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On Primitive Ideals in Graded Rings

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Abstract. Let $R = \bigoplus_{i=1}^{\infty} R_i$ be a graded nil ring. It is shown that primitive ideals in R are homogeneous. Let $A = \bigoplus_{i=1}^{\infty} A_i$ be a graded non-PI just-infinite dimensional algebra and let I be a prime ideal in A. It is shown that either $I = \{0\}$ or $I = A$. Moreover, A is either primitive or Jacobson radical.

Let $R = \bigoplus_{i=1}^{\infty} R_i$ be a ring graded by the positive integers. Then R is *graded nil* if all homogeneous elements in R are nil. It is well known that graded Jacobson radical rings are graded nil, but need not be nil. For example, as shown by Bartholdi [1], an affine "recurrent transitive" algebra without unit constructed from Grigorchuk's group of intermediate growth is graded nil and Jacobson radical but not nil, provided that the base field is an algebraic extension of F_2. Other examples of Jacobson radical and graded nil, but not nil rings, are polynomial rings over certain nil rings [3,10]. In this paper primitive ideals in graded rings are studied. All ideals are two-sided unless specified. The main results of this paper are the following.

Theorem 1 *Let $R = \bigoplus_{i=1}^{\infty} R_i$ be a graded nil ring and let I be a primitive ideal in R. Then I is homogeneous.*

As a corollary the following theorem may be stated.

Theorem 2 *Let $R = \bigoplus_{i=1}^{\infty} R_i$ be a graded nil ring and let I be an ideal of R. Suppose that $\bar{I}$ is the smallest homogeneous ideal of R containing I. Then R/I is Jacobson radical if and only if $R/\bar{I}$ is Jacobson radical.*

Let K be a field. An infinite dimensional K-algebra R is *just-infinite dimensional* if R/I is finite dimensional for every nonzero two-sided ideal I of R. A result of Reichstein, Rogalski and Zhang [7] states that just-infinite dimensional graded algebras are affine. Farkas and Small investigated just-infinite dimensional algebras over uncountable fields [4]. They showed that if K is an uncountable field and R is an affine, semiprimitive, just-infinite dimensional algebra, then either R is primitive or R satisfies a polynomial identity. Similar results concerning graded algebras over arbitrary fields can be derived.

Theorem 3 *Let K be a field, and let $R = \bigoplus_{i=1}^{\infty} R_i$ be a just-infinite dimensional graded K-algebra with Gelfand–Kirillov dimension greater than* 1. *Then R is either primitive or Jacobson radical. Moreover, if I is a prime ideal in R, then either $I = 0$ or $I = R$.*

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For elementary properties of Gelfand–Kirillov dimension we refer to [5]. The results of Small, Stafford and Warfield show that if R is a finitely generated semiprime algebra of GK dimension 1, then the center of R is a Noetherian domain of GK dimension 1 [8]. It is worth noticing that the center of a graded algebra is graded. Therefore a semiprime graded nil algebra of Gelfand–Kirillov dimension not exceeding 1 is locally nilpotent.

Theorem 4 *Let K be a field, and let $R = \bigoplus_{i=1}^{\infty} R_i$ be a just-infinite dimensional graded nil K-algebra. Then R is either primitive or Jacobson radical.*

The definition of a primitive ring becomes more complicated than in the usual case, because rings without unity are under consideration. The following definition will be used. This definition can be found in [11] (see also [2,6]).

Let R be a ring and let $b \in R$. We define a right ideal Q of R to be *b-modular* if $a - ba \in Q$ for all $a \in R$.

A right ideal Q of a ring R is *modular* if it is b-modular for some $b \in R$. If Q is a modular maximal right ideal of R, then for every $r \notin Q$ we have $rR + Q = R$ [11].

An ideal I of a ring R is *right primitive* in R if there exists a modular maximal right ideal Q of R such that I is the largest ideal contained in Q.

For each element $g \in R$, the degree of g, denoted by $\deg(g)$, is the minimal number d such that $g \in R_1 + R_2 + \cdots + R_d$. Given $a \in R$, let $\langle a \rangle$ denote the ideal generated by a in R. The Jacobson radical of R is denoted by $J(R)$.

We write $I \triangleleft R$ if I is a two-sided ideal of a ring R. Given a graded ring $R = \bigoplus_{i=1}^{\infty} R_i$ and $r \in R$, we say that r is *homogeneous* if $r \in R_i$ for some i. We say that an ideal I of R is a *homogeneous ideal* if I is generated by homogeneous elements.

Lemma 5 *Let $R = \bigoplus_{i=1}^{\infty} R_i$ be a ring graded by the positive integers. Let Q be a b-modular right ideal of R for some $b \in R$. Let $f, r \in R$ be such that $b - rf \in Q$. If r is homogeneous, then for every natural number i, there is an $f_i \in R$ such that $b - rf_i \in Q$ and $rf_i \in R_i + R_{i+1} + \cdots + R_{i+\deg(rf)-1}$.*

Proof We will proceed by induction on i. If $i = 1$, we put $f_1 = f$. Suppose that the result holds for some $i \geq 1$ and that $f_i = a_i + a_{i+1} + \cdots + a_{i+\deg(rf)-1}$, with $a_j \in R_{j-\deg(r)}$ for each j. Observe that $ra_i - bra_i \in Q$. Since $b - rf \in Q$, we get $ra_i - rfra_i \in Q$. Then the result holds for

$$f_{i+1} = fra_i + \sum_{j=i+1}^{i+\deg(rf)-1} a_j.$$

Indeed, $rfra_i \in R_{1+i} + \cdots + R_{\deg(rf)+i}$, hence $rf_{i+1} \in R_{i+1} + \cdots + R_{i+\deg(rf)}$, as required. ∎

Lemma 6 *Let $R = \bigoplus_{i=1}^{\infty} R_i$ be a ring graded by the positive integers. Let $I \triangleleft R$ and let $a_1 + \cdots + a_k \in I$, with $a_k \notin I$ and $a_i \in R_i$, for $1 \leq i \leq k$. Set $U = \langle a_k \rangle$. Suppose that $h \in U^l$ for some $l \geq 1$ and $h \in R_m + R_{m+1} + \cdots + R_{m+t}$, for some m, t and $t \geq k$.*

Then there exists $g \in U^{l-1}$ such that $h - g \in I$ and $g \in R_m + R_{m+1} + \cdots + R_{m+t-1}$.

Proof Let $h = \sum_{i=0}^{t} c_{m+i}$, with $t \geq k$ and $c_{m+i} \in R_{m+i}$. Note that $c_{m+t} = \sum_i p_i a_k q_i$ for some homogeneous $p_i, q_i \in R \cup \{1\}$ and $q_i \in U^{l-1}$, since $h \in U^l$. Set

$$g = h - c_{m+t} - \sum_i p_i \Big(\sum_{j=1}^{k-1} a_j \Big) q_i.$$

Observe that $h - g = \sum_i p_i (\sum_{j=1}^{k} a_j) q_i \in I$. Notice that,

$$\deg(p_i a_j q_i) \geq \deg(p_i a_k q_i) - (k-1) \geq \deg(c_{m+t}) - (k-1) = m + t - k + 1,$$

because $\deg(a_j) \geq \deg(a_k) - (k-1)$. Now $t \geq k$ yields $\deg(p_i a_j q_i) \geq m + 1$. On the other hand, $\deg(p_i a_j q_i) < \deg(p_i a_k q_i) = m + t$ for $j < k$. Hence, $g \in R_m + R_{m+1} + \cdots + R_{m+t-1}$, as required. ■

Lemma 7 *Let $R = \bigoplus_{i=1}^{\infty} R_i$ be a ring graded by the positive integers. Let $I \triangleleft R$ and let $a_1 + \cdots + a_k \in I$, with $a_k \notin I$ and $a_i \in R_i$, for $1 \leq i \leq k$. Set $U = \langle a_k \rangle$. Suppose that Q is a b-modular maximal right ideal in R for some $b \in R$. Let $r, f, g \in R$ be such that r is homogeneous, $g \in U^{\deg(rf)}$, and $b - rf \in Q$ and $b - g \in Q$.*

If $I \subseteq Q$, then for every $n > \deg(g)$ there is $g_n \in R$ such that $g_n \in R_n + R_{n+1} + \cdots + R_{n+k}$ and $b - rg_n \in Q$.

Proof Let f_i be as in Lemma 5 applied for this choice of f and r. Then $b - rf_i \in Q$ and $rf_i \in R_i + R_{i+1} + \cdots + R_{i+\deg(rf)-1}$. Let $g = \sum_{i=1}^{t} c_i$ with $c_i \in R_i$ and $c_i \in U^{\deg(rf)}$, where $t = \deg(g)$. For a natural number $n > \deg(g) = t$, set $h_n = \sum_{i=1}^{t} f_{n-i} c_i$. Observe that $h_n \in U^{\deg(rf)}$ and $b - rh_n \in Q$. Notice that $rh_n \in R_n + \cdots + R_{n+\deg(rf)-1}$, because $rf_{n-i} \in R_{n-i} + \cdots + R_{n-i+\deg(rf)-1}$. By applying Lemma 6 several times for $h = h_n$, we see that for each $n > t$ there is $g_n \in R$ such that $h_n - g_n \in I$ and $rg_n \in R_n + R_{n+1} + \cdots + R_{n+k}$, because $rh_n \in R_n + \cdots + R_{n+\deg(rf)-1}$. Now, $h_n - g_n \in I$ implies that $rh_n - rg_n \in Q$, and so $b - rg_n \in Q$, as required. ■

Let R be a ring, and let $b \in R$. Let Q be a b-modular right ideal in R. Let r be a homogeneous element in R such that $r \notin Q$. We say that v is an *inverse span for r* if for all sufficiently large n, there are $f_n \in rR$ such that $f_n \in R_n + R_{n+1} + \cdots + R_{n+v}$ and $b - f_n \in Q$.

Lemma 8 *Let $R = \bigoplus_{i=1}^{\infty} R_i$ be a ring graded by the positive integers, and let Q be a b-modular maximal right ideal in R for some $b \in R$. Let $I \triangleleft R$ be the largest ideal contained in Q and let $a_1 + \cdots + a_k \in I$, with $a_k \notin I$ and $a_i \in R_i$, for $1 \leq i \leq k$.*

Then k is an inverse span for all homogeneous $r \notin Q$.

Proof First observe that $rR + Q = R$ for every homogeneous $r \notin Q$, so there is $f \in R$ such that $b - rf \in Q$. Set $U = \langle a_k \rangle$. The ideal I is prime, since it is primitive. Consequently $U^{\deg(rf)} \not\subseteq I$, and so $U^{\deg(rf)} \not\subseteq Q$, since I is the largest two-sided ideal in Q. Hence $b - g \in Q$ for some $g \in U^{\deg(rf)}$. By Lemma 7, for every $n > \deg(g)$ there is an element $g_n \in R_n + R_{n+1} + \cdots + R_{n+k}$ such that $b - rg_n \in Q$. Consequently, k is an inverse span all homogeneous $r \notin Q$. ■

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Lemma 9 *Let $R = \bigoplus_{i=1}^{\infty} R_i$ be a ring graded by the positive integers. Let Q be a b-modular maximal right ideal in R for some $b \in R$. Let $r \in R$ be homogeneous with $r \notin Q$ and let v be an inverse span for all homogeneous elements $a \in rR$ such that $a \notin Q$. Suppose that there are homogeneous elements $p, p' \in rR$ such that $p, p' \notin Q$ and such that, for every natural number i, if $c - d \in Q$ with $c \in R_i \cap pR$ and $d \in R_i \cap p'R$, then $c \in Q$.*

Then $v - 1$ is an inverse span for r.

Proof Since v is an inverse span for p and p', for sufficiently large n, there are $g_n \in pR$ and $g'_n \in p'R$ with $g_n, g_{n'} \in R_n + R_{n+1} + \cdots + R_{n+v}$, and such that $b - g_n \in Q$ and $b - g'_n \in Q$. Let $g_n = p_{n,0} + p_{n,1} + \cdots + p_{n,v}$, $g'_n = p'_{n,0} + p'_{n,1} + \cdots + p'_{n,v}$, where $p_{n,i} \in pR \cap R_{n+i}$ and $p'_{n,i} \in p'R \cap R_{n+i}$ for all n, i. If for all sufficiently large n either $p_{n,v} \in Q$ or $p'_{n,v} \in Q$, then $v - 1$ is an inverse span for r, because $p, p' \in rR$. Suppose that there is m such that $p_{m,v} \notin Q$ and $p'_{m,v} \notin Q$. Now $p_{m,v} \in pR \cap R_{m+v}$ and $p'_{m,v} \in p'R \cap R_{m+v}$ yield $p_{m,v} - p'_{m,v} \notin Q$ since otherwise $p_{m,v} \in Q$, by the assumptions. For sufficiently large n there are $h_n \in cR$ such that $b - h_n \in Q$, and $h_n \in R_n + R_{n+1} + \cdots + R_{n+v}$, since v is an inverse span for $c = p_{m,v} - p'_{m,v}$. Now $h_n = \bar{g}_n + cr_n$ for some $cr_n \in R_{n+v}$ and $\bar{g}_n \in R_n + R_{n+1} + \cdots + R_{n+v-1}$. Thus (for sufficiently large n) $b - k_n \in Q$, where

$$k_n = h_n + (g'_m - g_m)r_n = \bar{g}_n + \sum_{i=0}^{v-1}(p'_{m,i} - p_{m,i})r_n.$$

Note that $k_n \in rR$, since $p, p' \in rR$, $\bar{g} \in cR$. Since $cr_n \in R_{n+v}$ and $c = p_{m,v} - p'_{m,v} \in R_{m+v}$, we get $r_n \in R_{n-m}$. Hence, for $i \leq v - 1$, we have $(p_{m,i} - p'_{m,i})r_n \in R_{n+i}$. Thus, $k_n \in R_n + R_{n+1} + \cdots + R_{n+v-1}$. Hence, $v - 1$ is an inverse span for r. ■

Lemma 10 *Let $R = \bigoplus_{i=1}^{\infty} R_i$ be a ring graded by the positive integers. Let Q be a b-modular maximal right ideal in R for some $b \in R$. Let $r \in R$ be homogeneous with $r \notin Q$. Suppose that v is an inverse span for all homogeneous elements $a \in rR$ such that $a \notin Q$, but $v - 1$ is not an inverse span for r. Let n be a natural number and let $e_0, e_1, \dots, e_n \in rR$ be homogeneous and such that $e_0, e_1, \dots, e_n \notin Q$.*

Then there is a natural number t and homogeneous elements $d_0, d_1, \dots, d_n \in R$ such that $e_0d_0 \notin Q$, $e_jd_j - e_0d_0 \in Q$ for all $j \leq n$, and $e_jd_j \in R_t$ for all $j \leq n$.

Proof We will proceed by induction on n. If $n = 1$ the result is true by Lemma 9. Suppose the result holds for some $n \geq 1$. By Lemma 9, applied to $p = e_n$, $p' = e_{n+1}$, there is a natural number m and homogeneous elements v, u, c, d in R such that $c = e_nv \in R_m$, $d = e_{n+1}u \in R_m$, $c - d \in Q$ and $c \notin Q$. By the inductive hypothesis applied to the elements $e'_0 = e_0, e'_1 = e_1, \dots, e'_{n-1} = e_{n-1}$ and $e'_n = c$, we get that there are $d'_0, d'_1, \dots, d'_n$ and a natural number t, such that $e'_0d'_0 \notin Q$, $e'_jd'_j - e'_0d'_0 \in Q$ and $e'_jd'_j \in R_t$, for all $j \leq n$. Now set $d_i = d'_i$ for $0 \leq i \leq n - 1$, $d_n = vd'_n$, $d_{n+1} = ud'_n$. Observe that $e_id_i = e'_id'_i$, and so $e_id_i \in R_t$, for all $0 \leq i < n$. Notice that $e_nd_n = e_nvd'_n = cd'_n = e'_nd'_n$. Hence $e_nd_n \in R_t$ and $e_nd_n - e_0d_0 = e'_nd'_n - e'_0d'_0 \in Q$. Observe that $e_{n+1}d_{n+1} = e_{n+1}ud'_n = dd'_n = (c + (d - c))d'_n = e'_nd'_n + (d - c)d'_n$. Now $c - d \in Q$ gives $e_{n+1}d_{n+1} - e'_nd'_n \in Q$, and so $e_{n+1}d_{n+1} - e_0d_0 \in Q$. Since c and d are homogeneous elements of the same degree and $e'_nd'_n \in R_t$, we get that $e_{n+1}d_{n+1} \in R_t$.

Therefore $e_0d_0 \notin Q$ and $e_jd_j - e_0d_0 \in Q$, $e_jd_j \in R_t$ for all $j \leq n+1$. This is the desired conclusion. ■

Lemma 11 *Let $R = \bigoplus_{i=1}^{\infty} R_i$ be a ring graded by the positive integers, and let Q be a b-modular maximal right ideal in R for some $b \in R$. Let $I \triangleleft R$ be the largest possible ideal contained in Q and let $a_1 + \cdots + a_k \in I$, with $a_k \notin I$ and $a_i \in R_i$, for $1 \leq i \leq k$. Suppose that for every homogeneous $p \notin Q$ there is a homogeneous element $p' \notin Q$ and $p' \in pR$ such that $pp' \in Q$.*

Then 0 is an inverse span for all homogeneous $r \in R$ such that $r \notin Q$.

Proof Let ν be minimal such that ν is an inverse span for all homogeneous $a \in R$ such that $a \notin Q$. We know that such ν exists by Lemma 4. Suppose to the contrary that $\nu > 0$. Then there is a homogeneous element $r \notin Q$ such that $\nu - 1 \geq 0$ is not an inverse span for r. For some α there are elements $f_\alpha, f_{\alpha+1}, f_{\alpha+2}, \ldots, f_{\alpha+\nu} \in rR$ such that $b - f_{\alpha+j} \in Q$ and $f_{\alpha+j} \in R_{\alpha+j} + R_{\alpha+j+1} + \cdots + R_{\alpha+j+\nu}$ for $0 \leq j \leq \nu$ (since ν is an inverse span for r). Now each element $f_{\alpha+j}$ for $0 \leq j \leq \nu$ can be written as $f_{\alpha+j} = g_j + c_j$ where $c_j \in R_{\alpha+j+\nu}$, and $g_j \in R_{\alpha+j} + \cdots + R_{\alpha+j+\nu-1}$. Fix a homogeneous element $w \in rR$ such that $w \notin Q$. If, for some $0 \leq j \leq \nu$, we have $c_j \notin Q$, then by the assumptions, there is homogeneous $e_j \notin Q$ such that $c_je_j \in Q$. If $c_j \in Q$, we put $e_j = w$. Observe that $c_je_j \in Q$, $e_j \in rR$, and $e_j \notin Q$ for all $0 \leq j \leq \nu$. By Lemma10, there is a natural number p and homogeneous elements $d_j \in R$ such that $e_0d_0 \notin Q$ and $e_jd_j \in R_p$ and $e_jd_j - e_0d_0 \in Q$ for all $0 \leq j \leq \nu$. Let $s = e_0d_0$ with $\deg(s) = p$. Since ν is an inverse span for s, we see that for sufficiently large n there are $\bar{f}_n \in sR$, $\bar{f}_n \in R_n + \cdots + R_{n+\nu}$, such that $b - \bar{f}_n \in Q$. Let $\bar{f}_n = \sum_{i=0}^{\nu} sb_i$, where $sb_i \in R_{n+i}$ for all $0 \leq i \leq \nu$. Now $b - f_{\alpha+j} \in Q$ implies $(b - f_{\alpha+j})e_jd_j \in Q$ for all $0 \leq j \leq \nu$. Notice that $s - e_jd_j \in Q$ gives $s - f_{\alpha+j}e_jd_j \in Q$ for all $0 \leq j \leq \nu$. Thus $\bar{f}_n - \bar{g}_n \in Q$ where $\bar{g}_n = \sum_{j=0}^{\nu} f_{\alpha+\nu-j}e_{\nu-j}d_{\nu-j}b_j$. Notice that $\bar{f}_n - \bar{g}_n \in Q$ implies $b - \bar{g}_n \in Q$. Observe that there are $t_l \in rR$ with $t_l \in R_{\alpha+l+\deg(e_l)} + R_{\alpha+l+\deg(e_l)+1} + \cdots + R_{\alpha+l+\deg(e(l))+\nu-1}$, such that $t_l - f_{\alpha+l}e_l \in Q$ for $0 \leq l \leq \nu$. Indeed, we can put $t_l = g_le_l$, because $f_{\alpha+l} = g_l + c_l$ and $c_le_l \in Q$, by the definition of e_l. Set $h_n = \sum_{j=0}^{\nu} t_{\nu-j}d_{\nu-j}b_j$. Now $b - \bar{g}_n \in Q$ and $t_l - f_{\alpha+l}e_l \in Q$; so $b - h_n \in Q$ for all $0 \leq l \leq \nu$. Observe that $h_n \in rR$, since $f_{\alpha+l} \in rR$ for all l, and so $t_l \in rR$ for all l. Note that for $0 \leq j \leq \nu$, we have $t_{\nu-j}d_{\nu-j}b_j \in R_c + \cdots + R_{c+\nu-1}$ where $c = (\alpha+\nu-j+\deg(e_{\nu-j})) + \deg(d_{\nu-j}) + \deg(b_j)$. Now $\deg(e_{\nu-j}) + \deg(d_{\nu-j}) = \deg(s)$ for all $0 \leq j \leq \nu$, and so

$$c = \alpha + \nu - j + \deg(s) + \deg(b_j) = \alpha + \nu - j + n + j,$$

since $sb_j \in R_{n+j}$. Therefore $c = \alpha + \nu + n$. Consequently,

$$h_n \in R_{\alpha+\nu+n} + \cdots + R_{\alpha+\nu+n+\nu-1}.$$

Note that $b - h_n \in Q$, and we can find such h_n for all sufficiently large n. Set $f'_n = h_{n-\alpha-\nu}$. Observe that $f'_n \in rR$, $b - f'_n \in Q$ and $f'_n \in R_n + \cdots + R_{n+\nu-1}$ for all sufficiently large n. It follows that $\nu - 1$ is an inverse span for r. ■

Proof of Theorem 1 Suppose to the contrary, that there are elements

$$a_1 \in R_1, \ldots, a_k \in R_k$$

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with $a_k \notin I$ such that $a_1 + \cdots + a_k \in I$. From the definition of a primitive ideal there is $b \in R$ and a b-modular maximal right ideal Q in R such that I is the largest ideal contained in Q. Let $r \notin Q$ be homogeneous. Let n be maximal such that $r^n \notin Q$. Since R is graded nil such an n exists. Set $r' = r^n$. Then $r' \notin Q$ and $rr' \in Q$, hence the assumptions of Lemma 11 hold. Therefore 0 is an inverse span for every homogeneous $a \notin Q$. In particular $b - rf \in Q$ for some homogeneous $f \in R$. Some power of rf is zero, since R is graded nil, and so $b \in Q$. Therefore, $R \subseteq Q$ since Q is b-modular, a contradiction. ■

Proof of Theorem 2 Suppose to the contrary, that $R/\bar{I}$ is Jacobson radical and R/I is not Jacobson radical. The Jacobson radical of R is the intersection of all primitive ideals in R [2, 11]. Hence, there is a primitive ideal P of R/I such that $P \neq R/I$. Let $C \neq R$ be the ideal such that $I \subset C$ and $P = C/I$. Observe that R/C is isomorphic to $(R/I)/(C/I) = (R/I)/P$. Hence, R/C is primitive. Therefore, C is a primitive ideal in R and $C \neq R$. Thus C is homogeneous by Theorem 1. Observe that $\bar{I} \subseteq C$, since $\bar{I}$ is the minimal homogeneous ideal containing I. By assumption $R/\bar{I}$ is Jacobson radical; so R/C is Jacobson radical, since $\bar{I} \subseteq C$. However, R/C is primitive, a contradiction. The other implication is clear. ■

Proof of Theorem 3 The Jacobson radical of a ring R is the intersection of all primitive ideals in R. Suppose that R is not Jacobson radical, so R has a primitive ideal I such that $I \neq R$. Then I is a prime ideal in R, and so R/I is prime. Assume that $I \neq \{0\}$. There is a natural number n such that R/I is n-dimensional vector space over K, since R is just-infinite dimensional. There is an integer i such that $\dim(R_i) > n$, since the Gelfand–Kirillov dimension of R is greater than 1. Thus $I \cap R_i \neq 0$, and so I contains a homogeneous element of degree i, say v. Let T be the ideal generated by v in R. Then R/T is finite dimensional and graded and hence nilpotent. It follows that R/I is nilpotent, since $T \subseteq I$, and this is impossible since R/I is prime. Therefore $I = \{0\}$, so R is primitive.

If R is a Jacobson radical ring and $I \neq 0$ is a prime ideal in R, then R/I is Jacobson radical and finite-dimensional vector space over K. Consequently, R/I is nilpotent. It follows that $I = R$, since R/I is prime. ■

Proof of Theorem 4 By Theorem 3, it suffices to consider the case when R has Gelfand–Kirillov dimension not exceeding 1. Then, as noted just before the statement of Theorem 4, R is locally nilpotent, and hence Jacobson radical. This finishes the proof. ■

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