Proceedings of the Royal Society of Edinburgh, **136A**, 733–758, *2006*

Branches of solutions to semilinear biharmonic equations on $\mathbb{R}^N$

Yinbin Deng
Department of Mathematics, Huazhong Normal University,
Wuhan 430079, People's Republic of China
(ybdeng@public.wh.hb.cn)

Yi Li
Department of Mathematics, Hunan Normal University,
Changsha 410081, Hunan, People's Republic of China, and
Department of Mathematics, University of Iowa,
Iowa City, IA 52242, USA
(yi-li@uiowa.edu)

(MS received 12 November 2003; accepted 31 August 2005)

For a large class of functions f, we consider the nonlinear biharmonic eigenvalue problem

$$\Delta^2 u(x) + f(x, u(x)) = \lambda u(x) \quad \text{for } x \in \mathbb{R}^N, \qquad \lim_{|x|\to\infty} u(x) = 0, \quad u \not\equiv 0.$$

We describe the behaviour of the branch of solutions emanating from an eigenvalue of odd multiplicity below the essential spectrum of the linearized problem. The discussion is based on the degree theory for C^2 proper Fredholm maps developed by Fitzpatrick, Pejsachowicz and Rabier.

1. Introduction

We consider a nonlinear biharmonic eigenvalue problem of the form

$$\Delta^2 u(x) + f(x, u(x)) = \lambda u(x) \quad \text{for } x \in \mathbb{R}^N, \qquad \lim_{|x|\to\infty} u(x) = 0, \quad u \not\equiv 0, \tag{1.1}$$

where $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a mapping satisfying the following conditions:

(H_1) $f(\cdot, 0) \equiv 0$;

(H_2) the function f satisfies the Carathéodory conditions;

(H_3) $f(x, \cdot) \in C^2(\mathbb{R})$ for almost all $x \in \mathbb{R}^N$ and, for all compact $K \subset \mathbb{R}$, the functions $\{\partial^2_{22} f(x, \cdot) : K \to \mathbb{R}\}$, $x \in \mathbb{R}^N$, are equi-continuous and $\partial^2_{22} f$ is bounded on $\mathbb{R}^N \times K$;

(H_4) $\partial_2 f(\cdot, 0)$ is bounded on $\mathbb{R}^N$, the limit

$$\alpha := \lim_{|x|\to\infty} \partial_2 f(x, 0)$$

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

exists, and there exist $0 \leqslant a(x), b(x) \in L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ such that

$$|\alpha u - f(x,u)| \leqslant a(x)|u| + b(x)|u|^{\sigma+1}$$

for some $\sigma > 1$ and $\alpha - \partial_2 f(x,0) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

As an example, we consider a mapping f of the form

$$f(x,s) = (p(x) + q(x)r(s))s$$

with the following properties:

(i) $p, q : \mathbb{R}^N \to \mathbb{R}$ are measurable and $r : \mathbb{R} \to \mathbb{R}$ is continuous;

(ii) $r(0) = 0$, $r(s)s$ is $C^1(\mathbb{R})$, $|r(s)s| \leqslant C|s|^{\sigma+1}$, and $r \geqslant 0$;

(iii) $\alpha = \lim_{|x|\to\infty} p(x)$ and $(\alpha - p(x)) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $q \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

It is easy to check that under the conditions (i)–(iii), the mapping f satisfies all the hypotheses (H_1)–(H_4).

Remark 1.1. When we consider the mapping $(x,s) \mapsto f(x,s)/s$, we always bear in mind the following mapping:

$$(x,s) \mapsto \begin{cases} \dfrac{f(x,s)}{s} & \text{if } s \neq 0, \\ \partial_2 f(x,0) & \text{if } s = 0. \end{cases}$$

Remark 1.2. Note that condition (H_3) is used to ensure that the Nemitsky operator $N : X \to Y$ associated with f is of class C^2 between the appropriate function spaces X and Y. This means that we can use the degree theory for proper C^2 Fredholm maps (see [8]). However, by using the degree theorem for C^1 Fredholm maps developed by Pejsachowicz and Rabier [18], we can replace (H_3) by

$(H_3)'$ $f(x,\cdot) \in C^1(\mathbb{R})$ for almost all $x \in \mathbb{R}^N$ and, for every compact $K \subset \mathbb{R}$, the functions $\{\partial_2 f(x,\cdot) : K \to \mathbb{R} \mid x \in \mathbb{R}^N\}$ are equi-continuous.

In fact, using the notation and arguments of § 3, hypotheses (H_1), (H_2), $(H_3)'$ and (H_4) are sufficient to ensure the following properties hold, for every compact $K \subset \mathbb{R}$:

(i) $\partial_2 f$ is bounded on $\mathbb{R}^N \times K$;

(ii) there exists a constant $C = C(K)$ such that for all $(x, s_1), (x, s_2) \in \mathbb{R}^N \times K$, we have $|f(x,s_1) - f(x,s_2)| \leqslant C|s_1 - s_2|$;

(iii) for every $u \in X$, there exists a constant $C = C(u)$ such that

$$|f(x,u(x))| \leqslant C|u(x)| \quad \text{a.e. on } x \in \mathbb{R}^N;$$

(iv) $\lim_{|x|\to\infty}\{\partial_2 f(x,u(x)) - \partial_2 f(x,0)\} = 0$.

Furthermore, $N \in C^1(X,Y)$.

Then, using a degree for proper C^1 Fredholm maps, theorem 1.3 remains true with (H_3) replaced by $(H_3)'$.

For $m \in \mathbb{N}$ and $p \geqslant 1$, we adopt the standard notation [2] for the Sobolev space $W^{m,p}(\mathbb{R}^N)$. Fixing a value $p \in (N/4, \infty) \cap (1, \infty)$ we set $X := W^{4,p}(\mathbb{R}^N)$, and we recall that the condition $\lim_{|x|\to\infty} u(x) = 0$ is satisfied for all $u \in X$ for such p.

We are interested in pairs $(\lambda, u) \in \mathbb{R} \times W^{4,p}(\mathbb{R}^N)$, solutions for the problem (1.1). Our aim is to show the existence of global branches (in the spirit of [23, 24]) of solutions of (1.1) bifurcating from a trivial solution $(\lambda_0, 0)$ in $\mathbb{R} \times X$.

For second-order elliptic differential equations on bounded domains, this kind of result goes back to the celebrated papers of Rabinowitz [24], Ambrosetti and Gamez [1] and Crandall and Rabinowitz [3]. Recently, there has been a resurgence of interest in the case of second-order elliptic partial differential equations such as

$$-\Delta u(x) + f(x, u(x)) = \lambda u(x) \quad \text{for } x \in \mathbb{R}^N, \qquad \lim_{|x|\to\infty} u(x) = 0, \quad u \not\equiv 0, \tag{1.2}$$

under various assumptions on f (see, for example, [5, 13, 14, 23]). In [23] (under appropriate conditions on f) the existence of global bifurcation for (1.2) was established. Some results on global bifurcation of the positive solution coming from $(\Lambda, 0)$, where Λ is the lowest eigenvalue of the linearization at $u = 0$, were also obtained there.

Let

$$\mathcal{Z} := \{(\lambda, u) \in (-\infty, \alpha) \times X \mid (\lambda, u) \text{ is a solution to (1.1)},\ u \neq 0\},$$

where α is defined in (H_4). Consider on $\mathcal{Z} \cup \{(\lambda_0, 0)\}$ the topology inherited from $\mathbb{R} \times X$ and let $\mathcal{C}_{\lambda_0}$ be the connected component of $\mathcal{Z} \cup \{(\lambda_0, 0)\}$ containing $(\lambda_0, 0)$, for $\lambda_0 \in (-\infty, \alpha)$.

Moreover, consider the linear biharmonic operator in $L^2(\mathbb{R}^N)$ defined by

$$Su := \Delta^2 u + \partial_2 f(\cdot, 0)u \quad \text{for } u \in \mathcal{D}(S) := W^{4,2}(\mathbb{R}^N). \tag{1.3}$$

Using this notation, we prove, in § 8, the following theorem.

THEOREM 1.3. *Suppose that the hypotheses* (H_1)–(H_4) *hold, and there exists* $\lambda_0 < \alpha$ *such that* $\dim \operatorname{Ker}(S - \lambda_0)$ *is odd.*

Then $\mathcal{C}_{\lambda_0}$ *has at least one of the following properties:*

(i) $\mathcal{C}_{\lambda_0}$ *is unbounded in* $\mathcal{Z}$*;*

(ii) *the closure of* $\mathcal{C}_{\lambda_0}$ *contains a point of the form* $(\lambda^*, 0)$ *with* $\lambda^* \neq \lambda_0$*;*

(iii) $\sup_{(\lambda,u)\in\mathcal{C}_{\lambda_0}} \lambda = \alpha$.

It is much more complicated to deal with the bifurcation for biharmonic equations. Firstly, there is no maximum principle for the biharmonic problem. So we cannot obtain asymptotic estimates of the solutions by the methods used to deal with the second-order elliptic problem. Secondly, we know little about the properties of the eigenfunctions of the biharmonic operator in $\mathbb{R}^N$. To overcome these difficulties, we first introduce the fundamental solutions for the linear biharmonic operator $\Delta^2 - \lambda$ if $\lambda < 0$. By applying some properties of Hankel functions, which are the solutions of Bessel's equation, we obtain the asymptotic representation of the fundamental solution of $\Delta^2 - \lambda$ at ∞ and 0. We then prove that, for $p > 1$,

$$\Delta^2 - \lambda : W^{4,p} \to L^p$$

is an isomorphism if $\lambda < 0$. Asymptotic estimates of the solutions of (1.1) can be obtained from the properties of the fundamental solutions of $\Delta^2 - \lambda$. We also establish some L^p theory for the biharmonic problem, so that a bootstrap argument can be used to deduce the regularity of the linear inhomogeneous biharmonic problem.

For the early results on the existence and other properties of solutions associated with biharmonic operators, please refer to [11, 12, 17, 19, 20] and references therein.

The organization of this paper is as follows. In § 2, we recall some notions about the degree theory of proper C^2 Fredholm mappings and, using this degree, we state a quite general version of the Rabinowitz-type global bifurcation theorem for C^2 proper Fredholm mappings proved by Pejsachowicz and Rabier in [8]. In § 3, we develop a functional framework, which will permit us to use this bifurcation theorem in order to handle the problem (1.1). Using hypotheses (H_1)–(H_3), we find a C^2 mapping $F : \mathbb{R} \times X \to L^p(\mathbb{R}^N)$, whose zeros are solutions of the problem (1.1) and such that $D_2F_{(\lambda,u)}(v) = \Delta^2 v + \{\partial_2 f(\cdot, u) - \lambda\}v$. In § 4, we introduce the fundamental solutions of $\Delta^2 - \lambda$ for $\lambda < 0$ and establish some properties of the fundamental solutions. In § 5, we show that the mapping $\Delta^2 - \lambda : W^{4,p} \to L^p$ is an isomorphism for all $p \geqslant 2$ if $\lambda < 0$. Asymptotic estimates of the solutions of the problem (1.1) are given in § 6. Using these estimates, we show that F is boundedly proper for $\lambda < \alpha$. In § 7, we establish the L^p theory for the biharmonic equations, and then we check that the choice of p in the definition of the space X does not affect the linearization spectrum. We conclude the discussion in this section by proving, using (H_4), that $D_2F_{(\lambda,u)}$ is a linear Fredholm operator with index 0 for every $\lambda < \alpha$. Finally, in § 8, we complete the proof of theorem 1.3.

2. Degree of Fredholm mappings

In this section, we outline the construction (see [6, 8] for details) of the degree of proper C^2 Fredholm mappings, and then we state the general bifurcation theorem used to prove our result.

Let X and Y be real Banach spaces. Denote by $L(X, Y)$ the space of bounded linear operators from X to Y with the usual norm. An operator in $L(X, Y)$ is called *Fredholm with index* 0 if its kernel has finite dimension and its image is closed with the same finite codimension in Y. We denote by $\phi_0(X, Y)$ the subset of $L(X, Y)$ consisting of those operators which are Fredholm with index 0 and by $GL(X, Y)$ the subset of $\phi_0(X, Y)$ consisting of the invertible ones. If $T \in GL(X)$ is a compact perturbation of the identity, we let $\deg_{\mathrm{LS}}(T)$ be the Leray–Schauder degree of $T : U \to X$ with respect to 0, where U is any neighbourhood of the origin. For an interval $I = [a, b]$ and a continuous path $\alpha : I \to \phi_0(X, Y)$ we call a continuous path $\beta : I \to GL(Y, X)$ a parametrix for α if each $\beta(\lambda)\alpha(\lambda)$ is a compact perturbation of the identity. Parametrices always exist [6]. If the ends of the path, $\alpha(a)$ and $\alpha(b)$, are invertible, then the parity of α in I, $\sigma(\alpha, I)$, defined by

$$\sigma(\alpha, I) = \deg_{\mathrm{LS}}(\beta(a)\alpha(a)) \deg_{\mathrm{LS}}(\beta(b)\alpha(b)),$$

is independent of the choice of parametrix [6].

Note that the Leray–Schauder degree is used in [6] only with linear compact perturbations of the identity, and, hence, $\sigma(\alpha, I) \in \{-1, 1\}$. The parity is an intersection index which, generically, is a mod 2 count of the number of intersections of $\alpha(I)$ with the set of singular operators. It is an additive homotopy invariant of paths in $\phi_0(X, Y)$ with invertible endpoints. Moreover, the parity is 1 if and only if the path is homotopic to a path of invertible operators.

With this in mind, we can describe briefly the construction of the degree, as follows.

Let $\mathcal{O}$ be an open, simply connected subset of X and $F : \mathcal{O} \to Y$ be a C^2 Fredholm mapping with index 0 (i.e. such that $DF_{(x)} \in \phi_0(X, Y)$ for $x \in \mathcal{O}$). A base point for the degree of F is any point $x_0 \in \mathcal{O}$ at which $DF_{(x_0)}$ is invertible.

Assume that there exists a base point p for F. Let Ω be a bounded open subset of $\mathcal{O}$ and such that F can be extended by continuity as a proper mapping to the closure $\bar{\Omega}$ of Ω; i.e. such that the pre-image $F^{-1}(K) \cap \bar{\Omega}$ of every compact set K in Y is also compact. Then, if $y \notin F(\partial\Omega)$ and if $DF_{(x)}$ is invertible for all $x \in F^{-1}(y) \cap \Omega$, the degree of F on Ω with respect to y and relative to p is defined by

$$d_p(F, \Omega, y) = \sum_{x \in F^{-1}(y) \cap \Omega} \sigma_p(x),$$

where $\sigma_p(x) = \sigma(DF \circ \gamma, [0, 1])$ is the parity of the derivative DF along any curve $\gamma : [0, 1] \to \mathcal{O}$ joining p to x. The fact that $\sigma_p(x)$ does not depend on the choice of γ follows immediately from the homotopy invariance of the parity and the simple connectedness of $\mathcal{O}$.

Using the general Sard–Smale theorem [21], the definition of degree is extended by regular value approximation to the case when $DF_{(x)}$ is not necessarily invertible for all $x \in F^{-1}(y) \cap \Omega$.

This base point degree satisfies the usual additivity, excision and normalization properties. Its most important property is the homotopy property [8].

Definition 2.1. Let X and Y be Banach spaces and I be an open interval of $\mathbb{R}$. We say that a mapping $F : I \times X \to Y$ is *boundedly proper* if the restriction of F to any closed bounded subset of $[a, b] \times X$ is proper for all a and b such that $\inf I < a \leqslant b < \sup I$ (i.e. for every compact subset K of Y and for every closed bounded subset B of $[a, b] \times X$, $F^{-1}(K) \cap B$ is compact).

Definition 2.2. Let X, Y be Banach spaces and I an open interval of $\mathbb{R}$. We say that a mapping $F : I \times X \to Y$ is *Fredholm with index* 0 if $D_2F_{(\lambda,u)}$ exists and $D_2F_{(\lambda,u)} \in \phi_0(X, Y)$ for all $(\lambda, u) \in I \times X$.

Now we can recall the global bifurcation theorem for Fredholm mappings [8] as follows. Let $p_1 : \mathbb{R} \times X \to \mathbb{R}$ denote the projection $p_1(\lambda, u) = \lambda$ for $(\lambda, u) \in \mathbb{R} \times X$.

Theorem 2.3 (Fitzpatrick *et al.* [8]). *Let X and Y be real Banach spaces, $I \subseteq \mathbb{R}$ be an open interval and $F : I \times X \to Y$ be a C^2 mapping with $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Suppose that F is boundedly proper and Fredholm with index 0. Moreover, assume that there exist $\lambda_0 \in I$ and $\epsilon > 0$ such that $0 < |\lambda - \lambda_0| \leqslant \epsilon$ implies that*

$$\lambda \in I, \quad D_2F_{(\lambda,0)} \in GL(X, Y), \quad \text{and} \quad \sigma(D_2F_{(\lambda,0)}, [\lambda_0 - \epsilon, \lambda_0 + \epsilon]) = -1.$$

Let $\mathcal{Z} = \{(\lambda, u) \in I \times X \mid F(\lambda, u) = 0$ and $u \neq 0\}$, and denote by $\mathcal{C}_{\lambda_0}$ the connected component of $\mathcal{Z} \cup \{(\lambda_0, 0)\}$ containing $(\lambda_0, 0)$. Then $\mathcal{C}_{\lambda_0}$ has at least one of the following properties:

(i) *$\mathcal{C}_{\lambda_0}$ is unbounded;*

(ii) *the closure $\mathcal{C}_{\lambda_0}$ contains a point of the form $(\lambda^*, 0)$ with $\lambda^* \in I \backslash [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$;*

(iii) *the closure of $p_1(\mathcal{C}_{\lambda_0})$ intersects the boundary of I.*

3. A functional framework

The aim of this section is to define a mapping $F : \mathbb{R} \times X \to Y$ whose zeros are solutions of the problem (1.1) and which satisfies the hypotheses of theorem 2.3.

To do this we choose $p \in (\frac{1}{4}N, \infty) \cap (1, \infty)$, and we set

$$X = W^{4,p}(\mathbb{R}^N) \quad \text{and} \quad Y = L^p(\mathbb{R}^N) \tag{3.1}$$

with the usual norms,

$$\|u\|_p = \left\{\int_{\mathbb{R}^N} |u|^p\right\}^{1/p} \quad \text{and} \quad \|u\|_X = \left\{\sum_{0 \leqslant |\mu| \leqslant 4} \|D^\mu u\|_p^p\right\}^{1/p},$$

where μ is a multi-index.

We recall the following properties of the space X (see [2, 9]).

(1) $X \hookrightarrow C(\mathbb{R}^N)$, continuously.

Moreover, the injection $W^{4,p}(B_R) \hookrightarrow C(\bar{B}_R)$ is completely continuous for every ball $B_R = \{x \in \mathbb{R}^N \mid |x| < R\}$.

(2) $X \hookrightarrow L^q(\mathbb{R}^N)$, continuously, for every $p \leqslant q \leqslant \infty$.

(3) $\lim_{|x| \to \infty} u(x) = 0$, for all $u \in X$.

Consider the mapping

$$F : \mathbb{R} \times X \to Y, \quad \text{by } (\lambda, u) \mapsto \Delta^2 u + f(\cdot, u) - \lambda u. \tag{3.2}$$

Here $\mathbb{R} \times X$ is equipped with the norm

$$\|(\lambda, u)\| = |\lambda| + \|u\|_X.$$

The first result shows that $F : \mathbb{R} \times X \to Y$ is a well-defined C^2 mapping.

LEMMA 3.1. *Let f be a mapping which satisfies the hypotheses (H_1)–(H_3) and let K be a compact subset of $\mathbb{R}$. The following conclusions then hold:*

(i) *$\partial_2 f$ is bounded on $\mathbb{R}^N \times K$;*

(ii) *there exists a constant $C = C(K)$ such that, for all $(x, s_1), (x, s_2) \in \mathbb{R}^N \times K$, we have*

$$|f(x, s_1) - f(x, s_2)| \leqslant C|s_1 - s_2|,$$
$$|\partial_2 f(x, s_1) - \partial_2 f(x, s_2)| \leqslant C|s_1 - s_2|;$$

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

(iii) *on letting $u \in X$, there exists a constant $C = C(u)$ such that*

$$|f(x, u(x))| \leqslant C|u(x)| \quad \text{a.e. on } \mathbb{R}^N.$$

Proof. (i) The conclusion is a consequence of (H_3).

(ii) For all $s_1, s_2 \in K$, we have

$$\begin{aligned} |f(x, s_1) - f(x, s_2)| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\{f(x, s_2 + t(s_1 - s_2))\}\,\mathrm{d}t \right| \\ &\leqslant |s_1 - s_2| \int_0^1 |\partial_2 f(x, s_2 + t(s_1 - s_2))|\,\mathrm{d}t \\ &\leqslant C|s_1 - s_2|. \end{aligned}$$

In the same way, we prove that

$$|\partial_2 f(x, s_1) - \partial_2 f(x, s_2)| \leqslant C|s_1 - s_2|.$$

(iii) Use the injection $X \hookrightarrow L^\infty(\mathbb{R}^N)$, the previous assertion of lemma 3.1(ii), and (H_1). □

To study the differentiability of the mapping F, we consider the nonlinear Nemitsky operator

$$N : X \to Y, \qquad u \mapsto f(\cdot, u). \tag{3.3}$$

THEOREM 3.2. *Let f be a mapping satisfying (H_1)–(H_4). The following conclusions then hold:*

(i) *the operator N (see (3.3)) is well defined;*

(ii) *N is C^2 and, for $u \in X$,*

$$\begin{aligned} DN_{(u)}(\xi) &= \partial_2 f(\cdot, u)\xi \qquad \forall \xi \in X, \\ D^2 N_{(u)}(\xi_1, \xi_2) &= \partial_{22} f(\cdot, u)\xi_1\xi_2 \quad \forall \xi_1, \xi_2 \in X; \end{aligned}$$

(iii) *the mapping F (see (3.2)) is well defined and C^2, and*

$$D_2 F_{(\lambda,u)} = \Delta^2 + (\partial_2 f(\cdot, u) - \lambda).$$

Proof. (i) The fact that the Nemitsky operator N is well defined is a consequence of (H_2), of [26, theorem 18.3, p. 152] (which ensures that $N(u)$ is measurable for every $u \in X$) and of lemma 3.1.

(ii) Let $u \in X$. For every $\xi \in X$, we have

$$\begin{aligned} \|N(u + \xi) - N(u) - \partial_2 f(\cdot, u)\xi\|_p &\leqslant \left\| \int_0^1 \{\partial_2 f(\cdot, u + t\xi) - \partial_2 f(\cdot, u)\}\xi\,\mathrm{d}t \right\|_p \\ &\leqslant \sup_{x\in\mathbb{R}^N} \sup_{t\in[0,1]} \{|\partial_2 f(\cdot, u + t\xi) - \partial_2 f(\cdot, u)|\}\|\xi\|_p \\ &\leqslant \sup_{x\in\mathbb{R}^N} \sup_{t\in[0,1]} \{|\partial_2 f(\cdot, u + t\xi) - \partial_2 f(\cdot, u)|\}\|\xi\|_X. \end{aligned}$$

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

Thus, it follows from lemma 3.1(ii) that

$$\lim_{\|\xi\|_X\to 0}\frac{\|N(u+\xi)-N(u)-\partial_2 f(\cdot,u)\xi\|_p}{\|\xi\|_X}=0.$$

Moreover, by lemma 3.1(i), we find that the following linear operator is well defined and bounded:

$$X\to Y,\qquad \xi\mapsto \partial_2 f(\cdot,u)\xi.$$

Hence, N is Fréchet differentiable.

Let $u\in X$. For every $\xi_1,\xi_2\in X$, we have

$$\begin{aligned}\|DN_{(u+\xi_2)}(\xi_1)&-DN_{(u)}(\xi_1)-\partial_{22}f(\cdot,u)\xi_1\xi_2\|_p\\ &\leqslant\left\|\int_0^1\{\partial_{22}f(\cdot,u+t\xi_2)-\partial_{22}f(\cdot,u)\}\xi_1\xi_2\,\mathrm{d}t\right\|_p\\ &\leqslant\sup_{x\in\mathbb{R}^N}\sup_{t\in[0,1]}\{|\partial_{22}f(\cdot,u+t\xi_2)-\partial_{22}f(\cdot,u)|\}\|\xi_1\xi_2\|_p\\ &\leqslant\sup_{x\in\mathbb{R}^N}\sup_{t\in[0,1]}\{|\partial_{22}f(\cdot,u+t\xi_2)-\partial_{22}f(\cdot,u)|\}C\|\xi_1\|_X\|\xi_2\|_X.\end{aligned}$$

Using hypothesis (H_3), we see that DN is Fréchet differentiable.

With analogous arguments, we prove that D^2N is continuous.

(iii) Using theorem 3.2(ii) and the fact that the mapping

$$\mathbb{R}\times X\to Y,\qquad (\lambda,u)\mapsto\Delta^2u-\lambda u$$

is C^∞, we conclude that F is C^2. □

REMARK 3.3. In § 6, we will prove that the mapping F defined by (3.2) is boundedly proper for $\lambda<\alpha$, and in § 7, we will prove that F is Fredholm with index 0.

4. Fundamental solutions for biharmonic operators

In this section, we deduce the fundamental solutions for the biharmonic operators $\Delta^2-\lambda$. It will be shown that the fundamental solutions of biharmonic operators can be expressed in terms of the fundamental solutions of the Helmholtz equation in $\mathbb{R}^N$ with complex coefficient. The main results of this section were proved in [4].

A fundamental solution of the Helmholtz equation in $\mathbb{R}^N$ is a solution of

$$(-\Delta-\mu)g_\mu=\delta,$$

where δ denotes the Dirac function. Of course, g_μ is not uniquely determined; we may add any solution of $(\Delta+\mu)u=0$. Let us try to make g_μ as simple as possible and ask for spherically symmetric solutions.

For $\mu\in\mathbb{C}$ (complex number set), we have to solve Bessel's equation:

$$z''(r)+\frac{N-1}{r}z'(r)+\mu z(r)=0,\quad r>0. \tag{4.1}$$

Defining $w(r)=z'(r)/r$, we get

$$w''(r)+\frac{N+1}{r}w'(r)+\mu w(r)=0.$$

Thus, $z_{N+2}(r) = z'_N(r)/r$, where $z_N(r)$ is the solution of (4.1), and we only need to know z_1 and z_2. For $N = 1$, we obtain

$$g_\mu(x) = \frac{\mathrm{i}}{2\sqrt{\mu}} \mathrm{e}^{\mathrm{i}\sqrt{\mu}|x|}.$$

Of course there is a second linearly independent solution of (4.1), namely $\bar{g}_\mu$. But for $\mu \in \mathbb{C} \setminus \mathbb{R}^+$, the latter is not square-integrable (since we always choose $\operatorname{Im} \sqrt{\mu} \geqslant 0$). So we only use g_μ.

For general N, with $\nu = \frac{1}{2}(N-2)$ (see [15] or [16]), we obtain

$$g_\mu^{(N)}(x) = \frac{\mathrm{i} c_N^\mu}{|x|^\nu} H_\nu^{(1)}(\sqrt{\mu}|x|), \tag{4.2}$$

where $H_\nu^{(1)} = J_\nu + \mathrm{i} Y_\nu$ is the first Hankel function and

$$c_N^\mu = \frac{\pi \mu^{\nu/2}}{2(2\pi)^{N/2}}$$

has to be adjusted so that $g_\mu^{(N)}(x)$ behaves like $g_0^{(N)}(x)$ for $|x| \to 0$. For $N = 2, 3$ we get

$$g_\mu^{(2)} = \tfrac{1}{4} \mathrm{i} H_0^{(1)}(\sqrt{\mu}|x|), \qquad g_\mu^{(3)}(x) = \frac{1}{4\pi|x|} \mathrm{e}^{\mathrm{i}\sqrt{\mu}|x|}, \tag{4.3}$$

and generally

$$g_\mu^{(N+2)}(r) = -\frac{(g_\mu^{(N)}(r))'}{2\pi r}, \quad r = |x|. \tag{4.4}$$

Now, with these preliminaries, we proceed to deduce the fundamental solutions for biharmonic operators in $\mathbb{R}^N$. A fundamental solution of the biharmonic operator $\Delta^2 - \lambda$ in $\mathbb{R}^N$ is a solution of

$$(\Delta^2 - \lambda) u = \delta. \tag{4.5}$$

Lemma 4.1. *The fundamental solution of (4.5) is*

$$G_\lambda^{(N)} = \frac{1}{2\sqrt{\lambda}} (g_{\sqrt{\lambda}}^{(N)} - g_{-\sqrt{\lambda}}^{(N)}), \tag{4.6}$$

where $g_\mu^{(N)}(x)$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^N$, which is given by (4.2).

Proof. Let $G_\lambda^{(N)}$ be a fundamental solution of (4.5). We then have

$$(\Delta^2 - \lambda) G_\lambda^{(N)} = \delta. \tag{4.7}$$

Taking Fourier transforms on both sides of (4.7), we obtain

$$(|\zeta|^4 - \lambda) \hat{G}_\lambda^{(N)}(\zeta) = \left(\frac{1}{2\pi}\right)^{N/2}.$$

Thus,

$$\hat{G}_\lambda^{(N)}(\zeta) = \left(\frac{1}{2\pi}\right)^{N/2} \frac{-1}{2\sqrt{\lambda}} \left(\frac{1}{|\zeta|^2 + \sqrt{\lambda}} - \frac{1}{|\zeta|^2 - \sqrt{\lambda}} \right). \tag{4.8}$$

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

Because $g_\mu^{(N)}$ is a fundamental solution of (4.1), we have

$$(|\zeta|^2+\mu)\hat{g}_\mu^{(N)}(\zeta) = -\left(\frac{1}{2\pi}\right)^{N/2},$$

and, hence,

$$\hat{g}_\mu^{(N)}(\zeta) = -\left(\frac{1}{2\pi}\right)^{N/2}\frac{1}{|\zeta|^2+\mu}.$$

Substituting into (4.8) with $\mu=\pm\sqrt{\lambda}$, we obtain

$$\hat{G}_\lambda^{(N)}(\zeta) = \frac{1}{2\sqrt{\lambda}}(\hat{g}_{\sqrt{\lambda}}^{(N)}(\zeta)-\hat{g}_{-\sqrt{\lambda}}^{(N)}(\zeta)). \tag{4.9}$$

Thus, we are led to

$$G_\lambda^{(N)}(x) = \frac{1}{2\sqrt{\lambda}}(g_{\sqrt{\lambda}}^{(N)}(\zeta)-g_{-\sqrt{\lambda}}^{(N)}(\zeta)).$$

□

We are interested in the case when $\lambda<0$. For convenience, we write $\lambda=-k^2$ with $k>0$. Using this notation, the fundamental solution of the biharmonic operator can be rewritten as

$$G_k^{(N)}(x) = \frac{1}{2\mathrm{i}k}(g_{\mathrm{i}k}^{(N)}-g_{-\mathrm{i}k}^{(N)}).$$

By using (4.4) we can easily deduce that

$$G_k^{N+2}(r) = -\frac{1}{2\pi|x|}(G_k^{(N)}(r))', \quad r=|x|. \tag{4.10}$$

We also can deduce from (4.7) that

$$G_k^{(N)}(x) = \frac{1}{2\mathrm{i}k}(g_{\mathrm{i}k}^{(N)}-\overline{g_{\mathrm{i}k}^{(N)}}), \tag{4.11}$$

where $\overline{g_{\mathrm{i}k}^{(N)}}$ is the conjugate of $g_{\mathrm{i}k}^{(N)}$.

For example, when $N=3$,

$$g_{\mathrm{i}k}^{(3)}(x) = \frac{1}{4\pi|x|}\exp\{\mathrm{i}\sqrt{\mathrm{i}k}|x|\} = \frac{1}{4\pi|x|}\exp\left\{-\frac{\sqrt{k}}{\sqrt{2}}|x|\right\}\left(\cos\frac{\sqrt{k}}{\sqrt{2}}|x|+\mathrm{i}\sin\frac{\sqrt{k}}{\sqrt{2}}|x|\right),$$

so we have

$$G_k^{(3)}(x) = \frac{1}{4\pi k|x|}\exp\left\{-\sqrt{\tfrac{1}{2}k}|x|\right\}\sin\sqrt{\tfrac{1}{2}k}|x|. \tag{4.12}$$

By (4.10), we obtain

$$G_k^{(5)}(x) = \frac{1}{8k\pi^2|x|^2}\left[\frac{\sqrt{k}}{\sqrt{2}}\sin\sqrt{\tfrac{1}{2}k}|x|+\frac{1}{|x|}\sin\sqrt{\tfrac{1}{2}k}|x|-\sqrt{\tfrac{1}{2}k}\cos\sqrt{\tfrac{1}{2}k}|x|\right]\exp\left\{-\sqrt{\tfrac{1}{2}k}|x|\right\}. \tag{4.13}$$

The asymptotic behaviour and some properties of $G_k^{(N)}(x)$ were given in [4]. For general $\nu=\frac{1}{2}(N-2)$, we have the following lemma.

LEMMA 4.2.

(i)

$$G_k^{(N)}(x) \in C^\infty(\mathbb{R}^N \setminus \{0\})$$

and

$$\Delta^2 G_k^{(N)}(x) + k^2 G_k^{(N)}(x) = 0 \quad \textit{for } x \neq 0. \tag{4.14}$$

(ii) *As* $|x| \to \infty$,

$$\exp\left\{\frac{\sqrt{k}}{\sqrt{2}}|x|\right\} G_k^{(N)}(x) \to 0 \quad \textit{and} \quad \exp\left\{\frac{\sqrt{k}}{\sqrt{2}}|x|\right\} |\nabla G_k^{(N)}(x)| \to 0. \tag{4.15}$$

(iii) *As* $|x| \to 0$, *with* $r = |x|$,

$$G_k^{(N)}(r) = \frac{2^{\nu-2}\Gamma(\nu-1)}{2(2\pi)^{N/2}} r^{2-2\nu} + O(r^{4-2\nu})$$

if $\nu = \frac{1}{2}(N-2) > 1$ *and* $\nu \notin \mathbb{N}$;

$$G_k^{(N)}(r) = \frac{2^{\nu-2}\Gamma(\nu-1)}{2(2\pi)^{N/2}} r^{2-2\nu} + O(r^{4-2\nu} + \ln r)$$

if $\nu = \frac{1}{2}(N-2) \geqslant 2$ *and* $\nu \in \mathbb{N}$;

$$G_k^{(N)}(r) \approx O(\ln r) \quad \textit{if } N = 4, \quad \nu = \tfrac{1}{2}(N-2) = 1,$$
$$G_k^{(N)} = O(1) \quad \textit{if } N = 2, 3 \quad \nu = 0, \tfrac{1}{2}.$$

(iv) $|G_k^{(N)}(r)| \leqslant C g_{-\delta}^{(N)}(r)$ *for some positive constants* C *and* $0 < \delta < \sqrt{k}/\sqrt{2}$.

It follows from properties (ii) and (iii) that

$$\left.\begin{array}{lll}
G_k^{(N)}(x) \in L^p(\mathbb{R}^N) & \text{for } 1 \leqslant p < +\infty, & \text{if } N = 2, 3, 4; \\
G_k^{(N)}(x) \in L^p(\mathbb{R}^N) & \text{for } 1 \leqslant p < \dfrac{N}{N-4}, & \text{if } N \geqslant 5; \\
|\nabla G_k^{(N)}(x)| \in L^p & \text{for } 1 \leqslant p < \dfrac{N}{N-3}, & \text{if } N > 3; \\
|\nabla G_k^{(N)}(x)| \in L^p & \text{for } 1 \leqslant p < +\infty, & \text{if } N = 3; \\
|\nabla G_k^{(N)}(x)| \in L^p & \text{for } 1 \leqslant p \leqslant +\infty, & \text{if } N = 2; \\
|\Delta G_k^{(N)}(x)| \in L^p & \text{for } 1 \leqslant p < \dfrac{N}{N-2}, & \text{if } N \geqslant 3; \\
|\Delta G_k^{(N)}(x)| \in L^p & \text{for } 1 \leqslant p < +\infty, & \text{if } N = 2.
\end{array}\right\} \tag{4.16}$$

Using this information about $G_k^{(N)}(x)$, we can express the solution of an inhomogeneous biharmonic equation as the convolution of the fundamental solution with the inhomogeneous term.

The following theorem comes from [4].

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

Theorem 4.3.

(i) *Let* $f \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and*

$$u = \int_{\mathbb{R}^N} f(z) G_k^{(N)}(x-z)\,\mathrm{d}z.$$

Then

$$\Delta^2 u + k^2 u = f(x).$$

(ii) *Let* u *be a distribution such that*

$$\Delta^2 u + k^2 u = f$$

and $f \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. *Then*

$$u = \int_{\mathbb{R}^N} f(z) G_k^{(N)}(x-z)\,\mathrm{d}z. \tag{4.17}$$

(iii) *There is no non-trivial distribution such that*

$$\Delta^2 u + k^2 u = 0, \quad u \in W^{2,2}(\mathbb{R}^N). \tag{4.18}$$

5. Isomorphism

In this section, we prove that the biharmonic operator

$$\Delta^2 - \lambda : W^{4,p} \hookrightarrow L^p$$

is an isomorphism for all $p \in [2, +\infty)$ if $\lambda < 0$. To this end, we need the following theorem on Young's inequality [25].

Theorem 5.1. *Let* $f \in L^p$ *and* $g \in L^q$ *with* $1/p + 1/q \geqslant 1$. *Then*

$$\int_{\mathbb{R}^N} f(x-z) g(z)\,\mathrm{d}z$$

converges for almost all $x \in \mathbb{R}^N$ *and defines an element of* $L^s(\mathbb{R}^N)$, *where*

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1 \quad \left(\text{with } s = +\infty \text{ when } \frac{1}{p} + \frac{1}{q} = 1\right).$$

Denoting this element by $f * g$, *we also find that* $f * g = g * f$ *and*

$$\|f * g\|_s \leqslant \|f\|_p \|g\|_q.$$

From the properties (see (4.16)) of the fundamental solution $G_k^{(N)}$ for $k > 0$, Young's inequality shows that the convolution $f * G_k^{(N)}$ defines an element of $L^s(\mathbb{R}^N)$ whenever $f \in L^p(\mathbb{R}^N)$, subject to the restrictions

$$p \leqslant \begin{cases} s \leqslant +\infty & \text{if } p > \frac{1}{4}N, \\ s < +\infty & \text{if } p = \frac{1}{4}N, \\ s < \dfrac{Np}{N-4p} & \text{if } 1 \leqslant p < \frac{1}{4}N. \end{cases} \tag{5.1}$$

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

Moreover, setting $T_k f = f * G_k^{(N)}$, we see that

$$T_k : L^p \to L^s$$

is a bounded linear operator under the restrictions (5.1).

Referring again to (4.16), for $i = 1, 2, \ldots, N$, the convolution $f * \partial_i G_k^{(N)}$ defines an element of $L^s(\mathbb{R}^N)$ whenever $f \in L^p(\mathbb{R}^N)$ subject to the restrictions

$$p \leqslant \begin{cases} s \leqslant +\infty & \text{if } p > \frac{1}{3}N, \\ s < \infty & \text{if } p = \frac{1}{3}N, \\ s < \dfrac{N \cdot p}{N - 3p} & \text{if } 1 \leqslant p < \frac{1}{3}N; \end{cases} \tag{5.2}$$

and, setting

$$S_k^i f = f * \partial_i G_k^{(N)} \quad \text{for } i = 1, 2, \ldots, N,$$

we see that

$$S_k^i : L^p \to L^s$$

is a bounded linear operator under the restrictions (5.2).

Also from (4.16), the convolution $f * \Delta G_k^{(N)}$ defines an element of $L^s(\mathbb{R}^N)$ whenever $f \in L^p(\mathbb{R}^N)$ subject to the restrictions

$$p \leqslant \begin{cases} s \leqslant +\infty & \text{if } p > \frac{1}{2}N, \\ s < +\infty & \text{if } p = \frac{1}{2}N, \\ s < \dfrac{Np}{N - 2p} & \text{if } 1 \leqslant p < \frac{1}{2}N; \end{cases} \tag{5.2$'$}$$

and, setting

$$S_k^{\Delta} f = f * \Delta G_k^{(N)},$$

we see that

$$S_k^{\Delta} : L^p \to L^s$$

is a bounded linear operator under the restrictions (5.2)$'$.

So we have obtained the following theorem.

THEOREM 5.2. *Let* $\lambda = -k^2 < 0$. *Then* $\Delta^2 - \lambda$ *is a bounded linear operator from* $W^{4,p}$ *into* L^p. *Furthermore,* $\Delta^2 - \lambda : W^{4,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ *is a self-adjoint operator.*

Define

$$S_p : W^{4,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N), \quad u \mapsto \Delta^2 u - \lambda u. \tag{5.3}$$

We then have the following lemma and theorem.

LEMMA 5.3. *Let* $\lambda = -k^2 < 0$. *Then*

$$S_2 : W^{4,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$$

is an isomorphism.

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

Proof. Because $L^2(\mathbb{R}^N)$ is a Hilbert space and S_2 is a self-adjoint operator which is positive, it follows from [25, lemma 3.2] that $S_2 : W^{4,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is an isomorphism. □

THEOREM 5.4. *Let $\lambda = -k^2 < 0$. Then the bounded linear operator*

$$S_p : W^{4,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$$

defined by (5.3) is an isomorphism, provided that $p \in [2, +\infty)$.

Proof. For any fixed $p \in [2, +\infty)$, by theorem 5.2,

$$S_p = \Delta^2 - \lambda : W^{4,p} \to L^p$$

is a bounded linear operator, and we shall show that

(i) $\operatorname{Ker} S_p = \{0\}$,

(ii) $\operatorname{Range} S_p = L^p(\mathbb{R}^N)$.

Proof of theorem 5.4(i). Suppose that $u \in W^{4,p}(\mathbb{R}^N)$ and $S_p u = 0$. This means that u is a solution of

$$\Delta^2 u - \lambda u = 0, \quad u \in W^{4,p}(\mathbb{R}^N). \tag{5.4}$$

Rewrite (5.4) in the form

$$-\Delta(-\Delta u) = \lambda u, \quad u \in W^{4,p}(\mathbb{R}^N), \qquad -\Delta u \in W^{2,p}(\mathbb{R}^N).$$

By a bootstrap argument, it follows that

$$u \in C^4(\mathbb{R}^N) \cap L^p(\mathbb{R}^N), \qquad \Delta u \in C^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N),$$

and $\lim_{|x|\to\infty} u(x) = 0$, $\lim_{|x|\to\infty} \Delta u(x) = 0$. Define

$$u_1 = (\Delta - \sqrt{\lambda})u \quad \text{and} \quad u_2 = (\Delta + \sqrt{\lambda})u.$$

Then

$$(\Delta + \sqrt{\lambda})u_1 = 0, \qquad (\Delta - \sqrt{\lambda})u_2 = 0, \tag{5.5}$$

and

$$u = \frac{1}{2\sqrt{\lambda}}(u_2 - u_1),$$

$$\lim_{|x|\to\infty} u_1(x) = 0, \qquad \lim_{|x|\to\infty} u_2(x) = 0.$$

For $\lambda < 0$, the solution of (5.5) can be expressed in terms of Hankel functions (see (4.2)). From the asymptotic behaviour of the Hankel functions [4]

$$H_\nu^{(1)}(r) = \left(\frac{1}{\pi r}\right)^{1/2} \exp\left[\mathrm{i}\left(r - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] + O\left(\frac{1}{r^{3/2}} \exp\left(\mathrm{i}\left(r - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right)\right)$$

as $r \to \infty$, we can deduce that

$$\exp\{\operatorname{Im}(\lambda)^{1/4}|x|\}u_i(x) \to 0, \quad i = 1, 2, \text{ as } |x| \to \infty.$$

Thus, we have

$$e^{\operatorname{Im}(\lambda)^{1/4}|x|}u(x) \to 0 \quad \text{as } |x| \to \infty. \tag{5.6}$$

It follows from (5.6) that $u \in L^r(\mathbb{R}^N)$ for all $r \in [2, +\infty)$. In particular, $u \in L^2(\mathbb{R}^N)$ and, hence, $u \in W^{4,2}(\mathbb{R}^N)$. Lemma 5.3 gives us $u \equiv 0$. □

Proof of theorem 5.4(ii). Let $f \in L^p(\mathbb{R}^N)$. We must show that there exists an element $u \in W^{4,p}(\mathbb{R}^N)$ such that $S_p u = f$. For this we consider a sequence $\{f_n\} \subset C_0^\infty(\mathbb{R}^N)$ such that $\|f_n - f\|_p \to 0$ as $n \to \infty$. Since $f_n \in L^2(\mathbb{R}^N)$, it follows from lemma 5.3 that there exists a unique element $u_n \in W^{4,2}(\mathbb{R}^N)$ such that

$$(\Delta^2 - \lambda)u_n = f_n.$$

We now show that

(a) $u_n \in W^{4,p}(\mathbb{R}^N)$,

(b) $\{u_n\}$ is a Cauchy sequence in $W^{4,p}(\mathbb{R}^N)$.

From theorem 4.3 and lemma 5.3, the inverse of the operator $S_2 : W^{4,2} \to L^2$ is known to be an integral operator, which we write as

$$T_k f_n = \int_{\mathbb{R}^N} G_k^{(N)}(x - z) f_n(z)\,\mathrm{d}z$$

with the kernel $G_k^{(N)}$. Furthermore, the estimates on $G_k^{(N)}$ imply that this integral operator acts as a bounded linear operator

$$T_{ks} : L^s(\mathbb{R}^N) \to L^s(\mathbb{R}^N)$$

for $s \in (1, +\infty)$ (see theorem 5.2). Since

$$u_n(x) = \int_{\mathbb{R}^N} G_k^{(N)}(x - z) f_n(z)\,\mathrm{d}z$$

and $f_n \in C_0^\infty(\mathbb{R}^N)$, we can conclude that $u_n \in L^s(\mathbb{R}^N)$ for all $s \in [1, +\infty]$. In particular, $u_n \in L^\infty(\mathbb{R}^N)$.

On the other hand, from lemma 5.3, $f_n \in L^2(\mathbb{R}^N)$ implies that u_n is the unique solution of

$$\Delta^2 u_n - \lambda u_n = f_n, \quad u_n \in W^{4,2}(\mathbb{R}^N). \tag{5.7}$$

Rewrite (5.7) in the form

$$-\Delta(-\Delta u_n) = f_n + \lambda u_n, \quad u_n \in W^{4,2}(\mathbb{R}^N), \qquad \Delta u_n \in W^{2,2}(\mathbb{R}^N). \tag{5.7$'$}$$

From $u_n \in W^{4,2}(\mathbb{R}^N)$ and Sobolev embedding, we deduce that

$$u_n \in \begin{cases} L^{2N/(N-8)}(\mathbb{R}^N), & \text{if } N > 8, \\ L^s(\mathbb{R}^N) \text{ for } s \in (1, +\infty), & \text{if } N = 8, \\ L^s(\mathbb{R}^N) \text{ for } s \in (1, +\infty], & \text{if } N < 8. \end{cases} \tag{5.8}$$

For $N > 8$, we have $f_n + \lambda u_n \in L^{2N/(N-8)}(\mathbb{R}^N)$ and $\Delta u_n \in W^{2,2}(\mathbb{R}^N)$. Using L^p-estimates for Δu_n, we have

$$\Delta u_n \in W^{2,2N/(N-8)}(\mathbb{R}^N),$$

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

and hence

$$u_n \in W^{4,2N/(N-8)}(\mathbb{R}^N).$$

Defining $q_1 = 2N/(N-8)$, and again using Sobolev's embedding, we deduce that

$$u_n \in L^{q_1 N/(N-4q_1)}(\mathbb{R}^N),$$

and, hence,

$$f_n + \lambda u_n \in L^{q_1 N/(N-4q_1)}(\mathbb{R}^N).$$

Because $q_1 N/(N-4q_1) > q_1$, a bootstrap argument can be used to deduce that $u_n \in W^{4,r}(\mathbb{R}^N)$ for all $2 \leqslant r < +\infty$. This implies that $u_n \in W^{4,p}(\mathbb{R}^N)$ for all $p \in [2, +\infty)$ after finitely many bootstrap iterations, and completes the proof of (a).

As for (b), we proceed as follows. By (a), we can now write $f_n = S_p u_n$. Since

$$[u_n - u_m](x) = \int_{\mathbb{R}^N} G_k^{(N)}(x-z)[f_n - f_m](z)\,\mathrm{d}z,$$

the boundedness of the operator $T_{kp} : L^p \to L^p$ implies that

$$\|u_m - u_n\|_p \leqslant K(p)\|f_m - f_n\|_p$$

and

$$(\Delta^2 - \lambda)(u_n - u_m) = f_n - f_m, \quad (u_n - u_m) \in W^{4,p}(\mathbb{R}^N), \quad \Delta(u_n - u_m) \in W^{2,p}(\mathbb{R}^N).$$

Thus,

$$-\Delta(-\Delta(u_n - u_m)) = (f_n - f_m) + \lambda(u_n - u_m), \qquad -\Delta(u_n - u_m) \in W^{2,p}(\mathbb{R}^N).$$

L^p-estimates yield that

$$\|\Delta(u_n - u_m)\|_{W^{2,p}(\mathbb{R}^N)} \leqslant C(p)\{\|u_n - u_m\|_p + \|f_n - f_m\|_p\},$$

and thus

$$\|u_n - u_m\|_{W^{4,p}(\mathbb{R}^N)} \leqslant C(p)\{(K(p)+1)\|f_n - f_m\|_p\}.$$

So $\{u_n\}$ is indeed a Cauchy sequence in $W^{4,p}(\mathbb{R}^N)$. There is an element $u \in W^{4,p}(\mathbb{R}^N)$ such that $\|u_n - u\|_{W^{4,p}} \to 0$ and so, by the continuity of $S_p : W^{4,p} \to L^p$, this implies that $S_p u = f$. □

6. Bounded properness

In this section, we show that the mapping F defined by (3.2) is boundedly proper for $\lambda < \alpha$. This will be derived from the following theorem.

Theorem 6.1. *Let f be a mapping satisfying the hypotheses (H_1)–(H_4), and consider*

$$F : \mathbb{R} \times W^{4,p} \to L^p,$$
$$(\lambda, u) \mapsto \Delta^2 u + f(\cdot, u) - \lambda u.$$

Then the restriction of F to $(-\infty, \alpha) \times W^{4,p}$ is boundedly proper (see definition 2.1).

Proof. Let $[a,b]\subset(-\infty,\alpha)$, and let B be a bounded closed subset of $[a,b]\times W^{4,p}$ and K a compact subset of Y.

We must prove that every sequence $\{(\lambda_n,u_n)\}$ of $F^{-1}(K)\cap B$ has a convergent subsequence. We set

$$F(\lambda_n,u_n)=w_n. \tag{6.1}$$

Without loss of generality we suppose that

$$\begin{aligned}&\lambda_n\to\lambda\in[a,b],\\&u_n\rightharpoonup u \text{ weakly in } W^{4,p},\\&w_n\to w \text{ strongly in } L^p \text{ with } w\in K.\end{aligned}$$

We can choose the subsequence such that

$$w_n\to w \quad \text{a.e. on } \mathbb{R}^N.$$

From (6.1), we have

$$\Delta^2u_n-(\lambda-\alpha)u_n=\alpha u_n-f(x,u_n)+w_n.$$

Since $\lambda<\alpha$, we will write $\lambda-\alpha=-k^2<0$ $(k>0)$ and, thus,

$$u_n(x)=\int_{\mathbb{R}^N}G_k^{(N)}(x-z)[\alpha u_n(z)-f(z,u_n(z))]\,\mathrm{d}z+\int_{\mathbb{R}^N}G_k^{(N)}(x-z)w_n(z)\,\mathrm{d}z. \tag{6.2}$$

Since $p>\frac{1}{4}N$, by Sobolev's embedding we deduce that

$$\|u_n(x)\|_{L^\infty}\leqslant\|u_n(x)\|_{W^{4,p}(\mathbb{R}^N)}\leqslant C$$

for all $n\geqslant1$. By applying (H_4) and lemma 4.2, we have, for some fixed $\delta\in(0,\sqrt{k}/\sqrt{2})$ (see lemma 4.2(iv)),

$$\begin{aligned}\left|\int_{\mathbb{R}^N}G_k^{(N)}(x-z)[\alpha u_n(z)-f(z,u_n(z))]\,\mathrm{d}z\right|&\leqslant C\int_{\mathbb{R}^N}g_{-\delta}^{(N)}(x-z)[a(z)|u_n(z)|+b(z)|u_n(z)|^{\sigma+1}]\,\mathrm{d}z\\&\leqslant C\max\{\|u\|_{L^\infty},\|u\|_{L^\infty}^{\sigma+1}\}\int_{\mathbb{R}^N}g_{-\delta}^{(N)}(x-z)[a(z)+b(z)]\,\mathrm{d}z\\&\leqslant C_1\int_{\mathbb{R}^N}g_{-\delta}^{(N)}(x-z)[a(z)+b(z)]\,\mathrm{d}z\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}^N}G_k^{(N)}(x-z)w_n(z)\,\mathrm{d}z\right|&=\left|\int_{\mathbb{R}^N}G^{(N)}(x-z)(w_n(z)-w(z)+w(z))\,\mathrm{d}z\right|\\&\leqslant C\int_{\mathbb{R}^N}g_{-\delta}^{(N)}(x-z)|w_n(z)-w(z)|\,\mathrm{d}z\\&\quad+C\int_{\mathbb{R}^N}g_{-\delta}^{(N)}(x-z)|w(z)|\,\mathrm{d}z.\end{aligned}$$

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

Substituting into (6.2), we have

$$\begin{aligned}|u_n(x)| \leqslant C_1 \int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)[a(z)+b(z)]\,\mathrm{d}z \\ + C\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)|w_n(z)-w(z)|\,\mathrm{d}z \\ + C\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)|w(z)|\,\mathrm{d}z,\end{aligned}$$

where $g_{-\delta}^{(N)}(x)$ is the fundamental solution of $-\Delta+\delta$ with $0<\delta<\sqrt{k}/\sqrt{2}$, and $a(x)$, $b(x)$ are given by (H_4). Now, for $R>0$ we have

$$\begin{aligned}\|u_n-u\|_{L^p}^p &= \int_{|x|\leqslant R}|u_n-u|^p\,\mathrm{d}x + \int_{|x|\geqslant R}|u_n-u|^p\,\mathrm{d}x \\ &\leqslant \int_{|x|\leqslant R}|u_n-u|^p\,\mathrm{d}x + \int_{|x|\geqslant R}|u_n|^p\,\mathrm{d}x + \int_{|x|\geqslant R}|u|^p\,\mathrm{d}x \\ &\leqslant \int_{|x|\leqslant R}|u_n-u|^p\,\mathrm{d}x + \int_{|x|\geqslant R}|u|^p\,\mathrm{d}x \\ &\quad + C_1\int_{|x|\geqslant R}\biggl[\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)[a(z)+b(z)]\,\mathrm{d}z\biggr]^p\,\mathrm{d}x \\ &\quad + C\int_{|x|\geqslant R}\biggl[\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)|w_n(z)-w(z)|\,\mathrm{d}z\biggr]^p\,\mathrm{d}x \\ &\quad + C\int_{|x|\geqslant R}\biggl[\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)|w(z)|\,\mathrm{d}z\biggr]^p\,\mathrm{d}x.\end{aligned}$$

The proof may now be completed using the following facts.

(i)

$$\lim_{R\to\infty}\int_{|x|\geqslant R}|u|^p\,\mathrm{d}x = 0$$

because $u\in L^p(\mathbb{R}^N)$.

(ii)

$$\lim_{R\to\infty}\int_{|x|\geqslant R}\biggl[\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)[a(z)+b(z)]\,\mathrm{d}z\biggr]^p\,\mathrm{d}x = 0$$

and

$$\lim_{R\to\infty}\int_{|x|\geqslant R}\biggl[\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)|w(z)|\,\mathrm{d}z\biggr]^p\,\mathrm{d}x = 0$$

because $-\Delta+\delta$ is an isomorphism from $W^{2,p}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$.

(iii)

$$\lim_{n\to\infty}\int_{|x|\geqslant R}\biggl[\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)|w_n(z)-w(z)|\,\mathrm{d}z\biggr]^p\,\mathrm{d}x = 0$$

because $-\Delta + \delta$ is an isomorphism from $W^{2,p}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$ and $w_n \to w$ strongly in $L^p(\mathbb{R}^N)$.

(iv)
$$\lim_{n\to\infty} \int_{|x|\leqslant R} |u_n - u|^p \,\mathrm{d}x = 0$$
by the compact embedding theorem for bounded domains.

By the second assertion of lemma 3.1, there exists a constant $C > 0$ such that

$$|f(x, u(x)) - f(x, u_n(x))| \leqslant C|u(x) - u_n(x)|$$

for all $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$. Thus,

$$\|f(\cdot, u_n) - f(\cdot, u)\|_{L^p} \leqslant C\|u_n - u\|_{L^p} \to 0 \quad \text{as } n \to \infty.$$

Now, consider the expression

$$\begin{aligned}(\Delta^2 + 1)u_n &= \Delta^2 u_n - \lambda u_n + f(x, u_n) - f(x, u_n) + \lambda u_n + u_n \\ &= F(\lambda_n, u_n) - f(x, u_n) + (\lambda + 1)u_n \to w - f(x, w) + (\lambda + 1)u \\ &\quad \text{strongly in } L^p \text{ as } n \to \infty.\end{aligned}$$

The operator $\Delta^2 + 1$ is an isomorphism of $W^{4,p}$ onto L^p (see theorem 5.4). So it follows that $\{u_n\}_{n=1}^{\infty}$ converges in $W^{4,p}(\mathbb{R}^N)$, which completes the proof of the theorem. □

7. The spectrum of the linear problem

In this section, we show that the features of the spectrum of the linearized problem that are important for our bifurcation results do not depend on the value of p used in the choice of the space $W^{4,p}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ for $p \geqslant 2$.

We present the results for any operator of the form $\Delta^2 u + Vu$, where $V \in L^{\infty}(\mathbb{R}^N)$. The linearization of (1.1) is obtained by setting $V = \partial_2 f(\cdot, 0)$.

For each $1 < q < +\infty$, we consider the family $(A_{q,\lambda})_{\lambda\in R}$ of bounded linear operators defined by

$$A_{q,\lambda} : W^{4,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N), \qquad u \mapsto \Delta^2 u + (V - \lambda)u,$$

and we set

$$\Sigma_q = \{\lambda \in \mathbb{R} \mid A_{q,\lambda} : W^{4,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N) \text{ is not an isomorphism}\}.$$

Remark 7.1. The mapping $(x, s) \hookrightarrow V(x)s$ satisfies the properties (H_1)–(H_3) and we assume that
$$\alpha = \lim_{|x|\to\infty} V(x) \in \mathbb{R}.$$

Remark 7.2. $\Sigma_q = \sigma(S_q)$, where S_q is the operator in $L^q(\mathbb{R}^N)$ defined by $S_q u = \Delta^2 u + Vu$ for $u \in \mathcal{D}(S_q) = W^{4,q}(\mathbb{R}^N)$ and $\sigma(S_q)$ is the spectrum of S_q in the usual sense.

In the following, the self-adjoint operator S_2 will be denoted by S.

The results below show that, for $q \in (1, +\infty)$ and $\lambda < \alpha$:

(i) $A_{q,\lambda}$ is a Fredholm operator of index zero;

(ii) $\dim \operatorname{Ker} A_{q,\lambda}$ is independent of q;

(iii) $L^q(\mathbb{R}^N) = \operatorname{Ker} A_{q,\lambda} \oplus \operatorname{Range} A_{q,\lambda}$.

In other words, the essential spectrum of S_q is contained in $[\alpha, +\infty)$, and for eigenvalues in $(-\infty, \alpha)$ the algebraic and geometric multiplicities are equal and independent of q.

Lemma 7.3. *Let $V \in L^\infty(\mathbb{R}^N)$, $q > 1$, $\lim_{|x|\to\infty} V(x) = 0$. The multiplication operator defined by*

$$W^{4,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N), \qquad u \mapsto Vu,$$

is then compact.

Proof. Let

$$\chi_\rho = \begin{cases} 1 & \text{for } |x| \leqslant \rho, \\ 0 & \text{for } |x| \geqslant \rho. \end{cases}$$

By the compactness of the Sobolev embedding on bounded domains, it follows that the operator

$$W^{4,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N), \qquad u \mapsto \chi_\rho V u,$$

is compact. But, for $u \in W^{4,q}(\mathbb{R}^N)$,

$$\begin{aligned} \|Vu - \chi_\rho V u\|_q^q &= \int_{|x|>\rho} |V|^q |u|^q \,\mathrm{d}x \\ &\leqslant \sup_{|x|>\rho} \{|V(x)|^q\} \|u\|_q^q \\ &\leqslant \sup_{|x|>\rho} \{|V(x)|^q\} \|u\|_{W^{4,q}}^q. \end{aligned}$$

Since $\lim_{\rho\to\infty}\{\sup_{|x|>\rho} |V(x)|^q\} = 0$, it follows that $u \mapsto Vu$ can be approximated by compact operators and so is itself compact. □

Lemma 7.4. *Let $V \in L^\infty(\mathbb{R}^N)$ and $q > 1$, $\lim_{|x|\to\infty} V(x) = \alpha$. Then, for $\lambda < \alpha$, the operator $A_{q,\lambda}$ defined by*

$$A_{q,\lambda} : W^{2,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N), u \mapsto \Delta^2 u - \lambda u + V(x)u,$$

is Fredholm with index zero.

Proof. We write

$$A_{q,\lambda} = \Delta^2 + (\alpha - \lambda) + (V(x) - \alpha).$$

From theorem 5.4 we have that

$$W^{4,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N), \qquad u \mapsto \Delta^2 u + (\alpha - \lambda)u,$$

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

is an isomorphism for all $\lambda < \alpha$, and so is a Fredholm operator with index 0. Moreover, we have $\lim_{x\to\infty}(V(x) - \alpha) = 0$. It follows from lemma 7.3 that the multiplication operator

$$W^{4,p}(\mathbb{R}^N) \to L^q(\mathbb{R}^N), \qquad u \mapsto (V - \alpha)u,$$

is a compact operator.

Recall that if $T : X \to Y$ is a bounded linear Fredholm operator and $K : X \to Y$ is a compact operator, then $T + K$ is Fredholm and $\operatorname{index}(T) = \operatorname{index}(T + K)$ (see, for example, [10, theorem 4.2, p. 189]). Thus, the operator $A_{q,\lambda}$ is a Fredholm operator with index 0 for all $\lambda < \alpha$. □

LEMMA 7.5 (Rabier and Stuart [22]). *Let* $h \in L^p(\mathbb{R}^N)$ *for some* $p \in [1, +\infty]$. *Consider the equation*

$$-\Delta u + u = h \tag{7.1}$$

in the sense of distributions.

(i) *There is a unique tempered distribution* $u = \Gamma(h)$ *satisfying (7.1).*

(ii) *If* $h \in L^p$ *for some* $p \in (1, +\infty)$, *then* $\Gamma(h) \in W^{2,p}(\mathbb{R}^N)$ *and there exists a constant* $C(N, p)$ *such that*

$$\|\Gamma(h)\|_{W^{2,p}} \leqslant C(N, p)\|h\|_{L^p} \tag{7.2}$$

for all $h \in L^p(\mathbb{R}^N)$.

(iii) *For* $p \in (1, +\infty)$, $-\Delta + 1 : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ *is an isomorphism.*

By applying this lemma, we can obtain L^p-estimates for the biharmonic equation.

LEMMA 7.6. *Let* $v, w \in L^p(\mathbb{R}^N)$ *for some* $p \in (1, +\infty)$ *be such that*

$$\int_{\mathbb{R}^N} v\Delta z \,\mathrm{d}x = \int_{\mathbb{R}^N} wz \,\mathrm{d}x \quad \text{for all } z \in C_0^\infty(\mathbb{R}^N). \tag{7.3}$$

Then $v \in W^{2,p}(\mathbb{R}^N)$ *and* $\Delta v = w$.

Proof. From (7.3) it follows that v is a distribution solution of

$$\Delta v = w \tag{7.4}$$

and thus

$$(-\Delta + 1)v = v - w \in L^p(\mathbb{R}^N).$$

By lemma 7.5(iii), $-\Delta + 1 : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is an isomorphism. So there exists $u \in W^{2,p}(\mathbb{R}^N)$ such that

$$(-\Delta + 1)u = v - w,$$

i.e.

$$-\int_{\mathbb{R}^N} u\Delta z \,\mathrm{d}x + \int_{\mathbb{R}^N} uz \,\mathrm{d}x = \int_{\mathbb{R}^N} vz \,\mathrm{d}x - \int_{\mathbb{R}^N} wz \,\mathrm{d}x$$

for all $z \in C_0^\infty(\mathbb{R}^N)$. From (7.3) we have

$$\int_{\mathbb{R}^N} (u-v)\Delta z \, dx = \int_{\mathbb{R}^N} (u-v) z \, dx \quad \text{for all } z \in C_0^\infty(\mathbb{R}^N),$$

and, hence,

$$\int_{\mathbb{R}^N} (u-v)(-\Delta z + z) \, dx = 0 \quad \text{for all } z \in C_0^\infty(\mathbb{R}^N). \tag{7.5}$$

Consider the equation

$$-\Delta z + z = |u-v|^{p-2}(u-v). \tag{7.6}$$

Because $|u-v|^{p-2}(u-v) \in L^{p'}(\mathbb{R}^N)$ with $1/p + 1/p' = 1$, it follows from lemma 7.5 that there exists $z \in W^{2,p'}(\mathbb{R}^N)$ such that (7.6) is satisfied. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{2,p'}(\mathbb{R}^N)$, we can take a sequence $\{z_n\} \subset C_0^\infty(\mathbb{R}^N)$ such that $z_n \to z$ in $W^{2,p'}(\mathbb{R}^N)$ as $n \to \infty$. From (7.5) and (7.6) we have

$$0 = \int_{\mathbb{R}^N} (u-v)(-\Delta z_n + z_n) \, dx \to \int_{\mathbb{R}^N} (u-v)(-\Delta z + z) \, dx = \int_{\mathbb{R}^N} |u-v|^p \, dx,$$

which implies that $u - v \equiv 0$ and, hence, $v \in W^{2,p}(\mathbb{R}^N)$. □

Lemma 7.7. *Let $w \in L^p(\mathbb{R}^N)$ and $u \in W^{2,p}(\mathbb{R}^N)$ for some $p \in (1, +\infty)$ such that*

$$\int_{\mathbb{R}^N} \Delta u \Delta z \, dx = \int_{\mathbb{R}^N} wz \, dx \quad \textit{for all } z \in C_0^\infty(\mathbb{R}^N).$$

Then $u \in W^{4,p}(\mathbb{R}^N)$ and u is a solution of

$$\Delta^2 u = w.$$

Proof. Taking $v = \Delta u$ in lemma 7.6, we obtain our lemma. □

Lemma 7.8. *Let $\lim_{|x|\to\infty} V(x) = \alpha$ and*

$$\alpha - V(x) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{7.7}$$

Consider $\lambda < \alpha$, $h \in \bigcap_{r \geqslant 2} L^r(\mathbb{R}^N)$ and $g \in W^{2,q}(\mathbb{R}^N)$ with $q \geqslant 2$, such that

$$\Delta^2 g + (V - \lambda) g = h. \tag{7.8}$$

Then $g \in \bigcap_{r \geqslant 2} W^{4,r}(\mathbb{R}^N)$.

Proof. Since

$$\Delta^2 g = h + (\lambda - V) g \quad \text{and} \quad g \in W^{4,q}(\mathbb{R}^N), \quad h \in \bigcap_{r \geqslant 2} L^r(\mathbb{R}^N),$$

it follows from lemma 7.7 and a bootstrap argument that $g \in W^{4,r}(\mathbb{R}^N)$ for all $r \geqslant q$. In particular, $g \in L^\infty(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$.

Now we prove that $g \in L^r(\mathbb{R}^N)$ for $r \in [2, q)$. To this end, we rewrite equation (7.8) in the form

$$(\Delta^2 + (\alpha - \lambda)) g = h + (\alpha - V(x)) g. \tag{7.9}$$

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

Since $\lambda < \alpha$, we have

$$g(x) = \int_{\mathbb{R}^N} G_k^{(N)}(x-z)[h(z) + (\alpha - V(z))g(z)]\,\mathrm{d}z,$$

where

$$k^2 = \alpha - \lambda \quad \text{with } k > 0$$

and $G_k^{(N)}$ is the fundamental solution of $\Delta^2 + k^2$.

By applying the properties of $G_k^{(N)}(x)$, we have

$$\begin{aligned} |g(x)| &\leqslant \int_{\mathbb{R}^N} |G_k^{(N)}(x-z)||h(z) + (\alpha - V(z))g(z)|\,\mathrm{d}z \\ &\leqslant C\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)|h(z)|\,\mathrm{d}z \\ &\quad + C\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)|\alpha - V(z)||g(z)|\,\mathrm{d}z \\ &\leqslant C\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)|h(z)|\,\mathrm{d}z \\ &\quad + C\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)|\alpha - V(z)|\,\mathrm{d}z \cdot \|g(z)\|_{L^\infty}, \end{aligned}$$

where $g_{-\delta}^{(N)}(x)$ is the fundamental solution of $-\Delta + \delta$ with $0 < \delta < \sqrt{k}/\sqrt{2}$. Since $-\Delta + \delta$ is an isomorphism from $W^{2,r}(\mathbb{R}^N) \to L^r(R^N)$ for all $r > 1$ and $h \in \bigcap_{r\geqslant 2} L^r(\mathbb{R}^N)$, $\alpha - V(x) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)|h(z)|\,\mathrm{d}z \in \bigcap_{r\geqslant 2} L^r(\mathbb{R}^N),$$

$$\int_{\mathbb{R}^N} g_{-\delta}^{(N)}(x-z)|\alpha - V(z)|\,\mathrm{d}z \in \bigcap_{r\geqslant 2} L^r(\mathbb{R}^N).$$

Thus, $g(x) \in \bigcap_{r\geqslant 2} L^r(\mathbb{R}^N)$. From (7.8) and lemma 7.7 we deduce that

$$g \in \bigcap_{r\geqslant 2} W^{4,r}(\mathbb{R}^N).$$

□

As a particular case of the previous lemma we have the following corollary.

COROLLARY 7.9. *Let* $\lim_{|x|\to\infty} V(x) = \alpha$ *and*

$$\alpha - V(x) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

Consider $\lambda < \alpha$ *and* $g \in W^{4,q}(\mathbb{R}^N)$ *such that*

$$\Delta^2 g + (V - \lambda)g = 0.$$

Then $g \in \bigcap_{r\geqslant 2} W^{4,r}(\mathbb{R}^N)$. *In particular,*

$$\mathrm{Ker}(A_{q,\lambda}) = \mathrm{Ker}(A_{2,\lambda}) \quad \textit{for all } q \in [2,\infty).$$

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

Lemma 7.10. *Let* $\lim_{|x|\to\infty} V(x) = \alpha$ *and*

$$\alpha - V(x) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

Then for all $\lambda \in \Sigma_q \cap (-\infty, \alpha)$, $q \geqslant 2$, *we have*

$$L^q(\mathbb{R}^N) = \mathrm{Ker}(A_{q,\lambda}) \oplus \mathrm{Range}(A_{q,\lambda}).$$

Proof. Let $\lambda \in \Sigma_q \cap (-\infty, \alpha)$, and consider $h \in \mathrm{Ker}(A_{q,\lambda}) \cap \mathrm{Range}(A_{q,\lambda})$. Thus, on the one hand we have, by corollary 7.9, that $h \in \bigcap_{r\geqslant 2} W^{4,r}(\mathbb{R}^N)$ and on the other hand there exists $g \in W^{2,q}(\mathbb{R}^N)$ such that $h = A_{q,\lambda}(g)$, and from lemma 7.8 we have $g \in \bigcap_{r\geqslant 2} W^{4,r}(\mathbb{R}^N)$. Hence,

$$h \in \mathrm{Ker}(A_{2,\lambda}) \cap \mathrm{Range}(A_{2,\lambda}).$$

Since $A_{2,\lambda}$ is a self-adjoint operator, the above intersection is reduced to $\{0\}$ and, hence, $h = 0$. Moreover, since $A_{q,\lambda}$ is Fredholm with index 0 (see lemma 7.4), we deduce that

$$\mathrm{Ker}(A_{q,\lambda}) \oplus \mathrm{Range}(A_{q,\lambda}) = L^q(\mathbb{R}^N)$$

for $\lambda < \alpha$. □

8. Proof of theorem 1.3

Throughout this section, we consider a mapping f satisfying hypotheses (H_1)–(H_4).

Lemma 8.1. *Consider the mapping* F *defined by (3.2). Then the restriction of* F *to* $(-\infty, \alpha) \times W^{4,p}(\mathbb{R}^N)$ *is Fredholm with index* 0.

Proof. Let $u \in W^{4,p}(\mathbb{R}^N)$. We have, by theorem 3.2, that

$$D_2F_{(\lambda,u)} = \Delta^2 + \partial_2 f(\cdot, u) - \lambda.$$

Since $\lim_{|x|\to\infty} u(x) = 0$, it follows from lemma 3.1 that

$$\lim_{|x|\to\infty} \{\partial_2 f(x, u(x)) - \partial_2 f(x, 0)\} = 0.$$

Hence,

$$\lim_{|x|\to\infty} \partial_2 f(x, u(x)) = \lim_{|x|\to\infty} \partial_2 f(x, 0) = \alpha.$$

The conclusion now follows from lemma 7.4. □

Remark 8.2. Let $\lambda \in \Sigma_q \cap (-\infty, \alpha)$. From lemma 7.10 we know that [7, condition (6.17)] is satisfied and, from [7, theorem 6.18], that λ is an isolated point in Σ_q.

Referring to lemma 8.1, we consider the smooth mapping

$$A : (-\infty, \alpha) \to \phi_0(W^{4,p}, L^p), \qquad \lambda \to A_\lambda := D_2F_{(\lambda,0)},$$

where $\phi_0(W^{4,p}, L^p)$ has been defined in § 2 and

$$D_2F_{(\lambda,0)}(u) = \Delta^2 u + \partial_2 f(\cdot, 0)u - \lambda u.$$

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

Notation. We set $\Sigma = \{\lambda \in \mathbb{R} \mid D_2F_{(\lambda,0)}$ is not an isomorphism$\}$. We have verified in § 7 that this set is independent of the choice of p in the spaces $W^{4,p}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ (see remarks 7.1 and 7.2).

Let $\lambda \in \Sigma \cap (-\infty, \alpha)$. We have noted in remark 8.2 that λ is an isolated point in Σ. Therefore, there exists a closed interval $J \subset (-\infty, \alpha)$ such that $J \cap \Sigma = \{\lambda\}$ and $\lambda \in J^0$. The parity of the restriction of A to J will be denoted by $\sigma(A, \lambda)$.

From lemmas 7.4 and 7.10, we see that the hypotheses of [7, theorem 6.18] are satisfied, and thus we consider that, for every $\lambda \in \Sigma \cap (-\infty, \alpha)$,

$$\sigma(A, \lambda) = -1 \iff \dim \operatorname{Ker}(A_\lambda) \text{ is odd.}$$

From corollary 7.9, we have $\operatorname{Ker}(A_\lambda) = \operatorname{Ker}(S - \lambda)$, and then, for every $\lambda \in \Sigma \cap (-\infty, \alpha)$,

$$\sigma(A, \lambda) = -1 \iff \dim \operatorname{Ker}(S - \lambda) \text{ is odd.}$$

We are now able to prove theorem 1.3, which was stated in § 1.

Proof of theorem 1.3. To prove this theorem, we verify the hypotheses of theorem 2.3.

By (H_1), $F(\lambda, 0) \equiv 0$.

From theorem 3.2 we see that F is a C^2 mapping.

From theorem 5.4, the restriction of F to $(-\infty, \alpha) \times W^{4,p}$ is boundedly proper.

From lemma 7.3, we may now deduce that the restriction of F to $(-\infty, \alpha) \times W^{4,p}(\mathbb{R}^N)$ is Fredholm with index 0.

Moreover, since $\lambda_0 < \alpha$, from the previous considerations we deduce that

$$\sigma(A, \lambda_0) = -1.$$

Hence, all the hypotheses of theorem 2.3 are fulfilled. □

Acknowledgments

This research was supported in part by the National Natural Science Foundation of China, the Excellent Teachers Foundation of the Ministry of Education of China, a Xiao-Xiang Grant from the Hunan Normal University and by the University of Iowa International Travel Fund. This work was completed when Y.D. visited the Département de Mathématiques, EPFL, Switzerland and the Department of Mathematics, University of Iowa, USA. Y.D. thanks Professor C. A. Stuart at EPFL for his kind support and useful discussion during his visit from October to December 2001, when the work on this paper was first started. Y.D. also thanks the Department of Mathematics, University of Iowa for their kind support and the hospitality during his visit from January to May 2003, when the rest of the paper was completed. Both authors thank the anonymous referee for reading this paper carefully and suggesting many useful comments.

References

1 A. Ambrosetti and J. L Gámez. Branches of positive solutions for some semilinear Schrödinger equations. *Math. Z.* **224** (1997), 347–362.

https://doi.org/10.1017/S0308210500004698 Published online by Cambridge University Press

2 H. Brézis. *Analyse fonctionnelle* (Paris: Masson, 1983).

3 M. G. Crandall and P. H. Rabinowitz. Bifurcation from simple eigenvalues. *J. Funct. Analysis* **8** (1971), 321–340.

4 Y. B. Deng and Y. Li. Exponential decay of the solutions for nonlinear biharmonic equations in $\mathbb{R}^N$. (Submitted.)

5 A. L. Edelson and C. A. Stuart. The principal branch of solutions of a nonlinear elliptic eigenvalue problem on $\mathbb{R}^N$. *J. Diff. Eqns* **124** (1996), 279–301.

6 P. M. Fitzpatrick and J. Pejsachowicz. An extension of the Leray–Schauder degree for fully nonlinear elliptic problems. In *Nonlinear Functional Analysis and Its Applications, Part 1, Berkeley, CA, 1983* (ed. F. E. Browder), Proceedings of Symposia in Pure Mathematics, vol. 45, part I, pp. 425–438. (Providence, RI: American Mathematical Society, 1986).

7 P. M. Fitzpatrick and J. Pejsachowicz. Parity and generalized multiplicity. *Trans. Am. Math. Soc.* **326** (1991), 281–305.

8 P. M. Fitzpatrick, J. Pejsachowicz and P. J. Rabier. The degree of proper C^2 Fredholm mappings. I. *J. Reine Angew. Math.* **427** (1992), 1–33.

9 D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, 2nd edn, Grundlehren der mathematischen Wissenschaften, vol. 224 (Springer, 1983).

10 I. Gohberg, S. Goldberg and M. A. Kaashoek. *Classes of linear operators, vol. I*, Operator Theory: Advances and Applications, vol. 49 (Birkhäuser, 1990).

11 H. C. Grunau and G. Sweers. Sharp estimates for iterated Green functions. *Proc. R. Soc. Edinb.* A **132** (2002), 91–120.

12 E. Jannelli. The role played by space dimension in elliptic critical problems. *J. Diff. Eqns* **156** (1999), 407–426.

13 H. Jeanjean, M. Lucia and C. A. Stuart. Branches of solutions to semilinear elliptic equations on $\mathbb{R}^N$. *Math. Z.* **230** (1999), 79–105.

14 H. Jeanjean, M. Lucia and C. A. Stuart. The branch of positive solutions to a semilinear elliptic equation on $\mathbb{R}^N$. *Rend. Sem. Mat. Univ. Padova* **101** (1999), 229–262.

15 N. N. Lebedev. *Special functions and their applications* (Englewood Cliffs, NJ: Prentice Hall, 1965).

16 R. Leis. *Initial-boundary value problems in mathematical physics* (Wiley, 1986).

17 E. S. Noussair, C. A. Swanson and J.-F. Yang. Transcritical biharmonic equations in $\mathbb{R}^N$. *Funkcial. Ekvac.* **35** (1992), 533–543.

18 J. Pejsachowicz and P. J. Rabier. Degree theory for C^1 Fredholm mappings of index 0. *J. Analysis Math.* **76** (1998), 289–319.

19 L. A. Peletier and R. C. A. M. Van der Vorst. Existence and nonexistence of positive solutions of nonlinear elliptic systems and the biharmonic equation. *Diff. Integ. Eqns* **5** (1992), 747–767.

20 P. Pucci and J. Serrin. Critical exponents and critical dimensions for polyharmonic operators. *J. Math. Pures Appl.* **69** (1990), 55–83.

21 F. Quinn and A. Sard. Hausdorff conullity of critical images of Fredholm maps. *Am. J. Math.* **94** (1972), 1101–1110.

22 P. J. Rabier and C. A. Stuart. Fredholm properties of Schrödinger operators in $L^P(\mathbb{R}^N)$. *Diff. Integ. Eqns* **13** (2000), 1429–1444.

23 P. J. Rabier and C. A. Stuart. Global bifurcation for quasilinear elliptic equations on $\mathbb{R}^N$. *Math. Z.* **237** (2001), 85–124.

24 P. H. Rabinowitz. Some global results for nonlinear eigenvalue problems. *J. Funct. Analysis* **7** (1971), 487–513.

25 C. A. Stuart. An introduction to elliptic equations on $\mathbb{R}^N$. In *Nonlinear Functional Analysis and Applications to Differential Equations, Trieste, 1997*, pp. 237–285 (River Edge, NJ: World Science Publishing, 1998).

26 M. M. Vainberg. *Variational methods for the study of nonlinear operators* (San Fransisco, CA: Holden-Day, 1964).

(*Issued 11 August 2006*)