HENSELIAN VALUED FIELDS AND inp-MINIMALITY

ARTEM CHERNIKOV AND PIERRE SIMON

Abstract. We prove that every ultraproduct of p-adics is inp-minimal (i.e., of burden 1). More generally, we prove an Ax-Kochen type result on preservation of inp-minimality for Henselian valued fields of equicharacteristic 0 in the RV language.

§**1. Introduction.** In his work on the classification of first-order theories [16] Shelah has introduced a hierarchy of combinatorial properties of families of definable sets, so-called dividing lines, which includes stable theories, simple theories, NIP, and NSOP. An important line of research in model theory is to characterize various algebraic structures depending on their place in this classification hierarchy (this knowledge can later be used to analyze various algebraic objects definable in such structures using methods of generalized stability theory). Here we will be concerned with valued fields and Ax-Kochen-type statements, i.e., statements of the form "a certain property of the valued field can be determined by looking just at the value group and the residue field." For example, a classical theorem of Delon [8] shows that given a Henselian valued field of equicharacteristic 0, if the residue field is NIP, then the whole valued field is NIP. More recent results of similar type are [3] demonstrating preservation of NIP for certain valued fields of positive characteristic, [17] demonstrating that the field of p-adics is strongly dependent, and [10] demonstrating that it is in Fact dp-minimal.

A motivating example for this article is to determine the model-theoretic complexity of the theory of an ultraproduct of the fields of p-adics $\mathbb{Q}_p$ modulo a nonprincipal ultrafilter on the set of prime numbers. Namely, let $K = \prod \mathbb{Q}_p / \mathcal{U}$, where $\mathcal{U}$ is a nonprincipal ultrafilter on the set of prime numbers. Note that the residue field k is a pseudo-finite field of characteristic 0 and that the value group Γ is a $\mathbb{Z}$-group. Besides, both k and Γ are interpretable in K in the pure ring language (e.g., by a result of Ax [2]). This implies that the theory of K is neither NIP, nor simple—the two classes of structures extensively studied in model theory. However it turns out that any ultraproduct of p-adics is NTP_2 [5]. The class of NTP_2 theories was introduced by Shelah [16, Chapter III] and generalizes both simple and NIP theories. We recall the definition.

DEFINITION 1.1. Let T be a complete first-order theory in a language L, and let $\mathbb{M} \models T$ be a monster model. Let κ be a cardinal (finite or infinite).

1. An inp-*pattern* of depth κ is given by $(\phi_i(x, y_i), \bar{a}_i, k_i : i \in \kappa)$, where $\phi_i(x, y_i)$ are L-formulas with a fixed tuple of free variables x and a varying tuple of parameter variables y_i, $\bar{a}_i = (a_{i,j} : j \in \omega)$ are sequences of tuples of elements from $\mathbb{M}$, and k_i are natural numbers such that:

   (a) For every $i \in \kappa$, the set $\{\phi_i(x, a_{i,j})\}_{j \in \omega}$ is k_i-inconsistent (i.e., no subset of size $\geq k_i$ is consistent).

   (b) For every $f : \kappa \to \omega$, the set $\{\phi_i(x, a_{i,f(i)})\}_{i \in \kappa}$ is consistent.

2. T is NTP_2 if there is a (cardinal) bound on the depths of inp-patterns.

Other algebraic examples of NTP_2 structures were identified recently, including bounded pseudo real closed and pseudo p-adically closed fields [15], certain model complete multivalued fields [13] and certain valued difference fields, e.g., the theory VFA_0 of a nonstandard Frobenius on an algebraically closed valued field of characteristic zero [6]. See also [7] and [12] for some general results about groups and fields definable in NTP_2 structures.

The notion of *burden* was introduced by Adler [1] based on Shelah's cardinal invariant κ_{inp} and provides a quantitative refinement of NTP_2. In the special case of simple theories burden corresponds to preweight, and in the case of NIP theories to dp-rank (e.g., see [5, Section 3] for the details and references).

DEFINITION 1.2.   1. T is *strong* if there are no inp-patterns of infinite depth.

2. T is *of finite burden* if there are no inp-patterns of arbitrary large finite depth, with x a singleton.

3. T is inp-*minimal* if there is no inp-pattern of depth 2, with x a singleton.

Note that inp-minimality implies finite burden implies strong (the last implication uses submultiplicativity of burden from [5]). All the examples mentioned above have been demonstrated to be strong of finite burden, with the exception of VFA_0: it remains open if VFA_0 is strong, see [6, Question 5.2]. Some results about strong groups and fields can be found in [7, Section 4] and [9].

Returning to ultraproducts of p-adics, we have the following more general result.

FACT 1.3 ([5]). *Let $\bar{K} = (K, k, \Gamma, \mathrm{val}, \mathrm{ac})$ be a Henselian valued field of equicharacteristic* 0, *considered as a three-sorted structure in the Denef-Pas language L_{ac}* (*i.e., there is a sort K for the field itself, as well as sorts k for the residue field and Γ for the value group, together with the maps $v : K \to \Gamma$ for the valuation and $\mathrm{ac} : K \to k$ for an angular component*).

1. *If k is NTP_2, then $\bar{K}$ is NTP_2.*

2. *If both k and Γ are strong* (*of finite burden*) *then $\bar{K}$ is strong* (*respectively, of finite burden*).

Any pseudofinite field is supersimple of SU-rank 1, so in particular is inp-minimal. Any ordered $\mathbb{Z}$-group is dp-minimal, so in particular is inp-minimal. It follows that any ultraproduct of p-adics is strong, of finite burden. However, Fact 1.3(2) gives a finite bound on the burden of $\bar{K}$ in terms of the burdens of k and Γ via a certain Ramsey number, and is far from optimal in general. It was conjectured in [5, Problem 7.13] that all ultraproducts of p-adics in the pure ring language are inp-minimal (note that in the Denef-Pas language, no valued field with an infinite

residue field can be inp-minimal as $\{ac(x) = a_i\}, \{val(x) = v_i\}$ with $(a_i), (v_i)$ pairwise different give an inp-pattern of depth 2).

In this article we establish an Ax-Kochen type result for inp-minimality in the RV language for valued fields, in particular confirming that conjecture.

THEOREM 1.4. *Let $\bar{K} = (K, \mathrm{RV}, \mathrm{rv})$ be a Henselian valued field of equicharacteristic 0, viewed as a structure in the RV-language (see Section 2). Assume that both the residue field k and the value group Γ are inp-minimal, and that moreover $k^\times / (k^\times)^p$ is finite for all prime p. Then $\bar{K}$ is inp-minimal.*

COROLLARY 1.5. *Any ultraproduct of p-adics is inp-minimal.*

Recall the following definition, see e.g., [18].

DEFINITION 1.6. A theory is *dp-minimal* if for every mutually indiscernible sequences of tuples $(a_i : i \in \omega), (a'_i : i \in \omega)$ and a singleton b in the home sort, one of this sequences must be indiscernible over b.

REMARK 1.7. An NIP theory is *dp-minimal* if and only if it is inp-minimal.

Johnson [14] shows that a dp-minimal not strongly minimal field admits a definable Henselian valuation. It follows that if K is dp-minimal, then $K^\times / (K^\times)^p$ is finite for all prime p (a Fact which Johnson states and uses). Combining this with Delon's result on preservation of NIP we have the following corollary (which also appears in Johnson's thesis [13]).

COROLLARY 1.8. *Under the same assumptions on $\bar{K}$, if both k and Γ are dp-minimal, then $\bar{K}$ is dp-minimal.*

There are three steps in the proof of the main theorem, corresponding to the sections of the article. First, we recall some Facts about the RV setting and show that the whole valued field is inp-minimal if and only if the RV sort is inp-minimal. Second, we show that the RV sort eliminates quantifiers down to the residue field k and the value group Γ. Using this quantifier elimination, in the last section, we show that the RV sort is inp-minimal if and only if both k and Γ are inp-minimal. Finally, we discuss some problems and future research directions.

§2. **Reduction to RV.** We recall some basic Facts about the RV setting, we are going to use [11] as a reference. Fix a valued field K, with value group Γ and residue field k. Let RV be the quotient group $K^\times / (1 + \mathfrak{m})$ where $\mathfrak{m} = \{x \in K : val(x) > 0\}$ is the maximal ideal of the valuation ring. We have a short exact sequence $1 \rightarrow k^\times \rightarrow \mathrm{RV} \stackrel{\mathrm{val}_{\mathrm{rv}}}{\rightarrow} \Gamma \rightarrow 0$.

Consider now the two-sorted structure $\bar{K} = (K, \mathrm{RV}, \mathrm{rv})$ in the language L_{RV^+} consisting of:

- the quotient map $\mathrm{rv} : K \rightarrow \mathrm{RV}$,
- on the sort K, the ring structure,
- on the sort RV, the structure $\cdot, 1$ of a multiplicative group, a symbol 0, a symbol ∞ and a ternary relation $\oplus$.
  The multiplicative group structure is interpreted as the group structure induced from $K^\times$ and $0 \cdot x = x \cdot 0 = 0, \infty = \mathrm{rv}(0)$. The relation $\oplus$ is interpreted as the partially defined addition inherited from K: $\oplus(a, b, c) \iff \exists x, y, z \in K \left(a = \mathrm{rv}(x) \wedge b = \mathrm{rv}(y) \wedge c = \mathrm{rv}(z) \wedge x + y = z \right)$.

REMARK 2.1.   1. One can define the set $\mathrm{WD}(x, y)$ of pairs of elements for which the sum is well defined as $\forall z, z'(\oplus(x, y, z) \wedge \oplus(x, y, z') \implies z = z')$. Given a pair of elements $x, y \in \mathrm{RV}$ such that $\mathrm{WD}(x, y)$ holds, we write $x + y$ to denote the unique element $z \in \mathrm{RV}$ satisfying $\oplus(x, y, z)$.

2. We have $\mathrm{WD}(\mathrm{rv}(a), \mathrm{rv}(b)) \iff \mathrm{val}(a + b) = \min\{\mathrm{val}(a), \mathrm{val}(b)\}$, in which case $\mathrm{rv}(a + b) = \mathrm{rv}(a) + \mathrm{rv}(b)$ (see [11, Proposition 2.4]).

3. The relation $\mathrm{val}_{\mathrm{rv}}(x) \leq \mathrm{val}_{\mathrm{rv}}(y)$ on RV is definable in this language [11, Proposition 2.8(1)]. Namely, let $d \in \mathrm{RV}$ be arbitrary with $\mathrm{val}_{\mathrm{rv}}(d) = 0$. Then $\mathrm{val}_{\mathrm{rv}}(x) > 0 \iff dx + 1 = 1$, and $\mathrm{val}_{\mathrm{rv}}(x) = 0 \iff \neg \mathrm{val}_{\mathrm{rv}}(x) > 0 \wedge \exists y(x \cdot y = 1 \wedge \neg \mathrm{val}_{\mathrm{rv}}(y) > 0)$. Then $\mathrm{val}_{\mathrm{rv}}(x) = \mathrm{val}_{\mathrm{rv}}(y) \iff \exists u(\mathrm{val}_{\mathrm{rv}}(u) = 0 \wedge x = u \cdot y)$ and $\mathrm{val}_{\mathrm{rv}}(x) < \mathrm{val}_{\mathrm{rv}}(y) \iff x \neq \infty \wedge x + dy = x$.

Let $\bar{\mathbb{K}} \succ \bar{K}$ be a monster model. We may always assume that $\bar{\mathbb{K}}$ admits a cross-section map $\mathrm{ac} : K \to k^\times$, so we can view $\bar{\mathbb{K}}$ also as a structure in the language L_{ac} with ac added to the language.

FACT 2.2 ([11], Proposition 5.1).   1. *Let K be a Henselian valued field with char $(k) = 0$, and suppose that $S \subseteq K$ is definable. Then there are $\alpha_1, \ldots, \alpha_k$ and a definable subset $D \subseteq \mathrm{RV}^k$ such that*

$$S = \{x \in K : (\mathrm{rv}(x - \alpha_1), \ldots, \mathrm{rv}(x - \alpha_k)) \in D\}.$$

2. *The RV sort is fully stably embedded (i.e., the structure on RV induced from $\bar{K}$, with parameters, is precisely the one described above).*

The following two lemmas are easy to verify (see [4], or the proof of [17, Claim 1.17] for the details).

LEMMA 2.3. *Let $(a_i)_{i \in I}$ be an L_{ac}-indiscernible sequence of singletons in $\mathbb{K}$, and consider the function $(i, j) \mapsto \mathrm{val}(a_j - a_i)$ for $i < j \in I$. Then one of the following cases occurs:*

1. *It is strictly increasing depending only on i (so the sequence is pseudo-convergent).*
2. *It is strictly decreasing depending only on j (so the sequence taken in the reverse direction is pseudo-convergent).*
3. *It is constant (we'll refer to such a sequence as a "fan").*

LEMMA 2.4. *Let $(a_i)_{i \in I}$ be an L_{ac}-indiscernible pseudo-convergent sequence from $\mathbb{K}$. Then for any $d \in \mathbb{K}$ there is some $i_* \in \bar{I} \cup \{+\infty, -\infty\}$ (where $\bar{I}$ is the Dedekind closure of I) such that the following holds (taking a_∞ from $\mathbb{K}$ such that $I \frown a_\infty$ is indiscernible).*

- *For $i < i_*$: $\mathrm{val}(a_\infty - a_i) < \mathrm{val}(d - a_\infty)$, $\mathrm{val}(d - a_i) = \mathrm{val}(a_\infty - a_i)$ and $\mathrm{ac}(d - a_i) = \mathrm{ac}(a_\infty - a_i)$.*
- *For $i > i_*$: $\mathrm{val}(a_\infty - a_i) > \mathrm{val}(d - a_\infty)$, $\mathrm{val}(d - a_i) = \mathrm{val}(d - a_\infty)$ and $\mathrm{ac}(d - a_i) = \mathrm{ac}(d - a_\infty)$.*

REMARK 2.5. Note also that for any nonzero $x, y \in K$, $\mathrm{rv}(x) = \mathrm{rv}(y)$ if and only if $\mathrm{val}(x - y) > \mathrm{val}(y)$; and for any $z \in K$ and $x, y \in K \setminus \{z\}$, $\mathrm{rv}(x - z) = \mathrm{rv}(y - z)$ if and only if $\mathrm{val}(x - y) > \mathrm{val}(y - z)$.

In the remainder of this section we will reduce inp-minimality of $\bar{K}$ to inp-minimality of the RV sort with the induced structure.

First we treat a key special case. Assume that there is an inp-pattern consisting of formulas $\psi(x, yz) = \phi(\text{rv}(x-y), z)$ and $\psi'(x, yz') = \phi'(\text{rv}(x-y), z')$ and *mutually L_{ac}-indiscernible* sequences $(c_i)_{i \in \mathbb{Z}}$, $(c_i')_{i \in \mathbb{Z}}$ with $c_i = a_i \hat{\ } b_i$ and $c_i' = a_i' \hat{\ } b_i'$ where ϕ and ϕ' are RV-formulas, $b_i \in \text{RV}^{|z|}$, $b_i' \in \text{RV}^{|z'|}$ and $a_i, a_i' \in K$. Without loss of generality both $\{\phi(\text{rv}(x - a_i), b_i)\}_{i \in \mathbb{Z}}$ and $\{\phi'(\text{rv}(x - a_i'), b_i')\}_{i \in \mathbb{Z}}$ are k-inconsistent, and let $d \models \phi(\text{rv}(x - a_0), b_0) \wedge \phi'(\text{rv}(x - a_0'), b_0')$. We may also add to the base elements $a_\infty, a_{-\infty}, a_\infty', a_{-\infty}'$ continuing our sequences on the left and on the right.

CLAIM 2.6. $\text{val}(d - a_i) \leq \text{val}(d - a_0')$ *and* $\text{val}(d - a_j') \leq \text{val}(d - a_0)$ *for all i and j. In particular,* $\text{val}(d - a_0) = \text{val}(d - a_0') = \gamma$ *for some $\gamma \in \Gamma$.*

PROOF. Assume that $\text{val}(d - a_i) > \text{val}(d - a_0')$ for some i. Then $\text{rv}(d - a_0') = \text{rv}(a_i - a_0')$. So $\models \phi'(\text{rv}(a_i - a_0'), b_0')$, and by mutual indiscernibility $a_i \models \{\phi'(\text{rv}(x - a_j'), b_j')\}_{j \in \omega}$—a contradiction. The other part is by symmetry. $\dashv$

CLAIM 2.7. $\gamma \leq \text{val}(a_0 - a_0')$.

PROOF. As otherwise $\text{val}(d - a_0) = \text{val}(d - a_0') = \gamma > \text{val}(a_0 - a_0')$, hence $\text{val}(a_0 - a_0') = \text{val}((d - a_0') - (a_0 - a_0')) = \text{val}(d - a_0)$—a contradiction. $\dashv$

We now consider several cases separately.

CASE A. $\text{val}(a_i - a_j')$ is constant, equal to some $\gamma' \in \Gamma$.

As in this case the two sequences are mutually indiscernible over γ', we may add it to the base.

Note that $\gamma \leq \gamma'$ by Claim 2.7. The following subcases cover all the possible situations, using mutual indiscernibility of the sequences over γ'.

SUBCASE 1. $\gamma < \gamma'$.

Then $\text{rv}(d - a_i) = \text{rv}(d - a_j') = \alpha$ for all i, j, for some some $\alpha \in \text{RV}$ with $\text{val}_{\text{rv}}(\alpha) = \gamma$. Note furthermore that for any $\alpha^* \in \text{RV}$ such that $\text{val}_{\text{rv}}(\alpha^*) < \gamma'$ we can find some $d^* \in K$ such that $\text{rv}(d^* - a_i) = \text{rv}(d^* - a_i') = \alpha^*$.

But then consider the array

$$\tilde{\phi}(\tilde{x}, b_i) = \phi(\tilde{x}, b_i) \wedge \text{val}_{\text{rv}}(\tilde{x}) < \gamma',$$

$$\tilde{\phi}'(\tilde{x}, b_i') = \phi'(\tilde{x}, b_i') \wedge \text{val}_{\text{rv}}(\tilde{x}) < \gamma',$$

where $\tilde{x}$ and b_i, b_i' are ranging over the RV sort and $\tilde{\phi}, \tilde{\phi}'$ are RV-formulas (we are abusing the notation by writing $\text{val}_{\text{rv}}(\tilde{x}) < \gamma'$ as a short-cut for $\text{val}_{\text{rv}}(\tilde{x}) < \text{val}_{\text{rv}}(a_\infty - a_\infty')$). We have $\models \tilde{\phi}(\alpha, b_0) \wedge \tilde{\phi}'(\alpha, b_0')$ and $\left\{\tilde{\phi}(\tilde{x}, b_i)\right\}_{i \in \mathbb{Z}}$, $\left\{\tilde{\phi}'(\tilde{x}, b_i')\right\}_{i \in \mathbb{Z}}$ are both inconsistent by the previous observation as the original array was inconsistent. This gives us an inp-pattern in the structure induced on the RV sort, and so implies that RV is not inp-minimal.

SUBCASE 2. $\gamma = \gamma'$, $\text{val}(a_i - a_j) > \gamma$ and $\text{val}(a_i' - a_j') > \gamma$ for all $i < j$.

It follows by Remark 2.5 that there are $\alpha, \alpha' \in \text{RV}$ with $\text{val}_{\text{rv}}(\alpha) = \text{val}_{\text{rv}}(\alpha') = \gamma$ such that $\text{rv}(d - a_i) = \alpha$ and $\text{rv}(d - a_i') = \alpha'$ for all i. Furthermore, $\text{rv}(a_i - a_j') = \alpha' - \alpha =: \beta$ for all i, j. It follows that our sequences are mutually indiscernible over β and we can add it to the base.

We then consider a new array

$$\widetilde{\phi}\left(\widetilde{x}, b_i\right) = \phi\left(\widetilde{x}, b_i\right) \wedge \mathrm{val}_{\mathrm{rv}}\left(\widetilde{x}\right) = \gamma,$$
$$\widetilde{\phi}'\left(\widetilde{x}, b_i'\right) = \phi'\left(\widetilde{x} - \beta, b_i'\right) \wedge \mathrm{val}_{\mathrm{rv}}\left(\widetilde{x}\right) = \gamma.$$

It follows that $\alpha \models \widetilde{\phi}\left(\widetilde{x}, b_0\right) \wedge \widetilde{\phi}'\left(\widetilde{x}, b_0'\right)$, so to contradict inp-minimality of RV it is enough to show that $\left\{\widetilde{\phi}\left(\widetilde{x}, b_i\right)\right\}_{i \in \mathbb{Z}}, \left\{\widetilde{\phi}'\left(\widetilde{x}, b_i'\right)\right\}_{i \in \mathbb{Z}}$ are both inconsistent. Let $\alpha^* \in$ RV with $\mathrm{val}_{\mathrm{rv}}\left(\alpha^*\right) = \gamma$ be arbitrary, and take $d^* \in K$ such that $\mathrm{rv}(d^* - a_0) = \alpha^*$. Using Remark 2.5 again, we then have $\mathrm{rv}\left(d^* - a_i\right) = \alpha^*$ and $\mathrm{rv}\left(d^* - a_i'\right) = \alpha^* + \beta$ for all i. Hence any α^* realizing a row in the new array gives d^* realizing a row in the original array.

SUBCASE 3. $\gamma = \gamma'$, $\mathrm{val}\left(a_i - a_j\right) > \gamma$ and $\mathrm{val}\left(a_i' - a_j'\right) = \gamma$ for all $i < j$.

In this case we still have some $\alpha \in$ RV such that $\mathrm{rv}\left(d - a_i\right) = \alpha$ for all i. On the other hand, it follows that $\mathrm{rv}\left(d - a_i'\right) = \mathrm{rv}\left(d - a_\infty\right) + \mathrm{rv}\left(a_\infty - a_i'\right)$.

We then consider a new array given by

$$\widetilde{\phi}\left(\widetilde{x}, b_i\right) = \phi\left(\widetilde{x}, b_i\right) \wedge \mathrm{val}_{\mathrm{rv}}\left(\widetilde{x}\right) = \gamma,$$

$$\widetilde{\phi}'\left(\widetilde{x}, \widetilde{b}_i'\right) = \phi'\left(\widetilde{x} + \mathrm{rv}\left(a_\infty - a_i'\right), b_i'\right) \wedge \mathrm{val}_{\mathrm{rv}}\left(\widetilde{x}\right) = \gamma \wedge \mathrm{WD}\left(\widetilde{x}, \mathrm{rv}\left(a_\infty - a_i'\right)\right),$$

so $\widetilde{b}_i' = \mathrm{rv}\left(a_\infty - a_i'\right)^\frown b_i'$. Note that $(b_i)_{i \in \mathbb{Z}}$ and $\left(\widetilde{b}_i'\right)_{i \in \mathbb{Z}}$ are mutually indiscernible sequences in RV. It follows that $\alpha \models \widetilde{\phi}\left(\widetilde{x}, b_0\right) \wedge \widetilde{\phi}'\left(\widetilde{x}, \widetilde{b}_0'\right)$, hence to contradict inp-minimality of RV it is enough to show that both $\left\{\widetilde{\phi}\left(\widetilde{x}, b_i\right)\right\}_{i \in \mathbb{Z}}, \left\{\widetilde{\phi}'\left(\widetilde{x}, \widetilde{b}_i'\right)\right\}_{i \in \mathbb{Z}}$ are inconsistent. Let $\alpha^* \in$ RV be arbitrary such that $\mathrm{val}_{\mathrm{rv}}\left(\alpha^*\right) = \gamma$ and $\mathrm{WD}(\alpha^*, \mathrm{rv}(a_\infty - a_i'))$ for all i. Let $d^* \in K$ be such that $\mathrm{rv}(d^* - a_\infty) = \alpha^*$. Then $\mathrm{rv}\left(d^* - a_i\right) = \alpha^*$ and $\mathrm{rv}\left(d^* - a_i'\right) = \alpha^* + \mathrm{rv}\left(a_\infty - a_i'\right)$ for all i. This implies that for any α^* realizing a row in the new array, the corresponding d^* realizes the same row in the original array.

SUBCASE 4. $\gamma = \gamma'$, $\mathrm{val}\left(a_i - a_j\right) = \mathrm{val}\left(a_i' - a_j'\right) = \gamma$ for all $i < j$.

Then $\mathrm{rv}\left(d - a_i\right) = \mathrm{rv}\left(d - a_\infty\right) + \mathrm{rv}\left(a_\infty - a_i\right)$ and $\mathrm{rv}\left(d - a_i'\right) = \mathrm{rv}\left(d - a_\infty\right) + \mathrm{rv}\left(a_\infty - a_i'\right)$ (as $\mathrm{val}\left(d - a_i'\right) = \mathrm{val}\left(d - a_i\right) = \mathrm{val}\left(d - a_\infty\right) = \mathrm{val}\left(a_\infty - a_i'\right)$, because the first three are equal to γ and the last one to γ').

We consider a new array given by

$$\widetilde{\phi}\left(\widetilde{x}, \widetilde{b}_i\right) = \phi\left(\widetilde{x} + \mathrm{rv}\left(a_\infty - a_i\right), b_i\right) \wedge \mathrm{val}_{\mathrm{rv}}\left(\widetilde{x}\right) = \gamma \wedge \mathrm{WD}\left(\widetilde{x}, \mathrm{rv}\left(a_\infty - a_i\right)\right),$$

$$\widetilde{\phi}'\left(\widetilde{x}, \widetilde{b}_i'\right) = \phi'\left(\widetilde{x} + \mathrm{rv}\left(a_\infty - a_i'\right), b_i'\right) \wedge \mathrm{val}_{\mathrm{rv}}\left(\widetilde{x}\right) = \gamma \wedge \mathrm{WD}\left(\widetilde{x}, \mathrm{rv}\left(a_\infty - a_i'\right)\right),$$

so $\widetilde{b}_i = \mathrm{rv}\left(a_\infty - a_i\right)^\frown b_i$ and $\widetilde{b}_i' = \mathrm{rv}\left(a_\infty - a_i'\right)^\frown b_i'$. Note that $\left(\widetilde{b}_i\right)_{i \in \mathbb{Z}}$ and $\left(\widetilde{b}_i'\right)_{i \in \mathbb{Z}}$ are mutually indiscernible sequences in RV. It follows that $\alpha \models \widetilde{\phi}\left(\widetilde{x}, \widetilde{b}_0\right) \wedge \widetilde{\phi}'\left(\widetilde{x}, \widetilde{b}_0'\right)$, so to contradict inp-minimality of RV it is enough to show that both $\left\{\widetilde{\phi}\left(\widetilde{x}, \widetilde{b}_i\right)\right\}_{i \in \mathbb{Z}}, \left\{\widetilde{\phi}'\left(\widetilde{x}, \widetilde{b}_i'\right)\right\}_{i \in \mathbb{Z}}$ are inconsistent. Let $\alpha^* \in$ RV be arbitrary such that $\mathrm{val}_{\mathrm{rv}}\left(\alpha^*\right) = \gamma$. Let d^* be such that $\mathrm{rv}(d^* - a_\infty) = \alpha^*$. Then $\mathrm{rv}\left(d^* - a_i\right) = \alpha^* + \mathrm{rv}\left(a_\infty - a_i\right)$ and $\mathrm{rv}\left(d^* - a_i'\right) = \alpha^* + \mathrm{rv}\left(a_\infty - a_i'\right)$ for all i,

assuming these sums are well defined (see Remark 2.1). But this implies that for any α^* realizing a row in the new array (hence all the sums above corresponding to this row are well defined by the choice of $\widetilde{\phi}, \widetilde{\phi}'$), the corresponding d^* realizes the same row in the original array.

SUBCASE 5. $\gamma = \gamma'$, $\text{val}(a_i - a_i) = \gamma$ and $\text{val}(a_i' - a_j') > \gamma$ for all $i < j$.

Follows from Subcase 3 by symmetry.

CASE B. Not Case A.

    CLAIM 2.8. *At least one of the sequences $(a_i)_{i\in\mathbb{Z}}$, $(a_i')_{i\in\mathbb{Z}}$ is not a fan.*

    PROOF. Assume that both are, say $\text{val}(a_i - a_j) = \alpha$ and $\text{val}(a_i' - a_j') = \alpha'$ for all $i < j$. It follows by mutual indiscernibility that $\text{val}(a_i - a_j') \leq \min\{\alpha, \alpha'\}$ for all i, j. But then $\text{val}(a_i - a_j') = \text{val}(a_0 - a_0')$ for all i, j, thus putting us in Case A. ⊣

So we may assume that $(a_i)_{i\in\mathbb{Z}}$ is a pseudo-convergent sequence (by Lemma 2.3, possibly exchanging (a_i) with (a_i') and reverting the ordering of the sequence).

SUBCASE 1. Some (equivalently, every) a_i' is a pseudo-limit of $(a_i)_{i\in\mathbb{Z}}$.

    Then $\text{rv}(d - a_i') = \text{rv}(d - a_\infty)$ for all i (by Claim 2.7).

    We define $\widetilde{\phi}'(\widetilde{x}, b_i') = \phi'(\widetilde{x}, b_i') \wedge \text{val}_{\text{rv}}(\widetilde{x}) < \text{val}(a_\infty - a_\infty')$.

    By Lemma 2.4 it follows that there is some $i^* \in \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ such that $\text{rv}(d - a_i) = \text{rv}(d - a_\infty)$ for $i > i^*$ and $\text{rv}(d - a_i) = \text{rv}(a_\infty - a_i)$ for $i < i^*$. Again by Claim 2.7, $i^* \leq 0$. Let's restrict $(a_i)_{i\in\mathbb{Z}}$ to $(a_i)_{i\in\omega}$.

    If $\text{val}(d - a_\infty) < \text{val}(a_\infty - a_0)$ then $\text{rv}(d - a_i) = \text{rv}(d - a_\infty)$ for all i. If $\text{val}(d - a_\infty) = \text{val}(a_\infty - a_0)$ then $\text{rv}(d - a_i) = \text{rv}(d - a_\infty)$ for all $i > 0$ and $\text{rv}(d - a_0) = \text{rv}(d - a_\infty) + \text{rv}(a_\infty - a_0)$. We thus define

$$\widetilde{\phi}\left(\widetilde{x}, \widetilde{b}_i\right) = (\text{val}(a_i - a_\infty) > \text{val}_{\text{rv}}(\widetilde{x}) \wedge \phi(\widetilde{x}, b_i)) \vee$$
$$\left(\text{val}(a_i - a_\infty) = \text{val}_{\text{rv}}(\widetilde{x}) \wedge \text{WD}(\widetilde{x}, \text{rv}(a_\infty - a_i)) \wedge \phi(\widetilde{x} + \text{rv}(a_\infty - a_i), b_i)\right)$$

with $\widetilde{b}_i = b_i \char94 \text{rv}(a_i - a_\infty)$. Then $\left(\widetilde{b}_i\right), (b_i')$ are mutually indiscernible sequences in RV and $\text{rv}(d - a_\infty) \models \widetilde{\phi}\left(\widetilde{x}, \widetilde{b}_0\right) \wedge \widetilde{\phi}'(\widetilde{x}, b_0')$. By inp-minimality of RV we have that either there is some $\alpha^* \models \left\{\widetilde{\phi}'(\widetilde{x}, b_i')\right\}_{i\in\omega}$, in which case we can find d^* with $\text{rv}(d^* - a_\infty) = \alpha^*$ and thus $d^* \models \{\phi'(\text{rv}(x - a_i'), b_i')\}_{i\in\omega}$, or that $\alpha^* \models \left\{\widetilde{\phi}\left(\widetilde{x}, \widetilde{b}_i\right)\right\}_{i\in\omega}$. Then it follows from the definition of $\widetilde{\phi}$ that there is d^* satisfying $\text{rv}(d^* - a_\infty) = \alpha^*$ and such that that $d^* \models \{\phi(\text{rv}(x - a_i), b_i)\}_{i\in\omega}$—a contradiction.

SUBCASE 2. Not Subcase 1.

Then we have the following observations.

    CLAIM 2.9. *For any $i, j \in \mathbb{Z}$ we have $\text{val}(a_\infty - a_i) > \text{val}(a_j' - a_i)$.*

    PROOF. Since a_j' is not a pseudo-limit of the sequence (a_i) (as we are not in Subcase 1), we must have $\text{val}(a_j' - a_{i_1}) < \text{val}(a_{i_2} - a_{i_1})$ for some $i_2 > i_1 \in \mathbb{Z}$. Then the claim follows by mutual indiscernibility. ⊣

CLAIM 2.10. *The sequence (a_i') must be pseudo-convergent.*

PROOF. If (a_i') was a fan, in view of Claim 2.9 we would have $\mathrm{val}(a_i - a_j')$ constant—a contradiction since we are not in Case A. Hence it is pseudo-convergent, after possibly reversing the order, by Lemma 2.3. ⊣

These two claims imply that the only possibility is that (a_i') is pseudo-convergent and that any a_i is a pseudo-limit of it. But then reversing the roles of the two sequences we are back to Subcase 1, concluding the analysis of the special case.

Now we reduce the case of a general inp-pattern to the special case treated above. Assume that there is an inp-pattern of depth 2. By Ramsey and compactness we may assume that the rows are mutually indiscernible in the L_{ac}-language. Though in Fact 2.2 the formula defining D may depend on the formula defining S, by indiscernibility, Ramsey and compactness we may assume that the formulas in our inp-pattern are in Fact of the form $\phi\left(\mathrm{rv}\left(x - y_1\right), \ldots, \mathrm{rv}\left(x - y_n\right), z\right)$ and $\phi'\left(\mathrm{rv}\left(x - y_1\right), \ldots, \mathrm{rv}\left(x - y_n\right), z'\right)$, for some $n \in \omega$, where ϕ and ϕ' are RV-formulas. Let d realize the first column of the inp-pattern.

CASE 1. $\mathrm{val}\left(d - a_{0,0}\right) < \mathrm{val}\left(a_{0,n} - a_{0,0}\right)$. Then $\mathrm{rv}\left(d - a_{0,0}\right) = \mathrm{rv}\left(d - a_{0,n}\right)$ and we define

$$\widetilde{\phi}\left(x, a_i \widetilde{b}_i\right) = \phi\left(\mathrm{rv}\left(x - a_{i,0}\right), \ldots, \mathrm{rv}\left(x - a_{i,n-1}\right), \mathrm{rv}\left(x - a_{i,0}\right), b_i\right)$$
$$\wedge \mathrm{val}\left(x - a_{i,0}\right) < \mathrm{val}\left(a_{i,n} - a_{i,0}\right)$$

with $\widetilde{b}_i = b_i \widehat{\ } \mathrm{rv}\left(a_{i,n} - a_{i,0}\right)$.

CASE 2. $\mathrm{val}\left(d - a_{0,0}\right) > \mathrm{val}\left(a_{0,n} - a_{0,0}\right)$. Then $\mathrm{rv}\left(d - a_{0,n}\right) = \mathrm{rv}\left(a_{0,n} - a_{0,0}\right)$ and we define

$$\widetilde{\phi}\left(x, a_i \widetilde{b}_i\right) = \phi\left(\mathrm{rv}\left(x - a_{i,0}\right), \ldots, \mathrm{rv}\left(x - a_{i,n-1}\right), \mathrm{rv}\left(a_{i,n} - a_{i,0}\right), b_i\right)$$
$$\wedge \mathrm{val}\left(x - a_{i,0}\right) > \mathrm{val}\left(a_{i,n} - a_{i,0}\right)$$

with $\widetilde{b}_i = b_i \widehat{\ } \mathrm{rv}\left(a_{i,n} - a_{i,0}\right)$.

CASE 3. $v\left(d - a_{0,n}\right) < v\left(a_{0,n} - a_{0,0}\right)$ and CASE 4. $v\left(d - a_{0,n}\right) > v\left(a_{0,n} - a_{0,0}\right)$ are symmetric to Cases 1 and 2 respectively.

CASE 5. $v\left(d - a_{0,0}\right) = v\left(d - a_{0,n}\right) = v\left(a_{0,n} - a_{0,0}\right)$. Then $\mathrm{rv}\left(d - a_{0,0}\right) = \mathrm{rv}\left(d - a_{0,n}\right) + \mathrm{rv}\left(a_{0,n} - a_{0,0}\right)$. We define $\widetilde{\phi}\left(x, a_i \widetilde{b}_i\right)$ to be the formula

$$\phi\left(\mathrm{rv}\left(x - a_{i,n}\right) + \mathrm{rv}\left(a_{i,n} - a_{i,0}\right), \ldots, \mathrm{rv}\left(x - a_{i,n-1}\right), \mathrm{rv}\left(x - a_{i,n}\right), b_i\right)$$
$$\wedge v\left(x - a_{i,n}\right) = v\left(a_{i,n} - a_{i,0}\right) \wedge \mathrm{WD}\left(\mathrm{rv}\left(x - a_{i,n}\right), \mathrm{rv}\left(a_{i,n} - a_{i,0}\right)\right),$$

with $\widetilde{b}_i = b_i \widehat{\ } \mathrm{rv}\left(a_{i,n} - a_{i,0}\right)$.

In any of the cases, we still have that $\left(\widetilde{b}_i\right)_{i \in \mathbb{Z}}, (b_i')_{i \in \mathbb{Z}}$ are mutually indiscernible, that $d \models \widetilde{\phi}\left(x, a_0 \widetilde{b}_0\right) \wedge \phi'\left(x, a_0' b_0'\right)$ and that $\left\{\widetilde{\phi}\left(x, a_i \widetilde{b}_i\right)\right\}_{i \in \mathbb{Z}}$ is inconsistent. Thus we get a new inp-pattern replacing $\{\phi\left(x, a_i b_i\right)\}$ by $\left\{\widetilde{\phi}\left(x, a_i \widetilde{b}_i\right)\right\}$, with $\widetilde{\phi}$ involving one less term of the form $\mathrm{rv}\left(x - y_i\right)$. Repeating the same operation n times for ϕ, and then for ϕ', we reduce the situation to the special case of formulas considered before.

§**3. Relative quantifier elimination for** RV**.** Now it will be more convenient to consider a valued field K in a slightly weaker language L_{RV}. Namely, we associate with it a three-sorted structure $\bar{K} = (K, \mathrm{RV}, \Gamma, \mathrm{val}_{\mathrm{rv}})$ such that on RV we have the multiplicative group structure $\cdot, 1$, a constant 0, a predicate for the residue field $k \subseteq \mathrm{RV}$ along with addition $\tilde{+}$ on k, and a map $\mathrm{val}_{\mathrm{rv}} : \mathrm{RV} \to \Gamma$.

The partial addition relation $\oplus$ on RV is definable in L_{RV} (using [11, Proposition 2.7]) as follows:

$$\oplus(x, y, z) \iff \left(\mathrm{val}_{\mathrm{rv}}(x) < \mathrm{val}_{\mathrm{rv}}(y) \wedge z = x\right) \vee \left(\mathrm{val}_{\mathrm{rv}}(y) < \mathrm{val}_{\mathrm{rv}}(x) \wedge z = y\right) \vee$$
$$\left(\mathrm{val}_{\mathrm{rv}}(x) = \mathrm{val}_{\mathrm{rv}}(y) \wedge \psi(x, y, z)\right),$$

where

$$\psi(x, y, z) = \left(\left(\frac{x}{y}\tilde{+}1 = 0 \wedge \mathrm{val}_{\mathrm{rv}}(z) > \mathrm{val}_{\mathrm{rv}}(x)\right) \vee \left(\left(\frac{x}{y}\tilde{+}1\right) y = z \wedge z \neq 0\right)\right).$$

The conclusion is that in particular if $(\mathrm{RV}, \Gamma, \mathrm{val}_{\mathrm{rv}})$ is inp-minimal as an L_{RV}-structure, then $(\mathrm{RV}, \cdot, \oplus)$ is inp-minimal as an L_{RV^+}-structure. In the next section we are going to demonstrate the former under the assumptions of the main theorem, but in order to do that we prove a relative quantifier elimination result for (a certain expansion of) the L_{RV} language.

Assumptions

- G is an abelian group such that G/nG is finite for all $n < \omega$.
- $K \subseteq G$ is a subgroup, with quotient $H = G/K$. Let $\pi : G \to H$ denote the projection map.
- M is the two-sorted structure with sorts G and H, and the following language.
  - On G: we have the group structure $+, -, 0$, a predicate $K(x)$ for the subgroup K, predicates $(P_n(x) : n < \omega)$ interpreted as $P_n(x) \leftrightarrow \exists y\, ny = x$, and constants naming a countable subgroup G_0 containing representatives of each class of G/nG, for each $n < \omega$ (such that moreover all classes of elements from K are represented by elements from $G_0 \cap K$).
  - On H: we have some language L_H (containing the induced group structure) and we assume that the structure (H, L_H) eliminates quantifiers.
  - On K: we have some language L_K such that (K, L_K) eliminates quantifiers and contains the language induced from G (via the group structure and predicates P_n).
  - We have the projection group homomorphism $\pi : G \to H$.
- Moreover, we assume that the language contains no other function symbols apart from π and the group structures on G and H.
- Finally, H is torsion-free.

PROPOSITION 3.1. *M has quantifier elimination.*

PROOF. We prove it by back-and-forth. So assume that M is $\aleph_1$-saturated and we have two substructures A and B from M and a partial isomorphism $f : A \to B$. So $A, B \supseteq G_0$ contain elements from both G and H, both are closed under the group operations, inverse and π.

Let $\alpha \in M$ be arbitrary, and we want to extend f to be defined on $A_1 = A(\alpha)$, the substructure generated by αA. We assume that $\alpha \notin A$.

STEP 1. If $\alpha \in H$, then we can extend f.

As $f|_{A \cap H}$ is L_H-elementary by quantifier elimination in (H, L_H), there is $\beta \in H$ and a partial L_H-automorphism g extending $f|_{A \cap H}$ and sending $A(\alpha) \cap H$ to $B(\beta) \cap H$. Then we extend f to F defined on $A(\alpha)$ by taking $F = f \cup g$ (note that, as there are no functions from H to G in the language, $A(\alpha) \cap G = A \cap G$).

So by iterating Step 1 we may assume that $\alpha \in G$ and that $\pi(a + n\alpha) \in A$ for all $a \in A$ and $n \in \mathbb{Z}$.

STEP 2. Assume that $\alpha \in K$. Then we can extend f.

As $f|_{A \cap K}$ is L_K-elementary by quantifier elimination, we can find $\beta \in K$ and a partial L_K-automorphism g extending it and sending $A(\alpha) \cap K$ to $B(\beta) \cap K$. Then we define F on $A(\alpha)$ by setting $F(a + n\alpha) = f(a) + g(n\alpha) = f(a) + ng(\alpha)$ for all $a \in A$, $n \in \mathbb{Z}$ (note that $n\alpha \in A(\alpha) \cap K$ for all $n \in \mathbb{Z}$ by the assumption) and F acts like f on $A(\alpha) \cap H = A \cap H$.

- F is well defined: Assume that $a + n\alpha = a' + n'\alpha$, so $A \ni a - a' = (n' - n)\alpha$, and thus $f(a) - f(a') = f(a - a') = f((n' - n)\alpha) = \ldots$ as $(n' - n)\alpha \in K \cap A$ and $g|_{A \cap K} = f|_{A \cap K} \ldots = g((n' - n)\alpha) = ng(\alpha) - n'g(\alpha)$. Then we have $F(a + n\alpha) - F(a' + n'\alpha) = f(a) + g(n\alpha) - f(a') - g(n'\alpha) = 0$.
- F extends f: immediate from the definition.
- Note that $F|_{A(\alpha) \cap K} = g$, as given $a + n\alpha \in A(\alpha) \cap K$ it follows that $a \in A \cap K$, and as $f|_{A \cap K} = g|_{A \cap K}$ we have $F(a + n\alpha) = f(a) + g(n\alpha) = g(a) + g(n\alpha) = g(a + n\alpha)$.
- $F|_G$ is a group homomorphism:

$$F(a + n\alpha + a' + n'\alpha) = F((a + a') + (n + n')\alpha) = f(a + a') + g((n + n')\alpha)$$
$$= f(a) + f(a') + g(n\alpha) + g(n'\alpha) = F(a + n\alpha) + F(a' + n'\alpha).$$

- F is onto $B(\beta)$: every element of $B(\beta)$ is of the form $b + n\beta$, so

$$F(f^{-1}(b) + n\alpha) = b + n\beta.$$

- F preserves π: On one hand $\pi(F(a + n\alpha)) = \pi(f(a) + ng(\alpha)) = \pi(f(a)) + n\pi(g(\alpha)) = \ldots$ as $g(\alpha) \in K \ldots = \pi(f(a)) + 0 = f(\pi(a)) = F(\pi(a))$ (recall that $\pi(a) \in A$). On the other hand we have $F(\pi(a + n\alpha)) = F(\pi(a) + n\pi(\alpha)) = F(\pi(a) + 0) = F(\pi(a))$.
- In particular, F preserves $K(x) = \{x \in G : \pi(x) = 0\}$.
- F preserves P_k: $P_k(F(a + n\alpha)) \Leftrightarrow P_k(f(a) + ng(\alpha)) \Leftrightarrow P_k(a + ng(\alpha))$ (as $f(a) = a \mod kG$) $\Leftrightarrow P_k(a + n\alpha)$ (as $g(\alpha) = \alpha \mod kG$ because all representatives of classes of $\alpha \in K$ are in $G_0 \cap K \subseteq A \cap K$, $P_k \cap K$ is L_K-definable and $g|_{A(\alpha) \cap K}$ is L_K-elementary).
- F preserves every $\phi(x_1, \ldots, x_k) \in L_K$: As $F|_{A(\alpha) \cap K} = g$ and g is an L_K-elementary map.
- F preserves every $\psi \in L_H$: As $\pi(a + n\alpha) = \pi(a) + n\pi(\alpha) \in A \cap H$ (as $\pi(\alpha) \in A$ by the assumption), and $F|_{A \cap H} = f|_{A \cap H}$ is L_H-elementary.

So F is a partial isomorphism as wanted.

By iterating Step 2 we may assume that $a + n\alpha \in K \Rightarrow a + n\alpha \in A$ for all $a \in A$ and $n \in \omega$.

STEP 3. Assume that $m\alpha \in A$ for some $m \geq 1$. Then we can extend f.

Let m be minimal with this property.

CLAIM 3.2. *There is $\beta \in G$ satisfying $m\beta = f(m\alpha)$ and $\beta = \alpha \mod kG$ for all $k \in \omega$.*

PROOF. By ω-saturation it suffices to shows this one k at a time. By assumption there is some $g \in G_0$ such that $P_k(\alpha - g)$, then

$$P_k(\alpha - g) \Rightarrow P_{mk}(m\alpha - mg) \Rightarrow P_{mk}(f(m\alpha) - mg)$$

(as $m\alpha, mg \in A$, $f(mg) = mf(g) = mg$ and f preserves P_l for all $l < \omega) \Rightarrow \exists \gamma \in G$ such that $mk\gamma = f(m\alpha) - mg$. Let $\beta = k\gamma + g$. Then $m\beta = f(m\alpha)$ and $\beta = g = \alpha \mod kG$, and the claim is proved. ⊣

We define F on $A(\alpha) \cap G$ by setting $F(a + n\alpha) = f(a) + n\beta$ and $F|_{A(\alpha) \cap H} = f|_{A(\alpha) \cap H}$ as $A(\alpha) \cap H = A \cap H$.

- F is well defined: If $a + n\alpha = a' + n'\alpha$ with $a, a' \in A$, then $(n - n')\alpha = a' - a \in A$. It follows that m divides $(n - n')$ by minimality (assume that $n - n' = km + m_1$, $|m_1| < m$, then $m_1\alpha = a' - a - km\alpha \in A$, contradiction), say $(n - n') = km$. Thus $f(a') - f(a) = f(a' - a) = f((n - n')\alpha) = f(km\alpha) = kf(m\alpha) = km\beta = (n - n')\beta$. But then $F(a + n\alpha) - F(a' + n'\alpha) = f(a) + n\beta - f(a') - n'\beta = 0$.
- F extends f is obvious from the definition.
- F is a group homomorphism from $A(\alpha)$ to $B(\beta)$:

  $$F((a + n\alpha) + (a' + n'\alpha)) = F((a + a') + (n + n')\alpha) = f(a + a') + (n + n')\beta$$
  $$= (f(a) + n\beta) + (f(a') + n'\beta) = F(a + n\alpha) + F(a' + n'\alpha).$$

- F preserves π: First observe that $\pi(m\beta) = \pi(f(m\alpha))$, so $m\pi(\beta) = \pi(f(m\alpha)) \overset{\text{as } m\alpha \in A}{=} f(\pi(m\alpha)) = f(m\pi(\alpha)) = mf(\pi(\alpha))$, and as H is torsion free this implies that $\pi(\beta) = f(\pi(\alpha))$. But then $F(\pi(a + n\alpha)) = f(\pi(a + n\alpha)) = f(\pi(a) + n\pi(\alpha)) = f(\pi(a)) + nf(\pi(\alpha)) = \pi(f(a)) + n\pi(\beta) = \pi(f(a) + n\beta) = \pi(F(a + n\alpha))$.
- In particular, F preserves $K(x) = \{x \in G : \pi(x) = 0\}$.
- F preserves $P_k(x)$: By the choice of β we have $\alpha = \beta \mod kG$ for all k, and for any $a \in A$ we have $f(a) = a \mod kG$ for all k (as $G_0 \subseteq A$ and f preserves P_k), hence $P_k(F(a + n\alpha)) \Leftrightarrow P_k(f(a) + n\beta) \Leftrightarrow P_k(a + n\alpha)$.
- F preserves L_K-formulas: As $a + n\alpha \in K \Rightarrow a + n\alpha \in A$ by the assumption and $F|_{A \cap K} = f|_{A \cap K}$ is L_K-elementary by elimination of quantifiers in (K, L_K).
- F preserves L_H-formulas: As $F|_{A(\alpha) \cap H} = f|_{A(\alpha) \cap H = A \cap H}$ by definition, and f is L_H-elementary.

So we may assume that:

1. $A \cap H$ is a relatively divisible subgroup of H (iterating Step 1);
2. $A \cap G$ is a relatively divisible subgroup of $A(\alpha) \cap G$ (iterating Step 3);
3. $\pi(a + n\alpha) \in A$ for all $a \in A, n \in \mathbb{Z}$ (iterating Step 1);
4. $a + n\alpha \notin K$ for all $a \in A, n \in \mathbb{Z} \setminus \{0\}$ (as $a + n\alpha \in K \Rightarrow a + n\alpha \in A$ by Step 2, so $n\alpha \in A$, so $\alpha \in A$ by divisibility of A—contradicting the assumption).

Step 4. General case.

CLAIM 3.3. *There is some $\beta \in G$ such that $\pi(\beta) = f(\pi(\alpha))$ and $\alpha = \beta \mod kG$ for all $k \in \omega$.*

Proof. By ω-saturation we only need to consider one value of k at a time. Let $g \in G_0$ be such that $P_k(g + \alpha)$ holds, then $\pi(g + \alpha)$ is k-divisible as well. As $g \in A \Rightarrow g + \alpha \in A(\alpha) \Rightarrow \pi(g + \alpha) \in A \cap H$ and $f|_{A \cap H}$ is L_H-elementary, it follows that $f(\pi(g + \alpha))$ is k-divisible as well. Take β to be $k\beta' - g$ where $\pi(\beta') = \frac{f(\pi(g + \alpha))}{k}$ (recall that H is torsion free). Now we have $P_k(g + \beta)$ and $\pi(\beta) = k\pi(\beta') - \pi(g) = f(\pi(g + \alpha)) - \pi(g) = f(\pi(g)) + f(\pi(\alpha)) - \pi(g) = f(\pi(\alpha))$ as $f(\pi(g)) = \pi(f(g))$ and $f(g) = g$, so the claim is proved.   ⊣

We define $F(a + n\alpha) = f(a) + n\beta$ and $F|_{A(\alpha) \cap H = A \cap H} = f|_{A \cap H}$.

- F is well defined: If $a + n\alpha = a' + n'\alpha$, then $(a - a') + (n - n')\alpha = 0 \in A$, which implies by the assumption that $n = n'$ and $a = a'$.
- F is a homomorphism: clear from definition and as f is a homomorphism on A.
- F preserves π (so in particular K):

$$\pi(F(a + n\alpha)) = \pi(f(a) + n\beta) = \pi(f(a)) + n\pi(\beta) = f(\pi(a))$$
$$+ nf(\pi(\alpha)) = f(\pi(a) + n\pi(\alpha)) = f(\pi(a + n\alpha)) = F(\pi(a + n\alpha)).$$

- F preserves P_k: $P_k(F(a + n\alpha)) \Leftrightarrow P_k(f(a) + n\beta) \Leftrightarrow P_k(a + n\beta)$ (as $f(a) = a \mod kG$ because we have all the representatives in G_0) $\Leftrightarrow P_k(a + n\alpha)$ (as $\alpha = \beta \mod kG$ by the choice of β).
- F preserves L_K-formulas and L_H-formulas: as in Step 3.   ⊣

Corollary 3.4. H and K are fully stably embedded, i.e., any subset of H (resp. K) definable with external parameters is already definable with internal parameters in L_H (resp., L_K)—this follows directly from the elimination of quantifiers.

§4. Reduction from RV to k and Γ.

Proposition 4.1. Let $M = (G, K, H)$ be a structure satisfying the assumptions from the previous section. Assume moreover that:

1. K (viewed as an L_K structure) and H (viewed as an L_H structure) are both inp-minimal;
2. for every n, there are only finitely many $x \in G$ for which $nx = 0$ (since H is torsion-free, such elements are in Fact in K).

Then M is inp-minimal.

Proof. We are working in a saturated extension of M. Assume that the conclusion fails, then we have an inp-pattern $\phi(x, y), \phi'(x, y'), \bar{a} = (a_i), \bar{a}' = (a_i')$ witnessing this, with $\bar{a}$ and $\bar{a}'$ mutually indiscernible. In particular they are mutually indiscernible over $G_0 \subseteq \mathrm{acl}(\emptyset)$ which contains representatives of each class of G/nG and all torsion of G, and rows are k_*-inconsistent. Let $b \models \phi(x, a_0) \wedge \phi'(x, a_0')$. It follows from quantifier elimination that $\phi(x, a_i)$ is equivalent to a disjunction of conjuncts of the form $\theta(t_{i,0}(x), \ldots, t_{i,l-1}(x), \alpha_i) \wedge \psi(\pi(x), b_i) \wedge \chi(x, c_i) \wedge \rho(x, e_i)$ where:

- the $t_{i,j}$ are terms with parameters in G, $\alpha_i \in K$ and θ is an L_K-formula;
- ψ is an L_H-formula and $b_i \in H$;
- $\chi(x, c_i)$ is of the form $\bigwedge_{j<k} n_j x + c_{i,j} = 0 \wedge \bigwedge_{j<k} m_j x + d_{i,j} \neq 0$ with $c_i = (c_{i,j})_{j<k} \,\hat{}\, (d_{i,j})_{j<k}$ from G;

- $\rho(x, e_i)$ is of the form

$$\bigwedge_{j<k} P_{m'_j}\left(n'_j x + e'_{i,j}\right) \wedge \bigwedge_{j<k} \neg P_{m''_j}\left(n''_j x + e''_{i,j}\right)$$

with $e_i = \left(e'_{i,j}\right)_{j<k}{}^\frown\left(e''_{i,j}\right)_{j<k}$.

Forgetting all but one disjunct satisfied by b, we may assume that $\phi(x, a_i)$ is equal to such a conjunction.

Any term $t_{i,j}$ is of the form $n_{i,j}x - g_{i,j}$ and the formula makes sense only when $n_{i,j}x - g_{i,j} \in K$, that is when $\pi(x) = \pi(g_{i,j})/n_{i,j}$. Choose some h_i such that $\pi(h_i) = \pi(g_{i,j})/n_{i,j}$ for some/all j. We can then replace $n_{i,j}x - g_{i,j}$ with $n(x - h_i) + h'_{i,j}$ with $h'_{i,j} \in K$. Adding $h'_{i,j}$ to α_i and changing the formula θ, we replace θ by a formula $\theta'(x - h_i, \alpha'_i), \theta' \in L_K$.

Recalling that G/nG is finite for every $n < \omega$, $\rho(x, e_i)$ is equivalent to some finite disjunction of the form $\bigvee_{i<N} P_{k_i}(x - g_i)$ where $g_i \in G_0$ (so for example to express $\neg P_k(nx + e)$ we have to say that x belongs to one of the finitely many classes mod kG satisfying this, and to express $P_k(nx + e) \wedge P_l(n'x + e')$ we have to say that x belongs to a certain subset of the classes mod klG).

Note that $\chi(x, c_0)$ is infinite as $\chi(x, c_0) \wedge \phi'(x, a_i)$ is consistent for every $i \in \omega$, while $\{\phi'(x, a_i)\}_{i \in \omega}$ is k_*-inconsistent. Thus $\chi(x, c_i)$ can only be of the form $\bigwedge_{j<k} n_j x + c_{i,j} \neq 0$ (as every equation of the form $nx + c = 0$ has only finitely many solutions by assumption (2)).

Thus we may assume that $\phi(x, a_i) = \theta(x - h_i, \alpha_i) \wedge \psi(\pi(x), b_i) \wedge \chi(x, c_i) \wedge P_l(x - g)$ where:

- $\alpha_i \in K$ and θ is an L_K-formula,
- ψ is an L_H-formula and $b_i \in \Gamma$,
- $\chi(x, c_i) = \left(\bigwedge_{j<k} n_j x + c_{i,j} \neq 0\right)$,
- $l \in \omega, g \in G_0$.

Similarly, we may assume that $\phi'(x, a'_i) = \theta'(x - h'_i, \alpha'_i) \wedge \psi'(\pi(x), b'_i) \wedge \chi'(x, c'_i) \wedge P_{l'}(x - g')$ with the same properties.

CASE 1. $b \in H$. Then by full stable embeddedness of H we can replace our array by $\widetilde{\phi}(x, \widetilde{a}_i)$ and $\widetilde{\phi}'(x, \widetilde{a}'_i)$ where $\widetilde{\phi}, \widetilde{\phi}' \in L_H$ and $\widetilde{a}_i, \widetilde{a}'_i \in H$ are such that $\widetilde{\phi}(x, \widetilde{a}_i) \cap H(x) = \phi(x, a_i) \cap H(x)$, and similarly for $\widetilde{\phi}'$. But this contradicts inp-minimality of (H, L_H).

CASE 2. $b \in K$. Similarly, by full stable embeddedness of K we can replace our array by $\widetilde{\phi}(x, \widetilde{a}_i)$ and $\widetilde{\phi}'(x, \widetilde{a}'_i)$ where $\widetilde{\phi}, \widetilde{\phi}' \in L_K$ and $\widetilde{a}_i, \widetilde{a}'_i \in K$ are such that $\widetilde{\phi}(x, \widetilde{a}_i) \cap K(x) = \phi(x, a_i) \cap K(x)$, and similarly for $\widetilde{\phi}'$. But this contradicts inp-minimality of (K, L_K).

CASE 3. $b \notin K \cup H$.

SUBCASE 3.1. Neither θ occurs in ϕ nor θ' occurs in ϕ' (i.e., ϕ is equivalent to the formula obtained from it by omitting θ).

Then we have $\phi(x, a_i) = \psi(\pi(x), b_i) \wedge \chi(x, c_i) \wedge P_l(x - g)$ and $\phi'(x, a'_i) = \psi'(\pi(x), b'_i) \wedge \chi'(x, c'_i) \wedge P_{l'}(x - g')$.

Consider $\widetilde{\psi}(x', b_i) := \psi(x', b_i) \wedge$ "$x' - \pi(g)$ is l-divisible" and $\widetilde{\psi}'(x', b_i') := \psi(x', b_i') \wedge$ "$x' - \pi(g')$ is l'-divisible"—this is an array in the structure induced on H. Note that $\pi(b) \models \widetilde{\psi}(x', b_0) \wedge \widetilde{\psi}'(x', b_0')$.

SUBCASE 3.1(a). K is infinite.

As H is inp-minimal, it follows without loss of generality that the set

$$\{\widetilde{\psi}(x', b_i) : i < \omega\}$$

has a solution h in H.

Say $h - \pi(g) = l\gamma$. Take $\beta \in G$ such that $\pi(\beta) = \gamma$. As K is infinite, there is an infinite sequence $(\beta_i)_{i \in \omega}$ in K such that all the differences $\beta_i - \beta_j$ are pairwise different. Let $e_i' = \beta + \beta_i$. Then we still have that $e_i' - e_j'$ are all pairwise different, and that $\pi(e_i') = \pi(\beta) + \pi(\beta_i) = \gamma$. Note that as by assumption there are only finitely many l-torsion elements in G, we may assume that $e_i' - e_j'$ is not l-torsion, for any $i \neq j$.

Finally, define $e_i = le_i' + g$. We have:

- all e_i's are pairwise different (as $e_i = e_j \Rightarrow (e_i' - e_j')$ is l-torsion, contradicting the choice of the elements b_i').
- $\pi(e_i) = l\pi(e_i') + \pi(g) = l\gamma + \pi(g) = h$.
- $P_l(e_i - g)$ holds as $e_i - g = le_i'$.

As the set $\bigvee_{i < k_* + 1} \left(\bigvee_{j < k} n_j x + c_{i,j} = 0 \right)$ is finite, then one of the e_i's realizes the first k_* elements of the first row—a contradiction.

SUBCASE 3.1(b). K is finite.

It follows that all of the fibers of π are finite.

CLAIM 4.2. *One of the partial types $\{\widetilde{\psi}(x', b_i) : i \in \omega\}$ or $\{\widetilde{\psi}'(x', b_i') : i \in \omega\}$ has infinitely many solutions in H.*

PROOF. By inp-minimality of H we find some $e_0' \in H$ a solution to one of the rows $\{\widetilde{\psi}(x', b_i) : i \in \omega\}$ or $\{\widetilde{\psi}'(x', b_i') : i \in \omega\}$. By Ramsey, mutual indiscernibility and compactness we can find some $e_0 \in H$ which is still a solution to the same row, and moreover $\bar{b}, \bar{b}'$ are mutually indiscernible over e_0, so we can add it to the base. Let $\widetilde{\psi}_1(x', b_i) := \widetilde{\psi}(x', b_i) \wedge x' \neq e_0$, and the same for $\widetilde{\psi}_1'$.

As by assumption and mutual indiscernibility $\psi(\pi(x), b_0) \wedge P_l(x - g) \wedge \phi'(x, a_i)$ is consistent for each $i \in \omega$, and $\{\phi'(x, a_i)\}_{i \in \omega}$ is k_*-inconsistent, it follows that for infinitely many $i \in \omega$, we can find pairwise different $f_i \models \psi(\pi(x), b_0) \wedge P_l(x - g) \wedge \phi'(x, a_i)$. As all fibers of π are finite, this implies that in Fact in H for infinitely many i's we can find pairwise different $f_i' \models \widetilde{\psi}(x', b_0) \wedge \widetilde{\psi}'(x', b_i')$. Thus $\widetilde{\psi}_1(x', b_0) \wedge \widetilde{\psi}_1'(x', b_i')$ is consistent for some i, and so $\widetilde{\psi}_1(x', b_0) \wedge \widetilde{\psi}_1'(x', b_0')$ is consistent by mutual indiscernibility over e_0. Repeating this argument, by induction on $s \in \omega$ we can choose $e_s \in H$ such that each e_{s+1} satisfies one of the rows of the array $\{\widetilde{\psi}_{s+1}(x', b_i) : i \in \omega\}$, $\{\widetilde{\psi}_{s+1}'(x', b_i') : i \in \omega\}$, with $\widetilde{\psi}_{s+1}(x', b_i) := \widetilde{\psi}_s(x', b_i) \wedge x' \neq e_s$ and $\widetilde{\psi}_{s+1}'(x', b_i') := \widetilde{\psi}_s'(x', b_i') \wedge x' \neq e_s$. In particular, all e_s are pairwise distinct, and by pigeonhole infinitely many of them realize the same row, so in particular the same row of the original array. ⊣

So let now $(e_i : i \in \omega)$ be an infinite list of pairwise different solutions of $\{\widetilde{\psi}(x', b_i) : i \in \omega\}$ in H. In particular $e_i - \pi(g) = l\gamma_i$ for some $\gamma_i \in H$ with $(\gamma_i : i \in \omega)$ pairwise different. Let $\beta_i \in G$ be arbitrary such that $\pi(\beta_i) = \gamma_i$. As all

fibers of π are finite, we may assume that all of β_i's are pairwise different as well. Finally, let $f_i := l\beta_i + g$. We have:

- $(f_i : i \in \omega)$ are pairwise different,
- $P_l(f_i - g)$ holds for all $i \in \omega$, as $f_i - g = l\beta_i$,
- $\pi(f_i) = l\pi(\beta_i) + \pi(g) = l\gamma_i + \pi(g) = e_i$.

As the set $\bigvee_{i < k_* + 1} \left(\bigvee_{j < k} n_j x + c_{i,j} = 0 \right)$ is finite, then one of the f_i's realizes at least k_* elements of the first row—a contradiction.

SUBCASE 3.2. θ occurs in ϕ and θ' occurs in ϕ'. I.e. ϕ (respectively, ϕ') is not equivalent to the formula obtained from it by omitting θ (respectively, θ').

Syntactically, this is only possible if $b - g_0 \in K, b - g_0' \in K$, hence both $\pi(b) \in$ $\mathrm{dcl}(g_0)$ and $\pi(b) \in \mathrm{dcl}(g_0')$. By mutual indiscernibility of the rows it follows that $\bar{a}, \bar{a}'$ are mutually indiscernible over $\pi(b)$ and we can add it to the base.

Then by mutual indiscernibility of $\bar{a}, \bar{a}'$ over $\pi(b)$, Ramsey, compactness and applying an automorphism, we can find some $f \in G$ such that $\pi(f) = \pi(b)$ and $\bar{a}, \bar{a}'$ are mutually indiscernible over f. So we can add f to the base as well.

Taking $c := b - f$ we have $c \in K$. Translating by f, we can consider a new array $\widetilde{\phi}(x, \widetilde{a}_i), \widetilde{\phi}'(x, \widetilde{a}_i')$ where $\widetilde{\phi}(x, a_i) = \theta(x + f - h_i, \alpha_i) \wedge \psi(\pi(x + f), b_i) \wedge$ $\chi(x + f, c_i) \wedge P_l(x + f - g)$, and analogously for $\widetilde{\phi}'$. Note that the first column is realized by $c \in K$. By Case 2, we can find some c' realizing, say, the first row of the new array. But then taking $b' := c' + f$ clearly b' realizes the first row of the old array.

SUBCASE 3.3. θ occurs in ϕ, but θ' does not occur in ϕ' (and the symmetric case by permuting the rows).

By assumption $\phi'(x, a_i') = \psi'(\pi(x), b_i') \wedge \chi'(x, c_i') \wedge P_{l'}(x - g')$. As in Subcase 3.1, it follows that $\pi(b) \in \mathrm{dcl}(a_0)$, say $\pi(b) = f(a_0)$ for some $\emptyset$-definable function f. We have $b \models \phi'(x, a_0')$. In particular, $\models \psi'(f(a_0), b_0') \wedge$ "$f(a_0) - \pi(g')$ is l'-divisible". By mutual indiscernibility of $\bar{a}, \bar{a}'$ it follows that

$$\models \psi'(f(a_i), b_j') \wedge \text{"}f(a_i) - \pi(g') \text{ is } l'\text{-divisible"}$$

for all $i, j \in \omega$.

We may also assume that all of $\{f(a_i) : i \in \omega\}$ are pairwise different. Otherwise, if $f(a_i) = f(a_j)$ for some $i < j$, by indiscernibility $\pi(b) = f(a_0) = f(a_\infty)$, and so $\bar{a}, \bar{a}'$ are mutually indiscernible over $\pi(b)$—and we can conclude as in Subcase 3.2. It follows that the partial type $\{\psi'(x', b_j') \wedge \text{"}x' - \pi(g') \text{ is } l'\text{-divisible"}\}$ has infinitely many solutions in H, witnessed by $\{f(a_i) : i \in \omega\}$. Now this implies that the second row of the original array $\{\phi'(x, a_i') : i \in \omega\}$ is consistent. Namely, if K is infinite, then we conclude as in Case 3.1(a) using one of the solutions, and if K is finite we conclude as in Case 3.1(b). $\dashv$

PROOF OF THEOREM 1.4. Given a valued field $\bar{K}$ satisfying the assumption of Theorem 1.4, via the reductions in Sections 2 and 3 it is enough to demonstrate that (RV, k, Γ) is inp-minimal. For this it is enough to show that the assumptions of Proposition 4.1 are satisfied for $G = \mathrm{RV}$, K a Morleyzation of k and H a Morleyzation of Γ. Both K and H are inp-minimal as Morleyzation obviously preserves inp-minimality, H is torsion-free since Γ is an ordered abelian group.

As Γ is an inp-minimal ordered group, it follows from [18, Lemma 3.2] that $\Gamma/n\Gamma$ is finite for all $n \in \omega$. Besides, we have that $k^\times/(k^\times)^p$ is finite for all prime p by assumption. Therefore also $\mathrm{RV}/n\,\mathrm{RV}$ is finite for all n. Finally, $k^\times$ has finite n-torsion for all n. ⊣

4.1. Remarks and questions. We do not know if the assumption that $k^\times/(k^\times)^p$ is finite for all p is in Fact necessary. It follows from the proof of [7, Corollary 4.6] that if k is an inp-minimal field, then there can be at most one prime p for which $k^\times/(k^\times)^p$ is infinite.

PROBLEM 4.3. *Let k be an* inp*-minimal field. Is it true that $k^\times/(k^\times)^p$ is finite for all prime p? Or at least, can we omit this extra assumption from Theorem 1.4?*

The answer is positive for a dp-minimal field by the results of Johnson [14] (so under the assumptions of Theorem 1.4, we have that $\bar{K}$ is dp-minimal if and only if both k and Γ are dp-minimal), but the proof relies on the construction of a valuation which doesn't seem to be available in the general inp-minimal case.

Another natural direction is to generalize Theorem 1.4 from the case of burden 1 to a general burden calculation.

PROBLEM 4.4. *Let $\bar{K} = (K, \mathrm{RV}, \mathrm{rv})$ be a Henselian valued field of equicharacteristic 0, viewed as a structure in the RV-language. Is it true that $\mathrm{bdn}(\bar{K}) = \max\{\mathrm{bdn}(k), \mathrm{bdn}(\Gamma)\}$?*[1]

REFERENCES

[1] H. ADLER, *Strong theories, burden, and weight*, preprint, 2007.

[2] J. AX, *On the undecidability of power series fields.* **Proceedings of the American Mathematical Society**, vol. 16 (1965), no. 4, p. 846.

[3] L. BÉLAIR, *Types dans les corps valués munis d'applications coefficients.* **Illinois Journal of Mathematics**, vol. 43 (1999), no. 2, pp. 410–425.

[4] A. CHERNIKOV, *Indiscernible sequences and arrays in valued fields.* **RIMS Kokyuroku**, vol. 1718 (2010), pp. 127–131.

[5] ———, *Theories without the tree property of the second kind.* **Annals of Pure and Applied Logic**, vol. 165 (2014), no. 2, pp. 695–723.

[6] A. CHERNIKOV and M. HILS, *Valued difference fields and NTP2.* **Israel Journal of Mathematics**, vol. 204 (2014), no. 1, pp. 299–327.

[7] A. CHERNIKOV, I. KAPLAN, and P. SIMON, *Groups and fields with* NTP$_2$. **Proceedings of the American Mathematical Society**, vol. 143 (2015), no. 1, pp. 395–406.

---

[1]While the article was under review, this question was answered positively by Touchard [19].

[8] F. Delon, *Types sur C$((x))$*, **Study Group on Stable Theories, Second Year**: **1978/79** (B. Poizat, editor), Exp. No. 5, Secrétariat Math., Paris, 1981.

[9] A. Dolich and J. Goodrick, *Strong theories of ordered Abelian groups*. **Fundamenta Mathematicae**, vol. 236 (2017), pp. 269–296.

[10] A. Dolich, J. Goodrick, and D. Lippel, *Dp-minimality*: *Basic facts and examples*. **Notre Dame Journal of Formal Logic**, vol. 52 (2011), no. 3, pp. 267–288.

[11] J. Flenner, *Relative decidability and definability in Henselian valued fields*, this Journal, vol. 76 (2011), no. 04, pp. 1240–1260.

[12] N. Hempel and A. Onshuus, *Groups in NTP$_2$*. **Israel Journal of Mathematics**, vol. 217 (2017), no. 1, pp. 355–370.

[13] W. A. Johnson, **Fun with fields**, Ph.D. thesis, UC Berkeley, 2016.

[14] W. Johnson, *The canonical topology on dp-minimal fields*. **Journal of Mathematical Logic**, vol. 18 (2018), no. 02, p. 1850007.

[15] S. Montenegro, *Pseudo real closed fields, pseudo p-adically closed fields and NTP$_2$*. **Annals of Pure and Applied Logic**, vol. 168 (2017), no. 1, pp. 191–232.

[16] S. Shelah, **Classification Theory and the Number of Nonisomorphic Models**, second ed., Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland, Amsterdam, 1990.

[17] ———, *Strongly dependent theories*. **Israel Journal of Mathematics**, vol. 204 (2014), no. 1, pp. 1–83.

[18] P. Simon, *On dp-minimal ordered structures*, this Journal, vol. 76 (2011), no. 2, pp. 448–460.

[19] P. Touchard, *Burden in henselian valued fields*, preprint, 2018, arXiv:1811.08756.

DEPARTMENT OF MATHEMATICS
  UNIVERSITY OF CALIFORNIA LOS ANGELES
    LOS ANGELES, CA 90095-1555, USA
E-mail: chernikov@math.ucla.edu

DEPARTMENT OF MATHEMATICS
  UNIVERSITY OF CALIFORNIA, BERKELEY
    BERKELEY, CA 94720-3840, USA
E-mail: simon@math.berkeley.edu