Proceedings of the Royal Society of Edinburgh, **151**, 573–600, *2021*
DOI:10.1017/prm.2020.31

Transition waves for lattice Fisher-KPP equations with time and space dependence

Ning Wang and Zhi-Cheng Wang
School Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic of China (nwang15@lzu.edu.cn; wangzhch@lzu.edu.cn)

Xiongxiong Bao
School of Science, Chang'an University, Xi'an, Shaanxi 710064, People's Republic of China (baoxx2016@chd.edu.cn)

(MS Received 4 September 2019; Accepted 1 April 2020)

This paper is concerned with the existence results for generalized transition waves of space periodic and time heterogeneous lattice Fisher-KPP equations. By constructing appropriate subsolutions and supersolutions, we show that there is a critical wave speed such that a transition wave solution exists as soon as the least mean of wave speed is above this critical speed. Moreover, the critical speed we construct is proved to be minimal in some particular cases, such as space-time periodic or space independent.

Keywords: Lattice differential equations; generalized transition waves; space periodic; time heterogeneous

2010 Mathematics Subject Classification: 34K31; 37L60; 35B51; 35C07; 35B40

1. Introduction

The current paper is to explore transition wave solutions of the space-time heterogeneous lattice differential equation

$$\dot{u}_i(t) = d_{i+1}u_{i+1}(t) - (d_{i+1} + d_i)u_i(t) + d_i u_{i-1}(t) + f_i(t, u_i(t)), \qquad (i, t) \in \mathbb{Z} \times \mathbb{R}, \tag{1.1}$$

where d_i is a positive constant and $f(t, s) := \{f_i(t, s)\}_{i\in\mathbb{Z}}$ is of Fisher-KPP type satisfying $f_i(t, 0) = 0$ and $f_i(t, 1) = 0$ for all $(i, t) \in \mathbb{Z} \times \mathbb{R}$.

Equation (1.1) comes directly from many biological models in patchy environments [39, 40], which describes the growth of population or biological invasion process. Chen and Guo [10, 11] and Zinner et al. [43] established the existence of travelling waves for (1.1) in homogeneous media, that is,

$$\dot{u}_i(t) = du_{i+1}(t) - 2du_i(t) + du_{i-1}(t) + f(u_i(t)), \qquad (i, t) \in \mathbb{Z} \times \mathbb{R}. \tag{1.2}$$

In this case, an entire solution $u(t) = \{u_i(t)\}_{i\in\mathbb{Z}}$ of (1.2) is called a *travelling wave solution* (connecting 0 and 1) if $u_i(t) \in (0,1)$ for all $t \in \mathbb{R}$ and $i \in \mathbb{Z}$, and there are a constant c and a function $\Phi(z)$ such that

$$u_i(t) = \Phi(i - ct), \quad (i,t) \in \mathbb{Z}\times\mathbb{R}$$

and

$$\lim_{z\to-\infty} \Phi(z) = 1 \quad \text{and} \quad \lim_{z\to+\infty} \Phi(z) = 0,$$

where c and $\Phi(\cdot)$ are called the *wave speed* and *wave profile* of the travelling wave solution, respectively. For equation (1.1) in space periodic environments, namely,

$$\dot{u}_i(t) = d_{i+1}u_{i+1}(t) - (d_i + d_{i+1})u_i(t) + d_{i-1}u_{i-1}(t) + f_i(u_i(t)), \quad (i,t) \in \mathbb{Z}\times\mathbb{R}, \tag{1.3}$$

where $d_i = d_{i-N}$ and $f_i(\cdot) = f_{i-N}(\cdot)$ for all $i \in \mathbb{Z}$, N is a positive integer, Guo and Hamel [**16**] gave the notion of pulsating waves of (1.3) and proved the existence of pulsating waves. Here a *pulsating wave* of (1.3) connecting 0 and 1, which is a generalization of the notion of travelling wave solutions to space periodic environments, is an entire solution $u(t) = \{u_i(t)\}_{i\in\mathbb{Z}}$ of (1.3) such that $u_i(t) \in (0,1)$ for all $t \in \mathbb{R}$ and $i \in \mathbb{Z}$, and there are a constant c such that

$$u_i\left(t + \frac{N}{c}\right) = u_{i-N}(t), \quad (i,t) \in \mathbb{Z}\times\mathbb{R}$$

and

$$u_i(t) \to 1 \text{ as } i \to -\infty, \quad u_i(t) \to 0 \text{ as } i \to +\infty, \quad \text{locally in } t \in \mathbb{R}.$$

For more results on travelling wave solutions of lattice differential equations, we refer to [**9**, **12**, **13**, **17**, **20**] and the references therein. In particular, there were many works focussing on lattice differential equations with delay, see [**14**, **22**, **25**, **41**] and the references therein.

However, the environment may be not homogeneous, even not periodic. For equation (1.1) in a general heterogeneous environment, the notions of travelling wave solutions and pulsating waves above are no longer suitable. To investigate the front propagating dynamics for lattice differential equations in general heterogeneous media, Cao and Shen [**7**, **8**] gave the following notion of *transition waves* for (1.1) recently, which is a discrete version of the notion of transition waves given by Shen [**36**] for reaction-diffusion equations in continuous media.

Definition 1.1 Cao and Shen [**7**, **8**]. *An entire solution $u(t) = \{u_i(t)\}_{i\in\mathbb{Z}}$ of (1.1) is called a transition wave (connecting 0 and 1) if $u_i(t) \in (0,1)$ for all $t \in \mathbb{R}$ and $i \in \mathbb{Z}$, and there exists $J : \mathbb{R} \to \mathbb{Z}$ such that*

$$\lim_{i\to-\infty} u_{i+J(t)}(t) = 1 \quad \text{and} \quad \lim_{i\to+\infty} u_{i+J(t)}(t) = 0$$

uniformly in $t \in \mathbb{R}$.

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

Under such a notion of transition waves, Cao and Shen [7] established the existence of transition waves for the following lattice Fisher-KPP equations in time heterogeneous media

$$\dot{u}_i(t) = u_{i+1}(t) - 2u_i(t) + u_{i-1}(t) + u_i(t)\tilde{f}(t, u_i(t)), \quad (i,t) \in \mathbb{Z} \times \mathbb{R}. \tag{1.4}$$

Furthermore, Cao and Shen [8] developed a method to test the stability and uniqueness of the transition waves established in [7]. Clearly, equation (1.4) studied by [7] does not involve the space heterogeneity of environment. Thus, it is worthwhile to further explore this topic for (1.1) in general space-time heterogeneous media, and this constitutes the purpose of this paper.

Here we would like to point out that, in contrast to those done for equation (1.1) in patch environment, the front propagating dynamics for reaction-diffusion equations in continuous media has been widely studied, see [**1**–**3**, **6**, **15**, **23**, **24**, **27**–**29**, **32**, **33**, **38**, **42**] for the studies of travelling wave solutions and pulsating waves in homogeneous media and periodic media, respectively. In fact, equation (1.1) can be regarded as the spatially discrete version of the following reaction-diffusion equation:

$$u_t = (d(x)u_x)_x + f(t, x, u). \tag{1.5}$$

To describe the front propagating dynamics of reaction-diffusion equations in general heterogeneous media, the concept of *generalized transition waves* of (1.5) has been introduced by Berestycki and Hamel [**4**, **5**]:

Definition 1.2. *A positive time-global solution u of (1.5) is called a generalized transition wave (connecting 0 and 1) if $u(t,x) \in (0,1)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}$ and there exists a function $c \in L^\infty(\mathbb{R})$ such that*

$$\lim_{x \to -\infty} u\left(x + \int_0^t c(s)\,\mathrm{d}s, t\right) = 1, \quad \lim_{x \to +\infty} u\left(x + \int_0^t c(s)\,\mathrm{d}s, t\right) = 0,$$

uniformly with respect to $t \in \mathbb{R}$. The function c is called the speed of the generalized transition wave u.

From now on, there is a great progress on the study of generalized transition waves of reaction-diffusion equations in general heterogeneous media, see [**18**, **28**–**31**]. Shen [**36**] also gave a definition of generalized transition waves for general time-dependent equations and there were many important developments on the front propagating dynamics of reaction-diffusion equations, see [**21**, **34**, **35**, **37**, **38**].

As mentioned before, in this paper, we investigate the front propagating dynamics of (1.1). In this paper, we do not directly use the definition of transition waves of (1.1) given by Cao and Shen [**7**, **8**]. Here we give a more precise definition in a similar way to definition 1.2, in which the front location function $J(t)$ is defined by the wave speed function $c(t)$.

Definition 1.3. *A positive time-global solution $U(t) := \{U_i(t)\}_{(i,t)\in\mathbb{Z}\times\mathbb{R}}$ of (1.1) is called a transition wave (connecting 0 and 1) if $U_i(t) \in (0,1)$ for all $t \in \mathbb{R}$ and*

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

$i \in \mathbb{Z}$, and there exists a function $c \in L^\infty(\mathbb{R})$ satisfying

$$\lim_{i\to-\infty} U_{i+\left\langle\int_0^t c(s)\,\mathrm{d}s\right\rangle}(t) = 1 \quad \text{and} \quad \lim_{i\to+\infty} U_{i+\left\langle\int_0^t c(s)\,\mathrm{d}s\right\rangle}(t) = 0$$

uniformly with respect to $t \in \mathbb{R}$. The function c is called the speed of the transition wave $U(t)$, and denote the integer-valued function $J : \mathbb{R} \to \mathbb{Z}$, $t \to \left\langle \int_0^t c(s)\,\mathrm{d}s \right\rangle$ as the associated front location function, where the function $\langle\cdot\rangle : \mathbb{R} \to \mathbb{Z}$ is the integral function.

It is clear that if the function $U(t)$ is a transition wave of (1.1) according to definition 1.3, then it must be a transition wave of (1.1) according to definition 1.1. However, the inverse may not hold. In fact, for a transition wave of (1.1) according to definition 1.1, the speed function may not be bounded (but the speed function of equation (1.4) studied by Cao and Shen [**7**] is bounded). Notice, if $\zeta(t)$ is a bounded integer-valued function, then $J(t) + \zeta(t)$ is also a front location function. Thus, the front location function is not unique. On the other hand, it is easy to check that if $\tilde{J}(t)$ is another front location function, then $J(t) - \tilde{J}(t)$ is a bounded integer-valued function. Hence, front location functions are unique up to addition by bounded integer-valued functions. The front location function $J(t)$ tells the position of the transition front $U(t)$ as time t elapses, while the uniform-in-t limits show the bounded interface width, that is,

$$\forall\ 0 < \epsilon_1 < \epsilon_2,\ \sup_{t\in\mathbb{R}} \operatorname{diam}\{i \in \mathbb{Z} | \epsilon_1 \leqslant U_i(t) \leqslant \epsilon_2\} < \infty.$$

This paper is organized as follows. In § 2, we introduce some important definitions and state our main results. Section 3 is devoted to investigating the existence of transition waves of (1.1) with spatially periodic and temporal heterogeneous media. In § 4, we start with checking if the c_* in theorem 2.2 coincides with the minimal speed known to exist in some particular cases.

2. Preliminaries and main results

In this section, we introduce the standing definitions and state the main results of this paper. Let

$$X := l^\infty(\mathbb{Z}) = \left\{ u = \{u_i\}_{i\in\mathbb{Z}} \,\middle|\, \sup_{i\in\mathbb{Z}} |u_i| < \infty \right\}$$

equipped with norm $\|\cdot\|_\infty$, where $\|u\|_\infty = \sup_{i\in\mathbb{Z}} |u_i|$. For given $u^1, u^2 \in X$, we write $u^1 \leqslant u^2$ if $u_i^1 \leqslant u_i^2$ for all $i \in \mathbb{Z}$, $u^1 < u^2$ if $u^1 \leqslant u^2$ but $u^1 \neq u^2$, and $u^1 \ll u^2$ if $u_i^1 < u_i^2$ for all $i \in \mathbb{Z}$. In order to derive the existence result, we first make some assumptions as follows:

(H0) $d_i = d_{i-N} > 0$ and $f_i(t,s) = f_{i-N}(t,s)$ where $i \in \mathbb{Z}$, $(t,s) \in \mathbb{R}^2$ and N is a positive integer.

(H1) $f(t,s) \in C^1(\mathbb{R}\times\mathbb{R}, X)$ with $f(t,s)$ and $\partial_s f(t,s)$ being bounded uniformly in $(t,s) \in \mathbb{R}^2$; and $\mu(t) := \{\mu_i(t)\}_{i\in\mathbb{Z}} = \{\partial_s f_i(t,0)\}_{i\in\mathbb{Z}}$ is uniformly Hölder continuous in $t \in \mathbb{R}$ and satisfies $0 < \inf_{(i,t)\in\mathbb{Z}\times\mathbb{R}} \mu_i(t) \leqslant \sup_{(i,t)\in\mathbb{Z}\times\mathbb{R}} \mu_i(t) < +\infty$.

(H2) For any $(i,t) \in \mathbb{Z}\times\mathbb{R}$, $f_i(t,0) = f_i(t,1) = 0$; $f_i(t,s) > 0$ for all $(i,t) \in \mathbb{Z}\times\mathbb{R}$ and $s \in (0,1)$; for each $i \in \mathbb{Z}$ and $(t,s) \in \mathbb{R}\times(0,+\infty)$, $f_i(t,s) \leqslant \mu_i(t)s$; and

$$\exists C > 0,\ \delta,\nu \in (0,1],\ \forall (i,t) \in \mathbb{Z}\times\mathbb{R},\ s \in (0,\delta),\ f_i(t,s) \geqslant \mu_i(t)s - Cs^{1+\nu}.$$

By (H1), for any given $u^0 \in X$ and $t_0 \in \mathbb{R}$, (1.1) has a unique (global) solution, denoted by $u(t;t_0,u^0)$ with $u(t_0;t_0,u^0) = u^0$. In the following, we give the definitions of the least mean (respectively the upper mean) of a function.

Definition 2.1 Nadin and Rossi [**29**]. *The least mean (respectively the upper mean) over $\mathbb{R}$ of a function $g \in L^\infty(\mathbb{R})$ is given by*

$$\lfloor g\rfloor := \lim_{T\to+\infty}\inf_{t\in\mathbb{R}} \frac{1}{T}\int_t^{t+T} g(s)\,\mathrm{d}s \quad \left(\text{resp. } \lceil g\rceil := \lim_{T\to+\infty}\sup_{t\in\mathbb{R}} \frac{1}{T}\int_t^{t+T} g(s)\,\mathrm{d}s\right).$$

As shown in proposition 3.1 of [**29**], the definitions of $\lfloor g\rfloor$ and $\lceil g\rceil$ do not change if one replaces

$$\lim_{T\to+\infty}\inf_{t\in\mathbb{R}} \frac{1}{T}\int_t^{t+T} g(s)\,\mathrm{d}s \quad \text{and} \quad \lim_{T\to+\infty}\sup_{t\in\mathbb{R}} \frac{1}{T}\int_t^{t+T} g(s)\,\mathrm{d}s$$

with

$$\sup_{T>0}\inf_{t\in\mathbb{R}} \frac{1}{T}\int_t^{t+T} g(s)\,\mathrm{d}s \quad \text{and} \quad \inf_{T>0}\sup_{t\in\mathbb{R}} \frac{1}{T}\int_t^{t+T} g(s)\,\mathrm{d}s$$

respectively in the above expressions. This shows that$\lfloor g\rfloor$ and $\lceil g\rceil$ are well defined for any $g \in L^\infty(\mathbb{R})$. In the following, we state our main results. The next theorem consists of a sufficient condition for the existence of generalized transition waves, expressed in terms of their speeds.

Theorem 2.2. *Assume that* (H0)–(H2) *hold. Then there exists $c_* \in \mathbb{R}$ such that for every $\gamma > c_*$,* (1.1) *has a transition wave with a speed $c \in L^\infty(\mathbb{R})$ such that $\lfloor c\rfloor = \gamma$.*

Remark 2.3. *If $N = 1$, then (1.1) can be represented in the form*

$$\dot u_i(t) = u_{i+1}(t) - 2u_i(t) + u_{i-1}(t) + f(t,u_i(t)), \quad (i,t) \in \mathbb{Z}\times\mathbb{R}. \tag{2.1}$$

*When the reaction term $f(t,s)$ of (2.1) is replaced by $s\tilde f(t,s)$, that is, equation (2.1) can be written as (1.4), the critical speed c_* has been characterized by Cao and Shen [**7**], in which they need an extra assumption on $\tilde f$:*

$$\frac{\partial \tilde f}{\partial s}(t,s) < 0 \quad \text{for } s \geqslant 0,\ t \in \mathbb{R}$$

and the condition $\inf_{(i,t)\in\mathbb{Z}\times\mathbb{R}} \mu_i(t) > 0$ $(= \inf_{t\in\mathbb{R}} \tilde f(t,0) > 0)$ given in (H1), which will be needed in lemma 3.7 below, could be relaxed by

$$\liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t \tilde f(\tau,0)\,\mathrm{d}\tau > 0.$$

*In this paper, we adopt a different approach to obtain the existence of transition waves of (1.1), which contain the result of Cao and Shen in [**7**] [see (E3) below].*

A natural question is to determine whether the c_* in theorem 2.2 is the minimal speed or not; that is, do transition waves with speed c such that $\lfloor c \rfloor < c_*$ exist? In § 4, we turn out that the c_* coincides with the minimal speed of transition waves for some particular cases, such as the coefficients are space-time periodic or space-independent. The answer in the general media is only partial.

3. Existence of transition waves

Throughout this section, we assume that (H0)–(H2) hold. To show the existence of transition waves, we present some important lemmas. The first lemma is to establish the comparison principle, in which the following assumption on integer-valued functions is needed.

(A) Let $J(t)$ be an integer-valued function on $t \in [t_0, T)$. For any $t' \in [t_0, T)$, $J(\cdot)$ satisfies either $J(t) \equiv J(t')$ for all $t \in (t', T)$, or there exists $t_1 \in (t', T)$ such that $|J(t_1) - J(t_0)| = 1$ and either $J(t) \equiv J(t')$ for $t \in (t', t_1)$ or $J(t) \equiv J(t_1)$ for $t \in (t', t_1)$.

Lemma 3.1. (1) *Let either $J_1(t)$ be an integer-valued functions on $t \in [t_0, T)$ satisfying the assumption* (A) *or $J_1(t) = -\infty$ on $t \in [t_0, T)$. Let either $J_2(t)$ be an integer-valued functions on $t \in [t_0, T)$ satisfying the assumption* (A) *or $J_2(t) = +\infty$ on $t \in [t_0, T)$, too. Suppose further that $J_2(t) - J_1(t) \geqslant 2$ on $t \in [t_0, T)$. Let $v^1(t) := \{v_i^1(t)\}$ and $v^2(t) := \{v_i^2(t)\}$ be defined in the set*

$$\overline{\Omega} := \{(i, t) \in \mathbb{Z} \times [t_0, T) | J_2(t) \geqslant i \geqslant J_1(t), t \in [t_0, T)\}.$$

Assume that $v^1(t) := \{v_i^1(t)\}$ and $v^2(t) := \{v_i^2(t)\}$ are bounded between 0 and 1 on $\overline{\Omega}$ and continuous in t. Suppose that $v^1(t) := \{v_i^1(t)\}$ and $v^2(t) := \{v_i^2(t)\}$ are of C^1 type in t and satisfy

$$\begin{aligned} &\dot{v}_i^1(t) - d_{i+1} v_{i+1}^1(t) + (d_i + d_{i+1}) v_i^1(t) - d_i v_{i-1}^1(t) - f_i(t, v_i^1(t)) \\ &\quad \geqslant \dot{v}_i^2(t) - d_{i+1} v_{i+1}^2(t) + (d_i + d_{i+1}) v_i^2(t) - d_i v_{i-1}^2(t) - f_i(t, v_i^2(t)) \end{aligned}$$

for $(i, t) \in \Omega$, where $\Omega := \{(i, t) \in \mathbb{Z} \times [t_0, T) | J_2(t) > i > J_1(t), t \in [t_0, T)\}$. Then $v_i^1(t) \geqslant v_i^2(t)$ for $t \in [t_0, T)$ and $J_2(t) > i > J_1(t)$ provided that $v_i^1(t_0) \geqslant v_i^2(t_0)$ for all $J_2(t_0) \geqslant i \geqslant J_1(t_0)$ and there holds one of the following four conditions:

(a) *$v_{J_1(t)}^1(t) \geqslant v_{J_1(t)}^2(t)$ and $v_{J_2(t)}^1(t) \geqslant v_{J_2(t)}^2(t)$ for any $t \in [t_0, T)$ if both $J_1(t)$ and $J_2(t)$ are the integer-valued functions on $t \in [t_0, T)$ satisfying the assumption* (A);

(b) *$v_{J_2(t)}^1(t) \geqslant v_{J_2(t)}^2(t)$ for any $t \in [t_0, T)$ and $\liminf_{i \to -\infty}(v_i^1(t) - v_i^2(t)) \geqslant 0$ uniformly on $t \in [t_0, t']$ for any $t' \in (t_0, T)$ if $J_1(t) = -\infty$ on $t \in [t_0, T)$ and $J_2(t)$ is an integer-valued function on $t \in [t_0, T)$ satisfying the assumption* (A);

(c) *$v_{J_1(t)}^1(t) \geqslant v_{J_1(t)}^2(t)$ for any $t \in [t_0, T)$ and $\liminf_{i \to +\infty}(v_i^1(t) - v_i^2(t)) \geqslant 0$ uniformly on $t \in [t_0, t']$ for any $t' \in (t_0, T)$ if $J_2(t) = +\infty$ on $t \in [t_0, T)$*

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

and $J_1(t)$ *is an integer-valued function on* $t \in [t_0, T)$ *satisfying the assumption* (A)*;*

(d) $\liminf_{i\to-\infty}(v_i^1(t) - v_i^2(t)) \geqslant 0$ *and* $\liminf_{i\to+\infty}(v_i^1(t) - v_i^2(t)) \geqslant 0$ *uniformly on* $t \in [t_0, t']$ *for any* $t' \in (t_0, T)$ *if* $J_1(t) = -\infty$ *and* $J_2(t) = +\infty$ *on* $t \in [t_0, T)$*.*

(2) *Let* $J_1(t)$ *and* $J_2(t)$ *be as in* (1)*. Let* $\overline{\Omega}$ *and* Ω *be defined in* (1)*. Suppose* $w^1(t) := \{w_i^1(t)\}$ *and* $w^2(t) := \{w_i^2(t)\}$ *be defined in* $(i, t) \in \overline{\Omega}$ *and be continuous in* t*. Suppose that* $w^1(t) := \{w_i^1(t)\}$ *and* $w^2(t) := \{w_i^2(t)\}$ *are of* C^1 *type in* t *and satisfy*

$$\begin{aligned}&\dot{w}_i^1(t) - d_{i+1}w_{i+1}^1(t) + (d_i + d_{i+1})w_i^1(t) - d_i w_{i-1}^1(t) - \mu_i(t)w_i^1(t)\\&\quad\geqslant \dot{w}_i^2(t) - d_{i+1}w_{i+1}^2(t) + (d_i + d_{i+1})w_i^2(t) - d_i w_{i-1}^2(t) - \mu_i(t)w_i^2(t)\end{aligned}$$

for $(i, t) \in \Omega$*. Then* $w_i^1(t) \geqslant w_i^2(t)$ *for* $t \in [t_0, T)$ *and* $J_2(t) > i > J_1(t)$ *provided that* $w_i^1(t_0) \geqslant w_i^2(t_0)$ *for all* $J_2(t_0) \geqslant i \geqslant J_1(t_0)$ *and one of the conditions* (a), (b), (c) *and* (d) *in* (1) *holds.*

Proof. We prove (1). Here we only consider the case that the condition (a) holds. In this case, we have that both $J_1(t)$ and $J_2(t)$ satisfy (A) respectively. Without loss of generality, we assume that there exists $t_1 \in (t_0, T)$ such that $J_1(t) \equiv J_1(t_0)$ and $J_2(t) \equiv J_2(t_0)$ for any $t \in (t_0, t_1)$, and

$$|J_1(t_1) - J_1(t_0)| = 1 \quad \text{or} \quad |J_2(t_1) - J_2(t_0)| = 1. \tag{3.1}$$

Let $\vartheta_i(t) = e^{ct}(v_i^1(t) - v_i^2(t))$, where $J_2(t_0) \geqslant i \geqslant J_1(t_0)$, $t \in [t_0, t_1)$ and c is a constant to be determined later. Then

$$\begin{aligned}\dot{\vartheta}_i(t) &= ce^{ct}(v_i^1(t) - v_i^2(t)) + e^{ct}(\dot{v}_i^1(t) - \dot{v}_i^2(t))\\&\geqslant d_{i+1}\vartheta_{i+1}(t) + d_i\vartheta_{i-1}(t) + (a_i(t) - (d_i + d_{i+1}) + c)\vartheta_i(t)\end{aligned} \tag{3.2}$$

for $J_2(t_0) > i > J_1(t_0)$ and a.e. $t \in [t_0, t_1)$, where

$$a_i(t) = \int_0^1 \frac{\partial f_i}{\partial s}(t, \tau v_i^1(t) + (1-\tau)v_i^2(t))\,\mathrm{d}\tau \quad \text{for } J_2(t_0) > i > J_1(t_0),\ t \in [t_0, t_1).$$

Let $p_i(t) = a_i(t) - (d_i + d_{i+1}) + c$. By the boundedness of $v^1(t)$ and $v^2(t)$ and the periodicity of d_i, then there is a $c > 0$ such that

$$\inf_{J_2(t_0)>i>J_1(t_0),t\in[t_0,t_1)} p_i(t) > 0.$$

In the following, one claims that $\vartheta_i(t) \geqslant 0$ for $J_2(t_0) > i > J_1(t_0)$ and $t \in [t_0, t_1)$.

Denote $p_0 = \sup_{J_2(t_0)>i>J_1(t_0),t\in[t_0,t_1)} p_i(t)$ and $d_{\max} := \max_{i\in\mathbb{Z}} d_i$. It is sufficient to prove the claim for $J_2(t_0) > i > J_1(t_0)$ and $t \in [t_0, t_0 + T_0]$ with $T_0 = \frac{1}{2}\min\{t_1 - t_0, 1/(p_0 + 2d_{\max})\}$. Assume, towards contradiction, that there exists $J_2(t_0) > \tilde{i} >$

$J_1(t_0)$ and $\widetilde{t} \in [t_0, t_0 + T_0]$ such that $\vartheta_{\tilde{i}}(\tilde{t}) < 0$. Thus

$$\vartheta_{\inf} = \inf_{J_2(t_0)>i>J_1(t_0), t\in[t_0,t_0+T_0]} \vartheta_i(t) < 0.$$

Hence, we can find some sequences $J_2(t_0) > i_n > J_1(t_0)$ and $t_n \in [t_0, t_0 + T_0]$ such that

$$\vartheta_{i_n}(t_n) \to \vartheta_{\inf} \quad \text{as } n \to \infty.$$

By virtue of the condition (a), (3.2) and the fundamental theorem of calculus for Lebesgue integrals, we get

$$\begin{aligned}
\vartheta_{i_n}(t_n) - \vartheta_{i_n}(t_0) &\geqslant \int_{t_0}^{t_n} [d_{i_n+1}\vartheta_{i_n+1}(t) + d_{i_n}\vartheta_{i_n-1}(t) + p_{i_n}(t)\vartheta_{i_n}(t)]\,\mathrm{d}t \\
&\geqslant \int_{t_0}^{t_n} [2d_{\max}\vartheta_{\inf} + p_{i_n}(t)\vartheta_{\inf}]\,\mathrm{d}t \\
&\geqslant T_0(2d_{\max} + p_0)\vartheta_{\inf} \quad \text{for } n \geqslant 1.
\end{aligned}$$

Recall that $\vartheta_{i_n}(t_0) \geqslant 0$, then

$$\vartheta_{i_n}(t_n) \geqslant T_0(2d_{\max} + p_0)\vartheta_{\inf} \quad \text{for } n \geqslant 1.$$

It follows that

$$\vartheta_{\inf} \geqslant T_0(2d_{\max} + p_0)\vartheta_{\inf} > \frac{1}{2}\vartheta_{\inf} \quad \text{as } n \to \infty,$$

which is a contradiction due to $\vartheta_{\inf} < 0$. Thus, $v_i^1(t) \geqslant v_i^2(t)$ for $J_2(t_0) \geqslant i \geqslant J_1(t_0)$ and $t \in [t_0, t_0 + T_0]$. Repeating the above procedure on $[t_0 + T_0, t_0 + 2T_0]$, $[t_0 + 2T_0, t_0 + 3T_0], \ldots$, we can get $v_i^1(t) \geqslant v_i^2(t)$ for $J_2(t_0) \geqslant i \geqslant J_1(t_0)$ and $t \in [t_0, t_1]$. Further using (3.1) and the continuous of $v_i^1(t)$ and $v_i^2(t)$ on $t = t_1$ for any given $J_2(t_1) \geqslant i \geqslant J_1(t_1)$, we get

$$v_i^1(t_1) \geqslant v_i^2(t_1) \quad \text{for all } J_2(t_1) \geqslant i \geqslant J(t_1).$$

Therefore, repeating the above method, we have

$$v_i^1(t) \geqslant v_i^2(t) \quad \text{for } J_2(t) \geqslant i \geqslant J_1(t) \quad \text{and} \quad t \in [t_0, T).$$

We can prove (2) by the similar arguments as above. The lemma is thus proved. □

REMARK 3.2. *In lemma 3.1(1), if there further exists $J_2(t_0) > i_0 > J_1(t_0)$ such that $v_{i_0}^1(t_0) > v_{i_0}^2(t_0)$, then one can easily get that $v_i^1(t) > v_i^2(t)$ for any $t \in (t_0, T)$ and $J_2(t) > i > J_1(t)$. Similarly, in lemma 3.1(2), if there further exists $J_2(t_0) > i_0 > J_1(t_0)$ such that $w_{i_0}^1(t_0) > w_{i_0}^2(t_0)$, then one can easily get that $w_i^1(t) > w_i^2(t)$ for any $t \in (t_0, T)$ and $J_2(t) > i > J_1(t)$.*

Here we would like to mention that, compared with the existing results on the comparison principle of the lattice differential equations, our theorem (lemma 3.1) is more general. In fact, the comparison principle for the lattice differential equations was usually established for the infinite lattices ($J_1(t) = -\infty$ and $J_2(t) = +\infty$)

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

*(see [**8**, lemma 2.1]) or finite lattices ($J_1(t) = 0$ and $J_2(t) = N$) (see [**22**, **26**]). Therefore, lemma 3.1 is of independent interest due to possible applications in future.*

Consider the following linearized equation of (1.1)around 0 :

$$\dot{u}_i(t) = d_{i+1}u_{i+1}(t) - (d_{i+1} + d_i)u_i(t) + d_i u_{i-1}(t) + \mu_i(t)u_i(t), \quad (i,t) \in \mathbb{Z} \times \mathbb{R}. \tag{3.3}$$

Since equation (3.3) is spatially periodic, we expect to look for the time global solution $u(t) := \{u_i(t)\}_{i\in\mathbb{Z}}$ of this equation under the form

$$u_i(t) = \mathrm{e}^{-\lambda i}\eta_{\lambda;i}(t), \qquad (i,t) \in \mathbb{Z} \times \mathbb{R}, \tag{3.4}$$

where $\lambda > 0$ is the spatial exponential decay rate and

$$\eta_{\lambda;i}(t) = \eta_{\lambda;i-N}(t) > 0, \quad (i,t) \in \mathbb{Z} \times \mathbb{R}. \tag{3.5}$$

Substituting (3.4) into (3.3), it reduces that $\{\eta_{\lambda;i}(t)\}_{(i,t)\in\mathbb{Z}\times\mathbb{R}}$ must satisfy

$$\begin{aligned} \dot{\eta}_{\lambda;i}(t) = d_{i+1}\,\mathrm{e}^{-\lambda}\eta_{\lambda;i+1}(t) - (d_{i+1} + d_i)\eta_{\lambda;i}(t) \\ + d_i\,\mathrm{e}^{\lambda}\eta_{\lambda;i-1}(t) + \mu_i(t)\eta_{\lambda;i}(t), \quad (i,t) \in \mathbb{Z} \times \mathbb{R}. \end{aligned} \tag{3.6}$$

The following lemma shows some properties of the solution of equation (3.6) and is the key point where the spatial periodicity hypothesis (H0) is used.

LEMMA 3.3. *For all $\lambda > 0$, there exists a time-global solution $\eta_\lambda(t) = \{\eta_{\lambda;i}(t)\}_{i\in\mathbb{Z}}$ of (3.6) with the form (3.5), which is unique up to a multiplicative constant and satisfies*

$$\eta_{\lambda;i}(t+T) \leqslant \max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t) \exp\left[\max_{i\in\mathbb{Z}} d_i(\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda})T + \int_t^{t+T} \max_{i\in\mathbb{Z}} \mu_i(s)\,\mathrm{d}s\right], \tag{3.7}$$

$$\begin{aligned} \eta_{\lambda;i}(t+T) \geqslant C_\lambda \max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t) \exp\Bigg[&\min_{i\in\mathbb{Z}}(d_{i+1}\,\mathrm{e}^{-\lambda} + d_i\,\mathrm{e}^{\lambda} - d_{i+1} - d_i)T \\ &+ \int_t^{t+T} \min_{i\in\mathbb{Z}} \mu_i(s)\,\mathrm{d}s\Bigg], \end{aligned} \tag{3.8}$$

where $(i,t) \in \mathbb{Z} \times \mathbb{R}$, $T \geqslant 0$, and the constant $C_\lambda > 0$ depends on λ.

Proof. Note that the equation (3.6) can be regarded as an ordinary differential system in $\mathbb{R}^N$ due to the spatial periodicity of its coefficients. Denote $m = \inf_{(i,t)\in\mathbb{Z}\times\mathbb{R}} \mu_i(t) > 0$. We know from [**16**, lemma 2.1] that for each $\lambda > 0$, there exists a unique element $\xi_\lambda^- = \{\xi_{\lambda,i}^-\}_{i\in\mathbb{Z}}$ of K_{per} with $\max_{i\in\mathbb{Z}} \xi_{\lambda,i}^- = 1$ such that

$$d_{i+1}\,\mathrm{e}^{-\lambda}\xi_{\lambda,i+1}^- - (d_{i+1} + d_i)\xi_{\lambda,i}^- + d_i\,\mathrm{e}^{\lambda}\xi_{\lambda,i-1}^- + m\xi_{\lambda,i}^- = \Phi(\lambda)\xi_{\lambda,i}^-, \quad i \in \mathbb{Z},$$

where $K_{per} := \{\xi^\lambda = \{\xi_i^\lambda\}_{i\in\mathbb{Z}} \in X, \;\; \xi_i^\lambda > 0 \;\text{ and }\; \xi_{i-N}^\lambda = \xi_i^\lambda \;\text{ for all }\; i \in \mathbb{Z}\}$ and $\Phi(\lambda) > 0$ is the principal eigenvalue of the $N \times N$ matrix $A_\lambda := [a_{\lambda;i,j}]$ defined

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

by

$$\begin{cases} a_{\lambda;i,i} = -(d_{i+1}+d_i)+m, & i\in\{1,\cdots,N\},\\ a_{\lambda;i,i+1} = d_{i+1}\,\mathrm{e}^{-\lambda}, & i\in\{1,\cdots,N-1\},\\ a_{\lambda;i+1,i} = d_{i+1}\,\mathrm{e}^{\lambda}, & i\in\{1,\cdots,N-1\},\\ a_{\lambda;1,N} = d_1\,\mathrm{e}^{\lambda}, & \\ a_{\lambda;N,1} = d_1\,\mathrm{e}^{-\lambda}, & \\ a_{\lambda;i,j} = 0, & |i-j|\geqslant 2 \text{ and } (i,j)\notin\{(1,N),(N,1)\}. \end{cases}$$

Then the function $\underline{v}_\lambda(t) := \{\underline{v}_{\lambda;i}(t)\}_{i\in\mathbb{Z}} = \{\mathrm{e}^{\Phi(\lambda)t}\xi^-_{\lambda,i}\}_{i\in\mathbb{Z}}$ solves

$$\dot{u}_i(t) = d_{i+1}\,\mathrm{e}^{-\lambda}u_{i+1}(t) - (d_{i+1}+d_i)u_i(t) + d_i\,\mathrm{e}^{\lambda}u_{i-1}(t) + mu_i(t), \quad (i,t)\in\mathbb{Z}\times\mathbb{R}$$

and it is a subsolution of (3.6). Similarly, there also exists a positive constant $\Psi(\lambda)$ and a positive vector $\xi^+_\lambda = \{\xi^+_{\lambda,i}\}_{i\in\mathbb{Z}} \in K_{per}$ with $\max_{i\in\mathbb{Z}}\xi^+_{\lambda,i} = 1$ such that

$$d_{i+1}\,\mathrm{e}^{-\lambda}\xi^+_{\lambda,i+1} - (d_{i+1}+d_i)\xi^+_{\lambda,i} + d_i\,\mathrm{e}^{\lambda}\xi^+_{\lambda,i-1} + M\xi^+_{\lambda,i} = \Psi(\lambda)\xi^+_{\lambda,i}, \quad i\in\mathbb{Z},$$

where $M = \sup_{(i,t)\in\mathbb{Z}\times\mathbb{R}}\mu_i(t)$. Clearly, $\overline{v}_\lambda(t) := \{\overline{v}_{\lambda;i}(t)\}_{i\in\mathbb{Z}} = \{\mathrm{e}^{\Psi(\lambda)t}\xi^+_{\lambda,i}\}_{i\in\mathbb{Z}}$ solves

$$\dot{u}_i(t) = d_{i+1}\,\mathrm{e}^{-\lambda}u_{i+1}(t) - (d_{i+1}+d_i)u_i(t) + d_i\,\mathrm{e}^{\lambda}u_{i-1}(t) + Mu_i(t), \quad (i,t)\in\mathbb{Z}\times\mathbb{R}$$

and it is a supersolution of (3.6). For each $n\in\mathbb{N}$, let $\eta^n_\lambda(t;k) = \{\eta^n_{\lambda;i}(t;k)\}_{i\in\mathbb{Z},t\geqslant -n}$ be the solution of (3.6) with initial datum $\eta^n_\lambda(-n;k) = k\xi^-_\lambda$, where $k\in[0,k'_n]$ and $k'_n := \mathrm{e}^{-\Phi(\lambda)n}$. By the comparison principle [see lemma 3.1(2)], we have

$$\eta^n_\lambda(0;k'_n) \geqslant \underline{v}_\lambda(0) = \xi^-_\lambda,$$

which implies that

$$0 = \max_{i\in\mathbb{Z}}\eta^n_\lambda(0;0) < 1 = \max_{i\in\mathbb{Z}}\xi^-_{\lambda,i} \leqslant \max_{i\in\mathbb{Z}}\eta^n_\lambda(0;k'_n).$$

Then by the continuous dependence of the solutions on initial values, there exists $k_n\in(0,k'_n]$ such that

$$\max_{i\in\mathbb{Z}}\eta^n_{\lambda;i}(0;k_n) = 1.$$

In fact, due to $m = \inf_{(i,t)\in\mathbb{Z}\times\mathbb{R}}\mu_i(t) > 0$, we further have $\max_{i\in\mathbb{Z}}\eta^n_{\lambda;i}(t;k_n) \leqslant 1$ for all $t\in[-n,0]$. By the comparison principle, we also have $\eta^n_{\lambda;i}(t;k_n) \leqslant k''\overline{v}(t)$ for all $t\geqslant 0$, where the positive constant k'' satisfies $k''\xi^+_{\lambda,i}\geqslant 1$ for all $i\in\mathbb{Z}$. Thus, we can find a positive, N-periodic in i solution $\eta_\lambda(t) = \{\eta_{\lambda;i}(t)\}_{i\in\mathbb{Z}}$ of (3.6), which is the limit of the sequence (up to extracting a subsequence) $\eta^n_\lambda(t)$ locally uniformly in $(i,t)\in\mathbb{Z}\times\mathbb{R}$ as $n\to\infty$.

Let us show that $\eta_\lambda(t)$ satisfies (3.7) and (3.8). For given $t_0 \in \mathbb{R}$, the function $\overline{\eta}_\lambda(t) := \{\overline{\eta}_{\lambda;i}(t)\}_{i\in\mathbb{Z},t\geqslant t_0}$ with

$$\overline{\eta}_{\lambda;i}(t) = \max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t_0) \exp\left[\max_{i\in\mathbb{Z}} d_i(\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda})(t - t_0) + \int_{t_0}^{t} \max_{i\in\mathbb{Z}} \mu_i(s)\,\mathrm{d}s\right]$$

is a supersolution of (3.6) on $t \geqslant t_0$. Then by lemma 3.1, we have $\eta_{\lambda;i}(t) \leqslant \overline{\eta}_{\lambda;i}(t)$ for any $i \in \mathbb{Z}$ and $t \geqslant t_0$, which further reduces to

$$\eta_{\lambda;i}(T + t_0) \leqslant \max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t_0) \exp\left[\max_{i\in\mathbb{Z}} d_i(\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda})T + \int_{t_0}^{T+t_0} \max_{i\in\mathbb{Z}} \mu_i(s)\,\mathrm{d}s\right] \quad (3.9)$$

for any $i \in \mathbb{Z}$ and $T \geqslant 0$ by letting $t = T + t_0$. By the arbitrariness of $t_0 \in \mathbb{R}$, we can derive the inequality (3.7) by replacing t_0 with t in (3.9).

Next, we show that (3.8) holds. We claim that there is $\tilde{C} > 0$ depending on λ such that

$$\eta_{\lambda;i}(t) \leqslant \tilde{C}\eta_{\lambda;i+J}(t + 1) \quad \text{for } (i,t) \in \mathbb{Z}\times\mathbb{R} \quad \text{and} \quad J \in \mathbb{Z}. \qquad (3.10)$$

Fix $J_0 \in \mathbb{Z}$ and let $(i_0, t_0) \in \mathbb{Z}\times\mathbb{R}$ be given. Applying lemma 3.1 to problem (3.6), one immediately has that

$$\eta_{\lambda;i_0+J_0}(t_0 + 1) \geqslant v_{i_0+J_0}(t_0 + 1), \qquad (3.11)$$

where $\{v_i(t)\}_{i\in\mathbb{Z}}$ satisfies (3.6) for $t > t_0$ with $v_{i_0}(t_0) = \eta_{\lambda;i_0}(t_0)$ and $v_i(t_0) = 0$ for all $i \neq i_0$. For all $i \in \mathbb{Z}$, call now $\gamma_i := z^i_{i+J_0}(1)$, where $\{z^i_j(t)\}_{j\in\mathbb{Z}}$ satisfies (3.6) for $t > 0$ with $z^i_i(0) = 1$ and $z^i_j(0) = 0$ for all $j \neq i$. One has $z^i_j(t) \geqslant 0$ for all $(j,t) \in \mathbb{Z}\times(0,+\infty)$, whence $\gamma_i \geqslant 0$. If $\gamma_i = 0$, then $(z^i_{i+J_0})'(1) = z^i_{i+J_0}(1) = 0$ and $z^i_{i+J_0-1}(1) = z^i_{i+J_0+1}(1) = 0$. By induction, $z^i_j(1) = 0$ for all $j \in \mathbb{Z}$. But

$$(z^i_j)'(t) \geqslant -2\max_{i\in\mathbb{Z}} d_i z^i_j(t)$$

for all $j \in \mathbb{Z}$. In particular,

$$z^i_i(1) \geqslant \mathrm{e}^{-2\max_{i\in\mathbb{Z}} d_i} z^i_i(0) = \mathrm{e}^{-2\max_{i\in\mathbb{Z}} d_i} > 0,$$

which gives a contradiction. As a consequence, each γ_i is positive. On the other hand, $\gamma_i = \gamma_{i-N}$ for all $i \in \mathbb{Z}$, due to the periodicity of (3.6). Therefore, $\Gamma_{J_0} := \min_{i\in\mathbb{Z}} \gamma_i > 0$. Eventually, one has

$$v_{i_0+J_0}(t_0 + 1) = \gamma_{i_0}\eta_{\lambda;i_0}(t_0).$$

Putting the last formula into (3.11) yields

$$\eta_{\lambda;i_0+J_0}(t_0 + 1) \geqslant \Gamma_{J_0}\eta_{\lambda;i_0}(t_0).$$

This together with the arbitrariness of i_0 and t_0 implies that

$$\eta_{\lambda;i+J_0}(t + 1) \geqslant \Gamma_{J_0}\eta_{\lambda;i}(t)$$

for all $(i,t)\in\mathbb{Z}\times\mathbb{R}$. Let $\tilde{C}=1/(\min_{J\in\{1,\ldots,N\}}\Gamma_J)$. Thus we get

$$\eta_{\lambda;i}(t)\leqslant \tilde{C}\eta_{\lambda;i+J}(t+1),\quad (i,t)\in\mathbb{Z}\times\mathbb{R}\quad\text{and}\quad J\in\mathbb{Z}.$$

Note that for given $t_0\in\mathbb{R}$, the function $\underline{\eta}_{\lambda}(t):=\{\underline{\eta}_{\lambda;i}(t)\}_{i\in\mathbb{Z},t\geqslant t_0}$ with

$$\begin{aligned}\underline{\eta}_{\lambda;i}(t)&=\min_{i\in\mathbb{Z}}\eta_{\lambda;i}(t_0)\\&\quad\times\exp\left[\min_{i\in\mathbb{Z}}(d_{i+1}\,\mathrm{e}^{-\lambda}+d_i\,\mathrm{e}^{\lambda}-d_{i+1}-d_i)(t-t_0)+\int_{t_0}^{t}\min_{i\in\mathbb{Z}}\mu_i(s)\,\mathrm{d}s\right]\end{aligned}$$

is a subsolution of (3.6) on $t\geqslant t_0$. Using again lemma 3.1, we have $\eta_{\lambda;i}(t)\geqslant\underline{\eta}_{\lambda;i}(t)$ for any $i\in\mathbb{Z}$ and $t\geqslant t_0$, which also reduces to

$$\begin{aligned}\eta_{\lambda;i}(T+t_0)&\geqslant\min_{i\in\mathbb{Z}}\eta_{\lambda;i}(t_0)\\&\quad\times\exp\left[\min_{i\in\mathbb{Z}}(d_{i+1}\,\mathrm{e}^{-\lambda}+d_i\,\mathrm{e}^{\lambda}-d_{i+1}-d_i)T+\int_{t_0}^{T+t_0}\min_{i\in\mathbb{Z}}\mu_i(s)\,\mathrm{d}s\right]\end{aligned}\tag{3.12}$$

for any $i\in\mathbb{Z}$ and $T\geqslant 0$ by letting $t=T+t_0$. Due to the arbitrariness of $t_0\in\mathbb{R}$, by replacing t_0 with t in (3.12), we have

$$\begin{aligned}\eta_{\lambda;i}(t+T)&\geqslant\min_{i\in\mathbb{Z}}\eta_{\lambda;i}(t)\\&\quad\times\exp\left[\min_{i\in\mathbb{Z}}(d_{i+1}\,\mathrm{e}^{-\lambda}+d_i\,\mathrm{e}^{\lambda}-d_{i+1}-d_i)T+\int_{t}^{t+T}\min_{i\in\mathbb{Z}}\mu_i(s)\,\mathrm{d}s\right]\end{aligned}$$

for any $i\in\mathbb{Z}$, $T\geqslant 0$ and $t\in\mathbb{R}$. Combining this inequality with (3.10), we derive

$$\begin{aligned}\eta_{\lambda;i}(t+T)&\geqslant\tilde{C}^{-1}\max_{i\in\mathbb{Z}}\eta_{\lambda;i}(t-1)\\&\quad\times\exp\left[\min_{i\in\mathbb{Z}}(d_{i+1}\,\mathrm{e}^{-\lambda}+d_i\,\mathrm{e}^{\lambda}-d_{i+1}-d_i)T+\int_{t}^{t+T}\min_{i\in\mathbb{Z}}\mu_i(s)\,\mathrm{d}s\right]\end{aligned}\tag{3.13}$$

for any $i\in\mathbb{Z}$, $T\geqslant 0$ and $t\in\mathbb{R}$. Using (3.7), we get

$$\max_{i\in\mathbb{Z}}\eta_{\lambda;i}(t-1)\geqslant\frac{\max_{i\in\mathbb{Z}}\eta_{\lambda;i}(t)}{\exp\left[\max_{i\in\mathbb{Z}}d_i\,(\mathrm{e}^{\lambda}+\mathrm{e}^{-\lambda})+\sup_{(i,t)\in\mathbb{Z}\times\mathbb{R}}\mu_i(t)\right]}$$

for $t\in\mathbb{R}$. This together with (3.13) implies that (3.8) holds and

$$C_\lambda:=\frac{\tilde{C}^{-1}}{\exp\left[\max_{i\in\mathbb{Z}}d_i\,(\mathrm{e}^{\lambda}+\mathrm{e}^{-\lambda})+\sup_{(i,t)\in\mathbb{Z}\times\mathbb{R}}\mu_i(t)\right]}.$$

It remains to prove the uniqueness result. Assume that $\eta^1(t)$ and $\eta^2(t)$ are two positive, N-periodic in i solutions to (3.6) satisfying (3.7) and (3.8). We first claim

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

that there exists $K > 1$ such that

$$K^{-1}\eta_i^2(t) \leqslant \eta_i^1(t) \leqslant K\eta_i^2(t), \quad \forall t \in \mathbb{R}, \quad i \in \mathbb{Z}. \tag{3.14}$$

Let $h > 0$ be such that

$$\eta_i^1(0) < h\eta_i^2(0), \quad \forall i \in \mathbb{Z}. \tag{3.15}$$

Applying lemma 3.1(2), we have

$$\eta_i^1(t) \leqslant h\eta_1^2(t), \quad \forall i \in \mathbb{Z}, \quad t \geqslant 0. \tag{3.16}$$

In fact, there also holds

$$\min_{i\in\mathbb{Z}} \eta_i^1(t) \leqslant h \max_{i\in\mathbb{Z}} \eta_i^2(t), \quad \forall t \leqslant 0. \tag{3.17}$$

Assume by contradiction that there exists $t_0 < 0$ such that $\min_{i\in\mathbb{Z}} \eta_i^1(t_0) > h \max_{i\in\mathbb{Z}} \eta_i^2(t_0)$. Applying lemma 3.1(2) again, we get $\eta_i^1(0) \geqslant h\eta_i^2(0)$ for all $i \in \mathbb{Z}$ which contradicts with (3.15). Using (3.8) with $T = 0$ to both $\eta^1(t)$ and $\eta^2(t)$, we can find two positive constants C_λ^1 and C_λ^2 satisfying

$$\min_{i\in\mathbb{Z}} \eta_i^1(t) \geqslant C_\lambda^1 \max_{i\in\mathbb{Z}} \eta_i^1(t) \quad \text{and} \quad \min_{i\in\mathbb{Z}} \eta_i^2(t) \geqslant C_\lambda^2 \max_{i\in\mathbb{Z}} \eta_i^2(t)$$

for all $t \in \mathbb{R}$. This together with (3.17) implies that

$$\eta_i^1(t) \leqslant \frac{h}{C_\lambda^1 C_\lambda^2} \eta_i^2(t), \quad \forall i \in \mathbb{Z}, \quad t \leqslant 0,$$

whence by (3.16), we obtain

$$\eta_i^1(t) \leqslant \max\left\{h, \frac{h}{C_\lambda^1 C_\lambda^2}\right\} \eta_i^2(t), \quad \forall i \in \mathbb{Z}, \quad t \in \mathbb{R}.$$

Thus, using symmetry, there is a possibly larger K such that (3.14) holds. Now, denote

$$k := \limsup_{t\to-\infty} \max_{i\in\mathbb{Z}} \frac{\eta_i^1(t)}{\eta_i^2(t)}.$$

By (3.14), one gets $k \in [K^{-1}, K]$. Let $t_n \in \mathbb{R}$ satisfy

$$\lim_{n\to\infty} t_n = -\infty \quad \text{and} \quad \lim_{n\to\infty} \max_{i\in\mathbb{Z}} \frac{\eta_i^1(t_n)}{\eta_i^2(t_n)} = k.$$

For $n \in \mathbb{N}$, consider the functions $\mu^n(t) := \{\mu_i^n(t)\}_{i\in\mathbb{Z}} = \{\mu_i(t + t_n)\}_{i\in\mathbb{Z}}$ and

$$\eta^{n,j}(t) := \left\{\eta_i^{n,j}(t)\right\}_{i\in\mathbb{Z}} = \left\{\frac{\eta_i^j(t + t_n)}{\max\limits_{l\in\mathbb{Z}} \eta_l^1(t_n)}\right\}_{i\in\mathbb{Z}} \qquad (j = 1, 2).$$

Applying (H1) and the Arzela–Ascoli theorem, we can get that $(\mu^n(t))_{n\in\mathbb{N}}$ converges (up to subsequences) to $\tilde{\mu}(t) := \{\tilde{\mu}_i(t)\}_{i\in\mathbb{Z}}$ locally uniformly in $(i, t) \in \mathbb{Z} \times \mathbb{R}$.

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

By (3.7) and (3.8), we have

$$\begin{aligned}
&C_\lambda^1 \exp\left[\min_{i\in\mathbb{Z}}(d_{i+1}\,\mathrm{e}^{-\lambda} + d_i\,\mathrm{e}^{\lambda} - d_{i+1} - d_i)t\right]\\
&\quad\leqslant \frac{\eta_i^1(t+t_n)}{\max_{l\in\mathbb{Z}}\eta_l^1(t_n)}\\
&\quad\leqslant \exp\left[\left(\max_{i\in\mathbb{Z}} d_i(\mathrm{e}^{\lambda}+\mathrm{e}^{-\lambda}) + \sup_{(i,t)\in\mathbb{Z}\times\mathbb{R}}\mu_i(t)\right)t\right],
\end{aligned}$$

which implies that for each $i\in\mathbb{Z}$, the sequences $(\eta_i^{n,1}(t))_{n\in\mathbb{N}}$ are locally uniformly bounded in $t\in\mathbb{R}$ and the derivatives $(\dot\eta_i^{n,1}(t))_{n\in\mathbb{N}}$ are then also locally uniformly bounded in $t\in\mathbb{R}$. By (3.14), the same matter holds true for $(\eta^{n,2}(t))_{n\in\mathbb{N}}$. Therefore, by the Arzela–Ascoli theorem again, $(\eta^{n,j}(t))_{n\in\mathbb{N}}$ converges (up to subsequences) to some functions $\tilde\eta^j(t) := \{\tilde\eta_i^j(t)\}_{i\in\mathbb{Z}}$ locally uniformly in $(i,t)\in\mathbb{Z}\times\mathbb{R}$, satisfying

$$\dot{\tilde\eta}_i^j(t) = d_{i+1}\,\mathrm{e}^{-\lambda}\tilde\eta_{i+1}^j(t) - (d_i+d_{i+1})\tilde\eta_i^j(t) + d_i\,\mathrm{e}^{\lambda}\tilde\eta_{i-1}^j(t) + \tilde\mu_i(t)\tilde\eta_i^j(t),$$

where $j=1,2$ and $(i,t)\in\mathbb{Z}\times\mathbb{R}$. Moreover,

$$\max_{i\in\mathbb{Z}}\frac{\tilde\eta_i^1(0)}{\tilde\eta_i^2(0)} = k \quad\text{and}\quad \tilde\eta_i^1(t)\leqslant k\tilde\eta_i^2(t),\quad \forall (i,t)\in\mathbb{Z}\times\mathbb{R}.$$

Then, there is $i_0\in\mathbb{Z}$ such that $\tilde\eta_{i_0}^1(0) = k\tilde\eta_{i_0}^2(0)$ and

$$\begin{aligned}
\dot{\tilde\eta}_{i_0}^1(0) - k\dot{\tilde\eta}_{i_0}^2(0) &= d_{i_0+1}\,\mathrm{e}^{-\lambda}\left(\tilde\eta_{i_0+1}^1(0) - k\tilde\eta_{i_0+1}^2(0)\right) + d_{i_0}\,\mathrm{e}^{\lambda}\left(\tilde\eta_{i_0-1}^1(0) - k\tilde\eta_{i_0-1}^2(0)\right)\\
&= 0
\end{aligned}$$

holds. Since each coefficient d_i is positive, one infers that

$$\tilde\eta_{i_0+1}^1(0) - k\tilde\eta_{i_0+1}^2(0) = \tilde\eta_{i_0-1}^1(0) - k\tilde\eta_{i_0-1}^2(0) = 0.$$

Repeating the above procedure, we have $\tilde\eta_i^1(0) - k\tilde\eta_i^2(0) = 0$ for any $i\in\mathbb{Z}$. By the classical theory for ordinary differential equations in Banach spaces [**19**], we have $\tilde\eta^1(t)\equiv k\tilde\eta^2(t)$ for all $t\in\mathbb{R}$. As a consequence, for any $\varepsilon>0$, we can find $n_\varepsilon\in\mathbb{N}$ such that

$$(k-\varepsilon)\eta_i^{n,2}(0) < \eta_i^{n,1}(0) < (k+\varepsilon)\eta_i^{n,2}(0),\quad i\in\mathbb{Z}$$

for all $n\geqslant n_\varepsilon$. Combining this with lemma 3.1, one has

$$(k-\varepsilon)\eta_i^{n,2}(t) < \eta_i^{n,1}(t) < (k+\varepsilon)\eta_i^{n,2}(t),\quad \forall i\in\mathbb{Z},\ t\geqslant 0$$

for all $n\geqslant n_\varepsilon$, which implies that

$$(k-\varepsilon)\eta_i^{2}(t) < \eta_i^{1}(t) < (k+\varepsilon)\eta_i^{2}(t),\quad \forall i\in\mathbb{Z},\ t\geqslant t_n$$

for all $n\geqslant n_\varepsilon$. Letting $n\to\infty$ and $\varepsilon\to 0^+$, we eventually obtain $\eta^1(t)\equiv k\eta^2(t)$ for all $t\in\mathbb{R}$. Lemma 3.3 thus follows. □

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

REMARK 3.4. *In the particular case $T = 0$, the inequality (3.8) expresses*

$$\min_{i\in\mathbb{Z}} \eta_{\lambda;i}(t) \geqslant C_\lambda \max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t), \quad \forall t \in \mathbb{R}. \tag{3.18}$$

Notice that, in contrast with (3.10), the two sides are evaluated at the same time. This particular instance of inequality (3.8) will be sometimes used in the following.

LEMMA 3.5. *For all $\lambda > 0$, there is a uniformly Lipschitz-continuous function $S_\lambda : \mathbb{R} \to \mathbb{R}$ and a constant $\beta_\lambda > 0$ satisfying that*

$$\left| S_\lambda(t) - \frac{1}{\lambda} \ln\left(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t)\right)\right| \leqslant \beta_\lambda, \quad \forall\, t \in \mathbb{R}. \tag{3.19}$$

Proof. Applying inequalities (3.7) and (3.8) yields

$$\begin{aligned}
\ln C_\lambda &+ \min_{i\in\mathbb{Z}} \left(d_{i+1}\,\mathrm{e}^{-\lambda} + d_i\,\mathrm{e}^{\lambda} - d_{i+1} - d_i\right) T \\
&\leqslant \ln\left(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t+T)\right) - \ln\left(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t)\right) \\
&\leqslant \left[\max_{i\in\mathbb{Z}} d_i \left(\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda}\right) + \sup_{(i,t)\in\mathbb{Z}\times\mathbb{R}} \mu_i(t)\right] T,
\end{aligned}$$

where $t \in \mathbb{R}$, $T \geqslant 0$ and $C_\lambda > 0$ is a constant given by lemma 3.3 associated with λ. Denote

$$\begin{aligned}
\tilde{\beta}_\lambda := \max\Bigg\{ &\left|\min_{i\in\mathbb{Z}} \left(d_{i+1}\,\mathrm{e}^{-\lambda} + d_i\,\mathrm{e}^{\lambda} - d_{i+1} - d_i\right)\right|, \\
&\max_{i\in\mathbb{Z}} d_i \left(\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda}\right) + \sup_{(i,t)\in\mathbb{Z}\times\mathbb{R}} \mu_i(t) \Bigg\}.
\end{aligned}$$

Thus we get

$$\left| \ln\left(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t+T)\right) - \ln\left(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t)\right)\right| \leqslant \tilde{\beta}_\lambda T + |\ln C_\lambda|$$

for all $t \in \mathbb{R}$ and $T \geqslant 0$. For any $n \in \mathbb{N}$, define S_λ on $[n, n+1]$ as the affine function satisfying

$$S_\lambda(n) = \frac{1}{\lambda} \ln\left(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(n)\right), \quad S_\lambda(n+1) = \frac{1}{\lambda} \ln\left(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(n+1)\right).$$

Then we have

$$\begin{aligned}
|S'_\lambda(t)| &= \left| \frac{1}{\lambda} \ln\left(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(n+1)\right) - \frac{1}{\lambda} \ln\left(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(n)\right)\right| \\
&\leqslant \frac{\tilde{\beta}_\lambda + |\ln C_\lambda|}{\lambda}, \quad t \in (n, n+1),
\end{aligned}$$

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

which implies that S_λ is uniformly Lipschitz-continuous over $\mathbb{R}$. Furthermore, for $t \in [n, n+1]$, we have

$$\begin{aligned}
&\left| S_\lambda(t) - \frac{1}{\lambda} \ln \left(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t) \right) \right| \\
&\quad \leqslant |S_\lambda(t) - S_\lambda(n)| + \left| \frac{1}{\lambda} \ln \left(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t) \right) - S_\lambda(n) \right| \\
&\quad = |S_\lambda(t) - S_\lambda(n)| + \left| \frac{1}{\lambda} \ln \left(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t) \right) - \frac{1}{\lambda} \ln \left(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(n) \right) \right| \\
&\quad \leqslant 2 \frac{\tilde{\beta}_\lambda + |\ln C_\lambda|}{\lambda}.
\end{aligned}$$

Thus, we confirm that $t \to S_\lambda(t) - 1/\lambda \ln(\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t))$ is uniformly bounded over $\mathbb{R}$. The lemma is thus proved. □

Now define a function

$$c_\lambda(t) := S'_\lambda(t), \quad \text{a.e.} \quad t \in \mathbb{R}. \tag{3.20}$$

It is clear that $c_\lambda(\cdot) \in L^\infty(\mathbb{R})$. In the following, we construct the transition wave of (1.1) by using $c_\lambda(t)$ as a possible speed.

Firstly, we show some properties of the least and upper means of the $(c_\lambda)_{\lambda>0}$. Owing to (3.19), we have

$$\lfloor c_\lambda \rfloor = \frac{1}{\lambda} \lim_{T\to+\infty} \inf_{t\in\mathbb{R}} \frac{1}{T} \ln \frac{\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t+T)}{\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t)} \tag{3.21}$$

and

$$\lceil c_\lambda \rceil = \frac{1}{\lambda} \lim_{T\to+\infty} \sup_{t\in\mathbb{R}} \frac{1}{T} \ln \frac{\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t+T)}{\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t)}$$

By (3.7) and (3.8), we get

$$\begin{aligned}
&\frac{1}{T} \ln C_\lambda + \min_{i\in\mathbb{Z}} (d_{i+1} \mathrm{e}^{-\lambda} + d_i \mathrm{e}^{\lambda} - d_{i+1} - d_i) + \frac{1}{T} \int_t^{t+T} \min_{i\in\mathbb{Z}} \mu_i(s) \, \mathrm{d}s \\
&\quad \leqslant \frac{1}{T} \ln \frac{\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t+T)}{\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t)} \\
&\quad \leqslant \max_{i\in\mathbb{Z}} d_i (\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda}) + \frac{1}{T} \int_t^{t+T} \max_{i\in\mathbb{Z}} \mu_i(s) \, \mathrm{d}s, \quad \forall t \in \mathbb{R},\ T \geqslant 0 \text{ and } \lambda > 0,
\end{aligned}$$

which further implies

$$\begin{aligned}
&\frac{1}{\lambda} \min_{i\in\mathbb{Z}} (d_{i+1} \mathrm{e}^{-\lambda} + d_i \mathrm{e}^{\lambda} - d_i - d_{i+1}) + \frac{1}{\lambda} \left\lfloor \min_{i\in\mathbb{Z}} \mu_i(t) \right\rfloor \\
&\quad \leqslant \lfloor c_\lambda \rfloor \leqslant \frac{1}{\lambda} \max_{i\in\mathbb{Z}} d_i (\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda}) + \frac{1}{\lambda} \left\lfloor \max_{i\in\mathbb{Z}} \mu_i(t) \right\rfloor, \quad \forall \lambda > 0
\end{aligned} \tag{3.22}$$

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

and

$$\frac{1}{\lambda}\min_{i\in\mathbb{Z}}(d_{i+1}\,\mathrm{e}^{-\lambda}+d_i\,\mathrm{e}^{\lambda}-d_i-d_{i+1})+\frac{1}{\lambda}\left\lceil\min_{i\in\mathbb{Z}}\mu_i(t)\right\rceil$$
$$\leqslant\lceil c_\lambda\rceil\leqslant\frac{1}{\lambda}\max_{i\in\mathbb{Z}}d_i(\mathrm{e}^{\lambda}+\mathrm{e}^{-\lambda})+\frac{1}{\lambda}\left\lceil\max_{i\in\mathbb{Z}}\mu_i(t)\right\rceil,\quad\forall\lambda>0.\tag{3.23}$$

Lemma 3.6. *The functions $\lambda\to\lambda\lfloor c_\lambda\rfloor$ and $\lambda\to\lambda\lceil c_\lambda\rceil$ are locally uniformly Lipschitz continuous on $(0,+\infty)$. So naturally, the functions $\lambda\to\lfloor c_\lambda\rfloor$ and $\lambda\to\lceil c_\lambda\rceil$ are continuous on $(0,+\infty)$.*

Proof. Fix $0<\varepsilon<\Lambda<+\infty$ and $\varepsilon\leqslant\lambda_0\leqslant\Lambda$. Let $\lambda_1\in[\varepsilon,\Lambda]$ be such that $|\lambda_1-\lambda_0|=\frac{\Lambda-\varepsilon}{2}$. For $j=0,1$, the function $v_{\lambda_j}(t):=\{v_{\lambda_j;i}(t)\}_{i\in\mathbb{Z}}=\{\mathrm{e}^{-\lambda_j i}\eta_{\lambda_j;i}(t)\}_{i\in\mathbb{Z}}$ satisfies (3.3). Rewriting $v_{\lambda_j}(t)=\{v_{\lambda_j;i}(t)\}_{i\in\mathbb{Z}}=\{\mathrm{e}^{w_{\lambda_j;i}(t)}\}_{i\in\mathbb{Z}}$, we have that

$$\dot w_{\lambda_j;i}(t)=d_{i+1}\,\mathrm{e}^{w_{\lambda_j;i+1}(t)-w_{\lambda_j;i}(t)}+d_i\,\mathrm{e}^{w_{\lambda_j;i-1}(t)-w_{\lambda_j;i}(t)}$$
$$-(d_{i+1}+d_i)+\mu_i(t),\quad(i,t)\in\mathbb{Z}\times\mathbb{R}.$$

For $\tau\in(0,1)$, the function $w(t):=\{w_i(t)\}_{i\in\mathbb{Z}}=\{(1-\tau)w_{\lambda_0;i}(t)+\tau w_{\lambda_1;i}(t)\}_{i\in\mathbb{Z}}$ satisfies

$$\begin{aligned}\dot w_i(t)&=d_{i+1}\left[(1-\tau)\,\mathrm{e}^{w_{\lambda_0;i+1}(t)-w_{\lambda_0;i}(t)}+\tau\,\mathrm{e}^{w_{\lambda_1;i+1}(t)-w_{\lambda_1;i}(t)}\right]\\&\quad+d_i\left[(1-\tau)\,\mathrm{e}^{w_{\lambda_0;i-1}(t)-w_{\lambda_0;i}(t)}+\tau\,\mathrm{e}^{w_{\lambda_1;i-1}(t)-w_{\lambda_1;i}(t)}\right]-(d_{i+1}+d_i)+\mu_i(t)\\&\geqslant d_{i+1}\,\mathrm{e}^{(1-\tau)[w_{\lambda_0;i+1}(t)-w_{\lambda_0;i}(t)]+\tau[w_{\lambda_1;i+1}(t)-w_{\lambda_1;i}(t)]}\\&\quad+d_i\,\mathrm{e}^{(1-\tau)[w_{\lambda_0;i-1}(t)-w_{\lambda_0;i}(t)]+\tau[w_{\lambda_1;i-1}(t)-w_{\lambda_1;i}(t)]}-(d_{i+1}+d_i)+\mu_i(t)\\&=d_{i+1}\,\mathrm{e}^{w_{i+1}(t)-w_i(t)}+d_i\,\mathrm{e}^{w_{i-1}(t)-w_i(t)}-(d_{i+1}+d_i)+\mu_i(t),\quad(i,t)\in\mathbb{Z}\times\mathbb{R},\end{aligned}$$

which implies that $\mathrm{e}^{w(t)}:=\{\mathrm{e}^{w_i(t)}\}_{i\in\mathbb{Z}}$ is a supersolution of (3.3). Since

$$\mathrm{e}^{w_i(t)}=\mathrm{e}^{-((1-\tau)\lambda_0+\tau\lambda_1)i}\eta^{1-\tau}_{\lambda_0;i}(t)\eta^{\tau}_{\lambda_1;i}(t),\quad(i,t)\in\mathbb{Z}\times\mathbb{R},$$

the function $\eta^{1-\tau}_{\lambda_0}(t)\eta^{\tau}_{\lambda_1}(t):=\{\eta^{1-\tau}_{\lambda_0;i}(t)\eta^{\tau}_{\lambda_1;i}(t)\}_{i\in\mathbb{Z}}$ is a supersolution of (3.6) with

$$\lambda=\lambda_\tau:=(1-\tau)\lambda_0+\tau\lambda_1.$$

Note that for given $t_0\in\mathbb{R}$, the function $\{\eta_{\lambda_\tau;i}(t+t_0)/\max_{i\in\mathbb{Z}}\eta_{\lambda_\tau;i}(t_0)\}_{i\in\mathbb{Z},t\geqslant0}$ is a solution of (3.6) with $\lambda=\lambda_\tau$ and initial datum $\{\eta_{\lambda_\tau;i}(t_0)/\max_{i\in\mathbb{Z}}\eta_{\lambda_\tau;i}(t_0)\}_{i\in\mathbb{Z}}\leqslant1$. Applying lemma 3.1 and the arbitrariness of t_0, we can get

$$\begin{aligned}\frac{\max_{i\in\mathbb{Z}}\eta_{\lambda_\tau;i}(t+T)}{\max_{i\in\mathbb{Z}}\eta_{\lambda_\tau;i}(t)}&\leqslant\frac{\max_{i\in\mathbb{Z}}\left\{\eta^{1-\tau}_{\lambda_0;i}(t+T)\eta^{\tau}_{\lambda_1;i}(t+T)\right\}}{\min_{i\in\mathbb{Z}}\left\{\eta^{1-\tau}_{\lambda_0;i}(t)\eta^{\tau}_{\lambda_1;i}(t)\right\}}\\&\leqslant\left(\frac{\max_{i\in\mathbb{Z}}\eta_{\lambda_0;i}(t+T)}{\min_{i\in\mathbb{Z}}\eta_{\lambda_0;i}(t)}\right)^{1-\tau}\left(\frac{\max_{i\in\mathbb{Z}}\eta_{\lambda_1;i}(t+T)}{\min_{i\in\mathbb{Z}}\eta_{\lambda_1;i}(t)}\right)^{\tau}\end{aligned}$$

for all $t \in \mathbb{R}$ and $T > 0$. Hence, using the inequality (3.18) for $\eta_{\lambda_0}(t)$ and $\eta_{\lambda_1}(t)$, we get

$$\frac{\max_{i\in\mathbb{Z}} \eta_{\lambda_\tau;i}(t+T)}{\max_{i\in\mathbb{Z}} \eta_{\lambda_\tau;i}(t)} \leqslant C_{\lambda_0}^{\tau-1} C_{\lambda_1}^{-\tau} \left(\frac{\max_{i\in\mathbb{Z}} \eta_{\lambda_0;i}(t+T)}{\max_{i\in\mathbb{Z}} \eta_{\lambda_0;i}(t)} \right)^{1-\tau} \times \left(\frac{\max_{i\in\mathbb{Z}} \eta_{\lambda_1;i}(t+T)}{\max_{i\in\mathbb{Z}} \eta_{\lambda_1;i}(t)} \right)^{\tau}, \tag{3.24}$$

where the constant $C_{\lambda_0} > 0$ and $C_{\lambda_1} > 0$ be given by lemma 3.3 associated with λ_0 and λ_1 respectively.

Defined by $F(\lambda) := \lambda \lfloor c_\lambda \rfloor$ for $\lambda > 0$. Following from (3.21) and (3.24), we obtain

$$F(\lambda_\tau) \leqslant \lim_{T\to+\infty} \inf_{t\in\mathbb{R}} \frac{1}{T} \left((1-\tau) \ln \frac{\max_{i\in\mathbb{Z}} \eta_{\lambda_0;i}(t+T)}{\max_{i\in\mathbb{Z}} \eta_{\lambda_0;i}(t)} + \tau \ln \frac{\max_{i\in\mathbb{Z}} \eta_{\lambda_1;i}(t+T)}{\max_{i\in\mathbb{Z}} \eta_{\lambda_1;i}(t)} \right).$$

Consequently, we have

$$F(\lambda_\tau) - F(\lambda_0) \leqslant \tau(\lambda_1 \lceil c_{\lambda_1} \rceil - \lambda_0 \lfloor c_{\lambda_0} \rfloor), \quad \forall \tau \in (0,1).$$

This together with (3.22) and (3.23) implies that

$$\begin{aligned} F(\lambda_\tau) - F(\lambda_0) &\leqslant \tau \Bigg(d_{\max}(\mathrm{e}^{\lambda_1} + \mathrm{e}^{-\lambda_1}) + \left\lceil \max_{i\in\mathbb{Z}} \mu_i(t) \right\rceil \\ &\quad - \min_{i\in\mathbb{Z}} (d_{i+1}\mathrm{e}^{-\lambda_0} + d_i\, \mathrm{e}^{\lambda_0} - d_i - d_{i+1}) - \left\lfloor \min_{i\in\mathbb{Z}} \mu_i(t) \right\rfloor \Bigg) \\ &\leqslant \tau \Bigg(2d_{\max}\, \mathrm{e}^{\Lambda} \,|\lambda_1 - \lambda_0| + 2d_{\max} \left(\mathrm{e}^{\Lambda} + \mathrm{e}^{-\varepsilon}\right) + 2d_{\max} \\ &\quad + \left\lceil \max_{i\in\mathbb{Z}} \mu_i(t) \right\rceil - \left\lfloor \min_{i\in\mathbb{Z}} \mu_i(t) \right\rfloor \Bigg) \\ &\leqslant K\tau \left(|\lambda_1 - \lambda_0| + 1\right) \\ &= K\tau \left(\frac{\Lambda - \varepsilon}{2} + 1 \right), \end{aligned}$$

where $d_{\max} = \max_{i\in\mathbb{Z}} d_i$ and

$$K = \max \left\{ 2d_{\max}\, \mathrm{e}^{\Lambda},\ 2d_{\max} \left(\mathrm{e}^{\Lambda} + \mathrm{e}^{-\varepsilon}\right) + 2d_{\max} + \left\lceil \max_{i\in\mathbb{Z}} \mu_i(t) \right\rceil - \left\lfloor \min_{i\in\mathbb{Z}} \mu_i(t) \right\rfloor \right\}.$$

This proves the Lipschitz continuity of F on $[\varepsilon, \Lambda]$ due to $|\lambda_\tau - \lambda_0| = \tau(\Lambda - \varepsilon)/2$. By the similar way as above, we can get the function $\lambda \lceil c_\lambda \rceil$ is locally uniformly Lipschitz continuous in $\lambda \in (0, +\infty)$. It then follows that the functions $\lambda \to \lfloor c_\lambda \rfloor$ and $\lambda \to \lceil c_\lambda \rceil$ are continuous on $(0, +\infty)$. This completes the proof. □

In order to define the critical speed c_*, we introduce the following set:

$$\kappa := \{\lambda > 0 : \exists \overline{k} > 0, \quad \forall 0 < k < \overline{k},\ \lfloor c_\lambda - c_{\lambda+k} \rfloor > 0\}. \tag{3.25}$$

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

LEMMA 3.7. *There exists $\lambda_* > 0$ such that $\kappa = (0, \lambda_*)$. Moreover, the function $\lambda \to \lfloor c_\lambda \rfloor$ is decreasing on κ.*

Proof. It can be proved by the similar arguments in [**30**, lemma 3.4]. For the completeness, we provide a proof in the following.

Fix $\lambda_0, \lambda_1 > 0$. For $\tau \in (0, 1)$, we set $\lambda_\tau := (1-\tau)\lambda_0 + \tau\lambda_1$. By calculating (3.24), we get

$$(1-\tau)\ln\frac{\max_{i\in\mathbb{Z}}\eta_{\lambda_0;i}(t+T)}{\max_{i\in\mathbb{Z}}\eta_{\lambda_0;i}(t)} + \tau\ln\frac{\max_{i\in\mathbb{Z}}\eta_{\lambda_1;i}(t+T)}{\max_{i\in\mathbb{Z}}\eta_{\lambda_1;i}(t)} - \ln\frac{\max_{i\in\mathbb{Z}}\eta_{\lambda_\tau;i}(t+T)}{\max_{i\in\mathbb{Z}}\eta_{\lambda_\tau;i}(t)}$$
$$\geqslant \ln\left(C_{\lambda_0}^{1-\tau}C_{\lambda_1}^{\tau}\right).$$

This together with (3.19) and (3.20) implies that

$$\int_t^{t+T}[(1-\tau)\lambda_0 c_{\lambda_0} + \tau\lambda_1 c_{\lambda_1} - \lambda_\tau c_{\lambda_\tau}]\,\mathrm{d}s \geqslant \ln\left(C_{\lambda_0}^{1-\tau}C_{\lambda_1}^{\tau}\right)$$
$$- 2\left[(1-\tau)\beta_{\lambda_0} + \tau\beta_{\lambda_1} + \beta_{\lambda_\tau}\right].$$

It follows that

$$\lambda_\tau\int_t^{t+T}(c_{\lambda_0} - c_{\lambda_\tau})\,\mathrm{d}s \geqslant \tau\lambda_1\int_t^{t+T}(c_{\lambda_0} - c_{\lambda_1})\,\mathrm{d}s + \ln\left(C_{\lambda_0}^{1-\tau}C_{\lambda_1}^{\tau}\right)$$
$$- 2\left[(1-\tau)\beta_{\lambda_0} + \tau\beta_{\lambda_1} + \beta_{\lambda_\tau}\right].$$

Dividing both sides by T, taking the infimum over $t \in \mathbb{R}$ and then taking the limit as $T \to +\infty$, we get

$$\lfloor c_{\lambda_0} - c_{\lambda_\tau}\rfloor \geqslant \tau\frac{\lambda_1}{\lambda_\tau}\lfloor c_{\lambda_0} - c_{\lambda_1}\rfloor, \quad \forall\tau \in (0,1). \tag{3.26}$$

Similarly, dividing both sides by $-T$, we obtain

$$\lfloor c_{\lambda_\tau} - c_{\lambda_0}\rfloor \leqslant \tau\frac{\lambda_1}{\lambda_\tau}\lfloor c_{\lambda_1} - c_{\lambda_0}\rfloor, \quad \forall\tau \in (0,1). \tag{3.27}$$

Furthermore, considering the upper mean, we analogously have

$$\lceil c_{\lambda_0} - c_{\lambda_\tau}\rceil \geqslant \tau\frac{\lambda_1}{\lambda_\tau}\lceil c_{\lambda_0} - c_{\lambda_1}\rceil, \quad \forall\tau \in (0,1)$$

and

$$\lceil c_{\lambda_\tau} - c_{\lambda_0}\rceil \leqslant \tau\frac{\lambda_1}{\lambda_\tau}\lceil c_{\lambda_1} - c_{\lambda_0}\rceil, \quad \forall\tau \in (0,1). \tag{3.28}$$

In the following, we show the properties of κ by using these inequalities and choosing suitable λ_0, λ_1 and τ.

Step 1. $\kappa \neq \varnothing$.

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

By (3.22) and (H1), we have

$$\lim_{\lambda\to 0^+}\lfloor c_\lambda - c_1\rfloor \geqslant \lim_{\lambda\to 0^+}\lfloor c_\lambda\rfloor - \lceil c_1\rceil = +\infty.$$

Then there exists $0<\lambda'<1$ satisfying $\lfloor c_{\lambda'}-c_1\rfloor > 0$. Applying (3.26) with $\lambda_0=\lambda'$ and $\lambda_1 = 1$, we get that $\lfloor c_{\lambda'}-c_{\lambda'+k}\rfloor > 0$ for all $0<k<1-\lambda'$. Therefore, $\lambda'\in\kappa$.

Step 2. κ is bounded from above.

It follows from (3.22) that

$$\lim_{\lambda\to +\infty}\lfloor c_1 - c_\lambda\rfloor \leqslant \lfloor c_1\rfloor - \lim_{\lambda\to +\infty}\lfloor c_\lambda\rfloor = -\infty.$$

Then there exists $\lambda'>1$ such that $\lfloor c_1-c_\lambda\rfloor<0$ for any $\lambda>\lambda'$. Then, for any $k>0$, using (3.27) with $\lambda_0=\lambda+k$, $\lambda_1=1$ and $\tau=k/k+\lambda-1$, we get

$$\lfloor c_\lambda - c_{\lambda+k}\rfloor \leqslant \frac{k}{(k+\lambda-1)\lambda}\lfloor c_1-c_{\lambda+k}\rfloor < 0,$$

which implies $\lambda\notin\kappa$. Thus κ is bounded from above by λ'.

Step 3. If $\lambda\in\kappa$, then $(0,\lambda]\subset\kappa$.

Let $0<\lambda'<\lambda$ and $k>0$. By using (3.26) and (3.27), we get

$$\lfloor c_{\lambda'}-c_{\lambda'+k}\rfloor \geqslant \left(\frac{k}{k+\lambda-1}\right)\left(\frac{\lambda+k}{\lambda'+k}\right)\lfloor c_{\lambda'}-c_{\lambda+k}\rfloor \geqslant \left(\frac{\lambda+k}{\lambda'+k}\right)\frac{\lambda}{\lambda'}\lfloor c_\lambda - c_{\lambda+k}\rfloor.$$

Thus, if $\lambda\in\kappa$, then $\lambda'\in\kappa$.

Step 4. If $\sup\kappa\notin\kappa$.

Let $\lambda^* := \sup\kappa$ and $k>0$. It follows from the definition of κ and λ^* that for each $n\in\mathbb{N}$, there exists $k_n\in(0,1/n)$ such that

$$\lfloor c_{\lambda^*+1/n} - c_{\lambda^*+1/n+k_n}\rfloor \leqslant 0.$$

For n large enough, there holds $1/n+k_n<k$. Using (3.26), we have

$$0\geqslant \lfloor c_{\lambda^*+1/n} - c_{\lambda^*+1/n+k_n}\rfloor \geqslant \left(\frac{k_n}{k-1/n}\right)\left(\frac{\lambda^*+k}{\lambda^*+\frac{1}{n}+k_n}\right)\lfloor c_{\lambda^*+1/n}-c_{\lambda^*+k}\rfloor$$

for n large enough. Consequently, we have

$$\lfloor c_{\lambda^*}-c_{\lambda^*+k}\rfloor \leqslant \lfloor c_{\lambda^*+1/n}-c_{\lambda^*+k}\rfloor + \lceil c_{\lambda^*}-c_{\lambda^*+1/n}\rceil \leqslant \lceil c_{\lambda^*}-c_{\lambda^*+1/n}\rceil$$

for n large enough. Using (3.28), we get

$$\lceil c_{\lambda^*}-c_{\lambda^*+1/n}\rceil \leqslant \frac{1/n}{\lambda_*+2/n}\lceil c_{\lambda^*/2}-c_{\lambda^*+1/n}\rceil \leqslant \frac{1/n}{\lambda_*+2/n}(\lceil c_{\lambda^*/2}\rceil - \lfloor c_{\lambda^*+1/n}\rfloor),$$

which tends to 0 as $n\to\infty$ by lemma 3.6. Thus we eventually have $\lfloor c_{\lambda^*}-c_{\lambda^*+k}\rfloor\leqslant 0$, which implies $\lambda^*\notin\kappa$.

Now we show that $\lambda\to\lfloor c_\lambda\rfloor$ is decreasing on κ. On the contrary, we suppose that there are $0<\lambda_1<\lambda_2<\lambda^*$ such that $\lfloor c_{\lambda_1}\rfloor\leqslant\lfloor c_{\lambda_2}\rfloor$. Since the function $\lambda\to\lfloor c_\lambda\rfloor$ is

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

continuous, it attains its minimum on $[\lambda_1, \lambda_2]$ at some λ'. Due to $\lfloor c_{\lambda_1} \rfloor \leqslant \lfloor c_{\lambda_2} \rfloor$, we can assume that $\lambda' \in [\lambda_1, \lambda_2)$. It follows from the definition of κ that there exists $\lambda'' \in (\lambda', \lambda_2)$ such that $\lfloor c_{\lambda'} - c_{\lambda''} \rfloor > 0$. Then, we have

$$\lfloor c_{\lambda''} \rfloor \leqslant \lfloor c_{\lambda'} \rfloor + \lceil c_{\lambda''} - c_{\lambda'} \rceil = \lfloor c_{\lambda'} \rfloor - \lfloor c_{\lambda'} - c_{\lambda''} \rfloor < \lfloor c_{\lambda'} \rfloor,$$

which contradicts with the definition of λ'. This completes the proof. □

Now we can define the critical speed

$$c_* := \lfloor c_{\lambda_*} \rfloor, \tag{3.29}$$

where λ_* is given in lemma 3.7.

In the following, we construct some suitable subsolutions and supersolutions of (1.1) to prove theorem 2.2. Let us first introduce a family of functions $(\varphi_\lambda)_{\lambda>0}$. For $\lambda > 0$, let η_λ be the function given by lemma 3.3. We normalize it by $\|\eta_\lambda(0)\|_\infty = 1$. Define $\varphi_\lambda(t) := \{\varphi_{\lambda;i}(t)\}_{i\in\mathbb{Z}}$ by

$$\varphi_{\lambda;i}(t) = \mathrm{e}^{-\lambda S_\lambda(t)} \eta_{\lambda;i}(t), \quad (i, t) \in \mathbb{Z} \times \mathbb{R}.$$

Using (3.19), we have

$$\mathrm{e}^{-\lambda\beta_\lambda} \frac{\eta_{\lambda;i}(t)}{\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t)} \leqslant \mathrm{e}^{-\lambda S_\lambda(t)} \eta_{\lambda;i}(t) \leqslant \mathrm{e}^{\lambda\beta_\lambda} \frac{\eta_{\lambda;i}(t)}{\max_{i\in\mathbb{Z}} \eta_{\lambda;i}(t)},$$

where $(i, t) \in \mathbb{Z} \times \mathbb{R}$ and $\beta_\lambda > 0$ is the constant given by lemma 3.5. Combining this inequality with (3.18), we get

$$M_\lambda := C_\lambda \mathrm{e}^{-\lambda\beta_\lambda} \leqslant \varphi_{\lambda;i}(t) \leqslant \mathrm{e}^{\lambda\beta_\lambda}, \quad (i, t) \in \mathbb{Z} \times \mathbb{R}, \tag{3.30}$$

where $C_\lambda > 0$ is the constant given by lemma 3.3.

Proof of theorem 2.2. Fix $\gamma > c_*$. Since the function $\lambda \to \lfloor c_\lambda \rfloor$ is continuous by lemma 3.6 and goes to $+\infty$ as $\lambda \to 0^+$ by (3.22), and $\kappa = (0, \lambda_*)$ by lemma 3.6, there exists $\lambda \in \kappa$ such that $\lfloor c_\lambda \rfloor = \gamma$. The function $\overline{v}(t) := \{\overline{v}_i(t)\}_{i\in\mathbb{Z}}$ defined by

$$\overline{v}_i(t) := \min\{\mathrm{e}^{-\lambda i} \eta_{\lambda;i}(t),\ 1\}, \quad (i, t) \in \mathbb{Z} \times \mathbb{R}$$

is acting as a supersolution of (1.1).

In the following, we construct a subsolution of (1.1). Recall the constant ν in (H2) and the definition of κ, there exists $\lambda' \in (\lambda, (1 + \nu)\lambda)$ such that $\lfloor c_\lambda - c_{\lambda'} \rfloor > 0$. Set

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

$\psi(t) := \{\psi_i(t)\}_{i\in\mathbb{Z}}$ with

$$\psi_i(t) = \mathrm{e}^{\sigma(t)-\lambda'(i-S_\lambda(t)+S_{\lambda'}(t))}\eta_{\lambda';i}(t), \quad (i,t)\in\mathbb{Z}\times\mathbb{R},$$

where $\sigma \in W^{1,\infty}(\mathbb{R})$ will be determined later. Clearly, there holds

$$\begin{aligned}\dot{\psi}_i(t) - d_{i+1}\psi_{i+1}(t) + (d_{i+1}+d_i)\psi_i(t) - d_i\psi_{i-1}(t) - \mu_i(t)\psi_i(t)\\ = (\sigma'(t) + \lambda'(c_\lambda(t) - c_{\lambda'}(t)))\psi_i(t), \quad (i,t)\in\mathbb{Z}\times\mathbb{R}.\end{aligned}$$

Here we recall a key property of least mean (see [**29**, lemma 3.2]):

$$\forall g\in L^\infty(\mathbb{R}), \quad \lfloor g\rfloor = \sup_{\sigma\in W^{1,\infty}(\mathbb{R})}\inf_{t\in\mathbb{R}}(\sigma'+g)(t). \tag{3.31}$$

Since $\lfloor\lambda'(c_\lambda - c_{\lambda'})\rfloor = \lambda'\lfloor c_\lambda - c_{\lambda'}\rfloor > 0$, by (3.31) we can choose $\sigma\in W^{1,\infty}(\mathbb{R})$ such that

$$K := \inf_{t\in\mathbb{R}}(\sigma'(t) + \lambda'(c_\lambda(t) - c_{\lambda'}(t))) > 0.$$

Thus, we get

$$\begin{aligned}\dot{\psi}_i(t) - d_{i+1}\psi_{i+1}(t) + (d_{i+1}+d_i)\psi_i(t) - d_i\psi_{i-1}(t)\\ \geqslant (\mu_i(t) + K)\psi_i(t), \quad (i,t)\in\mathbb{Z}\times\mathbb{R}.\end{aligned}$$

Now we define $v(t) := \{v_i(t)\}_{(i,t)\in\mathbb{Z}\times\mathbb{R}}$ by

$$v_i(t) = \mathrm{e}^{-\lambda i}\eta_{\lambda;i}(t) - m\psi_i(t), \quad (i,t)\in\mathbb{Z}\times\mathbb{R},$$

where m is a positive constant to be chosen. A direct computation gives

$$v(t) := \mathrm{e}^{-\lambda i}\eta_{\lambda;i}(t) - m\psi_i(t) = \mathrm{e}^{-\lambda(i-S_\lambda(t))}\left(\varphi_{\lambda;i}(t) - m\varphi_{\lambda';i}(t)\,\mathrm{e}^{\sigma(t)-(\lambda'-\lambda)(i-S_\lambda(t))}\right) \tag{3.32}$$

for all $(i,t)\in\mathbb{Z}\times\mathbb{R}$. Since $\varphi_\lambda(t)$ and $\varphi_{\lambda'}(t)$ satisfy (3.30) and $\sigma\in L^\infty(\mathbb{R})$, we can choose m large enough so that if $i - S_\lambda(t)\leqslant 0$, then $v_i(t)\leqslant 0$, and that $v_i(t)\leqslant\delta$ for all $(i,t)\in\mathbb{Z}\times\mathbb{R}$, where $\delta\in(0,1]$ is defined in (H2). If $v_i(t) > 0$ and therefore $i - S_\lambda(t) > 0$, we see that

$$\begin{aligned}&\dot{v}_i(t) - d_{i+1}v_{i+1}(t) + (d_{i+1}+d_i)v_i(t) - d_iv_{i-1}(t) - \mu_i(t)v_i(t)\\ &\quad\leqslant -mK\psi_i(t)\\ &\quad\leqslant -mK\psi_i(t)\frac{v_i^{1+\nu}(t)}{\mathrm{e}^{-(1+\nu)\lambda i}\eta_{\lambda;i}^{1+\nu}(t)}\\ &\quad= -mKv_i^{1+\nu}(t)\frac{\varphi_{\lambda';i}(t)}{\varphi_{\lambda;i}^{1+\nu}(t)}\,\mathrm{e}^{\sigma(t)-(\lambda'-(1+\nu)\lambda)(i-S_\lambda(t))}\\ &\quad\leqslant -mKv_i^{1+\nu}(t)M_{\lambda'}\,\mathrm{e}^{-(1+\nu)\lambda\beta_\lambda}\inf_{s\in\mathbb{R}}\mathrm{e}^{\sigma(s)},\end{aligned}$$

where we have used (3.30) and the fact that $\lambda' < (1+\nu)\lambda$. As a consequence, by hypothesis (H2), for m sufficiently large, $v(t)$ can be regarded as a subsolution of (1.1) in the set where it is positive.

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

Define a integer-valued function

$$J(t) := \langle S_\lambda(t) \rangle = \left\langle \int_0^t c_\lambda(s)\,\mathrm{d}s \right\rangle, \quad \forall t \in \mathbb{R}.$$

Using again (3.30), one has

$$\begin{aligned} & v_{i+J(t)}(t) \\ & \quad = \mathrm{e}^{-\lambda(i+J(t)-S_\lambda(t))} \left(\varphi_{\lambda;i+J(t)}(t) - m\varphi_{\lambda';i+J(t)}(t)\,\mathrm{e}^{\sigma(t)-(\lambda'-\lambda)(i+J(t)-S_\lambda(t))} \right) \\ & \quad \geqslant \mathrm{e}^{-\lambda i} \left(M_\lambda - m\,\mathrm{e}^{\lambda'\beta_{\lambda'}+\|\sigma\|_\infty-(\lambda'-\lambda)(i-1)} \right). \end{aligned}$$

Taking $N_0 \in \mathbb{Z}$ large enough, one has

$$\inf_{t\in\mathbb{R}} v_{N_0+J(t)}(t) \geqslant \mathrm{e}^{-\lambda N_0} \left(M_\lambda - m\,\mathrm{e}^{\lambda'\beta_{\lambda'}+\|\sigma\|_\infty-(\lambda'-\lambda)(N_0-1)} \right) =: \omega \in (0, \delta).$$

In fact, when $i \geqslant J(t) + N_0$, one has $0 < v_i(t) \leqslant \delta$. Now by the definition of $v_i(t)$, there exists an integer-valued function $J_-(t)$, satisfying (A) and $J(t) \leqslant J_-(t) \leqslant J(t) + N_0$ on $t \in \mathbb{R}$, such that $v_{J_-(t)}(t) \leqslant 0$ and $v_i(t) > 0$ for all $i > J_-(t)$ and $t \in \mathbb{R}$. Furthermore, by the definition of ω, there exists another integer-valued function $J_+(t)$, satisfying (A) and $J_-(t) < J_+(t) \leqslant J(t) + N_0$ on $t \in \mathbb{R}$, such that $v_{J_+(t)}(t) \geqslant \omega$ and $v_i(t) < \omega$ for all $J_-(t) \leqslant i \leqslant J_+(t) - 1$ and $t \in \mathbb{R}$. Consequently, defined the function $\underline{v}(t) = \{\underline{v}_i(t)\}_{(i,t)\in\mathbb{Z}\times\mathbb{R}}$ by

$$\underline{v}_i(t) := \begin{cases} v_i(t) & \text{if } i \geqslant J_+(t), \\ \omega & \text{if } i < J_+(t), \end{cases}$$

and it will serve as a subsolution of (1.1). Moreover, since $v_i(t) \leqslant \overline{v}_i(t)$ for $(i,t) \in \mathbb{Z} \times \mathbb{R}$ and $\overline{v}_{i+J(t)}(t) \geqslant \mathrm{e}^{-\lambda N_0} M_\lambda > \omega$ if $i < N_0$ and $t \in \mathbb{R}$, one sees that $0 \leqslant \underline{v}_i(t) \leqslant \overline{v}_i(t) \leqslant 1$ for all $(i,t) \in \mathbb{Z} \times \mathbb{R}$.

For each $n \in \mathbb{N}$, let $u^n(t) := \{u_i^n(t)\}_{i\in\mathbb{Z},t\geqslant -n}$ solve

$$\begin{cases} \dot{u}_i^n(t) = d_{i+1}u_{i+1}^n(t) - (d_{i+1}+d_i)u_i^n(t) + d_i u_{i-1}^n(t) + f_i(t, u_i^n(t)), & i \in \mathbb{Z},\ t > -n \\ u_i^n(-n) = \underline{v}_i(-n), & i \in \mathbb{Z}. \end{cases}$$

Now we show

$$0 \leqslant \underline{v}_i(t) \leqslant u_i^n(t) \leqslant \overline{v}_i(t) \leqslant 1, \quad \forall n \in \mathbb{N}, \forall i \in \mathbb{Z}, \forall t \geqslant -n. \tag{3.33}$$

Firstly, we show $u_i^n(t) \leqslant \overline{v}_i(t) \leqslant 1$ for all $n \in \mathbb{N}$, $i \in \mathbb{Z}$ and $t \geqslant -n$. Since $0 \leqslant \underline{v}_i(-n) \leqslant u_i^n(t) \leqslant \overline{v}_i(-n) \leqslant 1$, applying lemma 3.1(1) with $J_1(t) = -\infty$ and $J_2(t) = +\infty$ yields

$$0 \leqslant u_i^n(t) \leqslant 1, \quad i \in \mathbb{Z},\ t > -n.$$

By $f_i(t, u_i^n(t)) \leqslant \mu_i(t) u_i^n(t)$ for all $i \in \mathbb{Z}$, $t > -n$, applying lemma 3.1(2) with $J_1(t) = -\infty$ and $J_2(t) = +\infty$ yields

$$u_i^n(t) \leqslant \mathrm{e}^{-\lambda i}\eta_{\lambda;i}(t), \quad i \in \mathbb{Z}, t \geqslant -n.$$

Thus, we get $u_i^n(t) \leqslant \overline{v}_i(t)$ for $i \in \mathbb{Z}$ and $t \geqslant -n$. Next, we show $u_i^n(t) \geqslant \underline{v}_i(t)$ for all $n \in \mathbb{N}$, $i \in \mathbb{Z}$ and $t \geqslant -n$. Since $u_i^n(t) > 0$ for all $i \in \mathbb{Z}$ and $t \geqslant -n$, we have

$u^n_{J_-(t)}(t) \geqslant v_{J_-(t)}(t)$ for all $t \geqslant -n$. Applying lemma 3.1(1) with $J_1(t) = J_-(t)$ and $J_2(t) = +\infty$, we get $u^n_i(t) \geqslant v_i(t)$ for all $i \geqslant J_-(t)$ and $t \geqslant -n$. It follows that $u^n_i(t) \geqslant v_i(t)$ for all $i \geqslant J_+(t)$ and $t \geqslant -n$. Since $u^n_{J_+(t)}(t) \geqslant v_{J_+(t)}(t) \geqslant \omega$ for $t \geqslant -n$, again applying lemma 3.1(1) with $J_1(t) = -\infty$ and $J_2(t) = J_+(t)$ yields that $u^n_i(t) \geqslant \omega$ for $i \leqslant J_+(t)$ and $t \geqslant -n$. Thus, we have showed that $u^n_i(t) \geqslant \underline{v}_i(t)$ for $i \in \mathbb{Z}$ and $t \geqslant -n$. Thus, (3.33) holds.

Following from (3.33), one has that $u^n_i(-n+1) \geqslant \underline{v}_i(-n+1) = u^{n-1}_i(-n+1)$ for all $n \in \mathbb{N}^+$ and $i \in \mathbb{Z}$. It resorts from lemma 3.1 that $u^n_i(t) \geqslant u^{n-1}_i(t)$ for all $n \in \mathbb{N}^+$, $i \in \mathbb{Z}$ and $t \geqslant -n+1$. For each $(i,t) \in \mathbb{Z} \times \mathbb{R}$, the sequence $(u^n_i(t))_{n\in\mathbb{N},t\geqslant -n}$ is nondecreasing and bounded; call $U_i(t)$ its limit as $n \to +\infty$. On the other hand, for each $i \in \mathbb{Z}$, the functions $(u^n_i(t))_{n\in\mathbb{N},t\geqslant -n}$ are uniformly bounded between 0 and 1, and the derivatives $(\dot{u}^n_i(t))_{n\in\mathbb{N},t\geqslant -n}$ are then also uniformly bounded. Therefore, the convergence $u^n_i(t) \to U_i(t)$ as $n \to +\infty$ holds at least locally uniformly in t for each $i \in \mathbb{Z}$. For each n, we can integrate equation (1.1) in any given interval of time, and then pass to the limit as $n \to +\infty$. It follows that the functions $U_i(t)$ are of class C^1 and solve (1.5) for all $(i,t) \in \mathbb{Z} \times \mathbb{R}$. Furthermore, the above estimates imply that

$$\forall (i,t) \in \mathbb{Z} \times \mathbb{R}, \quad 0 \leqslant \underline{v}_i(t) \leqslant U_i(t) \leqslant \overline{v}_i(t) \leqslant 1.$$

One further sees that

$$\begin{aligned}\lim_{i\to+\infty} U_{i+J(t)}(t) &\leqslant \lim_{i\to+\infty} \overline{v}_{i+J(t)}(t) \leqslant \lim_{i\to+\infty} \mathrm{e}^{-\lambda(i+J(t))}\eta_{\lambda;i+J(t)}(t)\\ &\leqslant \lim_{i\to+\infty} \mathrm{e}^{-\lambda(i+S_\lambda(t))}\eta_{\lambda;i+J(t)}(t) = 0\end{aligned}$$

uniformly with respect to $t \in \mathbb{R}$. It remains to prove that

$$\lim_{i\to-\infty} U_{i+J(t)}(t) = 1$$

holds uniformly with respect to $t \in \mathbb{R}$. Set

$$\vartheta := \lim_{r\to-\infty} \inf_{i\leqslant r, t\in\mathbb{R}} U_{i+J(t)}(t).$$

Our aim is to show that $\vartheta = 1$. We know that $\vartheta \geqslant \omega > 0$, because $U_i(t) \geqslant \underline{v}_i(t) \geqslant \omega$ if $i < J(t) + M$. Let $i_n \in \mathbb{Z}$ and $t_n \in \mathbb{R}$ satisfy

$$\lim_{n\to\infty} i_n = -\infty, \quad \lim_{n\to\infty} U_{i_n+J(t_n)}(t_n) = \vartheta.$$

For $n \in \mathbb{N}$, let $k_n \in N\mathbb{Z}$ satisfy $y_n := i_n + J(t_n) - k_n \in \{1, \cdots, N\}$ and define $w^n(t) := \{w^n_i(t)\}_{(i,t)\in\mathbb{Z}\times\mathbb{R}} = \{U_{i+k_n}(t+t_n)\}_{(i,t)\in\mathbb{Z}\times\mathbb{R}}$. The functions $(w^n(t))_{n\in\mathbb{N}}$ are solutions of

$$\begin{aligned}\dot{w}^n_i(t) = d_{i+1}w^n_{i+1}(t) - (d_{i+1}+d_i)w^n_i(t) + d_i w^n_{i-1}(t)\\ + f_i(t+t_n, w^n_i(t)), \quad (i,t) \in \mathbb{Z} \times \mathbb{R}.\end{aligned}$$

Clearly, $(w^n(t))_{n\in\mathbb{N}}$ are uniformly bounded between 0 and 1, and the derivatives $(\dot{w}^n(t))_{n\in\mathbb{N}}$ are then also uniformly bounded. Therefore, by the Arzela–Ascoli theorem, $(w^n(t))_{n\in\mathbb{N}}$ converges (up to subsequences) to some function $w(t) := \{w_i(t)\}_{(i,t)\in\mathbb{Z}\times\mathbb{R}}$ locally uniformly in $(i,t)\in\mathbb{Z}\times\mathbb{R}$, satisfying

$$\begin{aligned}\dot{w}_i(t) - d_{i+1}w_{i+1}(t) + (d_{i+1}+d_i)w_i(t) - d_i w_{i-1}(t)\\ = f_i(t+t_n, w_i(t)), \quad (i,t)\in\mathbb{Z}\times\mathbb{R}.\end{aligned}$$

Furthermore, letting y be the limit of a converging subsequence of $(y_n)_{n\in\mathbb{N}}$, we find

$$\vartheta = \lim_{n\to\infty} U_{i_n+J(t_n)}(t_n) = \lim_{n\to\infty} w^n_{y_n}(0) = w_y(0)$$

and

$$w_i(t) = \lim_{n\to\infty} U_{i+k_n}(t+t_n) = \lim_{n\to\infty} U_{i+i_n+J(t_n)-y_n}(t+t_n) \geqslant \vartheta, \quad \forall (i,t)\in\mathbb{Z}\times\mathbb{R}.$$

By (H2) and $d_i>0$ for all $i\in\mathbb{Z}$, one infers that $w_{y+1}(0)=w_{y-1}(0)=\vartheta$. By induction, we have $w(0)\equiv\vartheta$ and $f_i(t,\vartheta)=0$ for all $(i,t)\in\mathbb{Z}\times\mathbb{R}$. This together with (H2) implies that either $\vartheta=0$ or $\vartheta=1$. In virtue of $\vartheta\geqslant\omega>0$, we eventually get $\vartheta=1$. Theorem 2.2 thus follows. □

4. Application to particular cases

In this subsection, we give further applications of the results of § 3 to some particular classes of heterogeneities already investigated in the literature.

(E1): If the terms $f(t,s) := \{f_i(t,s)\}_{i\in\mathbb{Z}}$ are also periodic in t with period $T>0$, the class of admissible speeds has been characterized by Cao and Shen [**8**], Fang, Yu and Zhao [**15**], Liang and Zhao [**24**], and Weinberger [**42**]. Following the method described above, we see that an entire solution of (3.3) in the form (3.4) given by $\eta_\lambda(t) = \mathrm{e}^{M(\lambda)t}\varphi_\lambda(t)$, where $M(\lambda)$ and $\varphi_\lambda(t) := \{\varphi_{\lambda;i}(t)\}_{(i,t)\in\mathbb{Z}\times\mathbb{R}}$ with $\varphi_{\lambda;i+N}(t) = \varphi_{\lambda;i}(t+T) = \varphi_{\lambda;i}(t)$, is the corresponding principal eigenvalue and principal eigenfunction of the problem

$$\begin{aligned}\dot{\varphi}_{\lambda;i}(t) = d_{i+1}\,\mathrm{e}^{-\lambda}\varphi_{\lambda;i+1}(t) - (d_{i+1}+d_i)\varphi_{\lambda;i}(t)\\ + d_i\,\mathrm{e}^{\lambda}\varphi_{\lambda;i-1}(t) + \mu_i(t)\varphi_{\lambda;i}(t) - M(\lambda)\varphi_{\lambda;i}(t)\end{aligned}$$

for all $(i,t)\in\mathbb{Z}\times\mathbb{R}$. Thus, $S_\lambda := M(\lambda)/\lambda t$ satisfies (3.19), whence the speed of wave for the linearized equation with decaying rate $c_\lambda \equiv M(\lambda)/\lambda$. Since the c_λ is a constant, we can get

$$\lceil c_\lambda - c_{\lambda+k}\rceil = \lfloor c_\lambda - c_{\lambda+k}\rfloor = \frac{M(\lambda)}{\lambda} - \frac{M(\lambda+k)}{\lambda+k}.$$

By (3.22), we have

$$\lim_{\lambda\to+\infty}\frac{M(\lambda)}{\lambda} = +\infty, \quad \lim_{\lambda\to0^+} M(\lambda) = \lim_{\lambda\to0^+}\lambda c_\lambda \geqslant \left\lfloor \min_{i\in\mathbb{Z}}\mu_i(t)\right\rfloor > 0.$$

This together with the strict convexity of $M(\lambda)$ with respect to λ (see [**24**] and [**42**]) implies that λ_* given by lemma 3.7 is the unique minimizer of $\lambda\to M(\lambda)/\lambda$. Therefore, the threshold λ_* we obtain for the decaying rates coincides with the

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

minimum point of $\lambda \to M(\lambda)/\lambda$. We eventually derive the existence of a generalized transition wave for any speed larger than $c_* := \min_{\lambda>0} M(\lambda)/\lambda$, which is exactly the sharp critical speed for travelling fronts obtained in [8, 15, 24, 42]. To sum up, the c_* we constructed in theorem 2.2 is the minimal speed in the space-time periodic monotone systems. It should be mentioned that the solutions we obtain in this paper must be transition waves, but not necessarily be pulsating waves.

(E2): Under the assumptions made by Guo and Hamel in [16], that is, f does not depend on t, the speed c_* derived in the present paper and that in [16] coincide, and thus it is minimal in the sense that there do not exist any generalized transitions wave with a lower speed.

(E3): Consider the case investigated by Cao and Shen in [7], namely, $d_i \equiv 1$ and f only depend on (t,u) and is replaced by $s\tilde{f}(t,s)$. One can easily check that $\eta_\lambda(t) \equiv \exp\left[\int_0^t \tilde{f}(s,0)\,\mathrm{d}s + (\mathrm{e}^{-\lambda} + \mathrm{e}^{\lambda} - 2)t\right]$. We can take a Lipschitz continuous function

$$S_\lambda(t) := \frac{1}{\lambda}\int_0^t \tilde{f}(s,0)\,\mathrm{d}s + \frac{1}{\lambda}\left(\mathrm{e}^{-\lambda} + \mathrm{e}^{\lambda} - 2\right)t,$$

which implies that

$$c_\lambda(t) := \frac{\mathrm{e}^{-\lambda} + \mathrm{e}^{\lambda} - 2 + \tilde{f}(t,0)}{\lambda}$$

is a speed of a wave with decaying rate λ. By [7, lemma 5.1], we have that the threshold λ_* given by lemma 3.7 of this paper satisfies

$$c_* := \lfloor c_{\lambda_*}(t) \rfloor = \frac{\mathrm{e}^{-\lambda_*} + \mathrm{e}^{\lambda_*} - 2 + \lfloor \tilde{f}(t,0) \rfloor}{\lambda_*} = \inf_{\lambda>0} \frac{\mathrm{e}^{-\lambda} + \mathrm{e}^{\lambda} - 2 + \lfloor \tilde{f}(t,0) \rfloor}{\lambda}.$$

Thus, in this paper, we get the same critical speed c_* as in [7], which was not proved to be minimal since the nonexistence of transition waves with lower speed was not investigated. Of course, if $\tilde{f}(t,0)$ is unique ergodic, then the speed c_* is minimal (see [7, remark 1.1]).

Acknowledgements

The first and second authors were partially supported by NNSF of China (11371179, 11731005). The third author was partially supported by NNSF of China (11701041) and the Fundamental Research Funds for the Central Universities, CHD (300102129201).

References

1 D. G. Aronson and H. F. Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Adv. Math.* **30** (1978), 33–76.

2 X. Bao and W.-T. Li. Propagation phenomena for partially degenerate nonlocal dispersal models in time and space periodic habitats. *Nonlinear Anal. Real World Appl.* **51** (2020), 102975.

3 X. Bao, W.-T. Li and Z.-C. Wang. Uniqueness and stability of time-periodic pyramidal fronts for a periodic competition-diffusion system. *Commun. Pure Appl. Anal.* **19** (2020), 253–277.

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

4 H. Berestycki and F. Hamel. Generalized travelling waves for reaction-diffusion equations. In *Perspectives in nonlinear partial differential equations*. Contemp. Math., vol. 446, pp. 101–123 (Providence: Amer. Math. Soc., 2007).

5 H. Berestycki and F. Hamel. Generalized transition waves and their properties. *Comm. Pure Appl. Math.* **65** (2012), 592–648.

6 H. Berestycki, F. Hamel and N. Nadirashvili. The speed of propagation for KPP type problems, I - periodic framework. *J. Eur. Math. Soc.* **7** (2005), 172–213.

7 F. Cao and W. Shen. Spreading speeds and transition fronts of lattice KPP equations in time heterogeneous media. *Discrete Cont. Dyn. Syst.* **37** (2017), 4697–4727.

8 F. Cao and W. Shen. Stability and uniqueness of generalized traveling waves of lattice Fisher-KPP equations in heterogeneous media. *Sci. Sin. Math.* **47** (2017), 1787–1808.

9 X. Chen, S.-C. Fu and J.-S. Guo. Uniqueness and asymptotics of traveling waves of monostable dynamics on lattices. *SIAM J. Math. Anal.* **38** (2006), 233–258.

10 X. Chen and J.-S. Guo. Existence and asymptotic stability of traveling waves of discrete quasilinear monostable equations. *J. Differ. Eqn.* **184** (2002), 549–569.

11 X. Chen and J.-S. Guo. Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics. *Math. Ann.* **326** (2003), 123–146.

12 Y.-Y. Chen, J.-S. Guo and F. Hamel. Traveling waves for a lattice dynamical system arising in a diffusive endemic model. *Nonlinearity* **30** (2017), 2334–2359.

13 X. Chen, J.-S. Guo and C.-C. Wu. Traveling waves in discrete periodic media for bistable dynamics. *Arch. Ration. Mech. Anal.* **189** (2008), 189–236.

14 J. Fang, J. Wei and X.-Q. Zhao. Spreading speeds and travelling waves for non-monotone time-delayed lattice equations. *Proc. R. Soc. A* **466** (2010), 1919–1934.

15 J. Fang, X. Yu and X.-Q. Zhao. Traveling waves and spreading speeds for time-space periodic monotone systems. *J. Funct. Anal.* **272** (2017), 4222–4262.

16 J.-S. Guo and F. Hamel. Front propagation for discrete periodic monostable equations. *Math. Ann.* **335** (2006), 489–525.

17 J.-S. Guo and C.-H. Wu. Recent developments on wave propagation in 2-species competition systems. *Discrete Cont. Dyn. Syst. Ser. B* **17** (2012), 2713–2724.

18 F. Hamel and L. Rossi. Transition fronts for the Fisher-KPP equation. *Trans. Am. Math. Soc.* **368** (2016), 8675–8713.

19 D. Henry. *Geometric theory of semilinear parabolic equations*. Lecture Notes in Math, vol. 840 (Berlin: Springer-Verlag, 1981).

20 A. Hoffman, H. J. Hupkes and E. S. Van Vleck. Entire solutions for bistable lattice differential equations with obstacles. *Mem. Am. Math. Soc.* **250** (2017), no. 1188, 119 pp.

21 J. Huang and W. Shen. Speeds of spread and propagation for KPP models in time almost and space periodic media. *SIAM J. Appl. Dyn. Syst.* **8** (2009), 790–821.

22 Y. Kyrychko, S. A. Gourley and M. V. Bartuccelli. Dynamics of a stage-structured population model on an isolated finite lattice. *SIAM J. Math. Anal.* **37** (2006), 1688–1708.

23 X. Liang and X.-Q. Zhao. Asymptotic speeds of spread and traveling waves for monotone semiflows with applications. *Commun. Pure Appl. Math.* **60** (2007), 1–40.

24 X. Liang and X.-Q. Zhao. Spreading speeds and traveling waves for abstract monostable evolution systems. *J. Funct. Anal.* **259** (2010), 857–903.

25 S. Ma and X. Zou. Propagation and its failure in a lattice delayed differential equation with global interaction. *J. Differ. Eqn.* **212** (2005), 129–190.

26 R. K. Miller and A. N. Michel. *Ordinary differential equations* (New York: Academic Press, 1982).

27 G. Nadin. Traveling fronts in space-time periodic media. *J. Math. Pures Appl.* **92** (2009), 232–262.

28 G. Nadin, J. M. Roquejoffre, L. Ryzhik and A. Zlatoš. Existence and non-existence of Fisher-KPP transition fronts. *Arch. Ration. Mech. Anal.* **203** (2012), 217–246.

29 G. Nadin and L. Rossi. Propagation phenomena for time heterogeneous KPP reaction-diffusion equations. *J. Math. Pures Appl.* **98** (2012), 633–653.

https://doi.org/10.1017/prm.2020.31 Published online by Cambridge University Press

30 G. Nadin and L. Rossi. Transition waves for Fisher-KPP equations with general time-heterogeneous and space-periodic coefficients. *Anal. PDE* **8** (2015), 1351–1377.

31 G. Nadin and L. Rossi. Generalized transition fronts for one-dimensional almost periodic Fisher-KPP equations. *Arch. Ration. Mech. Anal.* **223** (2017), 1239–1267.

32 J. Nolen, M. Rudd and J. Xin. Existence of KPP fronts in spatially-temporally periodic advection and variational principle for propagation speeds. *Dyn. PDE* **2** (2005), 1–24.

33 J. Nolen and J. Xin. Existence of KPP type fronts in space-time periodic shear flows and a study of minimal speeds based on variational principle. *Discrete Cont. Dyn. Syst.* **13** (2005), 1217–1234.

34 R. B. Salako and W. Shen. Long time behavior of random and nonautonomous Fisher-KPP equations. Part II. Transition fronts. *Stochastics Dyn.* **19** (2019), 1950046.

35 R. B. Salako and W. Shen. Long time behavior of random and nonautonomous Fisher-KPP equations. Part I. Stability of equilibria and spreading speeds. arXiv:1806.01354, 2018.

36 W. Shen. Traveling waves in diffusive random media. *J. Dyn. Differ. Eqn.* **16** (2004), 1011–1060.

37 W. Shen. Existence of generalized traveling waves in time recurrent and space periodic monostable equations. *J. Appl. Anal. Comput.* **1** (2011), 69–93.

38 W. Shen. Existence, uniqueness, and stability of generalized traveling waves in time dependent monostable equations. *J. Dyn. Differ. Eqn.* **23** (2011), 1–44.

39 N. Shigesada and K. Kawasaki, *Biological invasions: theory and practice*. Oxford Series in Ecology and Evolution (Oxford: Oxford University Press, 1997).

40 B. Shorrocks and I. R. Swingland. *Living in a patch environment* (New York: Oxford University Press, 1990).

41 Z.-C. Wang, W.-T. Li and J. Wu. Entire solutions in delayed lattice differential equations with monostable nonlinearity. *SIAM J. Math. Anal.* **40** (2009), 2392–2420.

42 H. F. Weinberger. On spreading speeds and traveling waves for growth and migration models in a periodic habitat. *J. Math. Biol.* **45** (2002), 511–548.

43 B. Zinner, G Harris and W. Hudson. Traveling wavefronts for the discrete Fisher's equation. *J. Differ. Eqn.* **105** (1993), 46–62.