E_1} + \|\mathbf{\Phi}\|_{\mathrm{Lip}(\mathcal{U})}.\|V\|_{E_1}.\|W\|_{E_1}$$

$$+ \|\tilde{\mathcal{L}}_\sigma^* \varphi W\|_{E_1'}.\|V\|_{E_1}, \tag{174}$$

$$\leqslant C\|\varphi\|_{\mathrm{Lip}(\Gamma\cup\tilde{\Gamma})}.\|V\|_{E_3}.\|W\|_{E_3}. \tag{175}$$

Hence by the Hahn–Banach theorem we get the existence of a continuous extension, which is unique because of the density of E_2 in E_3. □

In order to prove the existence part of Theorem 5.1 we will need another Green formula which involves the complete transposition of the operator L. At a smooth level, that is for any V, W in the space E_2, this Green identity reads

$$\sum_{\pm} \int_{\mathcal{U}_\pm} \mathcal{L}_\sigma V \cdot W = \int_{\mathcal{U}} \mathcal{L}_\sigma^* W \cdot V - \int_\Gamma a[\partial_X V] \cdot W + \int_\Gamma aV \cdot [\partial_X W] - T\int_{\tilde{\Gamma}} W \cdot V \tag{176}$$

where $\mathcal{L}_\sigma^* = a\partial_X^2 + \tilde{\mathcal{L}}_\sigma^*$ denotes the adjoint of the operator L_σ.

Lemma 5.5. *The map*

$$V \in E_2 \mapsto \tau := \begin{cases} a[\partial_X V] & \text{on } \Gamma \\ TV & \text{on } \tilde{\Gamma} \end{cases} \tag{177}$$

extends uniquely to a continuous linear map from E_4 to $H^{-\frac{3}{4}}(\Gamma \cup \tilde{\Gamma})$ and Green's identity (176) is still valid for any couple V, W in $E_4 \times H^{1,2}(\mathcal{U})$ in the generalized sense that

$$\langle \mathcal{L}_\sigma V, W\rangle_{H^{-1}(\mathcal{U}),H^1(\mathcal{U})} = \langle \mathcal{L}_\sigma^* W, V\rangle_{E_1',E_1} - \langle \tau, W|_{\Gamma\cup\tilde{\Gamma}}\rangle_{H^{-\frac{1}{2}}(\Gamma\cup\tilde{\Gamma}),H^{\frac{1}{2}}(\Gamma\cup\tilde{\Gamma})}$$

$$+ \int_\Gamma [\partial_X W] \cdot aV|_\Gamma. \tag{178}$$

Proof. By adapting the classical lifting method—see for instance [94]—we get that for any $\tilde{W}_1$ in $H^{\frac{3}{2}}(\Gamma \cup \tilde{\Gamma})$ and $\tilde{W}_2$ in $L^2(\Gamma)$ there exists a function W in $H^{1,2}(\mathcal{U})$ such that $W|_{\Gamma\cup\tilde{\Gamma}} = \tilde{W}_1$, $[\partial_X W]|_\Gamma = \tilde{W}_2$. The proof then follows the same lines as that of Lemma 5.4; we use Green's identity (176) for V in E_2 and a W obtained by lifting $\tilde{W}_1$ in $H^{\frac{3}{2}}(\Gamma \cup \tilde{\Gamma})$ and $\tilde{W}_2 = 0$, and then we use the Hahn–Banach theorem and density. □

Proceeding as previously, we get that the estimate (171) also holds for the adjoint operator. We infer the existence of a solution to the direct problem by using Riesz's theorem and Green's identity (178).

We now use again a sequence of approximation of the solution, and use the linearity of the problem together with the *a priori* estimate (172) to show that this sequence is a Cauchy sequence in the space of the functions V such that $\sqrt{t}\|V(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^3\times\mathbb{R})}$ is continuous on $(0,T)$. Completeness yields the conclusion.

We finally let σ go to infinity. The estimate (172) is uniform with respect to σ. Using weak compactness we can pass to the weak limit so that we get a solution to equation (165). □

5.3. Smoothness of the layer profile

We now study the smoothness of $V(t,x,X)$. It is convenient to recall here some notation given in the introduction. For any Fréchet space E of functions depending on t,x and possibly on X we will denote as $E_{\mathcal{D}}$ the set

$$E_{\mathcal{D}} := \left\{ f \in E / \exists C > 0 / \left(\frac{\mathcal{D}^k f}{C^k k!} \right)_{k\in\mathbb{N}} \text{ is bounded in } E \right\},$$

where $\mathcal{D}$ denotes the material derivative $\mathcal{D} := \partial_t + v^0 \cdot \nabla_x$. We define

$$\mathcal{A} := L^\infty\left((0,T); C^{0,r}\left(\mathbb{R}^3; p\text{-}\mathcal{S}(\mathbb{R}_\pm)\right)\right) \cap L^\infty\left((0,T); C^{s,r}\left(\mathcal{O}_\pm(t); p\text{-}\mathcal{S}(\mathbb{R}_\pm)\right)\right),$$
$$\mathcal{B} := L^\infty\left((0,T); C^{0,r}\left(\mathbb{R}^3\right)\right) \cap L^\infty\left((0,T); C^{s,r}\left(\mathcal{O}_\pm(t)\right)\right).$$

Moreover we recall from § 4.3.5 that the functions a and A are in $\mathcal{B}_{\mathcal{D}}$.

Theorem 5.2. *For any $f \in \mathcal{A}_{\mathcal{D}}$, for any $g_1, g_2 \in \mathcal{B}_{\mathcal{D}}$ there exists one solution $V \in \mathcal{A}_{\mathcal{D}}$ of*

$$\mathcal{L}V = f \quad \text{on } \mathcal{U}_\pm, \quad ([V],[\partial_X V]) = (g_1, g_2) \quad \text{on } \Gamma. \tag{179}$$

Proof. We first reduce the transmission conditions on the internal boundary Γ to homogeneous ones by defining the functions $\tilde{V}$ by

$$V := \tilde{V} \pm \frac{1}{2}\left((2g_1 + g_2)e^{\mp X} - (g_1 + g_2)e^{\mp 2X}\right) \quad \text{when } \pm X > 0. \tag{180}$$

Hence that the problem now reads as follows:

$$\mathcal{L}\tilde{V} = \tilde{f} \quad \text{on } \mathcal{U}_\pm, \quad [\tilde{V}] = [\partial_X \tilde{V}] = 0 \quad \text{on } \Gamma, \tag{181}$$

with $\tilde{f} \in \mathcal{A}_{\mathcal{D}}$. Let us stress that there is no loss of regularity in this lifting process, since the X-derivative has been applied innocuously to the second term in (180), as well as $\mathcal{D}$ since g_1, g_2 and a are in $\mathcal{B}_{\mathcal{D}}$.

The proof now reduces to showing that there exists only one solution $\tilde{V} \in \mathcal{A}_{\mathcal{D}}$ of (181).

We first prove that $\tilde{V}$ is in $\mathcal{C}_{\mathcal{D}}$ where we define $\mathcal{C} := L^\infty\left((0,T)\times\mathbb{R}^3; H^1(\mathbb{R})\right)$. Let us first establish an *a priori* estimate, applying for any $k \in \mathbb{N}$ the field $\mathcal{D}^k$ to the problem (181) to get

$$(\mathcal{E} - k)\tilde{V}^{[k]} = \tilde{f}^{[k]} + t\tilde{V}^{[k+1]} + tA\tilde{V}^{[k]} \quad \text{on } \mathcal{U}_\pm, \quad [\tilde{V}^{[k]}] = [\partial_X \tilde{V}^{[k]}] = 0 \quad \text{on } \Gamma, \tag{182}$$

where we denote by $\tilde{V}^{[k]} := \mathcal{D}^k \tilde{V}$ the kth iterated derivative of $\tilde{V}$ along $\mathcal{D}$ and where $\tilde{f}^{[k]} := \sum_{l=1}^3 \tilde{f}_l^{[k]}$, where $\tilde{f}_1^{[k]} := \mathcal{D}^k \tilde{f}$ whereas $\tilde{f}_2^{[k]}$ and $\tilde{f}_3^{[k]}$ denote respectively the

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commutators

$$\tilde{f}_2^{[k]} := [\mathcal{D}^k, \mathcal{E}] = \sum_{l=0}^{k-1} \binom{k}{l} \mathcal{D}^{k-l} a \cdot \partial_X^2 \tilde{V}^{[l]}, \quad \tilde{f}_3^{[k]} := -[\mathcal{D}^k, tA] = -\sum_{l=0}^{k-1} \binom{k}{l} \mathcal{D}^{k-l} tA \cdot \tilde{V}^{[l]}.$$

The last sums have to be omitted when $k = 0$. Here we have used that $[\mathcal{D}^k, t\mathcal{D}] = k\mathcal{D}^k$. We now multiply the first equation of (182) by $\tilde{V}^{[k]}$ and we now integrate with respect to X only. This yields for any $t, x \in (0, T) \times \mathbb{R}^3$ the estimate

$$\begin{aligned} &\int_{\mathbb{R}} a|\partial_X \tilde{V}^{[k]}|^2 + \left(k + \frac{3}{4}\right) \int_{\mathbb{R}} |\tilde{V}^{[k]}|^2 \\ &\quad \leqslant \left| \int_{\mathbb{R}} \tilde{f}^{[k]} \cdot \tilde{V}^{[k]} \right| + \int_{\mathbb{R}} t|\tilde{V}^{[k+1]} \cdot \tilde{V}^{[k]}| + C_1 \int_{\mathbb{R}} t|\tilde{V}^{[k]}|^2. \end{aligned} \tag{183}$$

Using the condition (24), the left hand side of (183) is larger than

$$c \int_{\mathbb{R}} |\partial_X \tilde{V}^{[k]}|^2 + \frac{3}{4}(k+1) \int_R |\tilde{V}^{[k]}|^2.$$

We now bound the right hand side of (183). Using the Cauchy–Schwarz and Young inequalities we have

$$\int_{\mathbb{R}} |\tilde{f}_1^{[k]} \cdot \tilde{V}^{[k]}| \leqslant \frac{4}{k+1} \int_{\mathbb{R}} |\tilde{f}_1^{[k]}|^2 + \frac{k+1}{4} \int_{\mathbb{R}} |\tilde{V}^{[k]}|^2. \tag{184}$$

Integrating by parts yields

$$\left| \int_{\mathbb{R}} \tilde{f}_2^{[k]} \cdot \tilde{V}^{[k]} \right| \leqslant \sum_{l=0}^{k-1} \binom{k}{l} \int_{\mathbb{R}} |\mathcal{D}^{k-l} a \partial_X \tilde{V}^{[l]} \partial_X \tilde{V}^{[k]}|. \tag{185}$$

Since a is analytic, there exists $C_a > 0$ such that for any $l \in \mathbb{N}$, $\|\mathcal{D}^l a\|_{\mathcal{B}} \leqslant (C_a)^l\, l!$, so using the Cauchy–Schwarz and Young inequalities yields

$$\left| \int_{\mathbb{R}} \tilde{f}_2^{[k]} \cdot \tilde{V}^{[k]} \right| \leqslant C_2 \left(\sum_{l=0}^{k-1} \frac{k!}{l!} C_a^{k-l} \|\partial_X \tilde{V}^{[l]}\| \right)^2 + \frac{c}{2} \int_{\mathbb{R}} |\partial_X \tilde{V}^{[k]}|^2, \tag{186}$$

where we define here $\|f\| := (\int_{\mathbb{R}} |f(t, x, X)|^2 dX)^{\frac{1}{2}}$. In a similar way, there exist $C_3, C_A > 0$ such that

$$\left| \int_{\mathbb{R}} \tilde{f}_3^{[k]} \cdot \tilde{V}^{[k]} \right| \leqslant C_3 \left(\sum_{l=0}^{k-1} \frac{k!}{l!} C_A^{k-l} \|\tilde{V}^{[l]}\| \right)^2 + \frac{1}{8} \int_{\mathbb{R}} |\tilde{V}^{[k]}|^2. \tag{187}$$

Finally for $0 < t < \underline{T} \leqslant T$ we have

$$\left| \int_{\mathbb{R}} t\tilde{V}^{[k+1]} \cdot \tilde{V}^{[k]} \right| \leqslant \frac{k+1}{8} \int_{\mathbb{R}} |\tilde{V}^{[k]}|^2 + \frac{4\underline{T}}{k+1} \int_{\mathbb{R}} |\tilde{V}^{[k+1]}|^2. \tag{188}$$

Hence

$$\frac{c}{2} \int_{\mathbb{R}} |\partial_X \tilde{V}^{[k]}|^2 + \frac{k+1}{4} \int_{\mathbb{R}} |\tilde{V}^{[k]}|^2 \leqslant \frac{4}{k+1} \int_{\mathbb{R}} |\tilde{f}_1^{[k]}|^2 + C_2 \left(\sum_{l=0}^{k-1} \frac{k!}{l!} C_a^{k-l} \|\partial_X \tilde{V}^{[l]}\| \right)^2$$

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$$+ C_3 \left(\sum_{l=0}^{k-1} \frac{k!}{l!} C_A^{k-l} \|\tilde{V}^{[l]}\| \right)^2$$
$$+ \frac{4\underline{T}}{k+1} \int_{\mathbb{R}} |\tilde{V}^{[k+1]}|^2 + C_1 \underline{T} \int_{\mathbb{R}} |\tilde{V}^{[k]}|^2$$

and thus we infer—keeping the notation C_1–C_3 for their square roots—that

$$\frac{c}{4} \|\partial_X \tilde{V}^{[k]}\| + \frac{\sqrt{k+1}}{8} \|\tilde{V}^{[k]}\| \leqslant \frac{2}{\sqrt{k+1}} \|\tilde{f}_1^{[k]}\| + C_2 \sum_{l=0}^{k-1} \frac{k!}{l!} C_a^{k-l} \|\partial_X \tilde{V}^{[l]}\|$$
$$+ C_3 \sum_{l=0}^{k-1} \frac{k!}{l!} C_A^{k-l} \|\tilde{V}^{[l]}\|$$
$$+ \sqrt{\frac{4\underline{T}}{k+1}} \|\tilde{V}^{[k+1]}\| + C_1 \sqrt{\underline{T}} \|\tilde{V}^{[k]}\|. \tag{189}$$

We introduce the functions

$$a_k(t,x) := \frac{\|\tilde{V}^{[k]}\|}{k! C^k}, \quad b_k(t,x) := \frac{\|\partial_X \tilde{V}^{[k]}\|}{k! C^k \sqrt{k+1}} \quad \text{and} \quad f_k(t,x) := \frac{\|\tilde{f}_1^{[k]}\|}{(k+1)! C^k},$$

where C is a positive real which will be chosen in a few lines. Dividing the estimate (189) by $k! C^k \sqrt{k+1}$ yields

$$\frac{c}{4} b_k + \frac{1}{8} a_k \leqslant 2 f_k + C_2 \sum_{l=0}^{k-1} \left(\frac{C_a}{C} \right)^{k-l} b_l + C_3 \sum_{l=0}^{k-1} \left(\frac{C_A}{C} \right)^{k-l} a_l$$
$$+ \sqrt{4\underline{T}} C a_{k+1} + C_1 \sqrt{\underline{T}} a_k. \tag{190}$$

We choose C large enough that $\max(\frac{C_2}{\frac{C}{C_A}-1}, \frac{C_3}{\frac{C}{C_a}-1}) \leqslant \min(\frac{c}{8}, \frac{1}{16})$ and then $\underline{T} > 0$ is chosen small enough that $\sqrt{4\underline{T}} C \leqslant \frac{1}{64}$ and that $C_1 \sqrt{\underline{T}} \leqslant \frac{1}{64}$. Hence summing over $k \in \mathbb{N}$, the estimates (190) yield the following *a priori* estimate: for any $t, x \in (0, \underline{T}) \times \mathbb{R}^3$,

$$\sum_{k \in \mathbb{N}} \left(\frac{c}{8} b_k + \frac{1}{32} a_k \right) \leqslant 2 \sum_{k \in \mathbb{N}} f_k. \tag{191}$$

We now define the iterative scheme $(\tilde{V}^n)_{n \in \mathbb{N}}$ by setting $\tilde{V}^0$ as the solution of

$$\mathcal{E} \tilde{V}^0 = \tilde{f} \quad \text{on } \mathcal{U}_\pm, \quad [\tilde{V}^0] = [\partial_X \tilde{V}^0] = 0 \quad \text{on } \Gamma,$$

and $\tilde{V}^{n+1}$ as the solution of

$$\mathcal{E} \tilde{V}^{n+1} = \tilde{f} + t(\mathcal{D} + A) \tilde{V}^n \quad \text{on } \mathcal{U}_\pm, \quad [\tilde{V}^{n+1}] = [\partial_X \tilde{V}^{n+1}] = 0 \quad \text{on } \Gamma.$$

It is not difficult to see that $\tilde{V}^0$ is in $\mathcal{C}_{\mathcal{D}}$. Then proceeding as in the proof of the estimate (191) we infer the convergence of the iterative scheme for any $t, x \in (0, \underline{T}) \times \mathbb{R}^3$ to a solution $\tilde{V}$ of the problem (181). Using several time slices yields that $\tilde{V}$ is in $\mathcal{C}_{\mathcal{D}}$.

Now to prove Theorem 5.2, we increase the smoothness with respect to x thanks to the operators Δ_j of spectral localization. We proceed as previously dealing with the additional spectral commutators as in § 5.1 (in particular using the commutator estimate Lemma 5.1).

If we denote by $\tilde{\mathcal{B}}$ the space of the functions $f(t,x,X)$ with $\mathcal{B}$ smoothness in t,x with values in $H^1(\mathbb{R})$, this yields that $\tilde{V}$ is in $\tilde{\mathcal{B}}_{\mathcal{D}}$. Then we prove by induction that $X^k\tilde{V}$ is in $\tilde{\mathcal{B}}_{\mathcal{D}}$ for all k in $\mathbb{N}$. Finally we use the equation to increase by induction the number of derivatives with respect to X and get that $\tilde{V}$ is in $\mathcal{A}_{\mathcal{D}}$. □

5.4. Other properties of the profile

Let us now prove that the normal projection $V\cdot n$ vanishes. We multiply equation (134) by n and take into account the equation (11) for $n(t,x)$, so we get

$$|n|^2\partial_X^2(V\cdot n)+\frac{X}{2}\partial_X(V\cdot n)-\frac{1}{2}(V\cdot n)=t(\partial_t+v^0\cdot\nabla_x)(V\cdot n). \tag{192}$$

Moreover also taking the scalar product of the transmission conditions on the internal boundary $X=0$ with n we get

$$(V\cdot n)|_{X=0^+}=(V\cdot n)|_{X=0^-}, \tag{193}$$

$$\partial_X(V\cdot n)|_{X=0^+}=\partial_X(V\cdot n)|_{X=0^-}. \tag{194}$$

Proceeding as in § 5.2 we get an energy estimate for the problem (192), (193), (194) from which we infer that $V\cdot n$ vanishes.

When $s\geqslant 2$ the first and second time derivatives of the functions n and ω_0 are in the space $L^\infty([0,T],C^{0,r}(\mathbb{R}^3))$. This allows us to get some estimates for the second time derivative $\partial_t^2 V$ by commuting ∂_t^2 with equations (19)–(25). Hence we can define a trace at $t=0$ of all the terms in equations (19)–(25). We hence get that the trace $V|_{t=0}$ satisfies (144), (145), (146).

6. Full expansion

We are now concerned with the following terms in the expansion with respect to νt of the solutions of the Navier–Stokes equations. Actually in this section we show that if the initial data is piecewise smooth on each side of the interface $\{\varphi^0=0\}$—that is, if we consider an initial velocity v_0 as described in the Definition 1.1 with $s=+\infty$—then it is possible to write a complete asymptotic expansion of the vorticity of the form

$$\omega^\nu(t,x)=\sum_{j\geqslant 0}\sqrt{\nu t}^{\,j}\;\Omega^j\left(t,x,\frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right)+O(\sqrt{\nu t}^{\,\infty}), \tag{195}$$

where the first profile Ω^0 is the one constructed in the previous section: $\Omega^0:=\Omega$.

Let us explain the underlying intuition for guessing the expansion (195): when the initial data is globally smooth, the vorticity ω^ν given by the NS equations admits a regular (Taylor-type) expansion:

$$\omega^\nu(t,x):=\sum_{j\geqslant 0}(\nu t)^j\;\omega^j(t,x)+O((\nu t)^\infty), \tag{196}$$

so we expect that when the initial data is piecewise smooth on each side of the interface $\{\varphi^0 = 0\}$, the same should still hold true when we incorporate the fast scale $\frac{\varphi^0(t,x)}{\sqrt{\nu t}}$, so we should write ω^ν through the ν-dependent profile:

$$\omega^\nu(t,x) = \Omega^\nu\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right), \tag{197}$$

where $\Omega^\nu(t,x,X)$ admits a regular expansion:

$$\Omega^\nu(t,x,X) := \sum_{j\geqslant 0} \sqrt{\nu t}^j\ \Omega^j(t,x,X) + O(\sqrt{\nu t}^\infty). \tag{198}$$

Once again, in order to prove this, we will consider the velocity formulation, looking first for a determination of the velocity profiles of the expansion

$$v^\nu(t,x) = v^0(t,x) + \sum_{j\geqslant 1} \sqrt{\nu t}^j\ V^j\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right) + O(\sqrt{\nu t}^\infty), \tag{199}$$

where the profile V^1 is the one constructed in the previous section: $V^1 := V$.

Then we will recover the vorticity profiles by plugging (195) and (199) into the relation (3) and setting the terms of order $\sqrt{\nu t}^j$ equal; this leads to

$$\Omega^j = \operatorname{curl}_x V^j + n \wedge \partial_X V^{j+1}. \tag{200}$$

Moreover we will look for a pressure expansion of the form

$$p^\nu(t,x) = p^0(t,x) + \sum_{j\geqslant 2} \frac{\sqrt{\nu t}^j}{t}\ P^j\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right) + O(\sqrt{\nu t}^\infty). \tag{201}$$

The profiles above are of the following form: for $\pm X > 0$,

$$U(t,x,X) := \underline{U}(t,x) + \tilde{U}(t,x,X), \tag{202}$$

where the function $\tilde{U}(t,x,X)$ is rapidly decreasing when $\pm X \to \infty$, and the letter U is the placeholder for the Ω^j, the V^j and the P^j. We will refer to the term $\underline{U}$ as the regular part and to the term $\tilde{U}$ as the layer part.

The layer part $\tilde{P}^2$ is equal to $\tilde{P}^2 = tP$ where P is the profile of the previous section. This possibility of being smoothly factorized by t is very particular to the order 2. This follows from the orthogonality property (135). Furthermore, to satisfy the pressure continuity we will have to add to this layer part $\tilde{P}^2$ a regular part $\underline{P}^2$, as anticipated in Remark 4.3.

As explained in the introduction, we will construct some profiles V^j and P^j, for $j \geqslant 2$, such that the resulting profiles in the right hand sides of (37)–(41) vanish, that is such that $F_a^j = 0$, $F_b^j = 0$, and $[V^j] = [a\partial_X V^j_{\tan} + \partial_n V^{j-1}_{\tan}] = 0$ and $[P^j] = 0$ for any j.

Actually, using (130)–(132) we obtain that for any $j \geqslant 1$, F_a^j and F_b^j are of the form

$$F_a^j = -\left(\mathcal{L} - \frac{j-1}{2}\right) V^j + f^j + \nabla_x P^j + \partial_X P^{j+1} n, \tag{203}$$

$$F_b^j = \partial_X (V^{j+1} \cdot n) + \operatorname{div}_x V^j, \tag{204}$$

where f^j would be decomposed into $f^j = \underline{f}^j + \tilde{f}^j$—as the profiles U were in (202)—and depends only on lower order profiles. The profiles F_a^j and F_b^j will be decomposed into $F_a^j = \underline{F}_a^j + \tilde{F}_a^j$ and $F_b^j = \underline{F}_b^j + \tilde{F}_b^j$ as well.

Here, it is understood that we substitute a for $|n|^2$, by considering the operator $\mathcal{L}$, in order to avoid a spurious degeneracy of the profile equations, as we have already done when we were constructing the first profile; cf. Remark 4.4. The decay property of the layer parts of the profiles entails that the error due to this substitution is $O(\sqrt{\nu t}^{\infty})$.

Let us now be more precise about the spaces that we will use for these profiles. Let us recall that $\mathcal{O}_\pm$ are the space–time domains $\mathcal{O}_\pm := \{(t,x) \in (0,T) \times \mathbb{R}^3 / x \in \mathcal{O}_\pm(t)\}$ and $\partial\mathcal{O}$ the interface between them. We will denote as $H^\infty(\mathcal{O}_\pm)$ the space of the functions $\underline{f}(t,x)$ which are H^∞, that is in H^s for any $s \in \mathbb{R}_+$, on both $\mathcal{O}_+$ and $\mathcal{O}_-$; as $(\mathbb{R} \oplus H^\infty)(\mathcal{O}_\pm)$ the space of the functions $\underline{f}(t,x)$ which are, on both $\mathcal{O}_+$ and $\mathcal{O}_-$, the sum of a constant and of a function in H^∞ and

$$\mathcal{A}^\infty := H^\infty(\mathcal{O}_\pm, p\text{-}\mathcal{S}(\mathbb{R})),$$

which therefore contains functions of t, x and X. It follows from the assumptions made in this section that v^0 is in $H^\infty(\mathcal{O}_\pm)$. On the other hand it is plain sailing to check that when the profiles V^j, for $j \geqslant 1$, and P^j, for $j \geqslant 2$, are respectively in $H^\infty(\mathcal{O}_\pm) \oplus \mathcal{A}^\infty$ and $(\mathbb{R} \oplus H^\infty)(\mathcal{O}_\pm) \oplus \mathcal{A}^\infty$, then the corresponding profiles F_a^j and F_b^j given by (203) and (204) are in $H^\infty(\mathcal{O}_\pm) \oplus \mathcal{A}^\infty$.

Then the main result of this section is the following.

Theorem 6.1. *There exist some profiles V^j, for $j \geqslant 1$, and P^j, for $j \geqslant 2$, respectively in $H^\infty(\mathcal{O}_\pm) \oplus \mathcal{A}^\infty$ and $(\mathbb{R} \oplus H^\infty)(\mathcal{O}_\pm) \oplus \mathcal{A}^\infty$, such that the corresponding profiles F_a^j and F_b^j given by (203) and (204) satisfy $F_a^j = 0$ and $F_b^j = 0$ for any $j \geqslant 1$, and $[V^j] = [a\partial_X V^j_{tan} + \partial_n V^{j-1}_{tan}] = 0$ and $[P^j] = 0$ for any $j \geqslant 2$.*

The remainder of this section is devoted to the proof of this result.

Let us denote, for any $j \geqslant 1$, by $(\mathcal{P}^j)$ the following problem:

$$(\mathcal{P}^j): \quad \tilde{F}_b^{j-1} = 0, \quad F_a^j = 0, \quad \underline{F}_b^j = 0, \quad [V^j] = [a\partial_X V^j_{\tan} + \partial_n V^{j-1}_{\tan}] = 0, \quad [P^j] = 0.$$

where we define $\tilde{F}_b^0 = \partial_X(V^1 \cdot n)$, and the requirement $[P^j] = 0$ is dropped out for $j = 1$.

We will determine iteratively the velocity and pressure profiles, V^j, for $j \geqslant 1$, and P^j, for $j \geqslant 2$, in order to satisfy the problems $(\mathcal{P}^j)$ for any $j \geqslant 1$.

For $j \geqslant 1$, we will split the problem $(\mathcal{P}^j)$ into three subproblems. In order to do this we first observe that the jump conditions $[V^j] = [a\partial_X V^j_{\tan} + \partial_n V^{j-1}_{\tan}] = 0$, $[P^j] = 0$ reduce to

$$[\underline{V}^j \cdot n] = -[\tilde{V}^j \cdot n], \tag{205}$$

$$[\tilde{V}^j_{\tan}] = -[\underline{V}^j_{\tan}], \tag{206}$$

$$[a\partial_X \tilde{V}^j_{\tan}] = -[\partial_n V^{j-1}_{\tan}], \tag{207}$$

$$[\underline{P}^j] = -[\tilde{P}^j]. \tag{208}$$

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We now introduce, for any $j \geqslant 1$, the following subproblems:

$$(\mathcal{P}^j_I): \quad \tilde{F}^{j-1}_b = 0,$$
$$(\mathcal{P}^j_{II}): \quad \underline{F}^j_a = 0, \quad \underline{F}^j_b = 0, \quad (205), (208),$$
$$(\mathcal{P}^j_{III}): \quad \tilde{F}^j_a = 0 \quad (206), (207).$$

Now, proving the problem $(\mathcal{P}^j)$ amounts to proving $(\mathcal{P}^j_I)$, $(\mathcal{P}^j_{II})$ and $(\mathcal{P}^j_{III})$.

However it is crucial to incorporate in the iteration process the following condition:

$$\forall t \in [0, T], \quad \forall s \in (-\eta, \eta), \quad \forall X, \quad \int_{\partial(\mathcal{O}_+(t))_s} \tilde{V}^j(t, x, X) \cdot n(t, x)\, d\sigma_{t,s}(x) = 0, \tag{209}$$

where we define $\partial(\mathcal{O}_+(t))_s := \{\sigma - sn(t, \sigma)/\sigma \in \partial\mathcal{O}_+(t)\}$ and $d\sigma_{t,s}$ is the surface measure on $\partial(\mathcal{O}_+(t))_s$.

We will prove the following by iteration:

(Π^j): $\exists (V^j)_{k \leqslant j}, (P^j)_{k \leqslant j}$ respectively in $H^\infty(\mathcal{O}_\pm) \oplus \mathcal{A}^\infty$ and $(\mathbb{R} \oplus H^\infty)(\mathcal{O}_\pm) \oplus \mathcal{A}^\infty$, and $\tilde{P}^{j+1} \in \mathcal{A}^\infty$ satisfying $(\mathcal{P}^k)_{k \leqslant j}$ and (209).

To determine the velocity and pressure profiles we proceed iteratively, determining at the step j the velocity profile V^j, the regular part of the pressure profile $\underline{P}^j$ and the layer part of the following pressure profile $\tilde{P}^{j+1}$, from the profiles already known from the previous steps. The step $j = 1$ was done in the previous section. We now explain how to do a step $j \geqslant 2$ when the previous ones are done. We therefore assume that (Π^{j-1}) is satisfied.

6.1. Determination of the normal component of the layer part $\tilde{V}^j$

We start with determining the layer part $\tilde{V}^j$ of the profile V^j. The problem $(\mathcal{P}^j_I)$ reads

$$\partial_X(\tilde{V}^j \cdot n) = -\operatorname{div}_x \tilde{V}^{j-1}. \tag{210}$$

Since we look for a layer profile $\tilde{V}^j$ in $\mathcal{A}^\infty$, the only solution is given by the following formula: for $\pm X > 0$,

$$\tilde{V}^j \cdot n := -\int_X^{\pm\infty} \operatorname{div}_x \tilde{V}^{j-1}. \tag{211}$$

Above, the integral refers to the third variable of the function $\operatorname{div}_x \tilde{V}^{j-1}(t, x, \cdot)$.

Let us now prove that, moreover, any layer profile $\tilde{V}^j$ with its normal component given by (211) satisfies (209). We use Fubini's principle to get that such a layer profile satisfies

$$\int_{\partial(\mathcal{O}_+(t))_s} \tilde{V}^j(t, x, X) \cdot n(t, x)\, d\sigma_{t,s}(x) = \int_X^{\pm\infty} A(t, s, Y) dY, \tag{212}$$

with

$$A(t, s, Y) := \int_{\partial(\mathcal{O}_+(t))_s} (\operatorname{div}_x \tilde{V}^{j-1})(t, x, Y) d\sigma_{t,s}(x).$$

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The integral of a function f over $\partial(\mathcal{O}_+(t))_s$ is linked to the integral over $\partial(\mathcal{O}_+(t))$ by the following relation:

$$\int_{\partial(\mathcal{O}_+(t))_s} f(x)d\sigma_{t,s} = \int_{\partial(\mathcal{O}_+(t))} f_\#(\sigma,s)\gamma_{t,s}(\sigma)d\sigma_{t,0}, \tag{213}$$

where $\gamma_{t,s}(\sigma)$ denotes the Jacobian of the transformation $\sigma \mapsto \sigma - sn(t,\sigma)$ which maps $\partial(\mathcal{O}_+(t))$ to $\partial(\mathcal{O}_+(t))_s$ and $f_\#(\sigma,s) := f(\sigma - sn(t,\sigma))$. Let us recall that the divergence splits into the following decomposition (see for example Theorem C.4.8 of [17]):

$$\begin{aligned}\operatorname{div}_x f &= (\gamma_{t,s}(\sigma))^{-1} \\ &\quad\times \Big(-\partial_s(\gamma_{t,s}(\sigma) f_\# \cdot n) + \operatorname{div}_{\partial(\mathcal{O}_+(t))}\big(\gamma_{t,s}(\sigma)(Id - s\,D_x n)^{-1}(f_\#)_{tan}\big)\Big),\end{aligned} \tag{214}$$

where $\operatorname{div}_{\partial(\mathcal{O}_+(t))}$ denotes the divergence operator on $\partial(\mathcal{O}_+(t))$.

Combining (212), (213) and (214), we get that $A = A_1 + A_2$ with

$$\begin{aligned}A_1 &:= -\partial_s \int_{\partial(\mathcal{O}_+(t))_s} \tilde{V}_\#^{j-1} \cdot n_\#\, d\sigma_{t,s} \quad \text{and} \\ A_2 &:= \int_{\partial\mathcal{O}_+(t)} \operatorname{div}_{\partial\Omega} \Big(\gamma_{t,s}(\sigma)(Id - sD_x n)^{-1}(\tilde{V}_\#^{j-1})_{tan}\Big)\, d\sigma_{t,0},\end{aligned}$$

where we define $\tilde{V}_\#^{j-1}(t,\sigma,s,\cdot) := \tilde{V}^{j-1}(t,\sigma - sn(t,\sigma),\cdot)$.

Using (209) at the order $j-1$ we obtain $A_1 = 0$. On the other hand $A_2 = 0$ thanks to Stokes's theorem. Going back to (212), this proves (209) at the order j.

6.2. Determination of $\underline{V}^j$ and $\underline{P}^j$

This section is devoted to the existence of $\underline{V}^j$ and $\underline{P}^j$ respectively in $H^\infty(\mathcal{O}_\pm) \oplus \mathcal{A}^\infty$ and $(\mathbb{R} \oplus H^\infty)(\mathcal{O}_\pm) \oplus \mathcal{A}^\infty$, satisfying the problem $(\mathcal{P}_{II}^j)$.

For the reader's comfort let us stress that, in this section, the functions depend only on t and x (and not on X). In particular the notation ∇ stands in this section for ∇_x.

Referring to (203), (204), the equations $\underline{F}_a^j = 0$ and $\underline{F}_b^j = 0$ read

$$\frac{j}{2}\underline{V}^j + t(\mathcal{D}\underline{V}^j + \underline{V}^j \cdot \nabla v^0) + \nabla \underline{P}^j = \underline{f}^j, \tag{215}$$

and

$$\operatorname{div} \underline{V}^j = 0. \tag{216}$$

Equations (215) together with (216) can be seen as some modified linearized Euler equations. The main difference from the classical linearized Euler equations is the extra factor t in front of the time derivative that makes the initial hypersurface $\{t=0\}$ characteristic. As a consequence, as for the Fuchsian equations, a growth condition, with respect to time, insures the existence and uniqueness of solutions, without prescribing any initial condition. Here we will use an L^2 setting, both in space and in time.

Here we are interested in the transmission problem made up of the equations (215), (216) on each side, $\mathcal{O}_\pm$, with the conditions (205) and (208) on the interface $\partial\mathcal{O}$. The main result in this section is the following:

Proposition 6.1. *There exist* $\underline{V}^j$ *and* $\underline{P}^j$ *respectively in* $H^\infty(\mathcal{O}_\pm) \oplus \mathcal{A}^\infty$ *and* $(\mathbb{R} \oplus H^\infty)(\mathcal{O}_\pm) \oplus \mathcal{A}^\infty$, *satisfying equations (215), (216) on each side,* $\mathcal{O}_\pm$, *with the transmission conditions (205)–(208) of the interface* $\partial\mathcal{O}$.

To be slightly more general we will actually consider the following problem:

$$\beta V + t(\mathcal{D}V + V \cdot \nabla v^0) + \nabla P = f \quad \text{for } (t, x) \text{ in } \mathcal{O}_\pm, \tag{217}$$

$$\operatorname{div} V = 0 \quad \text{for } (t, x) \text{ in } \mathcal{O}_\pm, \tag{218}$$

$$[V \cdot n] = g \quad \text{for } (t, x) \text{ in } \partial\mathcal{O}, \tag{219}$$

$$[P] = h \quad \text{for } (t, x) \text{ in } \partial\mathcal{O}, \tag{220}$$

where $\beta > \frac{1}{2}$ and the boundary data g satisfies the following condition of compatibility (between equation (218) and the condition (219)):

$$\int_{\partial\mathcal{O}_+(t)} g(t, .) d\sigma_{t,0} = 0, \tag{221}$$

for any time $t \in (0, T)$.

Equations (215), (216) are of the form (217), (218) with $\beta = \frac{j}{2}$ (so $\beta > \frac{1}{2}$, since $j \geqslant 2$) and the transmission condition (205) is of the form (219) with $g := -[\tilde{V}^j \cdot n]$, which satisfies the compatibility condition (221) thanks to (209).

Thus Proposition 6.1 will be deduced from the following result.

Proposition 6.2. *For any* $f, g, h \in H^\infty(\mathcal{O}_\pm)$, *with* g *satisfying the compatibility condition (221), there exist* V *and* P *respectively in* $H^\infty(\mathcal{O}_\pm) \oplus \mathcal{A}^\infty$ *and* $(\mathbb{R} \oplus H^\infty)(\mathcal{O}_\pm) \oplus \mathcal{A}^\infty$ *satisfying (217)–(220).*

6.2.1. A pressureless system. Let us first consider the pressureless version of the problem (217)–(220), that is the set of two equations

$$\beta V + t(\mathcal{D}V + \delta V \cdot \nabla v^0) = f, \quad \text{for } (t, x) \text{ in } \mathcal{O}_\pm, \tag{222}$$

where $\delta \in \{0, 1\}$. In the case where $\delta = 0$, the equation above makes sense for a scalar-valued unknown V. Such a case will be useful in the next section.

Since the domain is precisely transported by the vector field $\mathcal{D}$ and these equations are now local, they do not need any transmission condition and are totally independent.

More precisely we will prove the following result.

Lemma 6.1. *For* $\beta > \frac{1}{2}$, *for any* f *in* $H^\infty(\mathcal{O}_\pm)$ *there exists only one solution* V *in* $H^\infty(\mathcal{O}_\pm)$ *of (222).*

Proof. The basic energy estimate is formally obtained by multiplying equation (222) by V and by integrating over $\mathcal{O}_\pm$. This yields

$$T \int_{\mathcal{O}_\pm(T)} V^2(T, .) + \left(\beta - \frac{1}{2}\right) \int_{\mathcal{O}_\pm} V^2 \leqslant \int_{\mathcal{O}_\pm} |f \cdot V| + C \int_{\mathcal{O}_\pm} tV^2, \tag{223}$$

where $C := \|\nabla v^0\|_{L^\infty(\mathcal{O}_\pm)}$. The last term in (223) can be absorbed by the first one by a Gronwall lemma. On the other hand we use the Cauchy–Schwarz and Young inequalities

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to get that

$$\int_{\mathcal{O}_\pm} |f \cdot V| \leqslant C \int_{\mathcal{O}_\pm} f^2 + \frac{(\beta - \frac{1}{2})}{2} \int_{\mathcal{O}_\pm} V^2,$$

so we finally get the *a priori* estimate

$$T \int_{\mathcal{O}_\pm} V^2(T, .) + \frac{1}{2}(\beta - \frac{1}{2}) \int_{\mathcal{O}_\pm} V^2 \lesssim \int_{\mathcal{O}_\pm} f^2. \tag{224}$$

Classical variational methods then allow us to deduce that for any f in $L^2(\mathcal{O}_\pm)$ there exists only one solution V in $L^2(\mathcal{O}_\pm)$ of (222).

Now let us consider the regularity issues. The point here is that the boundary $\partial\mathcal{O}_\pm$ is totally characteristic for equation (222), which is local. As a consequence the regularity of the solution V can be established as if the equation holds in the full space $\mathbb{R}^3$. In effect, thanks to the universal extension operator, there exist $\overline{f}_\pm$ and $\overline{v}^0_\pm$ with

$$\|\overline{f}_\pm, \overline{v}^0_\pm\|_{H^s(\mathbb{R}^3\times(0,T)))} \lesssim \|f, v^0\|_{H^s(\mathcal{O}_\pm)},$$

such that, for any $t \in (0, T)$, $\overline{f}_\pm|_{\mathcal{O}_\pm} = f$ and $\overline{v}^0_\pm|_{\mathcal{O}_\pm} = v^0$. Proceeding as previously we get that the L^2 result is also true when the domain $\mathcal{O}_\pm$ is replaced by the full space $\mathbb{R}^3$: there exists only one solution $\overline{V}_\pm \in L^2\left((0, T) \times \mathbb{R}^3\right)$ of

$$t(\partial_t \overline{V}_\pm + \overline{v}^0_\pm \cdot \nabla \overline{V}_\pm + \overline{V}_\pm \cdot \nabla \overline{v}^0_\pm) + \beta \overline{V}_\pm = \overline{f}_\pm, \quad \text{in } \mathbb{R}^3 \times (0, T).$$

The restrictions $\overline{V}_\pm|_{\mathcal{O}_\pm}$ satisfy equation (222), so by uniqueness $V_\pm = \overline{V}_\pm|_{\mathcal{O}_\pm}$. We have therefore reduced the problem to proving the regularity of $\overline{V}_\pm$. The gain of dealing with the full space $\mathbb{R}^3$ is that we can now use a spectral localization, that is we apply the dyadic blocks Δ_j to the equation and then we obtain an energy estimate. We then get that:

Lemma 6.2. *For any $s \geqslant 0$, for any $f \in L^2((0, T), H^s(\mathcal{O}_\pm(t)))$ the solution V of (222) is also in $L^2((0, T), H^s(\mathcal{O}_\pm(t)))$.*

Now to get smoothness in time, we simply apply the time derivative ∂_t to equation (222). Indeed the time derivative $V^{[1]} := \partial_t V$ of the solution satisfies the equation

$$t(\partial_t V^{[1]} + v^0 \cdot \nabla V^{[1]} + V^{[1]} \cdot \nabla v^0) + (\beta + 1)V^{[1]} = f^{[1]}, \quad \text{in } \mathcal{O}_\pm, \tag{225}$$

with $f^{[1]} := \partial_t f - t((\partial_t v^0) \cdot \nabla V + V \cdot \nabla \partial_t v^0)$. Using Lemma 6.2 we get, for $s > 1$, an estimate of $V^{[1]}$ in $L^2\left((0, T), H^{s-1}(\mathcal{O}_\pm(t))\right)$ by the $L^2\left((0, T), H^{s-1}(\mathcal{O}_\pm(t))\right)$ norm of $\partial_t f$ (and of course depending also on v^0). We then proceed by induction to conclude the proof of Lemma 6.1. □

6.2.2. The auxiliary system. Let f, g, h be as in Proposition 6.2. Let $V \in H^\infty(\mathcal{O}_\pm(t))$. We define $\Pi[V] \in (\mathbb{R} \oplus H^\infty)(\mathcal{O}_\pm(t))$, the solution to

$$\begin{aligned}
&\Delta\Pi[V] = \operatorname{div} f - 2t\nabla v^0 : \nabla V, \qquad \text{for } x \text{ in } \mathcal{O}_\pm(t),\\
&[\partial_n \Pi[V]] = [f \cdot n] - (\beta + t\mathcal{D})g - t[V \cdot \left(\nabla v^0 + {}^t\nabla v^0\right) \cdot n] + C[V](t) \quad \text{for } x \text{ in } \partial\mathcal{O}(t),\\
&[\Pi[V]] = h \quad \text{for } x \text{ in } \partial\mathcal{O}(t),
\end{aligned}$$

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where the constant $C[V](t)$ is chosen in order to satisfy the condition of compatibility between the two first equations. Such a $\Pi[V]$ is unique up to a additive constant.

Moreover the operator which maps V to $\Pi[V]$ is a pseudo-differential linear operator of order 0 in V, which satisfies the transmission property, so it preserves piecewise smoothness at any order (we recall that in this section the interface is assumed to be smooth, so we only need here to refer the reader to classical works, for instance [107]).

In particular there exists $C > 0$ (depending only on the Euler solution, f and g) such that for any $V \in H^\infty(\mathcal{O}_\pm)$, for any $s \geqslant 0$,

$$\|\nabla \Pi[V]\|_{H^s(\mathcal{O}_\pm(t))} \leqslant C \left(\|V\|_{H^s(\mathcal{O}_\pm(t))} + 1 \right).$$

Then proceeding as in the previous subsection, we obtain that there exists $V \in H^\infty(\mathcal{O}_\pm)$ satisfying

$$\beta V + t(\mathcal{D} + V \cdot \nabla v^0) + \nabla \Pi(V) = f, \quad \text{for } x \text{ in } \mathcal{O}_\pm(t). \tag{226}$$

6.2.3. Proof of Proposition 6.2. Equations (217) and (220) are then satisfied with $P = \Pi(V)$. It remains to prove that equations (218) and (219) are satisfied as well.

First, taking the divergence of (226) we obtain that $(\beta + t\mathcal{D})\text{div}\, V = 0$, so $\text{div}\, V = 0$, by using the uniqueness part of Lemma 6.1.

Now we use (226) to estimate the jump of $V \cdot n$ across $\partial\mathcal{O}_\pm(t)$. Using that n satisfies equation (11) we get

$$(\beta + t\mathcal{D})([V \cdot n] - g) = -tC[V](t).$$

Thanks to the uniqueness part of Lemma 6.1 we infer that, for each time t, $[V \cdot n] - g$ does not depend on $x \in \partial\mathcal{O}_\pm(t)$. To prove that (219) is satisfied it is therefore sufficient to observe that $\int_{\partial\mathcal{O}_\pm(t)} ([V \cdot n] - g) d\sigma_t$ vanishes, because of Stokes's theorem and of the assumption that g satisfies the compatibility condition (221). The proof of Proposition 6.2 is then achieved.

6.3. Determination of the tangential component of the layer part $\tilde{V}^j$

The tangential part of $\underline{V}^j$ is in general discontinuous at $\{\varphi^0 = 0\}$. This jump discontinuity will have to be compensated by the layer part $\tilde{V}^j$, that we now look for.

Lemma 6.3. *There exists a couple $(\tilde{V}^j, \tilde{P}^{j+1})$ in $\mathcal{A}^\infty$ satisfying $(\mathcal{P}_I^j)$ and $(\mathcal{P}_{III}^j)$, and the condition (209).*

Proof. We have already determined above the normal component $\tilde{V}^j \cdot n$ of the velocity profile $\tilde{V}^j$ in order to verify $(\mathcal{P}_I^j)$ and the condition (209). The equation for $\tilde{F}_a^j$ reads

$$\left(\mathcal{L} - \frac{j-1}{2} \right) \tilde{V}^j = \partial_X \tilde{P}^{j+1} \cdot n + \tilde{f}^j. \tag{227}$$

This problem reduces to a problem for the tangential part $\tilde{V}^j_{\tan}$. In effect the pressure $\tilde{P}^{j+1}$ can be determined as a function of $\tilde{V}^j$:

$$|n|^2 \partial_X \tilde{P}^{j+1} = \left(\mathcal{L} - \frac{j-1}{2} \right) (\tilde{V}^j \cdot n) + 2t\tilde{V}^j \cdot \nabla_x v^0 \cdot n - \tilde{f}^j \cdot n. \tag{228}$$

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Here we have taken the scalar product of equation (227) with n and use the equation for n, that is equation (11). Then plugging (228) into equation (227) and using again the equation for n leads to

$$\left(\mathcal{L} - \frac{j-1}{2}\right)(\tilde{V}^j_{\tan}) = f^j_{\tan} + t\tilde{A}\tilde{V}^j \tag{229}$$

where $\tilde{A}\tilde{V}^j$ denotes the local zero-order operator

$$\tilde{A}\tilde{V}^j := 2(\tilde{V}^j \cdot \nabla_x v^0 \cdot n)n + \frac{1}{a}(\tilde{V}^j \cdot n)\left(\frac{1}{a}(n \cdot \nabla_x v^0 \cdot n)n - {}^t(\nabla_x v^0 \cdot n)\right). \tag{230}$$

Thanks to the analysis of § 5.2 there exists $\tilde{V}^j_{\tan}$ in $\mathcal{A}^\infty$ satisfying equations (229) on each side $\pm X > 0$ of the interface, with the transmission conditions (206), (207). Moreover one deduces from (228) a corresponding profile layer profile $\tilde{P}^{j+1}$ in $\mathcal{A}^\infty$. □

7. Some remarks

7.1. On the smoothing of the level function

We may be tempted to take into account the viscous smoothing of the level function φ^0, that is instead of the solution φ^0 of

$$Dv^0 := \partial_t \varphi^0 + v^0 \cdot \nabla_x \varphi^0 = 0.$$

to consider the solution φ^ν of the ‘viscous eikonal equation’

$$\partial_t \varphi^\nu + v^\nu \cdot \nabla_x \varphi^\nu = \nu \Delta_x \varphi^\nu, \tag{231}$$

with, again, the initial data $\varphi^\nu|_{t=0} = \varphi_0$. Actually, reading again the formulas (130), (131), (132) we notice that these terms are the ones in the prefactor of $\frac{1}{\sqrt{\nu t}}\partial_X U$ when one applies the transport–diffusion operator to a function of the form $U(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}})$.

In that case, proceeding as previously we have that φ^ν admits an expansion of the form

$$\varphi^\nu(t, x) \sim \varphi^\nu_a(t, x) := \varphi^0(t, x) + \nu t \boldsymbol{\Phi}\left(t, x, \frac{\varphi^0(t, x)}{\sqrt{\nu t}}\right) \tag{232}$$

where the profile $\boldsymbol{\Phi}(t, x, X)$ is defined as $\boldsymbol{\Phi}(t, x, X) := \underline{\boldsymbol{\Phi}}(t, x) + \tilde{\boldsymbol{\Phi}}(t, x, X)$, where the $\underline{\boldsymbol{\Phi}}$ are the solutions of

$$(1 + t\mathcal{D})\underline{\boldsymbol{\Phi}} = \Delta_x \varphi^0 \tag{233}$$

and $\tilde{\boldsymbol{\Phi}}$ is the solution of the equations

$$\left(\mathcal{E} - \frac{1}{2} - t\mathcal{D}\right)\tilde{\boldsymbol{\Phi}} = 0 \tag{234}$$

with the following transmission conditions across $X = 0$:

$$[\tilde{\boldsymbol{\Phi}}] = -(\underline{\boldsymbol{\Phi}}_+ - \underline{\boldsymbol{\Phi}}_-), \quad \text{and} \quad [\partial_X \tilde{\boldsymbol{\Phi}}] = 0, \tag{235}$$

where $\underline{\mathbf{\Phi}}_{\pm}$ are suitable extensions of $\underline{\mathbf{\Phi}}$. Let us show that $\Delta_x \varphi_a^\nu$ does not have a jump discontinuity. We have, at least in a neighborhood of $\{\varphi^0\}$, the following identity:

$$\Delta_x \varphi_a^\nu = \Delta_x \varphi^0 + a\partial_X^2 \tilde{\mathbf{\Phi}} + \sqrt{\nu t}(\Delta_x \varphi^0 \cdot \partial_X \tilde{\mathbf{\Phi}} + 2n \cdot \nabla_x \partial_X \tilde{\mathbf{\Phi}}) + \nu t \Delta_x \mathbf{\Phi}. \tag{236}$$

In the expression above the profiles are evaluated in $X = \frac{\varphi^0(t,x)}{\sqrt{\nu t}}$. Using equation (234) we have

$$[a\partial_X^2 \tilde{\mathbf{\Phi}}] = [(1 + t\mathcal{D})\tilde{\mathbf{\Phi}}]. \tag{237}$$

Since $D := \partial_t + v^0 \cdot \nabla_x$ is tangent to the hypersurface $\{\varphi^0(t,x) = 0\}$ we infer from (235) that

$$[a\partial_X^2 \tilde{\mathbf{\Phi}}] = -[(1 + t\mathcal{D})\underline{\mathbf{\Phi}}]. \tag{238}$$

Using equation (233) we get $[a\partial_X^2 \tilde{\mathbf{\Phi}}] = -[\Delta_x \underline{\varphi}^0]$, so

$$[\Delta_x \varphi_a^\nu] = \sqrt{\nu t}[(\Delta_x \varphi^0 . \partial_X \tilde{\mathbf{\Phi}} + 2n \cdot \nabla_x \partial_X \tilde{\mathbf{\Phi}})] + \nu t [\Delta_x \mathbf{\Phi}]. \tag{239}$$

Besides, differentiating with respect to x the transmission conditions (235) we get $[\Delta_x \varphi_a^\nu] = 0$.

One can see the link between the two points of view through a Taylor expansion: for any smooth profile U there exists a smooth profile $U^\flat$ such that

$$U\left(t, x, \frac{\varphi_a^\nu(t,x)}{\sqrt{\nu t}}\right) = U\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right) + \sqrt{\nu t} U^\flat \left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right). \tag{240}$$

In effect it is sufficient to define $U^\flat$ by $U^\flat(t,x,X) := U^\dagger(t,x,X,\mathbf{\Phi}(t,x,X))$ where $U^\dagger(t,x,X,h) := h^{-1}\left(U^\flat(t,x,X+h) - U^\flat(t,x,X)\right)$.

We believe that the point of view of the phase smoothing adopted in this section could be interesting with a view to future extensions to singular vortex patches. Let us say for instance that $\nabla_x \varphi_0$ has a jump discontinuity on the hypersurface ψ_0. Then the corresponding solutions φ^ν given by (231) admit an expansion of the form

$$\varphi^\nu(t,x) \sim \varphi^0(t,x) + \sqrt{\nu t}\varphi^1\left(t, x, \frac{\psi^\nu(t,x)}{\sqrt{\nu t}}\right) \tag{241}$$

where ψ^ν also solve the eikonal equation

$$\partial_t \psi^\nu + v^0 \cdot \nabla_x \psi^\nu = \nu \Delta_x \psi^\nu, \quad \psi^\nu|_{t=0} = \psi_0. \tag{242}$$

Let us assume that ψ_0 is smooth, so ψ^ν is well-approximated by the solution ψ^0 of the inviscid eikonal equation

$$\partial_t \psi^0 + v^0 \cdot \nabla_x \psi^0 = 0, \quad \psi^0|_{t=0} = \psi_0. \tag{243}$$

As a consequence, the expansion (241) can be simplified to

$$\varphi^\nu(t,x) \sim \varphi^0(t,x) + \sqrt{\nu t}\varphi^1\left(t, x, \frac{\psi^0(t,x)}{\sqrt{\nu t}}\right). \tag{244}$$

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Now we plug (244) into the expansion

$$\omega^\nu(t,x) \sim \Omega\left(t, x, \frac{\varphi^\nu(t,x)}{\sqrt{\nu t}}\right),$$

to get

$$\omega^\nu(t,x) \sim \tilde{\Omega}\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}, \frac{\psi^0(t,x)}{\sqrt{\nu t}}\right), \tag{245}$$

where $\tilde{\Omega}(t, x, X_1, X_2) := \Omega(t, x, X_1 + \varphi^1(t, x, X_2))$.

7.2. Tangent parallel layers do not interact

As mentioned in Remark 1.6, the proof à la Chemin of Theorem 1.1 also succeeds in covering the case where the initial vorticity ω_0 is discontinuous across two hypersurfaces $\{\varphi_0 = 0\}$ and $\{\varphi_0 = \eta\}$, where $\eta > 0$: it yields that the corresponding solution of the Euler equations has—at time t—a vorticity ω_0 that is piecewise smooth and discontinuous across the two hypersurfaces $\{\varphi^0(t,.) = 0\}$ and $\{\varphi^0(t,.) = \eta\}$, where φ^0 is again the solution of the transport equation (8), (9). For the corresponding solution of the Navier–Stokes equations two layers of width $\sqrt{\nu t}$ develop around the hypersurfaces $\{\varphi^0(t,.) = 0\}$ and $\{\varphi^0(t,.) = \eta\}$. When t proceeds there comes a time when the layers overlap. However they do not interact and the NS velocities can be described by the superposition of the two layers, that is by an expansion of the form

$$v^\nu(t,x) \sim v^0(t,x) + \sqrt{\nu t}\left(V\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right) + W\left(t, x, \frac{\varphi^0(t,x) - \eta}{\sqrt{\nu t}}\right)\right), \tag{246}$$

where the profile V is again the solution of equation (19) with the transmission conditions (25), whereas the extra profile W is the solution of equation (19), with the transmission conditions

$$[W] = 0 \quad \text{and} \quad [\partial_X W] = -\frac{n \wedge (\check{\omega}^0_+ - \check{\omega}^0_-)}{a},$$

where the brackets denote the jump discontinuity of $W(t,x,X)$ across $\{X = 0\}$, that is $[W] := W|_{X=0^+} - W|_{X=0^-}$, and $\omega^0_\pm$ are some well-chosen extensions of the restriction of the Euler vorticity to both sides $\{\pm(\varphi^0(t,.) - \eta) > 0\}$.

The point is that both profiles V and W satisfy the orthogonality condition $V \cdot n = W \cdot n = 0$, so the Burgers term

$$(V + W) \cdot n \partial_X (V + W)$$

which should induce a nonlinear coupling of the two layers identically vanishes (as the self-interaction did in § 4.3.2).

In the same vein—using this argument locally—we can infer that there is no nonlinear interaction between two vortex patches tangent at one point.

7.3. Well-prepared expansions

Let us also stress that it is also possible to construct some asymptotic expansions of the form

$$\omega^\nu(t,x) = \sum_{j\geqslant 0} \sqrt{\nu}^j\, \Omega^j\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu}}\right) + O(\sqrt{\nu}^\infty), \tag{247}$$

$$v^\nu(t,x) = v^0(t,x) + \sum_{j\geqslant 1} \sqrt{\nu}^j\, V^j\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu}}\right) + O(\sqrt{\nu}^\infty), \tag{248}$$

$$p^\nu(t,x) = p^0(t,x) + \sum_{j\geqslant 2} \sqrt{\nu}^j\, P^j\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu}}\right) + O(\sqrt{\nu}^\infty), \tag{249}$$

for both the Euler and the NS equations. Let us see how the construction is modified, at least for the first velocity profile.

For the Euler equation one gets for V^1 the profile equation

$$(D + A)V^1 = 0, \tag{250}$$

which is a transport equation along the flow of v^0 with an extra term, local, of order 0. Notably this equation involves X only as a parameter.

For the NS equations one get the profile equation

$$(D + A - a\partial_X^2)V^1 = 0 \tag{251}$$

which is hyperbolic in t, x and parabolic in t, X. In particular the hypersurface $\{t = 0\}$ is now non-characteristic.

Equations (250), (251) are therefore both well-posed when set for X in the whole real line. Let us explain an analogy which makes this sound natural. The expansions (247)–(249) could be seen as viscous and local counterparts of the asymptotic expansions of weakly nonlinear geometric optics (cf. for instance [100]). In this latter case the profiles are periodic with respect to the fast variable X and the small parameter $\sqrt{\nu}$ refers to short wavelengths. In this setting of geometric optics it is well-known—at least to experts—that a viscosity of size ν only parabolizes the profile equations corresponding to an ansatz of the form (247)–(249), without disturbing the well-posedness (on the contrary actually). We refer the reader to the papers [69, 37, 89] which illustrate this remark in some nearby settings.

Now if one looks for some solutions v^ν of the NS equations with a vortex patch v_I as initial data, one has to prescribe zero initial data for the layers, so the condition of compatibility between the transmission conditions and the initial condition on the 'corner' $\{t = X = 0\}$ are not satisfied even at order zero (cf. Remark 4.1). This destroys any hope for smoothness with respect to X, which leads to some difficulties in the analysis of the stability of the expansions. One way to get the compatibility conditions is to choose the initial condition, restricting ourselves to a kind of well-prepared initial data, assuming that the initial data is already of the form (247)–(249), that we could qualify by 'well-prepared' initial data. It is possible to prove the existence of such

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well-prepared initial data thanks to some Borel lemma. We refer the reader, for an insight into the method, to the papers [114, 113]. For such data one observes that the transmission conditions persist when time proceeds, and the lifetime of such expansions is that of the solution of the Euler equation ('the ground state') which traps the main part of the nonlinearity of the problem.

7.4. Weaker singularities

Let us mention that both kinds of expansions, (195)–(201) and (247)–(249), can be useful in the case of conormal singularities weaker than vortex patches. If for instance the vorticity is continuous through the initial internal boundary $\{\varphi_0(x) = 0\}$ but not the derivative of the vorticity, then the profiles Ω^0, V^1 and P^2 no longer depend on X. The layers appear only at the following orders (as for instance in (232)). More generally if for $k \in \mathbb{N}$ the vorticity is C^k through the initial internal boundary $\{\varphi_0(x) = 0\}$, it is possible to write a complete asymptotic expansion of the vorticity of the form

$$\omega^\nu(t,x) = \omega^0(t,x) + \sum_{j=1}^{k-1} (\nu t)^j \, \underline{\Omega}^j(t,x) + \sum_{j \geqslant k} \sqrt{\nu t}^j \, \Omega^j \left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}} \right) + O(\sqrt{\nu t}^\infty).$$

Such an observation was mentioned in the setting of the approximation of semi-linear symmetric hyperbolic systems of PDEs by the vanishing viscosity method in [113].

7.5. Stronger singularities

Let us now talk about singularities stronger than vortex patches, for which layers of larger amplitude are expected. Let us first deal with vortex sheets which involve a jump discontinuity of the velocity (instead of the vorticity). An initial velocity that is piecewise smooth with a jump discontinuity across an hypersurface $\{\varphi_0 = 0\}$ is an extremely unstable configuration for the Euler equations: in general the jump discontinuities of the velocity do not stay localized on a smooth hypersurface $\{\varphi^0(t,.) = 0\}$ when time proceeds. A few positive results are available with analytic initial data: local-in-time persistence was proved by Bardos et al. in [116] (see also an extension to global persistence for small analytic perturbation by Caflisch and Orellana in [21] and by Duchon and Robert in [54]), but the papers [22, 93, 121, 66] destroy any hope of extending out from the unphysical case of analytic data. Still a few physical phenomena are known to yield some stability, such as compressibility (see [42] for the two-dimensional supersonic case) and surface tension (see [84, 8]).

If for one of the previous reasons the velocity v^0 given by the Euler equation (or an appropriately modified inviscid system) stays piecewise smooth with discontinuity jumps only across a hypersurface $\{\varphi^0(t,.) = 0\}$, we expect the corresponding velocities given by the Navier–Stokes equation (or a modified viscous system) to be given by an expansion of the form

$$v^\nu(t,x) \sim v^0(t,x) + V\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}} \right),$$

since the perturbation term V should have to compensate the jumps of the inviscid solution v^0 through the transmission conditions $[V] = -[v^0]$ where the first brackets

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denote the jump discontinuity of V across $\{X = 0\}$ and the second ones the jump of v^0 across $\{\varphi^0(t, .) = 0\}$. In addition we have to prescribe $[\partial_X V] = 0$. Actually because of some nonlinear effects, we have to consider also the next term in the expansion:

$$v^\nu(t,x) \sim v^0(t,x) + V^0\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right) + \sqrt{\nu t}\, V^1\left(t, x, \frac{\varphi^0(t,x)}{\sqrt{\nu t}}\right) \tag{252}$$

to close the profile equations. In fact, plugging this ansatz into the NS equations and setting—with the notation of the introduction—the equations equal according to the orders of $\sqrt{\nu t}$ yields

$$V^0 \cdot n = 0, \quad \left(L + \frac{1}{2}\right) V^0 = t\{(V^0) \cdot \nabla_x (v^0 + V^0) + (V^1 \cdot n)\partial_X V^0\} \quad \text{and}$$
$$\partial_X V^1 \cdot n = -\text{div}_x V^0. \tag{253}$$

This equation presents the same difficulty as the Prandtl equations: to get rid of V^1 in the second equation we use the third equation, which leads to a loss of a derivative. It is likely that here again some positive results are possible in the case of analytic data. Indeed Caflisch and Sammartino in [20] have succeeded in proving well-posedness for analytic data in the well-prepared counterpart of equations (253) in two dimensions when the radius of curvature of the curve is much larger than the thickness of the layer.

A more radical solution for avoiding this loss of a derivative is to consider some cases without variation in the transverse directions. For instance we can consider the following very special case: we set the phase $\varphi^0(t, x) \equiv x_1$ and an inviscid velocity v^0 of the form $v^0(t, x) \equiv f(x_1)e_2$ where the function f is C_c^∞ on $\mathbb{R}_\pm$. The solutions of the Navier–Stokes equations are then simply

$$v^\nu(t,x) = v^0(t,x) + V\left(\frac{x_1}{\sqrt{\nu t}}\right) e_2,$$

where V is the solution of the elliptic equation $\partial_X^2 V + \frac{X}{2}\partial_X V = 0$ with the following transmission conditions on $\{X = 0\}$: $[V] = -[f]$ and $[\partial_X V] = 0$, where $[f]$ denotes the jump discontinuity of f across $\{x_1 = 0\}$.

If now we strengthen the amplitude again, looking for expansions of the form

$$v^\nu(t,x) = (\nu t)^{-\alpha}\, V\left(t, \frac{x_1}{\sqrt{\nu t}}\right) e_1,$$

with $\alpha > 0$, we get for V the equation

$$\partial_X^2 V + \frac{X}{2}\partial_X V + \alpha V = 0,$$

which is still coercive in $H^1(\mathbb{R})$ for $\alpha < \frac{1}{4}$, but admit 0 for an eigenvalue for $\alpha = \frac{k}{2}$, with $k \in \mathbb{N}^*$. The corresponding eigenfunctions are the Hermite functions $H_k := \partial_X^{k-1} H_1$ with $H_1 := e^{-\frac{X^2}{4}}$. Hence when strengthening the amplitude of the transition layer (that is when increasing α) there are still some non-trivial solutions of the profile equation, but the nature of the profile problem totally changes, from an elliptic problem with non-homogeneous condition transmissions to an eigenvalue problem.

In this latter case it is then possible to consider also some viscous perturbations singular with respect to several dimensions. For instance it is possible to exhibit some family of solutions of the two-dimensional NS equations with velocities of the form

$$v^\nu(t,x) = \frac{1}{\sqrt{\nu t}} V\left(t, \frac{x}{\sqrt{\nu t}}\right), \tag{254}$$

where the profile $V(t,X)$ vanishes when the fast variable $X := (X_1, X_2)$ goes to infinity. The corresponding vorticities are of the form

$$\omega^\nu(t,x) = \frac{1}{\nu t} \Omega\left(t, \frac{x}{\sqrt{\nu t}}\right) \quad \text{with } \Omega := \text{curl}_X V. \tag{255}$$

Plugging these ansätze into the NS equations, whose vorticity formulation reads as follows in two dimensions:

$$\partial_t \omega^\nu + v^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu, \tag{256}$$

and setting the terms of order $\frac{1}{(\nu t)^2}$ and those of order $\frac{1}{\nu t}$ equal, we get the following pair of equations:

$$V \cdot \nabla_X \Omega = 0 \quad \text{and} \quad \left(\Delta_X + \frac{1}{2} X \cdot \nabla_X + 1\right) \Omega = t \partial_t \Omega, \tag{257}$$

where we define $\Delta_X := \partial^2_{X_1} + \partial^2_{X_2}$ and $\nabla_X := (\partial_{X_1}, \partial_{X_2})$. We then see that the first equation in (257) is satisfied if the vorticity profile Ω is radially symmetric since in this case the corresponding velocity profile V is orthoradial. The initial hypersurface $\{t = 0\}$ is characteristic for the second equation in (257) which therefore has parasite solutions. Actually, omitting the X dependence we get the ODE $\Omega = t\partial_t \Omega$ whose solutions are $\Omega = C \ln t$. However only one, namely $\Omega \equiv 0$, has a correct behavior for t near 0. Now setting $t = 0$ in the second equation we get the equation

$$\left(\Delta_X + \frac{1}{2} X \cdot \nabla_X + 1\right) \Omega = 0$$

whose solution is given by $\Omega(X) = Ce^{-X^2}$; here C is determined by the conservation of the vorticity mass. The ansätze (254) describe the viscous smoothing of an initial Dirac mass at $x = 0$ and are usually referred to as the Oseen vortex. We refer the reader here to, for instance, the papers [59, 58], and to the references therein, for a much more precise study. We do not claim any novelty in this section but we think that it was thought-provoking to incorporate a little bit of this material here. In particular we found it interesting to gather the profile equation (257) and the profile equation (19) corresponding to the vortex patches, and to observe in particular the shift of the spectrum caused by the difference of amplitude, which made the analysis quite different.

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